SURVEYING

Theory and Practice

BOOKS BY

JOHN CLAYTON TRACY, Ph.B., C.E.

Professor of Civil Engineering, Emeritus
Yale University

PUBLISHED BY

JOHN WILEY & SONS, INC.

SURVEYING—THEORY AND PRACTICE

4¾ by 7½. 1279 *pages. Illustrated. Flexible binding.*

PLANE SURVEYING

4 *by* 6¾. *xxvii* + 794 *pages. Illustrated with line cuts. Flexible binding.*

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FIELD NOTE BOOK

4½ *by* 7⅛. 72 *ruled pages. Paper binding.*

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NEW YORK

JOHN WILEY & SONS, INC.

LONDON: CHAPMAN & HALL, LIMITED

Fifth Printing, June, 1955

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

This is a book of *fundamentals* in the practice as well as in the theory of surveying. It is intended for use as a textbook and reference book.

The theory of surveying can be learned in college, but the art of surveying can be acquired only by long practical experience. In gaining this experience the surveyor needs for ready reference a book of fundamentals in the practice of surveying, and it is greatly to his advantage if that reference book is combined with the textbook which he used in learning the elements of surveying. That such a combination is practicable and desirable seems evident from the continued demand for the author's textbook and pocket manual, *Plane Surveying,* first published in 1907. Many engineers as well as students still use that book.

The development and expansion of the theory and practice of surveying during the past forty years have been so great that an attempted revision of the author's *Plane Surveying* has resulted in a much more comprehensive book, which, in many respects, is new. This new book contains all of the old book that the test of time has shown should be retained; to that has been added more than five hundred pages of new subject matter. Much of the text taken from the original book has been revised and rearranged.

Theory and theoretical procedures are explained in great detail throughout this book. In addition there is a wealth of practical information and thousands of practical suggestions. To the experienced engineer it may seem that there is too much detail and that "nothing has been left to the common sense of the surveyor." The author believes, however, that nothing should be taken for granted, that anything worth telling the inexperienced surveyor is worth putting in print, and that, after all, common sense in surveying is closely related to common knowledge of little details, which someone must tell the inexperienced surveyor for the first time. If put in print where they will be seen again and again, they may be gradually absorbed, whereas if they are left to be "picked up" in practice many of them will not be learned at all.

During the years spent in the preparation of this book the author has had the generous cooperation of many specialists in surveying,

and, by permission, he has included in the book many parts or condensations of papers written by such specialists. In addition, the book contains many extracts of reports and specifications prepared by committees from various engineering societies and organizations.

Many parts of the book are so comprehensive that they might well be considered as separate manuals, such parts, for example, as the chapter on surveys for construction and the chapter on surveys of land boundaries. The chapters on vertical curves and on horizontal curves contain most, if not all, of the fundamentals that the average surveyor needs to know in designing and laying out curves. Attention is especially directed to Part IV, which furnishes a working knowledge of requirements and procedures that have become more or less standardized for certain types of surveys. (See page 935.) The greater portion of Part IV, including the chapters on horizontal control surveys and on aerial photographic surveys, is primarily for reference by practicing engineers and surveyors.

Since the surveyor is engaged continuously in an effort to avoid mistakes and reduce errors, several chapters are devoted to the subject of errors. Sources of errors, methods of eliminating errors, and procedures for attaining desired degrees of precision are treated, first in a general chapter on errors and then in chapters on errors in linear measurement, errors in leveling, and errors in transit work. Procedures for attaining high degrees of precision are explained in chapters on precise linear measurement, precise differential leveling, and precise measurement of angles.

A more definite idea of the general plan and scope of the book may be gained from the table of contents. Attention is directed, however, to the following features.

1. Great care has been taken to make the book convenient for ready reference. Chapters closely related to each other, such as chapters on leveling, follow each other in progressive order. Cross references are numerous and are generally to articles instead of to pages. Articles may be found by the use of the running heads at the tops of pages.

2. Each branch of field work in Part I is treated under three heads, viz.: (1) the use of the instrument; (2) the general method of procedure; (3) the practical details of field work.

3. Field work and office work are treated in separate parts of the book, but important relations between the two are emphasized throughout.

4. General statements of important principles and methods are almost invariably supplemented by special applications or practical illustrations, given, as a rule, in smaller type.

5. When there are two or more methods of doing the same thing, they are given one after the other and then compared.

6. Practical information and suggestions and all portions of the text that will be consulted more or less infrequently are given in smaller type in order not to obscure the more important fundamentals.

7. In the chapters on field notes, an attempt has been made to give a systematic treatment of the fundamentals of a very important part of field work. Not only are numerous forms and examples given throughout the book, but also different methods of keeping notes are explained and compared.

8. Methods of checking field work and office work are emphasized, and, wherever possible, they are included in methods of procedure so that work may be checked as it progresses rather than after it is finished.

9. In Part II general methods of office work are explained with emphasis on systematic procedure. Many practical suggestions are given for attaining both speed and accuracy in computation and in drafting.

10. No space has been wasted in illustrating surveying instruments, which, it may be assumed, the surveyor will see for himself. In Part III, however, a complete and critical study is made of the construction and the adjustment of instruments. Part III also includes directions and suggestions for standardizing tapes and a chapter on the care of instruments.

Acknowledgments

The author has found it impracticable to acknowledge adequately all the contributions which others have made directly or indirectly to this book. However, he especially desires to make the following grateful acknowledgments.

Professor Charles S. Farnham of the School of Engineering, Yale University, has rendered invaluable assistance to the author. Throughout the years spent in the preparation of this book he has given freely of his time. He has helped in gathering data and material from a great variety of sources, he has prepared many of the illustrative problems, he has assisted in the preparation of portions of the manuscript, and he has contributed knowledge gained from a long experience in teaching surveying. Without his assistance, this book would have been less complete and less useful.

Professor Walter L. Willig of the College of the City of New

York has reviewed the entire manuscript with great care and has made many important suggestions for changes and additions. He also has written the articles "Subdivisions Involving Curved Boundary Lines" and "Use of the Stereoscope."

Professor Charles A. Wheeler of Storrs, Connecticut, has read portions of the manuscript and has suggested improvements and additions.

Mr. Lester C. Higbee of Troy, New York, has contributed helpful information and has been of assistance in other ways.

Throughout this book credit is given for drawings and photographs used in cuts and for extracts from various professional papers, specifications, manuals, magazines, books, and other publications. However, the author desires to make further grateful acknowledgment to the instrument manufacturers, to the engineering societies and organizations, and to various authors whose drawings, photographs, and extracts have added so greatly to the usefulness of the book.

JOHN CLAYTON TRACY

NEW HAVEN, CONNECTICUT
September, 1946

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INTRODUCTION

To THE STUDENT:

The theory of plane surveying is surprisingly simple, but the application of the theory in practice is more difficult. There is not much to understand, but there is a great deal to remember. Success depends more upon training and judgment than upon a knowledge of theory. You can hardly expect in a college course to acquire the skill which comes from long-continued repetition, or the judgment which comes from wide experience. On the other hand, you should realize that the habits formed when first learning the use of surveying instruments and the methods of field work acquired in college are likely to cling to you long after graduation. It is the object of this book not only to explain the theory of surveying but also to help you to acquire those habits and methods of work that are in accord with the best practice. This does not mean that you can read over the practical suggestions and directions for field work once for all and then remember them ever after. On the contrary, the book should be consulted frequently. Study it first for general principles and for a few of the most important directions, and then, as you have opportunity, read and reread portions of the text in order to pick up the fine points of good field work.

Learn different methods of doing the same thing and study the advantages and disadvantages of each method. Study also the different sources of error, for field work is a continual fight against errors. In practice the surveyor aims to save time and expense. Working thus under pressure, it is a great advantage to be able in any given case to choose the most suitable method, and to know what sources of error may be ignored and what must be eliminated. Learn to check work while it is in progress, for a considerable portion of a surveyor's work is in making sure that the remaining portion is correct.

Finally, do not work too much by rule, and do not form the "handbook habit." The more you are able to work without help from this book the better the book will serve its purpose. Throughout the course maintain an attitude of questioning. Do not accept what an instructor says or what the book tells you as gospel truth. Constantly ask yourself *"Why"* and do not rest satisfied until you know *why*. This is the spirit which will enable you to get the most out of the course.

J. C. T.

PART I: FIELD WORK

CHAPTER 1

FUNDAMENTAL DEFINITIONS AND CONCEPTS

1. Surveying. Making such observations and measurements as will determine the relative positions of points on the earth's surface. Measurements may be either linear or angular.

2. The Earth's Surface is understood in geodesy to be the surface of a globe as determined by the mean sea level. This globe is not a true sphere but is slightly flattened at the poles.

2(*a*). THE EARTH'S AXES. The shorter axis, from pole to pole, is 41,709,784 ft., and the longer axis, in the plane of the equator, is 41,851,664 ft., a difference of 141,880 ft. or 26.87 miles. A model of the earth's surface with its major axis 25 ft. long would be flattened only about ½ in. at each pole.

2(*b*). THE SHAPE OF THE EARTH. "As a result of many precise measurements, the shape of the earth has been found to be such that with considerable exactness any section parallel to the equator is a circle, and any section through the poles is an ellipse; the figure is such as may be generated by revolving an ellipse about its minor axis and is called an *oblate spheroid.* To be still more exact, the equatorial section is not exactly circular but very slightly elliptical so that a section in any direction through the center would be an ellipse; such a figure is called an *ellipsoid.* Still further exactness indicates that the southern hemisphere is a trifle larger than the northern, and that all polar sections are therefore slightly oval, leading to the name *ovaloid.* As a matter of absolute precision, no geometrical solid represents the shape of the earth, and this has been recognized by applying the special name *geoid.*" (From *Geodetic Surveying* by Edward L. Ingram, C.E.)

2(*c*). MEAN SEA LEVEL has been determined by the U.S. Coast and Geodetic Survey at tidal stations spaced at intervals along the Atlantic, Gulf, and Pacific coasts. Self-registering gauges record continuously heights of the tide. The mean sea level is usually obtained by averaging the hourly heights of the tide over a considerable period of time. When the observations cover a number of years, it is expected that the disturbing influences caused by meteorological conditions will be largely averaged out. Note that the mean sea level as thus determined is *not* the mean between high and low tides as is sometimes stated.

"Mean sea level is really not a true level surface. We have found that mean sea level at Boston is one foot higher than mean sea level on the Gulf coast. We have also found that mean sea level on the Pacific coast is about two feet higher than the average for the Atlantic and Gulf coasts. The deviation of mean sea level from a true level surface raised an interesting and puzzling problem for the Coast and Geodetic engineers to solve. We have solved the problem, and we hope our solution is a sound one, by holding mean sea level at each of our fundamental tidal stations as zero and fitting all lines of levels to those zero elevations. This means that the level net has been slightly warped in its adjustment, but the effect on any short section of leveling, say 10 miles or 50 miles, is negligible." (From a paper by William Bowie in the *Annual Report of the Connecticut Society of Civil Engineers* for 1934.)

3. Level Surface. If the waters of the ocean could be freed from all disturbing influences of tides, wind, weather, and varying barometric pressures, they would come to rest in a true *level surface* (mean sea level); any surface parallel to this would also be truly level. A level surface is a curved not a plane surface. In ordinary surveying, however, a level surface at any point is considered as a plane surface perpendicular to a plumb line at that point.

3(*a*). PLUMB LINES. It is sometimes stated that all plumb lines converge at the center of the earth, but this is not stricly true. There are many deviations caused by the irregular distribution of the earth's mass; moreover the perpendiculars to the surface of an oblate spheroid do not converge at any point. These deviations of the plumb line are not taken into account in plane surveying.

4. Horizontal and Vertical Lines and Planes. A *horizontal plane* through any point on the earth's surface is a plane tangent to that surface at that point, and any *horizontal line* through that point must lie in that plane. A *vertical line* through the point is perpendicular to the horizontal plane, and any *vertical plane* through that point must pass through that vertical line. At any point the vertical line coincides with the plumb line, and the horizontal plane is perpendicular to it. Obviously, through any point on the earth's surface there can be only one horizontal plane and one vertical line, but there can be an infinite number of vertical planes and of horizontal lines.

5. Plane Surveying. In *plane surveying* that portion of the earth's surface covered by a survey is regarded as a plane surface, not curved as it really is. The relative positions of points horizontally are determined as if all the points were in the same horizontal plane, and maps are therefore *horizontal projections* of points and lines of the survey. This is equivalent to making the following assumptions: (1) That at every point of the survey the plumb line is perpendicular to the horizontal plane of the survey, and that consequently the plumb line at one point is parallel to the plumb line at any other point. (2) That the most direct line on the surface of the earth between two points is a straight line, not curved, and that the angle between two such lines that intersect is a plane angle not a spherical angle. (3) That north-south lines (meridians) through any two points are parallel. Obviously none of these assumptions is in accord with the fact.

5(*a*). ERRORS DUE TO ASSUMPTIONS. For ordinary surveys, extending over small areas, the methods of plane surveying based on the assumptions just stated involve no appreciable errors. Thus, for example, in a survey 10 miles square, the longest straight line (diagonal) is a little over 14 miles, and the

difference in length between this straight line and the corresponding arc on the earth's surface is only about 0.46 ft. For 20 miles the difference would be about 1.35 ft., and for 30 miles, it would be about 4.55 ft.

For convergence of north-south lines (meridians) see Art. 813.

6. Elevations. *Relative elevation* is the difference in elevation between any two points when one is compared with the other; *absolute elevation* of a point is its elevation with respect to some particular point or surface of reference known as the *datum.* When absolute elevations are with reference to the surface of the earth, the *datum surface* is the mean sea level (Art. 2(*c*)). The common term for a datum surface is *datum plane,* but this is a misnomer since such a surface is not a plane but a *level* surface (curved). Although it is natural to think of a level surface as flat, its curvature enters into the determination of elevations even in plane surveying, unless the extent of the survey is quite limited.

6(*a*). CURVATURE OF A DATUM SURFACE. In the usual method of determining elevations by means of an engineer's spirit level, the datum automatically becomes a curved (level) surface, because no matter at what point the instrument may be set up, the level line of sight through its telescope will be normal to the line of action of gravity (and hence normal to a plumb line) at that point. If this were not so, it would be difficult to use the level except within the limits of a small area. Assume, for example, that a horizontal line of sight at a particular point is 4.0 ft. above mean sea level, and that it is continued indefinitely as a straight line; at a point only a mile distant it would be about 4.6 ft. above mean sea level. This discrepancy of 0.6 ft., which includes a correction of about 0.1 for refraction, would be much too great to be within the limits of error commonly allowed in plane surveying.

6(*b*). NOTE. *The curvature of the earth's surface* is equal to the square of the distance divided by the mean diameter of the earth. If the distance D is in miles, and the curvature C, in feet, $C = 0.667D^2$. Hence for 2 miles the curvature would be: $C = 0.667 \times 2^2 = 2.668$ ft., or four times as great as for 1 mile.

7. Horizontal and Vertical Measurements. There are four kinds of fundamental measurements used in plane surveying: (1) horizontal distances; (2) horizontal angles; (3) vertical distances; (4) vertical angles. In some cases, linear measurements are made along an inclined surface, but such measurements are afterwards reduced to a horizontal or to a vertical basis. Likewise all angles are reduced to a horizontal or vertical basis. The *horizontal distance* between two points is the distance between the projections of the points on a horizontal plane; likewise the *horizontal angle* between two intersecting lines is the angle between the projections of these lines on a horizontal plane. A *vertical distance* in surveying is the distance between two points in a vertical line; a *vertical angle* is an angle between two intersecting lines in a vertical plane. In surveying, however, unless otherwise stated, a vertical angle to any point is the vertical angle between the line of sight to that point and a *horizontal* line or a horizontal plane. It may be an angle of

elevation above the plane or an angle of depression below. It is important that the student should gain at the outset true conceptions of these fundamental measurements as illustrated in Fig. 7(*a*).

7(*a*). ILLUSTRATION. Let *A* and *B* be points on a sloping surface. The distance *AB* as understood in surveying is not the air-line distance *AB* but its *horizontal projection AC*. Similarly, if *H, D,* and *G* are points in the same horizontal plane, and the points *H* and *G* are respectively the horizontal projections of any two points *E* and *F*, the angle *EDF*, as understood in surveying, is not the actual angle between the two lines *DE* and *DF* but the *horizontal* projection *HDG* of that angle. If the telescope of the surveying instrument is at *D*, and it is pointed first at *E* and then at *F* or vice versa, the instrument does not measure the actual angle between these two lines of sight but the horizontal projection of that angle.

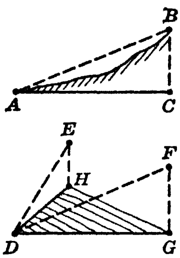


FIG. 7(*a*).

BC, EH, and *FG* are vertical distances and *CAB, HDE,* and *GDF* are *vertical angles*. If the telescope of a surveying instrument is at *D* and is pointed at *E* or at *F*, the angle to either line of sight as measured by the instrument is the angle it makes with a horizontal plane, i.e., a vertical angle.

If *AC* is at the mean sea level, *BC* is the *actual elevation* of *B* above the surface of the earth; it is also the *relative elevation* of *B* with respect to *C* or the *difference in elevation* of the two points (neglecting curvature and refraction).

8. Units of Measurements. (*a*). LINEAR UNITS. In the United States the most common unit of length in surveying is the *foot*. Decimal parts of a foot are generally used instead of inches and fractions of an inch. For example, in surveying 50 ft. 9 in. is usually given as 50.75 ft. It is seldom necessary to give measurements closer than the nearest thousandth of a foot; more often the nearest hundredth or even tenth of a foot will be sufficiently accurate. In laying off measurements for purposes of construction and in some other kinds of work, feet and inches are often used, and it may then become necessary to change decimals of a foot to inches or vice versa. A useful table for this purpose is given in the back of this book.

8(*b*). CHANGING DECIMALS OF A FOOT TO INCHES. For the student who is accustomed to fractions of an inch but not to decimals of a foot it is helpful to remember that 0.01 ft. is approximately ⅛ in. A difference, for example, between two measurements of 0.01 ft. seems almost inappreciable to the average student until he realizes that it is ⅛ in. Having fixed this relation in mind he can readily convert decimals of a foot to inches and vice versa.

Illustration. 0.04 ft. = 4 × ⅛ in. = ½ in. 0.07 ft. = 7 × ⅛ in. = ⅞ in. 0.1 ft. = 10 × ⅛ in. = 1¼ in. These values are only approximately correct, but near enough for most purposes, as seen from the following table.

Inches	*Approximate*	*To five places*
⅛	0.01 ft.	0.01042
½	0.04 "	0.04167
1	0.08 "	0.08333
1¼	0.10 "	0.10417

8(*c*). ANGULAR UNITS. Angles are measured in *degrees, minutes,* and *seconds.* 60 sec. = 1 min.; 60 min. = 1 deg. One degree is 1/360 part of the circumference of a circle. Usually angles are measured to the nearest minute or half minute.

8(*d*). UNITS OF AREA AND VOLUME. The units of area most used in surveying are the *square foot,* the *square yard,* and the *acre;* the units of volume are the *cubic foot* and the *cubic yard.*

8(*e*). STANDARD UNITS. In the United States, the yard, as legalized in the law of July 28, 1866, is defined by the relation:

$$\frac{1 \text{ yard}}{1 \text{ meter}} = \frac{3600}{3937}$$

The value of the meter is thus legalized at 39.37 in., whereas in Great Britain and France its exact value is given as 39.37079 in. Whether the inch, the foot, or the yard is the standard unit of length in the English system is a debatable subject. Fundamentally by definition the yard is basic; in manufacturing the inch is basic; in engineering the foot is basic.

8(*f*). LEGAL STANDARDIZATION. The National Bureau of Standards in Washington maintains standard bars and other standards by means of which measuring instruments and apparatus, including steel tapes, can be tested and standardized. The meter bar has been implicitly legalized by Congress in its appropriations for the International Bureau of Weights and Measures.

Although the British and American yards differ by a little over two parts in a million, this difference is impossible to detect in ordinary steel tapes. In comparing even the most precise geodetic tapes, one cannot tell from the lengths of the tapes whether they are of American or British origin, and the error in measurement of such tapes with the best geodetic apparatus is about the same order of magnitude as the difference between the two systems.

Although a standardization of a tape at the National Bureau of Standards would have no legal standing in other countries, from a scientific metrological standpoint the standardization would be valid elsewhere in that it would introduce no errors of measurement. The temperature of standardization would have to be noted carefully. In this country it is 68° F. In some countries it may still be 62° F., although the tendency is to adopt the standard temperature of 68° F. in accordance with international agreement for gauge standardization.

8(*g*). UNITS OF THE GUNTER'S CHAIN, once quite generally used in land surveying, is composed of 100 links and is 66 ft. long. The corresponding units are 1 *chain* = 66 ft., 1 *link* = 0.66 ft. or 7.92 in. 1 *rod* = 1/4 *chain* = 16½ ft. = 25 links. To reduce links to feet, subtract as many units as there are hundreds of links and take two-thirds of the remainder. *Example:* 50 links = (50 − 0.50) × 2/3 = 33 ft.

The length of the chain was originally chosen as 66 ft. because of the convenient relation of this length to the acre and to the mile, namely: 1 acre = 10 square chains = 160 sq. rods = 43,560 sq. ft.; 1 mile = 80 chains = 320 rods = 5280 ft. Not only are the units of the Gunter's chain used in certain types of land surveying, particularly when old deeds are involved, but they are also the legal units for U.S. Public Land Surveys as carried out by the Surveying Service of the General Land Office.

8(*h*). METRIC UNITS. In many countries the *meter* is the standard unit of length, and the metric system is in general use for scientific measurements throughout the world. The meter is also the unit of length used by the U.S. Coast and Geodetic Survey. If the value of the meter is taken as that

legalized in the United States, the following relations may be used for conversion.

1 meter	= 39.37 in. = 3.280833+ ft.
1 in.	= 0.0254 meter
1 ft.	= 0.3048+ meter
1 yd.	= 0.9144 meter
1 mile	= 1609.35+ meters

See Table XIV in the back of this book for conversion tables.

The angular units of measurement that correspond to the metric system of linear units are as follows: One circumference is divided into 400 *grades;* the grade is divided into 100 *centesimal min.;* the centesimal minute is divided into 100 centesimal sec. Conversional equivalents in English angular units are: 1 grade = 0.9°; 1 centesimal min. = 0.54 min.

8(*i*). THE VARA. The unit of length in Spanish measures is the *vara,* several values of which seem to be in use, namely: California, 1 vara = 33 in.; Texas, 1 vara = 33⅓ in.; Mexico, 1 vara = 32.9931 in.

9. Surveying—Character of the Work. The work of the surveyor may be divided into three parts: (1) *field work;* (2) *office work;* (3) *care and adjustment of instruments.*

9(*a*). FIELD WORK. The primary operations in field work are: (1) *measuring distances;* (2) *measuring angles;* and (3) *recording field observations* (*keeping field notes*).

9(*b*). OFFICE WORK. The indoor work of the surveyor consists chiefly in (1) *drafting* and (2) *computing.* The drafting consists mainly in plotting measurements to scale and, therefore, to a large extent, in reversing on paper the operations in the field. The computing is of two kinds: (1) that done for purposes of plotting and (2) that done in order to obtain certain data and information concerning the tract surveyed, such, for example, as its area.

9(*c*). THE CARE AND ADJUSTMENT OF INSTRUMENTS. It is necessary that certain parts of such surveying instruments as the level and the transit should be adjustable in order that they may be kept in condition for making accurate measurements. Keeping instruments in good adjustment may logically be regarded as a regular part of field work. It is doubtful, however, if one who is just learning to use instruments should undertake to adjust them. Methods of adjustment are best learned after one has had enough practice in the field to make him familiar with the use of instruments. For this reason, the construction, care, and adjustment of instruments are not included as a part of "Field Work" (Part I of this book), but instead are treated as a separate subject in Part III.

CHAPTER 2

FUNDAMENTALS OF FIELD WORK

In this chapter, the purpose is to give a general idea of the character of field work and to explain the primary operations involved.

10. Methods of Locating Points. The main purpose in all field work is to determine the relative positions of known points; or, conversely, to establish points whose relative positions are given. This work may be done by horizontal measurements, by vertical measurements, or by a combination of horizontal and vertical measurements.

10(*a*). Locating a point by horizontal measurements. In order to locate a point its position with respect to at least two fixed points of reference must be known. Let *A* and *B* be the two points of reference. The five most common methods are:

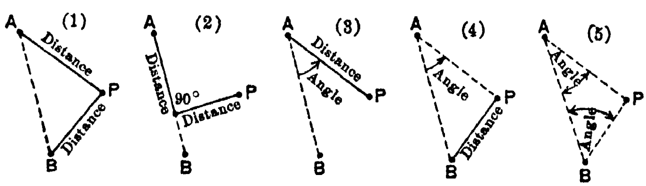


Fig. 10(*a*),1.

1. By the two distances *AP* and *BP*.
2. By the distances *AC* and *CP* when the angle at *C* is 90°. (*CB* may be used in place of *AC*.)
3. By the angle at *A* and the distance *AP*. (Or the angle at *B* and the distance *BP*.)
4. By the angle at *A* and the distance *BP*. (Or the angle at *B* and the distance *AP*.)
5. By two angles, one at *A*, the other at *B*.

Notice that in the first two methods no angles are measured, and that in the fifth method no distances are measured, the distance from *A* to *B* being known.

Compare the fourth method with the third, since in each method one angle and one distance are measured.

Two other methods used for locating a point are:

6. By the intersection of two straight lines, *AD* and *BC*, between four known points *A*, *B*, *C*, and *D*.

7. By two angles taken at the point to be located, and formed by lines to three known points *E*, *F*, and *G*.

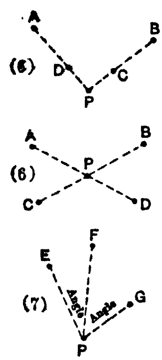


FIG. 10(*a*),2.

In these two methods no distances are measured, although of course the distances between the given points of reference are known. The last method is the only one of all seven methods in which an angle is measured *at the point to be located.*

10(*b*). LOCATING A POINT BY VERTICAL MEASUREMENTS. Any one of the seven methods just given may be used for locating a point vertically, provided it is possible to make corresponding measurements in a vertical plane. The most common problem in vertical measurements, however, is to determine the elevation of a point with respect to a point or plane of reference, the elevation of which is known. The vertical measurements involved in this problem will be explained in the chapters on leveling.

11. Control. (*a*). HORIZONTAL CONTROL. The line *AB* in any one of the first five methods of Fig. 10(*a*),1 is between two fixed points of reference and hence may be considered a *line of reference.* In any survey there are, in general, more than two points of reference; as many are established as are necessary for the proper location of all required points. These points of reference are often called *stations*, and each station is located with respect to at least two other stations by one of the methods of locating a point given in Art. 10. These stations and the lines between them form a system of horizontal control, so called because by means of this system of reference points and lines the horizontal locations of all required points are controlled.

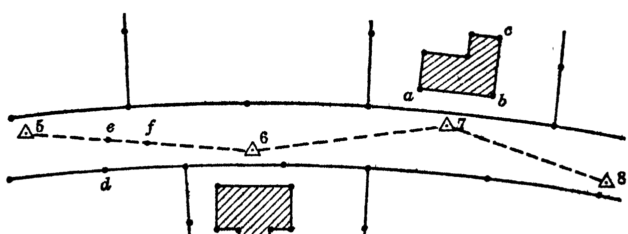


FIG. 11(*b*).

11(*b*). ILLUSTRATION. In Fig. 11(*b*) is represented a portion of the survey of a highway. Stations 5, 6, 7, and 8 are four of a large number of stations established along the highway at advantageous points, each of which is lo-

cated with respect to at least two other stations. For example, 7 may be located with respect to 5 and 6 by one of the methods of Art. 10, let us say, by the angle 5–6–7 and the distance 6–7 (third method).

The stations 5, 6, 7, 8 and the lines between them form a system of horizontal control for this particular portion of the survey. Details such as the boundary lines of the highway, dividing property lines, and houses are located with respect to this system of horizontal control. This is done by selecting enough points of each detail to determine its position when each of those points is located, directly or indirectly, with respect to the system of control. The dots in Fig. 11(*b*) suggest how these points may be selected. Each point must be located either directly with respect to two points of the system of control or indirectly with respect to two other points whose positions have been or will be determined from two points of the system. For example, points *a* and *b* could conveniently be located with respect to stations 6 and 7 or 7 and 8; point *c* could then be located with respect to *a* and *b*. It is frequently convenient to locate a point such as *d* with respect to a station and a point on a line between stations such as *e*, or even with respect to *two* points on a line between stations such as *e* and *f*. In either case, *d* is indirectly located with respect to two stations, since *e* and *f* must be located with respect to 5 and 6. The method selected for the location of any particular point of the survey will depend upon the judgment of the surveyor. The main purpose of the illustration is to show how horizontal control is fundamentally a system of reference points and lines, and that any other point may be located, directly or indirectly, with respect to two points of this system by one of the methods of Art. 10.

11(*c*). THE SYSTEM OF CONTROL shown in Fig. 11(*b*) is one of the simplest. Frequently a system of horizontal control is in the form of a polygon, or a large polygon subdivided into a number of smaller polygons, or a network of triangles. In any case the system is the skeleton around which the survey gradually is completed; and when a large area is covered, the skeleton may be quite complex. But even in such a case, any particular point of the survey may still be located by one of the simple methods given in Art. 10(*a*). The various systems of horizontal control will be treated more fully in subsequent chapters.

11(*d*). VERTICAL CONTROL. If elevations are required over any considerable area, points of reference called *bench marks* are usually established. The elevations of these bench marks are determined with respect to some chosen datum, and the bench marks then become a system of vertical control. The elevations of other points with respect to the same datum may then be determined by leveling from the most convenient bench mark as a point of reference.

12. Methods of Measuring Distances and Angles. (*a*). HORIZONTAL DISTANCES are usually measured either with a steel tape or with a cloth tape, depending upon the degree of precision desired. The Gunter's chain is still used to some extent for special types of work. Horizontal distances may also be determined with considerable precision by observing how much of a graduated rod held in a vertical position appears to be intercepted between two parallel horizontal cross hairs in the telescope of a surveying in-

strument (stadia method). When only approximate results are desired, distances may be paced or they may be measured by means of a speedometer, odometer, or similar device based on the revolutions of a wheel.

12(*b*). VERTICAL DISTANCES or differences in elevation may sometimes be measured directly with a tape, but usually they are determined by the use of a surveyor's level and leveling rod (leveling). Differences in elevation may also be determined within certain limits by means of a barometer, usually an aneroid barometer.

12(*c*). A DISTANCE, EITHER HORIZONTAL OR VERTICAL, MAY BE DETERMINED BY MEASURING AN ANGLE AND AT LEAST ONE SIDE OF A TRIANGLE of which another side is the required distance (geometric or trigonometric methods).

12(*d*). HORIZONTAL ANGLES are usually measured by means of a transit or other surveying instrument, so constructed that when a line of sight is directed first to one point and then to another, the horizontal projection of the angle between the two lines of sight may be read from the arc on a graduated horizontal circle. A horizontal angle may also be determined by means of a plane table and alidade. The two corresponding lines of sight are projected upon and drawn on the horizontal surface of the plane table, and the angle between these two projections may then be measured by any of the methods used in drafting. A third method of measuring a horizontal angle is by the use of a surveyor's compass. The direction (magnetic bearing) of each of two intersecting lines of sight is given by the magnetic needle, and from these two bearings the angle is easily calculated.

12(*e*). VERTICAL ANGLES are usually measured by means of the vertical arc or vertical circle on a transit. They may be determined similarly but with less accuracy by means of the vertical arc on a plane-table alidade.

12(*f*). NOTE. Angles, both horizontal and vertical, may be measured by means of a sextant, an instrument used in navigation and also in special types of surveying (Art. 733).

12(*g*). AN ANGLE, EITHER HORIZONTAL OR VERTICAL, MAY BE DETERMINED WHOLLY FROM LINEAR MEASUREMENTS by measuring the lengths of at least two sides of a right triangle, one angle of which is the required angle. The angle may then be computed by a trigonometric method.

13. Taking Measurements and Laying Off Measurements. In one method of procedure measurements are made of something that already exists; in the other method measurements are laid off for something that does not exist. For example, (1) a surveyor

may obtain measurements which will determine the position of an existing house or (2) he may lay off measurements that will establish the position of a house to be built. One process is the reverse of the other.

13(*a*). THE IMMEDIATE PURPOSE OF A SURVEY OF SOMETHING THAT ALREADY EXISTS is, generally speaking, to secure data for some kind of plot or map. The map may be one that shows only horizontal projections of points and lines, such as a map of existing boundary lines; or it may be one in which the main purpose is to show elevations, such as the profile of the center line of an existing highway; or it may be one which shows both horizontal projections and elevations, such as a topographic map with its surface contours. One of the first steps in any civil engineering project is likely to be the making of a survey and a map.

13(*b*). ONE PURPOSE OF LAYING OFF MEASUREMENTS FOR SOMETHING THAT DOES NOT EXIST may be to establish permanent points of reference, such as monuments to preserve boundary corners, or bench marks to preserve certain points of elevation. *Another purpose* may be to establish more or less temporary points for purposes of construction, such as points that may be used to determine lines or grades or both in the construction of a highway. "Staking out" work in this way is an important part of the surveying in a great variety of engineering projects.

14. General Character of Field Work. The character of the field work depends upon the purpose of the survey. The variety of work in surveying is indicated by the following branches: *land surveying* (rural and urban); *highway surveying; railroad surveying; mine surveying; hydrographic surveying* (of bodies of water).

The variety of methods that may be used in any branch of surveying is indicated by such terms as *linear measurement, angular measurement, direct measurement, indirect measurement, triangulation, differential leveling, profile leveling, precise leveling, compass surveying, transit surveying, stadia surveying, plane-table surveying, topographic surveying, photographic surveying* (on the ground), *aerial photographic surveying,* and *astronomical observations.*

14(*a*). SOME OF THE THINGS WHICH A SURVEYOR HAS TO DO IN THE FIELD WORK are to:

1. *Establish stations and bench marks* as points of reference in a system of horizontal or vertical control.
2. *Locate details* of the survey with respect to the stations and lines between stations, details such as boundary lines, streets, buildings, streams, bridges, and other important features of the area surveyed.
3. *Stake out* new boundaries; re-establish old boundaries; subdivide parcels of land.
4. *Give lines and elevations* for a great variety of construction work, such as that for buildings, foundations, masonry structures, highways, railways, sewers, and water supply systems.
5. *Establish* parallel lines, perpendiculars; measure to inaccessible points; find a distance between points when some obstacle prevents a direct measurement; run a straight line through an obstacle; and carry on a great variety of similar field work that is based on geometric or trigonometric principles.
6. *Determine elevations* of existing points or establish points at given elevations (leveling).
7. *Survey contours* of land areas, a kind of field work that involves both horizontal and vertical measurements (topographic surveying).
8. *Determine volumes* of earth, rock, masonry, concrete, and other material in connection with grading, excavation, and various other construction work.

9. *Make observations on the sun or a star* to determine a true north and south line (meridian); also occasionally to determine latitude, or longitude, or time.

15. Important Questions in Surveying which arise in connection with the field work are: (1) degree of accuracy required; (2) methods to be used; (3) instruments to be used; (4) errors, their sources, relative importance, and prevention; (5) speed; (6) systematizing the work; (7) justifiable cost of the survey. These questions are closely related to each other, and depend to a large extent on the purpose for which a survey is made.

16. Consideration of the Purpose of a Survey. The following illustrations indicate how the purpose of a survey helps to determine the degree of accuracy required and to decide other questions suggested in the preceding article.

Illustrations. A particular survey may be for the purpose of making a rough map, plotted to small scale, say 1 in. = 200 ft. The points will not be plotted nearer than 1/100 in.; hence it is useless to take measurements nearer than to the nearest 2 ft.

Another survey may be for a map also to small scale, but precise measurements are desired and will be printed on the map. Accurate measurements must be made accordingly.

A survey is made of a farm worth $20 an acre. The value of the land does not warrant great expense, probably not more than 10 to 25 cents per acre, but very accurate work costs money. Hence very accurate work cannot be done, and the method used would be a compass survey or some other quick method.

A survey is made of a city lot worth $1000 a front foot. Every fraction of a foot is valuable; hence time and expense are not spared in getting an accurate survey.

A surveyor stakes out an earth road, setting the stakes as nearly on line and giving the corresponding grades as nearly as he can without wasting time in splitting hairs. If, however, he is giving curb line and grade for a permanent pavement, the work must be done with greater precision.

17. Errors in Surveying. It is essential that the student, early in his course, should learn to distinguish between the different kinds of errors in surveying, to recognize their relative importance, and to use the customary methods of prevention, correction, and distribution. This subject is at once so large and so important that it will be treated in a separate chapter (Chapter 4).

18. Accuracy in Surveying. As indicated in Art. 16, the purpose of a survey and the value of the land largely determine the accuracy needed in the field work. In addition, the surveyor may be handicapped by rough country, too short a time for work, too small a party, poor instruments, bad weather, and many other unfavorable conditions.

18(*a*). LIMITS OF ERROR, therefore, cannot be given once for all. Before beginning a survey, a surveyor should, if possible, establish limits of error for himself, i.e., how close to read angles, how accurately to measure distances. In this he is guided by his own

experience and that of others. He should know what is considered good work under given conditions. Rules for limits of error, however, cannot be applied blindly. The best surveyor is not he who is extremely accurate in all his work, but he who does it just accurately enough for the purpose without waste of time or money.

18(*b*). CONSISTENT ACCURACY. In fixing limits of error for angles and distances, have the two consistent. There is no use in measuring angles accurately and corresponding distances roughly, or vice versa. Measure one as carefully or as roughly as the other.

Not all the work of a survey is done with the same precision. Transit lines, since they form the basis of all measurements, are usually run with care, for an error in angle or distance affects the whole survey, but an error in the angle or in the distance to a corner of a building, for example, affects only the location of that corner and not the whole survey. If land is valuable, boundaries are usually located with more care than buildings and other details for obvious reasons. *Make the precision of each step in the field work correspond to the importance of that step.*

19. Speed in surveying depends to a large extent on skill in handling instruments. Equally important are:

1. Planning the survey and laying out the work. A little foresight in the choice of methods, choice of instruments, choice of stations, and the use of common sense in deciding similar questions will save time.

2. Working with consistent accuracy. Much time is wasted, not only by students but also by experienced surveyors, in needless refinements.

3. Systematizing the work in order to avoid delays and to save unnecessary operations.

Speed is not necessarily the result of hurrying; it is more likely to be the result of economizing time along the lines suggested in this article.

20. System in surveying not only economizes time but also diminishes the chances of error. There are many things in the use of instruments and in other operations of field work which can be done always in the same way. If one is systematic, much of this work becomes second nature, and mistakes are thus avoided. Even when a mistake has been made one will often become conscious of it because of the violation of some fixed habit and so can find the error and correct it. For similar reasons system is equally valuable in recording notes and in office computations, and much of the value of a young surveyor's services lies in his knowledge of his employer's system and in his ability to work in that system.

CHAPTER 3

FUNDAMENTALS OF FIELD NOTES

In this chapter are given the general directions and suggestions that hold good for *all* kinds of field notes. Special directions and forms for field notes for leveling, compass surveying, transit surveying, stadia surveying, and topographic surveying are given in the corresponding chapters on those subjects.

21. Field Notes. Field notes are written records of field work, *made at the time the work is done.* Notes made afterwards, or notes copied from field notes, may be useful, but they are not field notes. Field notes must be as complete and accurate as possible, for the most careful and reliable field work will be of little or no value if the record of that work is unreliable. That is why engineers often prefer to keep the field notes themselves rather than to trust them to a less experienced assistant. The student must realize at the beginning of the course that not only is recording field notes as important as any part of the field work, but it is also perhaps the most difficult thing he will be called upon to do, and will require all the alertness, care, accuracy, and intelligence which he can command.

Note. To keep field notes satisfactorily one must have had long experience in practical work. It is not expected that the student can do it at the outset, indeed comparatively few men in actual practice are good notekeepers, but this is all the more reason why instructor and student alike should pay special attention to perfecting the student's ability in notekeeping.

22. Methods of Keeping Notes. Methods of notekeeping are different for different kinds of work, and will be given in detail as the work may require. It must be remembered, however, that there may be many methods equally good for a given kind of work, that an engineer must choose the one best adapted to his needs, and that, as a rule, of the many good methods in common use, only one or two illustrations are given in this book for each kind of work.

23. Field Notes—General Suggestions.

1. Use a good notebook that will stand hard usage.

2. Use a hard *sharp* pencil—2 H or 3 H; never a pencil soft enough to blur.

3. The notes should read from left to right, and from the bottom

up as in working drawings. In printing vertically hold the right-hand edge of the book towards you.

4. Trust nothing to memory. When in doubt as to whether or not an item will be needed, put it in.

5. Constantly ask yourself what will be needed in your notes in the office work. To answer this you must know the purpose of the work. If it is a survey and a map is to be plotted from the notes, you must know what the map is to be used for, and what, therefore, should go on the map; if leveling, what is to be done with the data obtained.

6. If a page of notes becomes illegible, make a copy of it while the data are fresh in mind, but mark it "COPY," and preserve the original page.

7. Make each page of the notes as complete as possible without undue crowding.

8. It is usually desirable at least to indicate all the numerical work of the calculations and reductions made in the field. If the numerical work itself is done in the notebook, it should not be allowed to obscure the main notes, and hence it is often best to put such computations on separate pages and cross-reference them. This does *not* mean that numerical work is to be *scribbled* in the notebook at random. On the contrary, it should be kept in as neat and systematic a manner as any other portion of the notes.

9. Names of the surveying party, equipment used, weather, date, and any special or unusual conditions under which the survey was made should be recorded whenever there is a possibility that such data may be needed for future reference. Data like these are sometimes of great importance, particularly when the field notes are presented as evidence in court.

10. Notes should be signed each day by the notekeeper.

11. The notes of a survey should be preceded by a title page, and if there are many pages, there should be an index. At the top of each page print a descriptive title so that one can tell at a glance what the notes on that page are about.

24. The Three Parts of Field Notes. Field notes like working drawings may be subdivided into three parts:

1. *Numerical values.* Records of all measurements (angles and distances).

2. *Sketches.* Records of outlines, relative locations, topographic features.

3. *Explanatory notes.* All notes of an explanatory nature which make clear those sketches and numerical values that might otherwise be obscure or misunderstood.

25. Numerical Values.

1. *Make plain figures.* Words can be guessed at, but not figures. Be particular to make distinct those numerals that look alike; form a 3 clearly different from a 5; do not let the top of a 7 be so short that the figure may be mistaken for a 1.

2. *Make large figures.* The tendency of the beginner is to make his numerals too small.

3. *Never write one figure on top of another.* A constant source of annoyance and error. If, for example, a 5 is written and then a 7 on top of it, how is one to know which is correct?

4. *Do not try to change one figure into another;* for example, a 3 into a 5 or a 2 into a 3. The reason is obvious.

5. *Erasing.* Avoid erasing. Draw a line through an incorrect value and write the correct value directly above or near by. This will show that a mistake was noticed in the field and corrected, thus giving double weight to the corrected value, whereas erasing or changing a figure leaves the impression of "doctoring" the notes. When it becomes necessary to alter a sketch or to make any part of the notes clearer, erasing is justifiable, provided nothing is lost which would give added weight to the notes.

6. *Measurements on sketches.* If measurements are put directly on the sketches, make it clear where they belong. The use of dimension lines, as in working drawings, is sometimes necessary. When the space on the sketch is too small for figures, a measurement may be placed to one side, and an arrow drawn to indicate where the measurement should go. When two or more successive distances are recorded along one side of the same straight line, enter also along the other side the sum total of these distances.

7. *Recording measurements.* In recording measurements as they are being taken in the field there are two sources of trouble: (*a*) measurements omitted because not taken; (*b*) measurements misunderstood as they are called out.

a. The notekeeper must therefore be on the alert to see that every measurement that should be recorded is taken and that every measurement taken is recorded.

b. The notekeeper should *repeat,* loudly enough to be heard distinctly, all measurements which others may call out to him to record. Measurements not taken in the presence of the notekeeper should be given to him in writing. For example, if the notekeeper is busy taking transit notes and the chainmen are at work elsewhere, they should report all measurements to him in writing at the first opportunity.

Every linear measurement should be recorded in such a way that

the last digit will indicate the degree of precision with which the measurement was made. For example, 164 ft. is to the nearest foot, 164.0 or 183.7 to the nearest tenth of a foot, 164.00 or 183.74 to the nearest hundredth. If the two ciphers were omitted in the next to the last measurement, there would be no way of knowing that it had been taken to the nearest hundredth of a foot. Similarly, every angular measurement should be recorded in accordance with the degree of precision with which it was taken. For example, 55° 00′ indicates that the measurement was to the nearest minute; an angle read to the nearest half minute is indicated thus: 10° 41′ 30″.

26. Sketches.

1. *Scale.* Sketches are almost never made to scale. In fact, it is often necessary to exaggerate certain portions of the sketch for the sake of clearness. It is well, however, when possible, to draw the sketches roughly to scale.

2. *Make sketches large, open, and clear.* Use the pages of the notebook generously—they are cheap compared with time wasted in deciphering crowded notes.

3. *Use of straightedge.* Sketches generally are made freehand, but frequently time may be saved and the sketch greatly improved by using a straightedge for the straight lines. A small triangle carried in the pocket will be found useful.

4. *Angles.* The protractor is seldom used in the field. Angles are either guessed at or no attempt is made to draw them true size.

5. *Exaggeration of details,* i.e., portions of a sketch drawn larger at one side of the main sketch, or on another page, is often necessary.

6. *What a sketch should show* must be determined on the spot by the notekeeper, and for the most part before he begins the sketch.

7. *Do not attempt to represent too large a portion of the survey in a single sketch.* Better two open and clear sketches that supplement each other than one which is so crowded that parts of it are obscure or ambiguous.

8. *When to make a sketch.* In general, if measurements cannot be easily described, or if the description (in words) of a series of measurements would take more time or space than would be required for a sketch, make a sketch. Always make a sketch when it will help to settle beyond question any doubt which otherwise might arise in the interpretation of the notes.

27. Explanatory Notes.

1. *Object.* To make clear that which is not perfectly evident from numerals and sketches, and to record such information con-

cerning important features of the ground covered and the work done as might be of possible use later.

2. *Lettering.* The common practice is to print notes, using some good style of freehand lettering. Compactness and legibility are gained by such lettering well done. In many cases ordinary handwriting can be used provided it is as legible as good freehand printing. One method used for freehand lettering is given in the next article.

3. *Place the notes* in vacant spaces where they will not interfere with or obscure numerals, dimension lines, or sketches.

4. *When needed.* Ask yourself in a given case if the measurements you have put down, or the sketch you have made, could possibly need oral explanation should some one else use the notes. If in doubt, put the explanation in writing in the notes themselves.

5. *Kind of explanations needed.* Do not consider that the field notes for any part or detail of the survey are complete until you are satisfied that you have recorded the answer to every pertinent question concerning the detail that you yourself can ask. If it is a fence, shall the *kind* of fence be noted? If it is a road, shall the type of road, its present condition, and similar information be given? If it is a house, a bridge, or other structure, what information concerning it may possibly be useful? Thus the surveyor in keeping field notes must put himself in the place of one who will use them, and, with the purpose of the survey constantly in mind, ask himself all sorts of questions in order that he may not fail to record information that later may be needed.

28. Tabulated Field Notes. Frequently so many measurements are taken in a small area that it is difficult if not impossible to record all the numerical values on a sketch without undue crowding. For example, a large number of angles may be measured at a single station and a correspondingly large number of linear measurements taken from the same station. The common method of recording in such cases is to *tabulate* numerical values on one page of the notebook, usually a left-hand page, and, on the opposite page, to indicate by a sketch, and by explanatory notes if necessary, the points or lines between which the measurements were made. In some types of work, leveling for example, numerical values are almost invariably tabulated. There are usually several different forms of tabulated notes that are in common use for each kind of field work. Some of these standard forms will be given in the chapters on leveling, compass surveying, transit surveying, topographic surveying, and other special types of field work.

29. Freehand Lettering. The style of lettering recommended for field notes is the "single-stroke" lettering. It is a surprisingly easy system to learn, and it is especially adapted to rapid notetaking because of its simplicity. In this article the essential features of this system are given in so far as they apply to field notes. For a more thorough treatment of single-stroke lettering the student is referred to books on engineering drawing and on lettering for engineering purposes.*

The student is urged to practice until he can letter almost as rapidly as he can write. When he can do this and do it well he has acquired something of great value, not only for recording field notes but also for other lines of work. Even a few hours of intelligent practice, in which the *direction and sequence* of the strokes are carefully observed, will yield results well worth attaining.

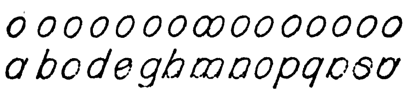


FIG. 29(*a*).

29(*a*). LOWER-CASE LETTERS. In Fig. 29(*a*) it is seen that fifteen of the lower-case letters of the alphabet can be formed with a small ellipse as a basis; the other nine can be formed with straight lines. This method of forming letters, however, is not recommended, but is inserted here as introductory to a method somewhat similar. In Fig. 29(*b*) this other method is shown.

29(*b*). INCLINED LOWER-CASE LETTERS. The methods given in this article for forming inclined letters hold good also for upright letters. Inclined letters, however, can be made more rapidly and with greater uniformity by the average man. In Fig. 29(*b*) the arrows in the first and fourth rows indicate the directions of the strokes; in the second and fifth lines the completed letters are shown; in the third and sixth lines *incorrect* methods of forming letters are illustrated. The main points to keep in mind concerning the different letters are:

1. The *i, j, l,* and *c* are the only letters made with one continuous stroke.

2. The strokes in each letter are as far as possible *downward.* The slope of the main stem with few exceptions is about 1 to 2½. (For height of letters see Art. 29(*g*).)

3. The order or sequence of strokes is from left to right, except in *s, u, k, x,* and *z.*

* For those who desire quickly to become skillful in the art of single-stroke lettering, *Kirby's Lettering Tablets,* published by John Wiley and Sons, Inc., New York City, are recommended. The sheets in these tablets are so ruled that they aid the beginner in learning how to proportion letters, and simple but adequate directions tell him how to form them.

4. In the first group, *a, d, g,* and *q,* the only difference between the letters is in the third stroke. In each of these letters the second or middle stroke is flatter than the first in order that there may be a clear space between the second stroke and the vertical stem. In order to avoid the effect shown in the third row which comes from not doing this, it may be well for the beginner to exaggerate, somewhat, the "hump" in the first stroke and then make the second stroke almost straight.

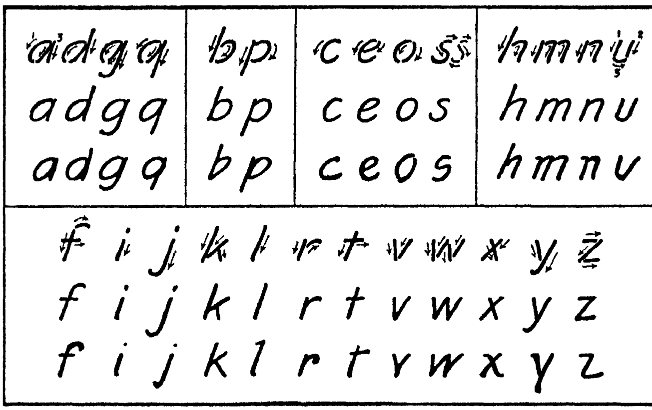


FIG. 29(*b*).

5. In the letters *b* and *p* the oval is wider at the top than at the bottom.

6. The basis of each of the letters in the third group, *c, e, o,* and *s,* is an inclined ellipse. The *e* is made with two strokes; if made with one it is likely to have a different slant from the other letters, as shown in the third line. Notice that the first stroke of the *s* is started in the opposite direction from that naturally expected. Some draftsmen prefer to make the *s* in two strokes instead of three as shown in the supplementary *s.* The *o* is made with two strokes, each of which should begin and end almost horizontally in order that the two strokes may meet in a smooth curve and not in a sharp apex; this also gives proper width to the *o* without making it round.

7. In the letters of the fourth group, *h, m, n,* and *u,* the first effort should be to get the vertical strokes parallel. The second stroke in *h, m,* and *n* should not be started too far up the first stroke, and it should make almost a sharp turn at the upper corner. The first stroke should not begin with a curve. Notice that the

third stroke of the *u* is at the bottom, the other two strokes being made first to secure parallelism and proper width of letter.

8. In the letters *g, q, b, p, j, l,* and *y* the portion above or below the main body of the letter is a little less than one-half the total height of the letter; in the letter *t* it is still less.

9. In the letter *k* the third stroke does *not* start at the intersection of the other two. Make the second stroke strike pretty near the bottom of the first and begin it well to the right so that the letter will not be too narrow. Notice that the second stroke of the *r* is *upward.*

10. In the letters *v, w, x,* and *y* make the strokes in one direction almost vertical, and the strokes in the opposite direction at a considerable slant, so that the main axis of the letter may have the slope corresponding to that of other letters. The intersection of the two strokes in *x* is a little more than halfway up.

11. In the letters *k, v, w, x, y,* and *z* make the angles at the intersection of lines clear and sharp.

12. In the letters *g, j,* and *y* the stroke extending below the line is straight until just at the end it turns sharply to the left, forming a *small* hook.

13. The cross strokes in the *f* and *t* are on a line with the tops of most of the other letters.

29(*c*). NUMERALS. Good numerals are even more important than good letters in field notes; they must be made unmistakable. The main points to be kept in mind are:

1. Make figures somewhat larger than lower-case letters.

2. In the 2 the down curve is *reversed* and the base is *straight.* Avoid the effect shown in the third line of Fig. 29(*c*). Some draftsmen prefer an abrupt turn or angle in the downstroke as shown in the supplementary 2.

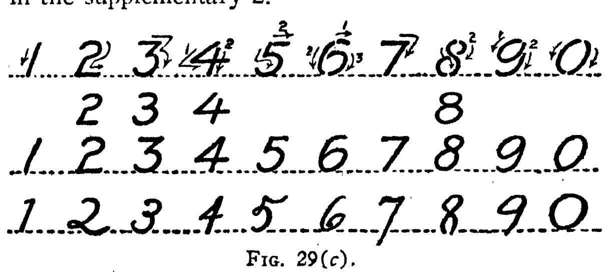


FIG. 29(*c*).

3. Of the two kinds of 3's that with its upper line horizontal and straight is perhaps the easier to make. The lower loop should begin with a sharp turn *upward,* and its height should be more than half the total height of the numeral.

4. Of the two kinds of 4's that shown in the top row seems to be preferred. The first stroke begins at about 45°, and should extend more than halfway down the figure before turning into the horizontal; otherwise the top of the 4 will be too small as shown in the third line.

5. In the 5 the first stroke should turn abruptly about two-fifths of the way down, and extend almost horizontally for quite a little distance. It really curves slightly *above* the horizontal to form the upper portion of a loop similar to that of the 3. The second stroke is *straight* and horizontal; make it fairly long.

6. In the 6 the tendency is to make the lower loop too small.

7. The 7 is formed by two straight lines—no curves—with the down stroke at about 45°.

8. In pencil the 8 is best made with two strokes. Sometimes it is convenient to form it with two ovals, the upper one being the smaller. In inking, the pen is lifted where the second stroke crosses the first.

9. The 9 is an inverted 6, but the upper loop is nearly finished with one stroke. The tendency is to make the loop too small.

10. The 0 is made with two strokes each of which begins and ends almost horizontally to give smoothness to the curve and width to the whole numeral without making it round.

Note. The student may find that in keeping field notes in pencil some of the numerals made in Fig. 29(*c*) with two or more strokes can be made in one stroke. In making figures in ink, however, he will get better results to follow the strokes as given.

29(*d*). Capital letters are shown in Fig. 29(*d*). Many of the directions already given for the lower-case letters apply to the capital letters also. Some additional points are:

1. The height of capital letters is about one and one-half times that of lower-case letters. (See Art. 29(*g*).)

2. The width of most of the capital letters is about equal to the height of lower-case letters.

3. The third or horizontal stroke in *E, F,* and *H* is slightly more than halfway up. This is true also of the middle horizontal line of *B* and the corresponding lines of *P* and of *R*.

4. The second stroke in *L* and in *T* is comparatively short.

5. In *A, V, W, X,* and *Y* the chief difficulty is in getting the ***axis*** of the letter at the standard slope. This is accomplished by making certain lines almost vertical, and opposite lines at a correspondingly greater slant, as shown in the figure. The horizontal stroke in *A* is quite near the bottom.

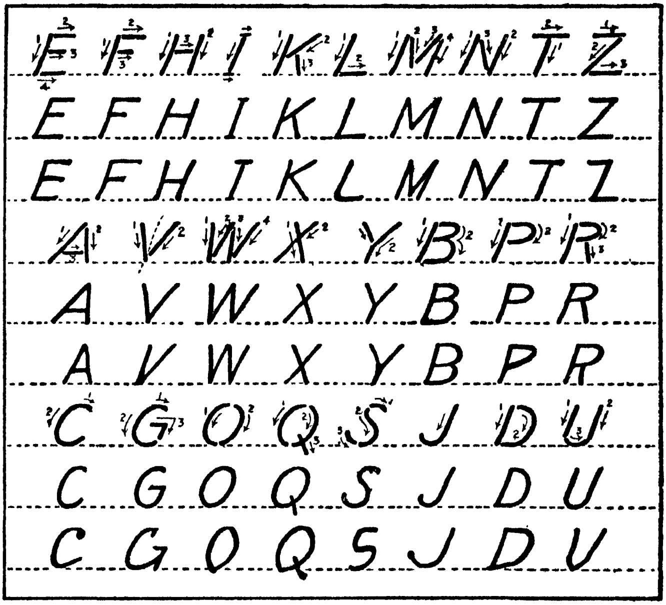


FIG. 29(*d*).

6. The width of *M* and of *W* is greater than the width of other letters. The second and third strokes of the *M* meet in a point *above* the bottom line of the letter.

7. The letters *C, G, O, Q,* and *S* are based on the ellipse. The tendency is to make these letters too wide.

8. In practicing lettering observe carefully the order and direction of the strokes, which are for the most part *downward* and from *left to right,* and notice what errors to avoid as illustrated in the lines 3, 6, and 9.

29(*e*). SPACING THE LETTERS. Crowd the letters close together and thus compress words into small space; otherwise one of the chief advantages of lettering will be lost. The tendency is to make letters too wide and too far apart. To make narrow compressed letters seem large and clear increase the height. Thus, for example, in Fig. 29(*e*) the third line takes no more space lengthwise than the second, but the words are clearer because the height of the letters is greater. The first line illustrates how space may be wasted by not compressing the letters.

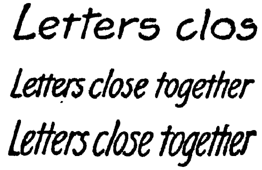


FIG. 29(*e*).

29(*f*). Spacing of words. The space between two successive words should be at least as much as that occupied by a letter of average width.

29(*g*). Height of letters. A good height for lower-case letters is 1/16 in., the height of numerals being slightly greater. This makes the height of capital letters a little less than 1/8 in. The effect when these heights are used may be judged by the field notes shown in Chapter 24.

30. Final Suggestions on Taking Notes. Remember that the extra time required in making field notes plain will probably be saved when the notes are plotted. Never let anything appear in the notes that can be misunderstood or misinterpreted, especially numerals. Usually there will be more to the notes for a given portion of the work than at first appears; the sketch will grow more complex, and there will be unexpected measurements to be recorded. Make a generous allowance for this in starting a page of notes so that they will not become crowded. In level notes and in most of the forms of transit notes it is better to omit every other line, thus leaving room for corrections. Be neat and systematic in keeping notes for field work, and the habits thus formed will be of great value in many other lines of work.

31. Symbols. Certain symbols or abbreviations, more or less in common use for field notes, will be given later, particularly in Art. 153 and in Chapter 24. The surveyor should, however, feel free to use those that seem best adapted to his purpose. For example, a conventional symbol for a transit station is a tiny circle; for a stadia station, a tiny square; for a triangulation station, a tiny triangle. In each case a dot in the center represents the station point. When, however, all the stations in any survey are of one type, as, for example, transit stations, such distinctions are of course unnecessary. Any one of the symbols can be used, therefore, or one entirely different can be adopted, such as a dot marked "Sta." However, it is best not to use in the same survey different symbols to represent the same thing—choose one and stick to it. Moreover, certain symbols and abbreviations are so generally understood, such as B.M. for bench mark, that they are almost invariably used.

32. Notebooks. The type of field notebook most often used is either a bound book with stiff covers or a looseleaf notebook. Books of the latter type permit the removal of notes as soon as they are completed and the arrangement of pages in any desired order. The requirements for a good notebook are durability under hard usage; of paper not easily torn, with a surface that will "take" a hard pencil and withstand erasures; of a size convenient to fit

easily into a pocket. Pages should be ruled to suit the type of field work. Books ruled for various kinds of notes may be purchased from dealers in engineering supplies. For students, a cheap form of notebook will often answer every purpose.*

33. Suggestions for Class Work. Field notes for class work present some difficulties not encountered in practice, particularly when a set of notes is to be used later in office work by a large number of students. Time spent by students in copying notes is largely wasted from an educational point of view. Notebooks with carbon sheets may be used, but they are not entirely satisfactory. Frequently different parts of a survey are being carried on by different parties of students, and then it becomes necessary to adopt a method of procedure whereby the field notes of each party will be so correlated with the notes of other parties that together they will form a set of field notes as unified and comprehensive as if they had all been taken by one party. Thus the whole problem of field notes in class work can be solved only by careful and ingenious planning by the instructors in charge of the work. Only the most general suggestions can be given here:

1. When a class is divided into parties for field work that will require a considerable period of time, a party notebook for each party is desirable. A separate-leaf notebook may be advantageous, especially when used later in class office work. Such a book can be made from a standard bound notebook by removing the binding thread or clamps.

2. Each page of a party notebook should be dated and signed by the notekeeper as the work progresses. It may be necessary to consult him later should questions concerning the notes arise.

3. It is desirable that the party notebook should be inspected at the end of each day, or oftener, by the instructor in charge of the party. Changes, corrections, or additions by the instructor may well be made with a colored pencil.

4. It may be desirable to have one or more master sheets on which are kept, from day to day, such records as will indicate the progress of each party, and such other information as may be necessary to coordinate the work of all parties.

5. For special exercises not included in a regular survey, for example, exercises in the use of instruments, it is desirable that the student have his own notebook. In addition to the suggestions already given for field notes, the following are offered to the student as an aid in keeping notes for special exercises.

33(*a*). Notes for special exercises.

1. At the beginning of the notes, state what the exercise is, as, for example: *"To lay off an angle of 38° 20′ with a tape."*

2. Avoid recording the notes of more than one exercise on a page.

3. When computations are entered in the notebook, they should be neat and orderly—avoid "scribbling."

4. Notes should be entered directly in the book and not taken on scraps of paper and copied later.

5. Consider the recording of notes for special exercises as an opportunity for practice in notekeeping, and keep as good notes in every respect as if they were those of an actual survey.

* Notebooks intended particularly for student use may be obtained from John Wiley and Sons, Inc.

CHAPTER 4

ERRORS

In the first part of this chapter, the aims are to define the different kinds of error and to show how these errors affect measurements; also to explain how the most probable or best value of a quantity may be obtained and how the precision of measurements may be judged. In the second part of the chapter a brief review of the method of least squares is given in so far as that method is likely to be used in plane surveying.

Methods of preventing or eliminating errors, and the relative importance of errors from different sources, will be treated more in detail in subsequent chapters.

34. Error. No measurement in surveying can be exact. The difference between a measurement and the true value of the quantity measured is the *true* error of that measurement. *The true error of a measurement is never known,* for the true value of the quantity measured is never known. How then can a surveyor judge of the precision of his measurements, and how can he determine the most probable or best values of the quantities measured? These are the main questions to be considered in any discussion of errors.

35. Sources of Error. (1) Natural; (2) instrumental; (3) personal.

1. *Natural.* Temperature, wind, refraction, action of gravity, obstacles to measurement.

2. *Instrumental.* Imperfections in the construction and adjustment of instruments. Expansion, contraction, and other possible changes in the instruments.

3. *Personal.* Imperfections of human sight and touch. Actual mistakes.

36. Three Classes of Errors. (1) *Mistakes;* (2) *constant errors,* also known as systematic errors; (3) *accidental errors.* If a measurement of a quantity is greater than the true value of that quantity, the error is *plus;* if less than the true value, the error is *minus.* The total error is not due to any one cause, but it is the *algebraic* sum of errors due to different causes.

Mistakes are errors which have their source in the mind of the observer. Some writers do not include mistakes as a class of errors on the ground that errors may be considered as legitimate, mistakes as illegitimate.

Constant errors, or systematic errors, are those which occur from well-understood causes and can be greatly reduced by taking proper precautions to eliminate the causes.

Accidental errors are those which remain after mistakes and constant errors have been eliminated. The effect of accidental errors may be reduced by computing the arithmetical mean of several measurements of a distance or an angle. This is not true of constant errors.

An accidental error is as likely to be plus as minus, but a constant error under given conditions always has the same sign and the same magnitude.

Thus, for example, if a tape is ¼ in. too long because of high temperature, this is a constant error, because under the same conditions, i.e., the same temperature, the tape will always be ¼ in. too long. Under *changed* conditions, i.e., low temperature, the tape may become too short; then the sign of the error is changed. This illustrates how a *constant* error from a given source may be plus or it may be minus, but under the *same* conditions it cannot be plus *or* minus. Suppose, on the other hand, that an observer in measuring between two points makes an error in reading the tape, so small that he cannot detect it. This *accidental* error is as likely to be plus as minus, and if he repeats the measurement he is likely to make a similar error and that, too, is as likely to be plus as minus.

If a measurement is repeated a number of times under precisely the same conditions, the *constant* error is the same for each measurement, both in sign and magnitude. It is not reduced or affected in any way by taking the mean of all the measurements. The accidental error of each measurement, on the contrary, is as likely to be plus as minus, and the accidental error of the mean, therefore, is likely to be less than the accidental error of any one measurement. Theoretically, that part of the total error due to accidental errors would entirely disappear from the means of an indefinitely large series of observations, taken under precisely the same conditions; that part due to constant errors would still remain.

36(*a*). Illustrations. A few illustrations of the three classes of errors will help to make the distinctions clear.

Mistakes. Errors due to carelessness, inexperience, and mental confusion, such as mistakes in reading the tape, the vernier, or the leveling rod; keeping the wrong count of tape lengths; turning the wrong tangent screw; sighting at the wrong station.

Constant Errors. Errors due to such well-understood sources as temperature, sag, pull, imperfect graduation, and imperfect adjustments of instruments.

Accidental Errors. Errors due to imperfections of human sight and touch which render it impossible to make exact readings, to know when the cross hair exactly bisects a signal, to handle instruments delicately; sudden and

imperceptible changes in an instrument itself; indeterminate *variations* in pull, temperature, and certain other sources of constant errors.

36(*b*). REMARK. When mistakes are very small it is sometimes difficult to distinguish them from accidental errors. It is to be noticed also that although temperature is a source of constant error, *variations* of temperature too small to be measured may be classed as accidental errors; the same is true of small variations in some of the other sources of constant errors.

Notice also that constant errors from different sources tend to balance each other when some are plus and some are minus, as, for example, when the error in chaining due to high temperature tends to balance error due to sag. Again, constant errors from the same source may tend to balance each other in the mean of measurements taken under *different* conditions, as, for example, when, in chaining, part of the measurements of a line is made when the temperature is above the standard temperature for the tape, and part when it is below.

36(*c*). CUMULATIVE AND COMPENSATING ERRORS. When the value of a quantity is found by adding together the measurements of several smaller quantities, any source of constant error becomes a source of *cumulative* errors, because, under given conditions, a constant error always has the same sign. On the other hand, any source of accidental error becomes a source of *compensating* errors, since for each of the separate measurements the sign is as likely to be plus as minus. *Compensating* errors tend to balance each other, whereas the cumulative errors from any one source affect the total result in the same way.

36(*d*). ILLUSTRATIONS. Any constant error in the length of a tape becomes cumulative when the tape is applied several times in measuring a line, i.e., when the tape lengths are added together, for the total error due to that cause will vary in direct proportion to the number of tape lengths. On the other hand, errors which occur in marking where the end of the tape comes each time that it is stretched are compensating, for the chances are that if one tape length is marked too long the error will be balanced by another tape length marked too short.

Errors due to the settling of the instrument in leveling are cumulative, because the effect is always the same—the calculated elevations of succeeding points are too high. Errors due to the difficulty of telling when the cross hair exactly bisects the target are compensating.

37. Discrepancies. A discrepancy is merely the difference between two measurements of the same quantity; it is *not* an error. A large discrepancy indicates a *mistake* in the work, but, large or small, it is no measure of the true error. A discrepancy may be small yet the error may be great because each of the two measurements contains an error that may be large. This is an important point too often overlooked.

In ordinary surveying most of the distances, angles, and elevations are measured but once. An important measurement, however, is often repeated, and if the discrepancy is too large, a third measurement is taken, and so on until serious discrepancies are eliminated.

37(*a*). ILLUSTRATION. A surveyor measures a line, the true length of which is 400.00 ft., with a 100-ft. tape that, unknown to him, is 0.09 ft. short. The first time he gets 400.53 ft. and the second time 400.47 ft. The

discrepancy is only 0.06 ft., indicating that he has made no serious mistake, but the true error of the first measurement is 0.53 ft. and of the second 0.47 ft. The mean of these two errors is 0.50 ft., which is more than eight times as great as the discrepancy.

38. Importance of Eliminating Constant Errors. The discrepancy between the two measurements in the preceding illustration did not indicate the true error, because of a *constant* error in the length of the tape which no amount of repeating the measurement under the same conditions could reveal. The surveyor should have known that his tape was 0.09 ft. short, and have corrected his measurements accordingly; then the discrepancy would have indicated more nearly the precision of his work. This illustrates why it is the first business of the surveyor to eliminate constant errors as far as practicable. A mistake, as revealed by a discrepancy, may then be corrected by again repeating the measurement. When constant errors and mistakes have been eliminated, only accidental errors remain, and these may be reduced by taking the arithmetical mean.

As pointed out in Art. 36(*c*), under certain conditions constant errors become *cumulative,* whereas accidental errors tend to be *compensating.* Hence the obvious importance of eliminating constant errors in chaining, in leveling, and in all similar work involving the summation of separate measurements.

39. Prevention and Elimination of Mistakes and Errors. (*a*). MISTAKES are best discovered by systematic checks. *Checking* work, an essential part of all surveying, consists in comparing results with known conditions.

Illustrations. The discrepancy between duplicate measurements should be small. The sum of two or more measurements should equal another measurement, as, for example, when the sum of two angles should equal a third angle. The sum of two or more measurements should fulfill some known condition, as, for example, the interior angles of a triangle should add up to 180°.

39(*b*). CONSTANT ERRORS. The first step in the elimination of constant errors is to use *accurate* tapes, levels, transits, and other measuring instruments, and to know what degree of precision at best may reasonably be expected in the use of each. The second step is to keep instruments in adjustment and the whole surveying equipment in good condition. Sources of constant error, however, will always remain. Some of these errors may be largely prevented or reduced by certain methods of field work that will be explained in subsequent chapters. Other errors may be reduced by applying to measurements corrections based on known values for sources of constant errors, such as the *error* in the length of a tape tested at a standard temperature and pull, say 68° and 10 lb., and the change

in the length due to any variation from that temperature and pull. Constant errors from some sources are best eliminated partly by methods of field work and partly by subsequent corrections.

39(*c*). Accidental errors cannot be eliminated, but they follow a mathematical law—the law of probability—according to which they may be greatly reduced by comparing the results of a series of measurements (Art. 42).

40. Relative Importance of Errors from Different Sources. It is evident that constant and cumulative errors as a rule are more important than accidental and compensating errors, and that the more measurements there are of a quantity the less the mean is affected by accidental errors. Beyond this it is difficult to make any general statement. Any error large enough to be *appreciable* should usually be avoided or corrected; but appreciable is a relative term, and an error which is appreciable under certain conditions is inappreciable under other conditions. Thus, for example, the error in chaining due to changes in temperature is an appreciable error, provided the precision required warrants the use of a thermometer; otherwise it may be considered inappreciable. Errors in plane surveying so small that they are ignored would often be considered of great magnitude in geodetic surveying.

40(*a*). Remark. The more the surveyor knows about the sources and nature of errors the more likely he is to judge correctly their relative importance. He must know that some sources of constant or cumulative errors are more important than others, and that certain errors, though they may be compensating, are not to be ignored. A more thorough treatment of these questions is given in the chapters, "Errors in Linear Measurement," "Errors in Leveling," and "Errors in Transit Work."

41. Probable Values and the Adjustment of Measurements. Probable values are found from observed values. When the relations between observed values are complex, a knowledge of the theory of least squares is necessary. Some simple cases which frequently occur in surveying may be given, however, without entering upon that subject.

There are two general cases, namely: (1) when the measurements are of one and the same quantity; (2) when the measurements are of different but related quantities.

41(*a*). Several measurements of the same quantity. The most probable value of the quantity is the arithmetical mean of all the measurements, *provided the measurements are made with equal care.*

Illustrations. Four measurements of a line in feet are 160.02, 160.04, 160.51, and 160.03. Obviously the third measurement was not made with equal care, some mistake was made, and the measurement should be discarded. The most probable value is the arithmetical mean, i.e., (160.02 + 160.04 +

160.03) ÷ 3 = 160.03 ft. Four measurements of an angle are 30° 29′, 30° 31′, 30° 31′, and 30° 30′. The most probable value is the arithmetical mean, i.e., the sum of the four measurements divided by four, or 30° 30′ 15″.

41(*b*). MEASUREMENTS OF SEVERAL RELATED QUANTITIES. When quantities are related because of *two or more* conditions which they must fulfill, the problem of finding the most probable values of these related quantities properly belongs to the theory of least squares. When, however, there is only *one* condition fixing the relation between the quantities, the most probable values, as a rule, can easily be found. Two cases under this head which often arise in surveying are: (1) when the sum of several measurements should equal an *exact* quantity, and (2) when the sum should equal some other *measurement*. In the first case the difference between the sum and the *exact* quantity is a true error; in the second case the difference between the sum and a *measured* quantity is a *discrepancy*.

41(*c*). CASE I. *When the sum of several measurements should equal an exact quantity, distribute the true error equally among all the measurements.* The results will be the most probable values of the measurements.

Illustration. The three angles of a triangle add up to 179° 59′. The *exact* quantity fixing the relations between the measurements is 180°. The *true* error is 1′ or 60″. Hence add to each of the three angles one-third of 60″, or 20″.

In the same way the most probable values of the interior angles of any polygon may be found by finding the true error of their sum according to the rule in Art. 298(*a*), and distributing this error equally among all the angles. Notice, however, that the true error of any one angle is not known.

41(*d*). CASE II. *When the sum of several measurements should equal some other measurement, distribute the amount of the discrepancy equally among all the measurements, including that one to which the sum should be equal.* In this case the check measurement (i.e., the total measurement) cannot be exact, and, if all the measurements are made with the same care, it is likely to contain an error equal to that contained in each of the smaller measurements. *If, however, the correction is added to each of the smaller measurements, it must be subtracted from the total measurement, and vice versa.*

41(*e*). ILLUSTRATION. If in the illustration of case I the three angles had been the three parts of a *fourth* angle instead of the three angles of a triangle, and if the measured value of the fourth angle were 180°, then 180° − 179° 59′ = 1′ = 60″ is the discrepancy. The correction is 60″ ÷ 4 = 15″ instead of 20″, and this correction is added to each of the three smaller angles but subtracted from the total angle. The most probable value of the latter is, therefore, 180° − 15″ = 179° 59′ 45″.

41(*f*). ILLUSTRATION. The measured value of an angle is 40°. When divided into two angles the measurements are respectively 24° 30′ and 15° 29′. The discrepancy is 40° − (24° 30′ + 15° 29′) = 1′ = 60″. Three measure-

ments of equal care are involved; hence the correction is 60″ ÷ 3 = 20″ and the most probable values are 24° 30′ 20″, 15° 29′ 20″, and 39° 59′ 40″.

41(*g*). Remark. The essential difference between case I and case II is that in case I the quantity used as a check has a *true* value which cannot be changed, whereas in case II the check value is itself a measurement, and likely, therefore, to contain an error equal to or greater than that contained in each of the smaller measurements.

41(*h*). The method of case II does not apply to measurements of successive distances, the sum of which should equal a *total* distance, because, unless the measurements are of equal length, they should not be given equal weight. Even if they are equal, moreover, no one of them should be given the same weight as the measurement of the total distance. This is because in linear measurement the error from any source of constant error increases as the distance measured increases.

42. The Theory of Errors or Method of Least Squares.* The method of least squares is a method of determining the most probable or best result of observations. It is based on the assumption that accidental errors will entirely disappear from the mean of an indefinitely large series of independent observations of the same quantity made under precisely the same conditions, and that therefore *if constant errors are eliminated from these observations* the arithmetical mean is the most *probable value* of that quantity. In practice, however, (1) an *indefinitely* large number of observations cannot be made, (2) no two observations can be made under *exactly* the same conditions, and (3) constant errors cannot be *entirely* eliminated. It follows, therefore, that the most *probable* value of a quantity is strictly a *theoretical* value, and that the greater the number of observations, and the more completely constant errors are eliminated from these observations, the nearer the most *probable* value of a quantity will approach the *true* value as a limit. It never reaches this limit (or if it did, there would be no means of knowing it), and therefore, no matter how accurate the observations may be, the probable value always contains an error.

42(*a*). A residual is the difference between the value of a quantity as obtained by a single observation and the most probable value of that quantity as obtained by a number of observations. In any given case there are as many residuals as there are observations.

42(*b*). The probable error of a quantity is a magnitude such that the chances are even that the true error contained in that quantity is greater or less than the probable error.

Illustration. A series of measurements of the length of a line gives for the mean or most probable value 528.91 ft., and the probable error of that mean,

* It is the purpose of this article and those that follow to give only so much of the theory of errors as may be needed in the simple cases which arise in plane surveying. For a more comprehensive treatment of the subject consult one of the standard works on the method of least squares. Although this subject is not ordinarily studied at the beginning of an elementary course in surveying, it is a part of any general discussion of errors such as this chapter contains, and it is, therefore, included for future reference.

determined by a method to be explained later, is ±0.034 ft. This means that it is an even wager that the true error contained in 528.91 ft. does not exceed ±0.034 ft. Notice that this is quite different from saying that the true error does not exceed ±0.034 ft.

43. Use Made of Most Probable Values and Probable Errors. *Most probable values* are used in surveying for the adjustment of measurements, and *probable errors* to indicate the precision of measurements.

Most Probable Value. The simplest and most common cases of adjusting measurements by finding the arithmetical mean have already been given in Art. 41.

43(*a*). PROBABLE ERROR OF THE MEAN. Suppose that a second series of measurements of the length of the line in Art. 42(*b*) gives 528.94 ft. for the mean and ±0.065 ft. for the probable error of that mean. The results of the two sets of measurements are:

Most probable value	*Probable error*
528.91	±0.034
528.94	±0.065

The probable error of the mean of all the measurements in the first series is approximately half that of the mean as determined by the second series. This indicates that the precision of the first result (528.91) may be regarded as twice that of the second (528.94).

43(*b*). THE PROBABLE ERROR OF A SINGLE OBSERVATION is found from the residuals of a number of observations, as explained in the next article. If it is necessary to take a number of observations in order to determine the probable error of a single observation, why not be content to determine the probable error of the mean of all the observations and stop there? Of what use is the probable error of a single observation? The answer is that, having determined the probable error of a single observation under given conditions, this probable error indicates the precision which may be expected in any single observation made under the same conditions.

Illustration. In measuring an angle a number of times the probable error of a single observation is found to be ±8″ (Art. 47). This means that for any other angle measured by the same observer with the same transit under practically the same conditions a probable error may be expected of about ±8″. If this is well within the allowable error, the observer is justified in proceeding with single observations of all angles that can be measured under the given conditions.

43(*c*). PROBABLE ERRORS AND WEIGHTS. Probable errors afford a means of finding the weights which should be given different sets of observations, for it has been found that the weights of observations vary inversely as the squares of their probable errors.

Illustration. In Art. 47 the probable error of the mean of the first set of observations is 2.8″, and of the second set 4.6″. Hence the weights are as $1/(2.8)^2$ to $1/(4.6)^2$, or about as 3 to 1.

44. To Determine Probable Errors. The formulas given below for calculating probable errors are based on the following assumptions: (1) That small errors are more frequent than large ones. (2) That positive and negative errors are equally frequent. (3) That very large errors do not occur.

Additional points to be kept in mind are: (1) Constant errors must be eliminated as far as practicable. (2) Observations should be made under as nearly the same conditions as possible. (3) At least three observations should be made. If there are only two observations, the *discrepancy* will indicate the precision as truly as will the probable error. The more observations there are the more the significance that may be attached to the probable error, although a small number of observations will give surprisingly satisfactory results if the other conditions already mentioned are fulfilled.

44(*a*). REMARK. The student is warned against attaching importance to probable errors unless he is reasonably sure that the above conditions are fulfilled, especially the condition pertaining to the elimination of constant errors. Since the method of least squares is based on the arithmetical mean as the most probable value, and since, as pointed out in Art. 36, constant errors cannot be eliminated by taking the mean of observations, the more completely the constant errors are eliminated, or corrected, the more applicable is the method of least squares. If, for example, a tape is too long, all measurements made with it will be too short, and there is no magic in least squares which will correct the results. The probable error in this case loses its significance unless each of the separate measurements from the mean of which the probable error is derived is first corrected for the constant error in the length of the tape.

45. Formulas for Probable Errors.* Let n = the number of the observations; v = the difference between any one observation and the mean of all the observations, i.e., the *residual* (see Art. 42(*a*)); E = the probable error of a single observation; E_0 = the probable error of the mean of all the observations; Σ = a symbol signifying *sum* of.

Then, from the theory of least squares,

$$E = 0.6745\sqrt{\frac{\Sigma v^2}{n-1}} \quad \text{or} \quad \frac{0.6745}{\sqrt{n-1}}\sqrt{\Sigma v^2} \tag{1}$$

$$E_0 = \frac{E}{\sqrt{n}} = 0.6745\sqrt{\frac{\Sigma v^2}{n(n-1)}} \quad \text{or} \quad \frac{0.6745}{\sqrt{n(n-1)}}\sqrt{\Sigma v^2} \tag{2}$$

Let Σp = sum of the weights and R_0 the probable error of the general or weighted mean. Then

$$R_0 = 0.6745\sqrt{\frac{\Sigma p v^2}{(n-1)\Sigma p}} \tag{3}$$

* See any treatise on the theory of least squares.

Let R_s = the probable error of the sum or difference of several independent quantities $z, z_1, z_2, \ldots z_n$, and $r, r_1, r_2, \ldots r_n$ = the probable errors of $z, z_1, z_2, \ldots z_n$, respectively.

$$R_s = \sqrt{r^2 + r_1^2 + r_2^2 \ldots + r_n^2} \tag{4}$$

The probable error of a product Az, where A is a known quantity, z an observed quantity, and r the probable error of z, is

$$R_p^2 = A^2r^2 \quad \text{or} \quad R_p = Ar \tag{5}$$

Let SS_1 = an area found by multiplying together the measured sides S and S_1, and r and r_1 = the probable errors of S and S_1, respectively. Then the probable error of the computed area SS_1 is

$$R_a = \sqrt{S^2r_1^2 + S_1^2r^2} \tag{6}$$

46. Abbreviated Methods. In applying the formulas of the preceding article approximate values will often be all that are required. For this purpose the two following equations may be substituted for equations 1 and 2, respectively:

$$E = \frac{0.8453\Sigma v}{\sqrt{n(n-1)}} \tag{7}$$

$$E_0 = \frac{0.8453\Sigma v}{n\sqrt{n-1}} \tag{8}$$

When there are only two observations and D is the discrepancy between them,

$$E = 0.47D \text{ or roughly } \tfrac{1}{2}D, \text{ and } E_0 = 0.337D \text{ or roughly } \tfrac{1}{3}D \tag{9}$$

In equations 1 and 2 work may be saved by using the coefficients given in the following table:

n	Single Observations, $\frac{0.6745}{\sqrt{n-1}}$	Series of Observations, $\frac{0.6745}{\sqrt{n(n-1)}}$	n	Single Observations, $\frac{0.6745}{\sqrt{n-1}}$	Series of Observations, $\frac{0.6745}{\sqrt{n(n-1)}}$
			11	0.2133	0.0643
2	0.6745	0.4769	12	0.2034	0.0587
3	0.4769	0.2754	13	0.1947	0.0540
4	0.3894	0.1947	14	0.1871	0.0500
5	0.3372	0.1508	15	0.1803	0.0465
6	0.3016	0.1231	16	0.1742	0.0435
7	0.2754	0.1041	17	0.1686	0.0409
8	0.2549	0.0901	18	0.1636	0.0386
9	0.2385	0.0795	19	0.1590	0.0365
10	0.2248	0.0711	20	0.1547	0.0346

A table of squares and square roots greatly facilitates the work. Other tables similar to that given above will be found in standard textbooks on least squares.

47. Illustration of the Method of Least Squares Applied to Angles.*

First Observer				Second Observer			
No.	Angle	v	v^2	No.	Angle	v	v^2
1	60° 20′ 20″	4	16	1	60° 20′ 50″	14	196
2	30	6	36	2	21 00	24	576
3	10	14	196	3	20 40	4	16
4	20	4	16	4	20	16	256
5	40	16	256	5	30	6	36
6	10	14	196	6	10	26	676
7	20	4	16	7	20	16	256
8	40	16	256	8	21 00	24	576

First Observer:

8)190 $\qquad \Sigma v^2 = 988$

Mean = 60° 20′ 24″ −

$$E_0 = \frac{0.6745}{\sqrt{n(n-1)}}\sqrt{\Sigma v^2}$$

$$= 0.0901 \times (\pm 31.4) = \pm 2.8''$$

$$E = E_0\sqrt{n} = \pm 2.8'' \times 2.8 = \pm 8.0''$$

Second Observer:

8)162 170 $\qquad \Sigma v^2 = 2588$

Mean = 60° 20′ 36″ +

$$E_0 = \frac{0.6745}{\sqrt{n(n-1)}}\sqrt{\Sigma v^2}$$

$$= 0.0901 \times (\pm 50.9) = \pm 4.6''$$

$$E = E_0\sqrt{n} = \pm 4.6'' \times 2.8 = \pm 12.9''$$

47(*a*). Illustration. The results given above were obtained by two different observers using different transits. The probable error of the mean in the first case is $\pm 2.8''$; in the second it is $\pm 4.6''$. This indicates that the relative precision of the work of the two observers is as 2.8 to 4.6, or nearly as 5 to 3.

To find the most probable value of the angle by comparing the mean values of the two series of observations it is necessary to weight their mean values. Since weights are inversely proportional to the squares of probable errors (Art. 43(*c*)), the weights are as $\frac{1}{2.8^2}$ to $\frac{1}{4.6^2}$, or nearly as 3 to 1.

The most probable value is $60°\ 20' + \frac{24'' \times 3 + 36'' \times 1}{4} = 60°\ 20'\ 27''$.

The probable error of the above value may be obtained as follows:

$24'' = z_1$ and $36'' = z_2$, $\qquad \pm 2.8'' = r_1$ and $\pm 4.6'' = r_2$, $\qquad \frac{3}{4} = A_1$ and $\frac{1}{4} = A_2$

From (5) $\begin{cases} \text{Prob. error of } A_1z_1 \text{ or } 24 \times \frac{3}{4} = A_1r_1 = (\frac{3}{4} \times \pm 2.8'') = \pm 2.1'' = R_p, \\ \quad \text{"} \quad \text{"} \quad \text{"} \quad A_2z_2 \text{ or } 36 \times \frac{1}{4} = A_2r_2 = (\frac{1}{4} \times \pm 4.6'') = \pm 1.15'' = R_{p1}. \end{cases}$

The probable error of the sum of $(24'' \times \frac{3}{4}) + (36'' \times \frac{1}{4})$ is, from (4),

$$R_s = \sqrt{R_p^2 + R_{p1}^2} = \sqrt{2.1^2 + 1.15^2} = \pm 2.4''$$

* For an illustration of the method of least squares applied to linear measurements, see Art. 97.

CHAPTER 5

LINEAR MEASUREMENTS

IN this chapter are given general directions and suggestions for measuring distances, particularly *horizontal* distances. The major part of the chapter is devoted to the use of the steel tape. Indirect methods of measuring distances, such as those used in triangulation and in stadia surveying, are treated in subsequent chapters. For descriptions of different kinds of tapes and for methods of standardizing tapes, consult Chapter 52, "Chains, Tapes, and Rods." For the care and the repair of steel tapes, consult Chapter 55, "The Care of Instruments."

48. Pacing. It is of great advantage in surveying and in other engineering work to be able to pace distances with reasonable accuracy, for the engineer can use this method repeatedly for all sorts of horizontal linear measurements in which only approximate results are desired. Moreover, he is frequently called upon to make a rough survey as quickly as possible, and in such an emergency a survey in which distances are obtained by pacing and directions are observed by means of a hand compass may best answer the purpose.

48(*a*). LENGTH OF PACE. In learning to pace distances one must determine what length of pace to use and how to keep paces as nearly equal as possible. These two questions should be considered together. For example, a pace 3 ft. in length can be used for a short distance, but over a long distance it may become so tiring that one is likely unconsciously to shorten his steps. Hence a length of pace more nearly that of one's natural step is preferable, and this, for a man of average height, is usually a little greater than 2½ ft. The length of one's natural step may be determined by walking on fairly level ground over various lines of known lengths, and this length of step, whatever it may be, can then be used as the standard pace. Some engineers prefer arbitrarily to adopt a standard pace of 2½ ft., i.e., 4 paces to each 10 ft., or 40 paces to each 100 ft., in order to simplify the conversion of the sum total of any number of paces to corresponding distance in feet. One who uses this method must train himself to maintain an even step of 2½ ft. regardless of the length of his natural step. That 2½ ft. is a good average step is indicated by its adoption by the U.S. Army.

48(*b*). DIFFICULTIES IN PACING. One can soon learn to pace distances along level, unobstructed ground with a degree of accuracy equivalent approximately to 1 ft. per hundred feet. Pacing over rough or uneven ground is much more difficult. Not only is one likely to shorten or lengthen one's steps going up and down slopes, but also a distance measured along a slope must, in some way, be reduced to the length of its horizontal projection, either by changing the length of pace or by making allowance in some other way. One can learn only from practice and experience how to do this. Similarly, when pacing a line that passes through an obstacle, one must devise some method of ascertaining the length of that part of the line covered by the obstacle; in this, one will be helped by a knowledge of the various methods used in measuring with a tape when there is an obstacle between two points on the line.

48(*c*). COUNTING PACES. The distance covered by either foot from where it leaves the ground to where it again rests is called a *stride* and obviously

is equal in length to *two* paces. It is somewhat simpler to count strides instead of paces unless some mechanical device is used for keeping tally. Common mechanical devices are the *tally register*, in which each pace is registered by pressure from the hand; the *passometer*, an instrument shaped like a watch, which, carried in the pocket or attached to one leg, registers automatically the number of paces; and the *pedometer*, a device similar to the passometer except that, adjusted to the length of pace, it registers the total distance covered by any number of paces.

49. Linear Measurements by Means of a Wheel. A horizontal distance may be measured by counting the number of revolutions that a wheel of known circumference makes in passing over the distance. The well-known speedometer works on this principle. The *odometer* is an instrument for registering the number of revolutions of a wheel. A device composed of a wheel and an odometer set in a light framework can frequently be used to advantage in measuring distances. The precision is greater than that for pacing, and over smooth ground it may almost equal that obtained by seemingly much more accurate methods of measurement. To keep the wheel from slipping, the rim of the wheel is sometimes omitted or the spokes are made to project beyond the rim. The allowable limits of error which permit linear measurements by means of a wheel do not ordinarily make it necessary to reduce slope distances to a horizontal basis unless a slope is unusually steep. The obvious disadvantage of a wheel for measuring distances is that it can be used only over ground where a wheel can roll readily.

50. Chaining. A term which originally meant measuring with a chain, but is now used to denote measuring with either a chain or a tape.

Remark. Although the chain is no longer in general use, the term *chaining* is well established, has historical significance, and a meaning that is unmistakable. Little is gained and perhaps something is lost by changing to the term *taping,* which, incidentally, is as suggestive of the athletic field as it is of the field of surveying. *Tapeman* for *chainman* is still worse in its implications.

51. Importance of Chaining. Linear measurement is the basis of all surveying. No matter how accurately angles may be measured, the survey can be no more precise than the chaining. Not only students but also chainmen in actual work are likely to look upon chaining as drudgery calling for no special ability. As a matter of fact, wide-awake, careful chainmen are scarce and are valued accordingly. It is not easy to avoid mistakes, nor is it a simple thing to chain a line when great precision is required. To chain a line twice and get the same result is no indication of great precision unless certain constant errors have been eliminated. It is worth while, therefore, to study the subject of chaining carefully, especially anything that will help to avoid mistakes and eliminate errors.

52. Care of the Steel Tape. Steel tapes are easily broken. The chief danger is in pulling on the tape when there is a loop or kink in it; consequently guard against this. Do not jerk the tape needlessly, or step on it when it is on soft ground, or allow vehicles to run over it, or bend it into or around sharp corners. If the tape gets wet, wipe it dry before putting it away. (See also Art. 755.)

53. To Do Up a Steel Tape in a Figure 8. If a steel tape is gathered in in 5-ft. lengths *without being allowed to turn over,* it will fall of itself into concentric folds. After the tape is all in, these folds may be crossed and tied at the center, forming a figure 8. This form is very convenient, and in some respects it is better than winding the tape on a reel. Any quick method of bringing the 5-ft. marks together in one hand may be used. The following is recommended: Stretch the tape straight on the ground its entire length. Stand at one end squarely facing the tape, so that you look along its length. Take the end of the tape in the right hand, *palm of the hand up,* and swing the right hand behind you, *pulling the tape through the left hand,* held palm up. As each 5-ft. mark comes along, the left hand seizes it and lays it over the palm of the right hand *without permitting the tape to turn over.* The right hand closes on it and swings behind for another 5-ft. length, and thus the tape is gathered in in 5-ft. lengths. The swing behind is about as long as the average man can take readily. Once the knack is acquired it is hardly necessary to watch for the 5-ft. marks, and the whole thing is done very rapidly.

53(*a*). Note. After a steel tape has been done up in a figure 8 it may be twisted into a still more convenient form as follows: Tie the ends of the tape, but leave untied the middle of the 8 where the sides cross, thus allowing the tape to hang in one loop. Grasp opposite sides of the loop where they cross in the 8 and pull them apart, thus stretching the 8 crosswise. If properly done the tape will turn into two circular loops which may be brought together as one circle about the size of one loop of the 8. To undo the tape reverse the process, allowing the tape to spring back into a figure 8. Several trials may be necessary before the "trick" is learned, but the form is such a convenient one for carrying tapes that every chainman should learn to use it.

54. To Undo a Steel Tape (when it is in a figure 8). Hold in one hand and let out in 5-ft. lengths, *one at a time,* the reverse of doing it up. Attempting to let it out all at once will almost surely result in an annoying tangle.

55. To Read a Tape. The *zero point* of a tape and the corresponding final graduation point at the other end may be (1) at the extreme ends of the holding rings, (2) at the extreme ends of the steel ribbon, or (3) on the ribbon near the ends. When in doubt compare the end foot with an intermediate foot. Too often mistakes occur from taking the wrong point even when the chainmen know perfectly well where the zero point and the final graduation point are. An intermediate graduation point is at the *center* of the corresponding graduation mark.

Many errors in chaining are due to mistakes in reading the tape. These are most likely to occur when the tape used is so graduated that the decimal part of any measurement must be obtained from

the decimal subdivisions in an end foot, there being no such subdivisions in any of the intermediate feet. In order to avoid mistakes certain systematic methods of procedure in reading the tape should be followed, and these methods will be explained in subsequent articles.

55(*a*). To read a Gunter's chain. When a Gunter's chain or a 66-ft. tape graduated in links is used, each *link* is 1/100 of a *chain* (Art. 8(*g*), and links are recorded as decimal parts of a chain. Thus, 8 chains and 39 links are 8.39 chains, and 8 chains and 8 links are 8.08 chains, *not* 8.8 chains. When one has learned to measure distances correctly with a tape, one should be able to do so with a chain. Hence nothing further will be given concerning the use of the chain.

56. General Method of Chaining. The two chainmen are called *head chainman* and *rear chainman*. In ordinary work, however, whichever chainman happens to be nearer the point toward which the chaining is being done acts as head chainman. The method of chaining depends upon whether the distance to be measured is less or greater than the length of tape as well as upon how the tape is graduated.

57. To Measure a Distance Less Than the Length of the Tape. When the tape is graduated continuously from zero with every foot subdivided into tenths or into tenths and hundredths, the method obviously is simple. One chainman holds the zero point of the tape at one end of the line to be measured, and the other chainman reads the tape at the other end. When, however, only the foot at one end or at each end of the tape is subdivided, both chainmen must take part in the reading.

Assume that it is desired to measure a distance from *A* to *B* that is less than the length of the tape, and that the decimal part of a foot in the measurement must be read from the subdivisions on the first foot of the tape, i.e., between the zero point and the first foot mark. One chainman holds the zero point at *A*; the other chainman notes what whole foot mark is just beyond *B*, holds that mark at *B*, and calls out its number. The chainman at *A* tightens the tape, thus moving the zero point back from *A*; he then *mentally subtracts 1 ft.* from the number called out by the chainman at *B* and to the result he adds the decimal part of the foot from the first foot mark to *A*. As he calls out the final reading, the chainman at *B* should listen carefully to make sure that 1 ft. has been subtracted from the reading that he first gave, and if it has been, he calls out "Check" or "All right."

57(*a*). Illustration. Assume that the distance from *A* to *B* is 60.85 ft. One chainman holds the zero point at *A*. The other chainman, stretching the tape, finds that the 61-ft. mark falls just beyond *B*, brings that mark back to *B*, and, holding it there, calls out "Holding 61." The chainman at *A* tightens the tape, thus moving the zero point beyond *A*, mentally subtracts

1 ft. from 61, and to 60 thus obtained he adds 0.85, which he finds to be the distance between the first foot mark and *A*. He then calls out "60.85" as the final reading. The chainman at *B* notes that 1 ft. has been subtracted from his own reading of 61 and calls out "Check."

57(*b*). SYSTEMATIC PROCEDURE. Students are urged to follow carefully the method just given, at least until they have had experience enough to adopt some other equally good. Unless some such systematic method is adopted and generally followed, there is sure to be trouble. It is particularly important to note that the chainman who is at the end of the tape always makes the final reading in order that there may be no question as to whether the decimal part of a foot is added or subtracted. Suppose, for example, that in the preceding illustration the chainman at the end of the tape simply calls out "0.85"; if the other chainman has to make the final reading, he may think that it is 0.85 ft. from the zero point or 0.85 ft. "off" and will then give as the final reading 60.15 instead of 60.85. It should be understood, moreover, that the chainman who is not at the end of the tape *must never subtract a foot* before calling out the original reading in even feet. If, for example, the chainman at *B* in the preceding illustration should subtract one foot from 61 and call out 60, the chainman at the end of the tape, following the usual procedure, would subtract another foot and give as a final reading 59.85 instead of 60.85 ft. Should this happen, the chainman at *B* should be alert enough to "catch" the mistake—but it should not happen.

57(*c*). TO LAY OFF A DISTANCE LESS THAN THE LENGTH OF THE TAPE ALONG A GIVEN LINE AND FROM A GIVEN POINT. The procedure is the reverse of that used in measuring the distance between two given points. It may be illustrated as follows: Let it be required to lay off 41.72 ft. The chainman at the zero end of the tape notes the point in the subdivided end foot that is 0.72 ft. from the *first* foot mark (not from zero) and holds that point at the given end of the distance to be laid off. The other chainman adds one to 41, and, calling out "Holding on 42," marks opposite that graduation the required point. The first chainman checks the even number of feet by noting that the second chainman called out "42" and not "41" or some other number. Before the second chainman can mark the required point, it may be necessary to "line in" that point in order to bring it on the given line.

57(*d*). MISTAKES. The mistakes that are likely when only the end foot of a tape is subdivided occur mainly because part of any measurement is read by one chainman and part by the other, and also because it is necessary to subtract or add one foot to the number of feet read on the tape. To avoid these mistakes, more or less elaborate precautions are necessary, and they complicate methods of procedure. Moreover, in measuring between two given points, time is lost in stretching the tape in its trial position and then shifting it to its final position. The chances for mistakes are fewer and the methods of chaining simpler when every foot of the tape is subdivided. The first cost of such a tape is greater, but it is more than made up by the time saved and the mistakes avoided.

57(*e*). IN MEASURING A NUMBER OF SHORT DISTANCES FROM THE SAME POINT, as from a station, it is best to keep the zero end of the tape ahead or away from the point.

57(*f*). TO AVOID MISUNDERSTANDINGS BY THE NOTEKEEPER, certain precautions should be taken habitually, particularly in calling out any measurement that sounds like some other. For example, if a chainman calls out "Fifty-three" the notekeeper might record it as 53.0 or 50.3 or 50.03. Not only should the chainman call the number in a way to avoid any misunderstanding, but the notekeeper should also invariably repeat the number, preferably in a different form of wording, so that the chainman can correct any misunderstanding, should one occur. For example:

Chainman: For 53.00, "Fifty-three *even*."
Notekeeper: "Fifty-three exact," or "Fifty-three—zero—zero."
Chainman: For 50.30, "Fifty—point—three—zero."
Notekeeper: "Fifty and thirty hundredths" or "5—0—3—0."
Chainman: For 50.03, "Fifty—point—zero—three."
Notekeeper: "Fifty and three hundredths."

A notekeeper may also misunderstand from what point or to what point a measurement is taken. Here again both the notekeeper and the chainman must be on guard in order to avoid a mistake in recording. It is one thing to take a measurement correctly; it is something additional to make sure that it is entered in the notebook correctly.

57(*g*). TAPES WITH A SUBDIVIDED FOOT AT EACH END make it easier to use either end, whichever is the more convenient, in getting the decimal part of a measurement. This advantage is offset to some extent at least because mistakes in reading the even number of feet are more likely to occur than when only the foot at the zero end is subdivided. This may be illustrated as follows. A distance from *A* to *B* of 25.50 ft. is measured with the subdivided foot at the 100-ft. end of the tape held at *A*. The chainman at *B* observes that the 74-ft. graduation is just beyond *B*. Unless the tape is graduated from zero in *both* directions he must subtract 74 from 100 before calling out the reading of 26 ft.; he may make a mistake in subtraction or he may forget to subtract. The latter mistake is most likely to occur when the distance measured is equal to approximately half the length of the tape. For example, to read 49 instead of 51, or 45 instead of 55. Whenever there is any doubt concerning the correct number for a foot mark *read the number for the foot mark first on one side and then the other* of the required reading.

When a tape is graduated on one side from zero at one end and on the other side from zero at the other end, the two sets of graduations running in opposite directions are likely to lead to confusion in reading numbers at foot marks. To avoid mistakes, observe the number on each side of the required reading as suggested in the preceding paragraph.

57(*h*). TAPES WITH AN EXTRA FOOT. The use of a tape with an extra foot may be illustrated by a 101-ft. tape. Starting with the zero point one foot from one end, graduations for feet are numbered from 0 to 100. The extra foot at the zero end is subdivided. In the use of this tape the number of even feet in a measurement can be read directly without mentally subtracting one foot. The same purpose may be accomplished with a 100-ft. tape by graduating it in a similar manner for 0 to 99; the zero mark is one foot from one end and that foot is subdivided. The method of measuring with either of these tapes is the same in principle as that of measuring with an "architect's" scale on which the space between the zero point and the end is equal to one unit of measurement and that space is subdivided. Either type is preferable to that on which the end foot is subdivided but the zero point is at the end instead of one foot from the end. When a tension handle is used with an extra end foot type of tape, it is generally better to carry the 100-ft. end forward.

58. Summary of Common Mistakes in Reading Tapes. (1) Use of the wrong point on the tape for "zero" or "100." (2) Mistaking a numeral that is indistinct from wear or other cause. (3) Reading a numeral upside down, for example, a 6 for a 9 or vice versa. (4) Mental transposition of figures, for example, 27.87 for 27.78. (5) Failure to subtract or to add one foot to the even number of feet when only the end foot of a tape is subdivided, for example, 68.52 for 67.52 (Art. 57(*a*)). (6) Reading the tenths and hundredths in the end foot from the zero mark instead of from the first foot mark or vice versa. Reading from the wrong end of the tape, for example, 49 instead of 51 or 55 instead of 45. (7) Failure by the chainman who is reading the decimal part of a foot at the end of a tape to understand correctly the number of even

feet called out by the other chainman (Art. 57(b)). (8) Reading the wrong foot on a fully graduated tape, particularly when the decimal is like 0.95, near the higher foot mark.

It is to be noted that even when no mistake has been made in reading, one may still occur in calling out that reading to the note-keeper. For example, 53.0, 50.3, and 50.03 may all sound alike (Art. 57(f)).

59. To Measure a Distance Greater than the Length of the Tape. Four things must be kept in mind: (1) The chaining must be done in a direct line. (2) The tape must be stretched tight for each tape length. (3) The end of each tape length must be marked on the ground in such a way that it may be used as the beginning of the next tape length. (4) The number of tape lengths must be counted correctly. The best method of procedure is that which does all this in the shortest time *consistent with the degree of precision required.* For purposes of alignment a range pole may be set "on line," preferably at the far end. The rear chainman, sighting on this pole, "lines in" the head chainman as he marks the end of each tape length; in more accurate work this is done by means of the transit. To mark the tape lengths, chaining pins may be used. For work of high precision, stakes may be set, and a knife mark or tack on each stake indicates the exact point. On ground too hard for pins or stakes, nails or spikes, set in cleared smooth spots with their heads flush with the ground, are often used. They may be left there for possible checking. On hard smooth surfaces such as pavements, points may be scratched or marked with crayon. Some reliable method should be adopted for keeping count of tape lengths. When chain pins are employed, this may be done by counting the number of pins that have been used in marking the tape lengths. Two good chainmen will soon discover for themselves what method of procedure in chaining is best adapted to the work that they may be doing. One method suited for ordinary work is suggested in the following illustration:

59(a). ILLUSTRATION. Required to chain from *A* to *B*, distance 634.3 ft. with a 100-ft. tape. Rear chainman (R.C.) holds the 100-ft. end of the tape at *A* and motions the head chainman (H.C.) into line, sighting on a pole held or set at *B*. R.C. sees that the tape is straight and tight, and calls out "Stick," or "All right," and H.C. "sticks" the pin at the end of the tape and calls out "Stuck." R.C. *drops* his end of the tape and H.C. moves on, dragging the tape after him and keeping himself approximately in line by looking at *A* over the pin already set. When the 100-ft. end of the tape reaches the pin, R.C., who is following, calls out "Chain," a signal for H.C. to stop. R.C. then holds the 100-ft. graduation at the pin and the process is repeated. The short distance from the last pin to *B* (34.3 ft.) is measured by one of the methods already explained for distances less than a tape length. Counting the last pin that was set, the rear chainman, if he gathered up the pins as he went along, should have six pins, which tells him that up to that point the tape was stretched six times, or it is 600 ft.

59(*b*). Zero end ahead. In the illustration just given, the *zero* end of the tape is kept *ahead,* making it possible to use the zero end in measuring the short distance between the last pin and *B* without dragging the tape beyond *B*. Similarly, it is more convenient to have the zero end ahead in laying off one or more intermediate points at given distances from the starting point.

59(*c*). In setting chaining pins, it is usually best to stick each pin into the ground slanting, at right angles to the line so that it slides under the tape, with its *center* opposite the graduation point on the tape. On the whole, this is better than sticking a pin with its *edge* against the end of the tape or opposite a graduation. Similarly, in the other methods of marking tape lengths, it is best to have the end of each tape length marked by the center of the stake, spike, scratch or crayon mark.

59(*d*). Lining in. In ordinary work "lining in" may be done as follows. The rear chainman moves his hand right or left to signal the head chainman into line—a vigorous motion, move quite a distance—a slight motion, a small distance. Much time may be saved if the head chainman will line himself in, approximately, by looking backward over the head of the rear chainman to the point or station from which the measurement is being made. In more precise work the transitman "lines in" the head chainman, in which case the rear chainman must keep out of the line of sight. The head chainman frequently carries a range pole to be sighted on line. Do not split hairs in lining in. (See "Error in Alignment," Art. 72.)

59(*e*). Keeping count of tape lengths. When chaining pins are used, some chainmen prefer to start with one pin in the hands of the rear chainman, so that the number of tape lengths up to the point marked by any pin will be shown by the number of pins held by the rear chainman without counting the pin left in the ground at that point. For 1000 ft. this method would require eleven pins, and hence surveyors' pins or arrows are usually sold in sets of eleven. When pins are not used, chainmen should agree upon some good method of avoiding mistakes. Keeping count of tape lengths may seem to the beginner too simple a process to require special attention, but the experienced chainman knows, alas, how easy it is to make a mistake of an even tape length, such as "dropping" 100 ft. from a measurement. One good precaution is for both chainmen to keep tally independently. Probably the best precaution of all is for the head chainman to mark the figure for each hundred with crayon on the pavement, or on a stone picked up as he moves forward, or to scratch the figure in the ground. When marking, he calls the hundred and the rear chainman looks at his figure and calls the check.

59(*f*). Chainmen should form the habit of estimating the distance measured whenever this is practicable, even when the distance is less than the length of the tape. It is absurd, for example, to call a distance 40 ft. when obviously it is nearer 60 ft., or a distance 300 ft. when it is readily seen that it must be 400 ft. When it becomes "second nature" for a chainman to check distances roughly by eye, he has acquired, perhaps, the best method of all for avoiding serious mistakes. The transitman may check distances also by self-stadia observations on the foot graduations of a range pole—a check against mistakes over about 10 ft.

60. Chaining on a Slope. In measuring a slope there are two ways of getting the *horizontal* distance, which is the distance usually required in surveying: (1) to measure along the slope and correct this measurement by calculation; (2) to take such precautions that the tape, instead of resting on the ground, will be *horizontal* every time it is stretched for a measurement.

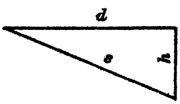


Fig. 60(*a*).

60(*a*). First method. The calculated correction for slope. Let *s* represent a chained distance on a slope, *d* its horizontal projection, *h* the difference in elevation between the two ends of the slope (determined by means of a hand level or by a regular level),

and c the correction for slope, i.e., $s - d$. The required length is $d = \sqrt{s^2 - h^2}$, and the correction c is $s - d$ equal to $s - \sqrt{s^2 - h^2}$. An approximate value for c is $h^2/2s$. When s is 100 ft. and h is less than 30 ft., the error in this approximation is less than 0.1 ft., and decreases rapidly as h decreases. For $h = 20$ ft., it is 0.02 ft. per 100 ft.; for $h = 10$ ft., it is only 0.001 ft. From this last value, it is evident that for grades of 10 per cent or under, the error is so small as to be generally negligible; for a grade of 5 per cent or under, the true value of c and the approximate value are almost identical. When the angle of inclination A is known, $d = s \cos A$, or $s - s$ versine A. The slope-measurement method is particularly useful when the surface, like that of a pavement, is smooth and straight. Under certain conditions, it is better to measure the angle of slope than to determine the difference in elevation between the two ends of the slope. This may be done with a clinometer or with a transit.

When there are many corrections for slope to be calculated, it is often better to do this calculation in the office than in the field. Calculations may be facilitated by the use of tables in which corrections for different slopes are given (Table III, p. 1104).

60(*b*). Second method. Chaining with tape horizontal. Usually it is easier and better to chain downhill than to chain uphill. The uphill end of the tape is held on the ground; the downhill end is held high enough off the ground to make the tape horizontal, and the point on the ground under the downhill end is found by using the plumb bob. The process is repeated in a series of steps, the tape corresponding to a tread, and the plumb line to a riser. Chaining uphill is the reverse of this process, but it is more difficult because the chainman at the downhill end of the tape must plumb that end over the point on the ground and at the same time hold it steady against the pull by the other chainman. In either case there is but one good way of holding the downhill end of the tape. Stand with one side of the body toward the other chainman, the tape passing across in front so that when it is stretched, the hand and arm holding the tape are pulled against the other side of the body. In this way the weight of the body resists the pull, steadies the end of the tape, and leaves one hand free to manipulate the plumb line.

When the slope is so steep that the downhill end of the tape cannot be held high enough to keep the tape horizontal, the distance must be measured in sections each of which is short enough to permit easy horizontal measurement of the corresponding portion of

the slope. When the slope is nearly uniform, the sections may be of equal length ranging from 10 ft. or less to about 90 ft., depending upon the steepness of the slope. The more general method may be illustrated as follows.

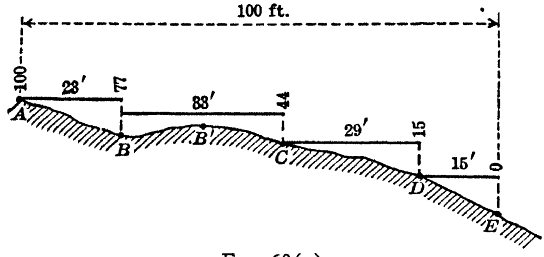


FIG. 60(*c*).

60(*c*). ILLUSTRATION. Required to chain horizontally on a slope, beginning at *A*. The rear chainman (R.C.) holds the 100-ft. end of the tape at *A* and the head chainman (H.C.) carries his end downhill to a point on line approximately 100 ft. from *A*. He then returns to a point *B*, which is about as far from *A* as he thinks he can stand and still hold the corresponding portion of the tape horizontal. He observes that the nearest foot mark is 77, and, holding the portion of the tape between this mark and the 100-ft. mark horizonal, he uses the plumb line to set a pin at *B* directly below 77. Leaving the tape approximately in its original position along the ground, R.C. comes forward to hold the 77 mark at *B*, while H.C. proceeds to some point *C*, say 33 ft. from *B*, where he plumbs down from the 44-ft. mark to establish the point *C*. R.C. then holds the 44-ft. mark at *C* while H.C. plumbs down from, say, the 15-ft. mark to *D*. R.C. then holds the 15-ft. mark at *D* while H.C. plumbs down from the zero mark to establish *E*, the point that is a tape length from *A*. The tape, which all this while has been left approximately in its original position, is now dragged its full length downhill, thus bringing the 100-ft. mark at *E*, and the process of "breaking tape" can then be repeated for the next 100 ft. In this way the entire distance down the slope, whatever it may be, can be chained, and as each tape length is laid off, the tape itself serves to make the sum of the secfions into which that tape length was subdivided invariably 100 ft. When pins are used to keep count of tape lengths, pains should be taken to avoid counting any pin that marks a point between the ends of the tape.

It would appear from the figure that in measuring from *B* to *C* the tape would necessarily be held off the ground at both points. The use of *two* plumb lines which this would involve adds to the difficulty of measurement, and this difficulty may be avoided in this case by establishing some intermediate point B′ where the tape can be held on the ground.

Assume that the horizontal distance between the two ends of the line to be chained is 375.6 ft. When the 300-ft. point has been reached, R.C. holds at that point the "75.6" graduation, and the remaining distance of 75.6 ft. may then be chained by breaking tape as many times as may be necessary. When only the end foot of the tape is subdivided, R.C. holds the "76" graduation at the 300-ft. point before beginning to break tape.

60(*d*). PRACTICAL SUGGESTIONS. There is usually a tendency to hold the downhill end of the tape too low; this may frequently be avoided by observing horizontal lines such as those on buildings. In rough country or in work of rather high precision some device such as a hand level should be used. The error due to sag can be avoided to some extent by increasing the pull. Use short lengths of the tape, the steeper the slope the shorter the length. On very steep slopes use 10-ft. lengths or shorter. Dropping pebbles, chain pins, or plumb bob from the raised end of the tape to find the point underneath will give only approximate results. The plumb bob may be lowered slowly until its point is just above the ground, and, when it has stopped swinging, it may be lowered gently to mark the point. The transit pole, held vertical by means of a plumb line, is sometimes used. Accurate results may be ob-

tained by driving stakes in line, short distances apart, and measuring between them, using the plumb line; this will often require more than two chainmen. Better results still will follow if a tripod is placed over the downhill stake to support the end of the tape and hold the plumb line steady. The tripod belonging to the transit or level, detached from the instrument, will answer the purpose, or, if much of such chaining is to be done, construct two rough tripods. Four men with two tripods can chain a steep slope accurately and quickly. While three of the men are measuring between two stakes, a fourth man can be setting the tripod which is not in use over the next downhill stake. For work of high precision special tripods may be used, made of wood or metal, with copper strips fastened to their tops to facilitate marking distances (Art. 100(*a*)).

61. Precautions to Insure Accuracy. In ordinary work no account is taken of small errors in the length of the tape, and temperature, pull, and sag are neglected. Ordinarily chain pins or other approximate methods of marking points are used. The rear chainman lines the head chainman in, and slopes are measured by holding the tape horizontal judged by the eye alone. In work of high precision the length of tape is carefully tested; the temperature is noted during chaining and the correction applied; the pull is kept standard with the spring balance; the tape is supported at short intervals to eliminate sag, or the sag correction is determined; some device, as a fine scratch on top of a stake, marks the chain lengths in place of the chain pins. The chainmen are lined in from the transit, or stakes are set in line for the use of the chainmen; the tape is kept horizontal by leveling, or if measurements are made along the slope, the corrections for slope are determined with corresponding care. Whatever degree of precision may be required, the chainmen should know what methods to use and what precautions to take to insure that degree of precision. (See the next chapter.)

62. Errors. Some of the things which chainmen should know concerning errors in chaining are:

1. Sources and relative importance of errors.
2. Methods for eliminating or correcting errors.
3. Customary limits of error for different kinds of work.
4. Field work requirements corresponding to different limits of error.

These subjects are discussed more fully in the next chapter.

63. Summary. (From preceding pages.) (*a*). SOME THINGS A CHAINMAN SHOULD DO.

1. Find the exact points which mark the ends of a tape, i.e., the zero point and the 100-ft. point on a 100-ft. tape (Art. 714(*h*)).
2. Find by what method the tape is graduated, and note how it must be read.
3. Agree on a set of signals with the other members of the party.
4. Find with a spring balance what an 8- or 10-lb. pull feels like.
5. See that the tape is straight, and exert a steady standard pull while measuring.
6. Ascertain the allowable limit of error, and work accordingly.

7. Line in and stick the chain pins correctly, or mark points accurately by some method (Art. 59(*c*)).

8. Keep the right count of tape lengths (Art. 59(*e*)).

9. Check a reading by observing the number on each side of it; always note in which direction numbers are increasing.

10. When only the end foot is subdivided the chairman at that *end* of the tape should invariably make the final reading (Art. 57(*b*)). Keep in mind the mistakes likely to occur in reading this type of tape (Art. 58).

11. Form the habit of checking measurements roughly by eye (Art. 59(*f*)).

12. The head chainman should keep himself approximately in line by looking backward over the line. He should lift his end of the tape to straighten it, and then let it down slowly on the line to be sure that it remains straight. In accurate work, after the pin is stuck, he should lift the chain as before and let it down again to test the position of the pin. Frequently this test will show that the pin has not been set in exactly the right place. Similar precautions should be taken when any other method of marking points is used.

13. Chain downhill on a slope rather than uphill. Keep the tape horizontal—the tendency is to hold the downhill end too low. Use short lengths on steep slopes and use the method of "breaking tape." Take pains in plumbing down. Neutralize the effect of sag by increasing the pull.

14. In work of high precision, use the spring balance and thermometer, keep the tape horizontal by leveling, and eliminate sag by supporting the tape whenever possible. In such work it is more necessary than ever that the tape should be standardized.

15. In calling out a measurement to a notekeeper or other member of the party, make sure that he gets it by having him call it back to you. Remember the types of called numbers that are most likely to be misunderstood and take corresponding precautions (Art. 57(*f*)). Make certain that a notekeeper understands between *which* two points a measurement is taken.

16. Keep in touch with the work so as to notice any measurements inadvertently omitted, and also to anticipate directions from the head of the party.

17. Be constantly on the alert to preserve the tape from injury. Avoid kinks and bends by keeping the tape fully extended. When a short distance is to be measured, there is a strong temptation to pull the tape back into a pile of loops in order to use a short portion of it. Many tapes are broken in this way. If necessary, turn the tape around end for end, but keep it extended. When not in use, stretch it beside a fence or in some out-of-the-way place where it will not be stepped on.

63(*b*). Some things a chainman should not do.

1. Don't jerk the tape, or pull on it when it is kinked, or let vehicles run over it, or bend it sharply into or around corners, or step on it. Don't lose the handles; tie them on.
2. Don't split hairs lining in.
3. Don't disturb the chaining pin, or let it be pulled over.
4. Don't let go of the tape until the other chainman signals.
5. Don't stick the pin until the other chainman signals.
6. Don't pull up the pin until sure that it will no longer be needed.
7. Don't hit the leg of the transit when measuring near it.
8. The rear chainman should not get in the line of sight when the transitman is lining in the head chainman.
9. If the tape is in a leather or metallic box, do not keep pulling the tape out and winding it up. If much work is to be done, it may be better to detach the box.

64. Chaining with the Tape Supported off the Ground. Thus far in this chapter it has been assumed that the chaining is done with the tape on the ground or, on slopes, with one end on the ground. In highly accurate chaining, however, the tape is generally kept off the ground by stakes, or by small tripods ("bucks"), or by supports of some other type, in order to reduce certain errors. The tape may be supported at two points only (the *two-point* method, sometimes called the "buck method"), or it may be sup-

ported at more than two points. The field procedure for this kind of chaining is explained in detail in Art. 99.

65. Chaining with a High Degree of Precision. In order to chain with a high degree of precision the chainmen should have a thorough knowledge of: (1) the various sources of error in chaining; (2) how to correct or to eliminate the error from each source; (3) to what extent the error from each source should be eliminated in order to attain a desired degree of precision. The theoretical knowledge required to answer these three implied questions may be gained from a study of the next chapter. The practical field methods used in chaining with a high degree of precision are based on this theoretical knowledge. The methods are best understood after one has had experience in ordinary chaining and after a thorough study of the whole subject of errors in chaining. To chain a line with a high order of precision requires great skill and care; it is one of the basic problems in chaining. The field procedure for this precise chaining is explained in detail in Chapter 8.

CHAPTER 6

ERRORS IN LINEAR MEASUREMENTS

In this chapter sources of error in chaining are pointed out, the relative importance of errors from different sources is discussed, and methods of correction or elimination are given. Limits of precision are suggested for different kinds of work, and field work requirements are indicated for different limits of precision. A general classification of errors is given (Art. 36), and it is necessary in studying this chapter to keep clearly in mind the difference between *accidental* and *constant* errors, and between *cumulative* and *compensating* errors. The distinction between *discrepancy* and *error* is also an important one, since the discrepancy between duplicate measurements may or may not indicate the precision of the work (Art. 37).

66. Sources of Error in Chaining.

1. Error in the length and in the graduations of the tape.
2. Errors due to temperature and to other natural sources.
3. Errors in manipulation: tape not horizontal, not stretched tight, imperfect alignment, sag, pull, and marking tape lengths.
4. Personal errors due to limitations of human powers of observation, particularly the inability to see exactly. Mistakes in reading the tape. Blunders, such as occur in counting tape lengths. (Mistakes and blunders are treated in the preceding chapter.)

67. Error Expressed by a Ratio. $1/500$, $1/1000$, $1/12{,}000$, etc., mean an error of 1 part in 500, or 1 part in 1000, or 1 part in 12,000, and so on.

Illustration. An error of 0.05 ft. in a measurement of 714.85 ft. is: $0.05 \div 714.85 = 1/14{,}297$, or about $1/14{,}300$.

68. Error in the Length of the Tape. *Cumulative* and important. *Plus* or *minus* according to whether the tape is too short or too long. If a tape is *too short*, the measured distance between two fixed points will be *longer* than the true distance (error plus); and contrariwise, if the tape is *too long*, the measured distance will be *shorter* than the true distance (error minus).

68(*a*). Illustration. The *true* distance between two points is 400 ft. If, however, this distance is measured with a 100-ft. tape which is 0.02 ft. too short, i.e., length = 99.98 ft., the result will be $400 \div 0.9998 = 400.08$ ft. If, on the other hand, the tape is 0.02 ft. too long, i.e., length = 100.02 ft., the result will be $400 \div 1.0002 = 399.92$ ft.

68(*b*). To apply the tape correction. The tested length of a 100-ft. tape multiplied by 0.01 gives the length of the tape in terms of the standard, i.e., in terms of 100 ft. Thus a tape 0.1 ft. too long is 100.1 ft. long, or 1.001 times what it should be. Call this constant for any tape *C*, then use the following rule.

Rule. Multiply all measurements by C to reduce to true measurements.

68(*c*). Illustration. In the preceding illustration the constant for the tape which is 0.02 ft. too short is 0.9998, and for the tape which is 0.02 ft. too long it is 1.0002; hence

Measured distance	×	Constant	=	True distance
400.08	×	0.9998	=	400 ft.
399.92	×	1.0002	=	400 ft.

In establishing one point at a required distance from another point the rule is reversed. *Rule. Divide the required distance by C; the result is the actual (false) distance to measure off.*

68(*d*). Illustration. It is required to establish one point 400 ft. from another point with a tape 0.02 ft. too short, i.e., $C = 0.9998$. Then the *false* distance which must be measured off is $400 \div 0.9998 = 400.08$ ft.

68(*e*). Standardization of the tape means: the comparison of the tape with a standard of length in order to determine the error, if any, in the length of the tape, and thus to ascertain the corresponding correction constant *C*. This is the basic test in the standardization of tapes. Other tests, made for various purposes in standardizing tapes, are explained in detail in Chapter 52.

69. Error Due to the Tape Not Being Horizontal. Error is *cumulative, plus,* and important. Eliminated by the method of Art. 60(*a*) or 60(*b*).

70. Error When the Tape Is Not Stretched Tight and Straight. (Ends in line, tape lying on surface of ground, but not stretched tight.) *Cumulative, plus,* and important.

71. Error Due to Temperature. Error is *cumulative—plus* when the temperature is below the standard temperature for the tape, *minus* when it is above. This is one of the most important sources of error. The length of a 100-ft. steel tape may be changed more than ½ in. by an extreme change in temperature from 20° to 90° F. The following rate of change is, perhaps, the easiest to remember: *The length of a 100-ft. steel tape changes approximately 0.01 ft. for every 15° F. change of temperature.* Tapes are standardized at 68° F. In ordinary work temperature may be neglected, although, if the variation from 68° is extreme, it may have to be taken into account even in that type of work. In measurements of high precision the error due to temperature must, as far as possible, be eliminated by noting the temperature and applying the correction. It is not always easy to obtain the temperature of a tape, particularly if it is in the sun, or partly in the sun and partly in the shade. For this reason, measurements of high precision, such as those of base lines, should preferably be made on a cloudy day or even at night in order that the tape may be nearer the temperature of the atmosphere.

71(*a*). The correction for temperature to be applied to any measured distance *D* is:

$$C_t = \text{Coefficient of expansion multiplied by } D(t_1 - t_0) \qquad \text{(I)}$$

in which t_1 is the temperature of the tape when the measurement is

taken and t_0 the temperature of the tape when it was standardized. If the temperature of the tape is higher than the standard temperature, the correction C_t is to be *added* to the measured distance D; if the temperature of the tape is lower, the correction is to be *subtracted.*

The average coefficient of thermal expansion for steel tapes is 0.0000065 per 1° F.; for invar tapes it is 0.0000002 for 1° F., or about one-thirtieth that for steel. The change from Fahrenheit to centigrade or vice versa may be made from the relation 1° C. = 1.8° F. Hence the corresponding *average* coefficients are: for steel tapes, $0.0000065 \times 1.8 = 0.0000117$ per 1° C.; and for invar tapes, 0.00000036 per 1° C.

A table of temperature corrections per foot of tape based on the coefficient of expansion for the tape facilitates computations. Such a table for a steel tape is given on p. 1103.

71(*b*). NOTE. It has been found that an invar tape with a coefficient of thermal expansion as small as 0.0000002 is likely to be noticeably more unstable in length than one with a coefficient of 0.000001 or 0.0000015, and for this reason there is, at the present time, a tendency away from invar tapes with very low coefficients.

71(*c*). NOTE. If the coefficient of expansion of a steel tape is not known, use 0.0000065 per 1° F. or $100 \times 0.0000065 = 0.00065$ ft. change per 1° F. for a tape 100 ft. long. Thus $70 \times 0.00065 = 0.0455$ ft. for extremes of 70° change for a 100-ft. tape.

71(*d*). REMARKS. The error due to temperature may be quite large, and in certain lines of work it is too often neglected by surveyors. A city block, for example, 400 ft. to 500 ft. long, measured with the same tape may be found apparently 2 in. or 3 in. shorter in summer than in winter, if it is in a climate where the thermometer ranges from 0° to 90° F.

71(*e*). AN EXPERIMENT. In order that the effect of even a small change in temperature may be perfectly evident, and incidentally to show also the effect of the sun's rays, it is recommended that students be required to make the following experiment. About noon on a clear day a 100-ft. steel tape is stretched across supporting stakes, set on line 20 ft. apart, with their tops on a level. A 10-lb. pull is kept by means of a spring balance, and a fine scratch is made on the top of the stake at each end of the tape. At about 5 P.M. of the same day the tape is again stretched under the same conditions. The scratches made at noon are often found to be ⅛ in. or more farther apart *apparently* than when they were made. This shrinkage of ⅛ in. in the tape corresponds to a fall of 15° in temperature, but the change in the atmosphere may have been much less than this. A large part is due to the sun, which at noon shone on the tape with much greater intensity than at five o'clock. Still more striking results may be obtained by repeating the experiment with the tape stretched in contact with the warm ground or a hot pavement.

72. Error in Alignment. Error is *cumulative* and *plus*—relatively unimportant. Much time is often wasted in splitting hairs when lining in with the transit. Setting pins as much as 6 in., first one side and then the other of the line, affects measurements but little.

72(*a*). NOTE. Let l = the length of a tape. Assume that one end of the tape is "on line" and the other a distance x "off line." The error e for a tape length is approximately $e = x^2/2l$ (Art. 60(*a*)). The effect on chaining is the same as if the tape were too short by an amount e, and hence the error

is cumulative and plus. Assuming that the front end of the tape is brought on line at the end of the next tape length, there is an equal error for the second tape length, so that the distance x really causes an error of $2c$. When $l = 100$ ft. and $x = 1.5$ ft., c is approximately 0.01 ft., that is, the error is only 0.01 ft. per 100 ft. or 0.02 ft. in two tape lengths. As a general rule, the end of the tape held by the head chainman is not likely to be 1.5 ft. off line so that the error is ordinarily even less.

73. Errors Due to Variation in Pull. If the pull is always too large or always too small, the error becomes cumulative; otherwise it may be compensating. Pull is unimportant except in work of high precision. Ordinarily it is guessed at; it should be strong but steady, 8 to 12 lb. One can guess nearer if he actually experiments with a spring balance a few times. An ordinary 100-ft. steel tape stretches 0.001 to 0.002 ft. for each pound of pull in excess of the pull used in standardizing the tape. In precise work errors due to variation in pull may be eliminated by the use of spring balances (Art. 95(e) and (f)).

73(a). CORRECTION FOR PULL.

$$C_p = \frac{L(P - P_0)}{AE} \tag{II}$$

C_p = amount the tape is lengthened in feet; L = length of tape in feet; P = applied pull in pounds; P_0 = pull at which the tape was standardized in pounds; A = cross-sectional area of tape in square inches (approximately 0.0025 sq. in. for a light tape weighing about a pound); E = modulus of elasticity (28,000,000 to 30,000,000 in pounds per sq. in.). The cross-sectional area of any tape may be calculated from its length and weight from the relation: 3.5265 cu. in. of steel weighs 1 lb. For example, a tape 100 ft. long that weighs 1 lb. has a cross-sectional area of $A \times 1200 = 3.5265$ cu. in., or $A = 0.002939$. If the tape weighs 2 lb., its cross-sectional area will, of course, be twice as great.

The correction C_p may be determined experimentally for the entire length of a tape, or for any portion of it, by direct measurement of the changes in length caused by variations in pull as observed on a spring balance. Obviously it will require a greater pull to stretch a heavy tape a given amount than to stretch a lighter tape the same amount.

73(b). EXAMPLE. A 100-ft. steel tape weighing 1 lb. was standardized at a tension of 10 lb. How much will it be lengthened by an additional pull of 10 lb.?

$$C_p = \frac{L(P - P_0)}{AE} = \frac{100(20 - 10)}{0.0029 \times 29{,}000{,}000} = 0.0117$$

73(c). EXAMPLE. Same as (b) except that the tape weighs 3 lb. and has a corresponding area of about 0.0088 sq. in. $C_p = 0.00392$ ft.

Note. If the length of a measurement is 25 ft., the tape in that distance will stretch proportionately, i.e., one-fourth as much.

74. Error Due to Sag. *Cumulative* and *plus.* When a tape or any portion of a tape is supported at two points, the curve of the sag between the two points of support is a catenary. The difference in length between this catenary arc and its chord is the error due to sag. This error is important in accurate chaining, particularly when a heavy tape is used.

There are three methods of eliminating error due to sag: (1) By applying a correction that is calculated from the equation of the curve. (2) By exerting a *compensating pull* on the tape sufficient to cause an elongation of the tape equal to the shortening due to sag. The tension in the tape caused by this pull is called *normal tension.* If a tape is heavy, it is often difficult if not impracticable to exert a compensating pull. (3) By providing supports for the tape sufficiently close together to eliminate any appreciable sag.

74(*a*). THE CORRECTION FOR SAG may be calculated from the following formula based on the assumption that the curve of sag approximates a parabola:

$$C_s = \frac{w^2L^3}{24P^2} = \frac{W^2L}{24P^2} \qquad \text{(III)}$$

P = the applied pull in pounds; w = weight of tape per foot in pounds; L = distance between supports in feet; W = the total weight of the tape between supports. It is to be noted that the correction for sag varies with the cube of the distance between supports, provided the pull is kept constant in amount.

Illustrative Examples. A 100-ft. tape weighing 1 lb. is supported only at its ends. What is the correction for sag for a pull of 10 lb.?

$$C_s = \frac{W^2L}{24P^2} = \frac{1 \times 100}{24 \times 10^2} = 0.042 \text{ ft.}$$

The correction for a 100-ft. tape weighing 3 lb. is:

$$C_s = \frac{3^2 \times 100}{24 \times 10^2} = 0.375 \text{ ft.}$$

The correction for an unsupported length of 25 ft. is for the two tapes respectively:

$$C_s = \frac{L}{24}\left(\frac{wL}{P}\right)^2 = \frac{25}{24}\left(\frac{0.01 \times 25}{10}\right)^2 = 0.00065 \text{ ft. for the lighter tape}$$

$$C_s = \frac{25}{24}\left(\frac{0.03 \times 25}{10}\right)^2 = 0.00586 \text{ ft. for the heavier tape}$$

Mental checks: The correction for 25 ft. will be $(^{25}/_{100})^3 = (^1/_4)^3 = {}^1/_{64}$ of the correction for 100 ft. Hence, $^1/_{64} \times 0.042 = 0.00065$, and $^1/_{64} \times 0.375 = 0.00586$.

When the distance between the two supports is measured with the tape, the distance read on the tape is the length of the catenary. When the correction for sag is applied to this reading, the error due to sag has been eliminated and, if there were no other source of error, the reading would be the true distance between the sup-

ports. If, however, the pull exerted is not equal to the tension for which the tape was standardized, or if the temperature is not the same as that used in the standardization, the correction for pull, or for temperature, or for both may be necessary. Correction for pull, however, can generally be avoided by keeping the tension at the standard for the tape.

The correction for sag when the tape, under a standard pull, is supported at its ends only can be computed once for all. The correction for sag when the tape is suported at one end and at an intermediate point obviously cannot be taken as any known proportion of the correction for the whole length of the tape, but must be computed separately.

The U.S. Bureau of Standards will determine by test the correction to the total length of a tape when supported at the ends only, and the correction to the length of a subinterval under the same conditions as to tension and points of support, or under specified different conditions as to tension and points of support, as explained in Art. 715. The surveyor can easily make these tests himself if he has the equipment for standardizing tapes (Art. 716).

74(*b*). THE COMPENSATING PULL. When a tape or any portion of a tape is supported at two points only, the compensating pull can be computed from the following formula:

$$P_n = \frac{0.204wL\sqrt{AE}}{\sqrt{P_n - P_0}} = \frac{0.204W\sqrt{AE}}{\sqrt{P_n - P_0}} \qquad \text{(IV)}$$

P_n = the compensating pull that produces *normal tension;* P_0 = pull in pounds for which the tape was standardized; w = weight of tape per foot; L = distance between supports in feet; W = total weight of tape between supports; A = cross-sectional area of tape in square inches (see Art. 73(*a*)); E = 28,000,000 to 30,000,000 lb. The equation may be derived by equating the correction for sag to the correction for pull as given in equations III and II. The equation is generally solved by trial.

When a tape is heavy and the unsupported length is relatively great, say 100 ft., the compensating pull is too great to be practicable as shown in the following example.

Illustrative Example. A 100-ft. tape weighing 3 lb. is supported only at its ends. What is the compensating pull?

$$P_n = (0.204 \times 3\sqrt{0.00293 \times 3 \times 30{,}000{,}000}) \div \sqrt{P_n - P_0} = \frac{314.2}{\sqrt{P_n - 10}} = 49.8 \text{ lb.}$$

For a tape weighing 1 lb. the compensating pull would be 19.6 lb.

Illustrative Example. A 100-ft. tape weighs 2 lb. What is the compensating pull for 25 ft. of unsupported length? *Ans.* 16.6 lb.

The U.S. Bureau of Standards will determine by test the compensating pull or normal tension under a specified condition of support, or the surveyor can make this test for himself (Art. 715).

74(*c*). COMPENSATING PULL WHEN THE TAPE IS SUPPORTED AT THREE OR MORE EQUIDISTANT POINTS. The U.S. Bureau of Standards will determine by test the compensating pull or normal tension for a specified condition of support, including equidistant supports (Art. 715). The Bureau will also, on application, furnish details of carrying out rapidly by graphical methods the solution of the formula, given below, for the general case of a tape supported at equidistant points when the observed temperature t is not the same as the standard temperature t_s.

$$P_1^3 = [P_0 - AEk(t_1 - t_0)]P_1^2 - \frac{AEw^2d^2}{24}$$

P_1 = pull to be applied to the tape when it is used at the observed temperature t_1 and supported at equidistant points at a distance d apart, in order that the tape will have its nominal length; P_0 the tension at which the tape has its nominal length when supported throughout at standard temperature t_0; A = cross-sectional area of the tape; E = Young's modulus of elasticity; k = coefficient of thermal expansion; w = weight of tape per unit of length.

75. Errors in Marking Tape Lengths. *Compensating,* except for very short lines; important. When chain pins are used, observe the directions in Art. 59(*c*) for setting pins. In accurate work the error is greatly reduced by making fine scratches to mark the tape lengths. Disturbing pins after they are set must be classed as a blunder. One of the greatest sources of error in marking tape lengths is in plumbing down from an end of the tape held off the ground.

76. Errors in Reading the Tape. *Compensating.* Errors in reading the tape are not likely to be important unless large enough to be classed as actual mistakes or blunders. In very accurate work, however, such as the measurements of base lines, they become important, and some device provided with a microscope or magnifying glass is often used for observing graduation marks on the tape. To avoid error due to parallax, sighting should be along the center line of the graduation mark and not oblique to it. Mistakes and blunders in reading the tape are treated in the preceding chapter, and common mistakes are summarized in Arts. 57(*d*) and 58.

77. Sources of Errors in Chaining. Summary.

1. Incorrect length of tape	Cumulative	+ or −
2. Tape not stretched *horizontally*	Cumulative	+
3. Tape not stretched tight and straight, but both ends in line	Cumulative	+
4. Error due to temperature	Cumulative	+ or −
5. Error in alignment	Cumulative	+
6. Variation in pull	Compensating	±
7. Error due to sag	Cumulative	+
8. Error in marking tape lengths	Compensating	±
9. Disturbing pins after they are set	Blunder	
10. Errors in reading the tape	Mistake	
11. Incorrect count of tape lengths	Blunder	

78. Relative Importance of Sources of Error. As an aid to the discussion of this subject a table has been prepared showing for

each source of error the conditions which will cause an error of 0.01 ft. (or about ⅛ in.) in a distance of 100 ft. Some of the values given will differ for different tapes, but they are approximately correct for the average 100-ft. steel tape.

Conditions	*Error*
1. *Length of tape.* A difference of ⅛ in. from standard length	0.01 ft.
2. *Tape not horizontal.* One end 1.4 ft. higher than the other end	0.01 "
3. *Tape not stretched tight.* Center of tape 0.7 ft. or 8⅜ in. out of line	0.01 "
4. *Temperature.* For every 15° change	0.01 "
5. *Alignment.* One end of tape in line, the other 1.4 ft. out of line	0.01 "
6. *Sag.* The middle of the tape 0.61 ft. or 7⅜ in. below the ends	0.01 "
7. *Pull.* For every 15 lb. pull	0.01 "
8. *Marking tape lengths.* Error of ⅛ in. in marking or plumbing	0.01 "
9. *Reading tape.* Error of ⅛ in. in reading	0.01 "

78(*a*). Errors most likely to occur. From inspection of this table it is evident that an error of 0.01 ft. in 100 ft. is much more likely to occur from some sources than from others. It is quite likely that the temperature will vary 15°, but not at all likely that the pull will vary 15 lb. Chainmen are more likely to chain along a slope without realizing that one end of the tape is 1.4 ft. lower than the other than they are to allow one end of the tape to get 1.4 ft. out of line when chaining between two points.

A steel tape, unknown to the surveyor, may be ⅛ in. too long or too short; and though it may not occur to him to test the length of the tape, he is very careful not to make an error of ⅛ in. in marking tape lengths (an error less important because compensating).

In ordinary work certain errors like that due to temperature are unimportant, because it is not necessary to take them into account to gain the required precision, and for that class of work they are said to be inappreciable (Art. 40); but for city work and other accurate work these same errors become appreciable and important.

78(*b*). In judging the relative importance of errors the following additional points are to be kept in mind:

1. Cumulative errors are more important than compensating errors, but, as shown above, not all cumulative errors are equally important.

2. In a short line a compensating error may occur only once or twice and thus fail to be compensating. For example, if the length of a line is a little less than two tape lengths, any error in marking the end of the first tape length is not compensating. The more tape lengths there are in a line the more likely are such errors to be truly compensating, and the more times a line is measured the more likely are accidental errors to disappear from the mean (Art. 39(*c*)).

3. One cumulative error sometimes balances another cumulative

error. For example, a strong pull may offset sag, or high temperature may offset a slight shortage in the length of a tape.

4. Other things being equal, it is most important to guard against those errors which are most likely to occur.

79. Sources of Error Most Important in Ordinary Chaining. Summary. Error in the length of the tape. Tape not stretched tight and straight. Tape not level on steep slopes. Error due to temperature if the variation from 68° F. is great. (Approximately 0.01 ft. for a 100-ft. tape for each 15° F. of variation.) Error due to sag, particularly if the tape is of the heavy type and the unsupported portion of the tape is long. (The error may exceed 0.3 ft. if the full length of a 100-ft. tape is unsupported.) Error in marking tape lengths may be great, particularly in plumbing down. Mistakes or blunders in reading the tape and keeping count of tape lengths—likely to be the most serious of all errors.

80. Limits of Error. The allowable error in chaining is sometimes determined by the time and money available for a survey and not by the importance of obtaining accurate results. Assuming, however, that the surveyor can proceed regardless of time or expense, he should for any particular piece of work determine the limit of error within which the chainmen must come. The purpose of a survey determines this, but different parts of the same survey may require different degrees of accuracy. For example, a survey is made of a city lot with buildings on it, to be plotted to a scale of 1 in. = 20 ft. Unless the dimensions of the buildings are to be put on the map or kept for record, no great precision is necessary in measuring them, for 1/20 in. on the map equals a foot, and an error of less than ½ ft. will hardly show. On the other hand, the lengths of boundary lines should be determined as accurately as possible, because these lengths should be recorded on the map (by numerals and dimension lines) regardless of scale, and if incorrect they may give rise to boundary disputes. In general, then, *the greatest pains should be taken in measuring distances to be preserved in records or on maps in actual figures, while distances that will be scaled from a map, or distances needed simply to plot a map, may be measured less carefully.* It is one thing to know the precision required, it is another thing to obtain it. Some conditions affecting accuracy of the work are: (1) correctness and fineness of the graduations on the tape; (2) nature of the ground chained, hilly or level, open or covered with trees and underbrush; (3) speed, i.e., time spent as determined by the importance of the work or the allowable cost; (4) weather—clear or cloudy, very hot or very cold, calm or windy; (5) the effectiveness of devices for

stretching and supporting the tape, and for measuring temperature; (6) the ability of the chainmen, and their willingness to take pains.

Note. When both linear and angular measurements are combined in a survey, it is obviously futile to measure distances with greater precision than that obtained in measuring angles; hence the precision required in linear measurements may depend upon the accuracy with which angles are measured. Conversely, the precision required in angular measurement may depend upon the accuracy with which distances are measured.

81. Probable Error of a Line *N* Tape Lengths Long. (*a*). A line N tape lengths long may be regarded as the *sum* of N measurements. If the probable error of each measurement (a tape length long) is the same, and this probable error is represented by r, then from formula 4 (Art. 45), $R_s = \sqrt{Nr^2}$ or $R_s = r\sqrt{N}$. Thus it is seen that the probable error of the total observed length of a line may be found by multiplying the probable error of one tape length by the square root of the number of tape lengths, i.e., length of the line in tape lengths. The more completely the constant errors are eliminated, and the more completely the other assumptions of least squares are fulfilled, the nearer this formula is to the truth. Its use, therefore, is chiefly confined to work of great precision.

81(*b*). PROBABLE DISCREPANCY. In ordinary chaining the accuracy of the work is judged in most cases by the discrepancy between two measurements of the line made with the same care. In such a case constant errors do not affect the discrepancy. (Why?) And since this discrepancy is largely due to accidental errors, it will *not* vary in proportion to the length of the line. On the other hand, the number of tape lengths is usually too small to warrant the application of the theory of least squares without reservation. All that can be said is that if the discrepancy between two measurements for 100 ft. (or tape length) is a certain amount d, it is likely that the discrepancy for any other length will be $D = d\sqrt{N}$, where N is the number of tape lengths. This is considerably nearer the truth than to say $D = dN$.

81(*c*). REMARK. In rough chaining constant errors are likely to predominate, but in work of great precision, where pains have been taken to eliminate constant errors, accidental errors are likely to predominate. Hence in rough work the *true* or unknowable error is likely to vary more nearly as the *length* of the line, but as the precision becomes greater and greater the *true* error will vary more and more nearly as the square root of the number of tape lengths. Notice that this applies to the *true* or unknowable errors and not to discrepancies.

82. To Determine Coefficients of Precision. (*a*). When the chaining is done under such conditions as to warrant the use of the method of least squares, the value of r may be determined by measuring a line of average length a number of times under the conditions which will prevail in the measurement of other lines. Then from the formulas in Art. 45 determine the error of a single observation. Substitute this value for R_s in the formula $R_s = r\sqrt{N}$, substitute for N its value, and solve for r. (See Art. 97 for a complete illustration.)

Another method is to make duplicate measurements of several lines of different lengths, and find r from the formula $r = 0.4769\sqrt{\frac{\Sigma pd^2}{n}}$, where d = difference between two measurements of same line, Σ = sum of, n = number of lines, and p is the weight for any one line, which varies inversely as its length, or $p = 1/l$. (See any standard treatise on the theory of least squares.)

82(*b*). FOR ORDINARY CHAINING where the use of the method of least squares is hardly justified, the value of d in $D = d\sqrt{N}$ may be obtained by measuring a line of average length a number of times under the prevailing conditions of field work, and, substituting for D the mean of the discrepancies, solve for d.

82(*c*). ILLUSTRATION. The average length of lines in a survey will be, say, about 400 ft. The chaining is to be done under favorable conditions. Four measurements of a line, made as nearly as possible with the same speed and under the same conditions that will prevail in the survey, give 400.10, 399.95, 400.05, 400.15. By comparing the first measurement with each of the other three, the second with the succeeding two, the third with the fourth, six discrepancies are obtained, and the mean is

$$(0.15 + 0.05 + 0.05 + 0.10 + 0.20 + 0.10) \div 6 = 0.1083$$

Then $0.1083 = d\sqrt{4}$, or $d = 0.0542$. For any other line the discrepancy may be expected to be about $0.0542\sqrt{N}$, or for a line, say, 900 ft. long $0.0542\sqrt{9} = 0.1626$ ft.

83. Customary Limits of Error. The method suggested in the preceding illustration for comparing measurements of different lines is for guidance only, and in determining the coefficient d it is absurd to carry results to several decimal places as was done in that illustration. When it is not worth while to determine d in the field it may be assumed as 0.01, 0.02, 0.03, or some other quantity which accords with experience. In order to give some definite idea of what constitutes good chaining under favorable conditions, coefficients are suggested in Table I, in Art. 84, for different classes of work. For hilly, rough country they must be modified, especially if underbrush, bad weather, and other unfavorable conditions prevail. Under such conditions coefficients called fairly good in the table may become good or even excellent, and the same is true, also, when the purpose of the survey does not require very accurate results.

The impossibility of fixing any standard for chaining owing to varying conditions is of course apparent, and the coefficients given are for guidance only. They should be modified, when necessary, to accord with the dictates of common sense and experience. It is assumed that the duplicate measurements, from which the discrepancy in any given case is determined, are made under approximately the same conditions—same chainmen, same tape, same temperature, and same speed.

The upper part of the table suggests coefficients, and the lower part gives the allowable discrepancies for lines of different lengths and for different coefficients.

Illustration. A line is chained twice under favorable conditions. A steel tape and plumb bob are used, the tape being kept horizontal by the eye. The two measurements are 828.91 and 828.88. The discrepancy is 0.03. In Table I the nearest value for 800 ft. is 0.028, which corresponds to a *coefficient* of 0.01. From the upper part of the table it is seen that 0.01 is *excellent* for the conditions under which the chaining was done. If a thermometer and spring balance had been used and other precautions taken to correspond, it is seen from the table that 0.01 is considered *good* or *fairly good.*

84. Degrees of Precision. (a). The method of expressing errors by ratios (Art. 67) is used in two ways to indicate degrees of precision, namely: (1) to express the ratio of the probable error of a single measurement or the probable error of the mean of several measurements to the length of the line chained; (2) to express the ratio of a discrepancy to the length of the line chained. For reasons given in Art. 81(b) the ratio of the discrepancy to the *length* of the line may be considered to indicate closely the variation to be expected *in comparatively short lines,* but as the lines become longer the discrepancy will vary more nearly as the *square*

root of the length. Table II has been made out on the assumption that the conditions for chaining are favorable; for unfavorable conditions the values must be modified. The warning against the injudicious use of Table I holds good for this table also. For lines less than 1000 ft. long the values in the two tables correspond closely; hence it makes little difference which is used, although Table II is in the more convenient form. For lines over 1000 ft. use Table I.

TABLE I

COEFFICIENTS OF PRECISION FOR CHAINING

Conditions: fairly open, level ground	Coefficients for Different Results		
	Excellent	Good	Fairly good
Measurements made with			
Chain, plumb bob, chain pins, average speed	0.03 to 0.04	0.05 to 0.08	0.09 to 0.1
Steel tape, plumb bob, chain pins, average speed	0.01 to 0.02	0.02 to 0.04	0.04 to 0.05
Steel tape, spring balance, thermometer, sag eliminated	0.002 to 0.005	0.005 to 0.01	0.01 to 0.02

ALLOWABLE DISCREPANCIES IN CHAINING

Length	Discrepancies for Different Values of Coefficient *d*									
100	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**
300	0.017	0.035	0.052	0.07	0.09	0.10	0.12	0.14	0.16	0.17
400	0.020	0.040	0.060	0.08	0.10	0.12	0.14	0.16	0.18	0.20
600	0.024	0.049	0.073	0.10	0.12	0.15	0.17	0.20	0.22	0.24
800	0.028	0.057	0.085	0.11	0.14	0.17	0.20	0.23	0.25	0.28
1000	0.032	0.063	0.095	0.13	0.16	0.19	0.22	0.25	0.28	0.32
1200	0.035	0.069	0.104	0.14	0.17	0.21	0.24	0.28	0.31	0.35
1400	0.037	0.075	0.112	0.15	0.19	0.22	0.26	0.30	0.34	0.37
1600	0.040	0.080	0.120	0.16	0.20	0.24	0.28	0.32	0.36	0.40
1800	0.042	0.085	0.127	0.17	0.21	0.26	0.30	0.34	0.38	0.42
2000	0.045	0.089	0.134	0.18	0.22	0.27	0.31	0.36	0.40	0.45
3000	0.055	0.110	0.163	0.22	0.27	0.33	0.38	0.44	0.49	0.55
4000	0.063	0.126	0.190	0.25	0.32	0.38	0.44	0.51	0.57	0.63
5000	0.071	0.141	0.212	0.28	0.35	0.42	0.49	0.57	0.64	0.71
6000	0.077	0.155	0.232	0.31	0.39	0.46	0.54	0.62	0.70	0.77
7000	0.084	0.167	0.251	0.33	0.42	0.50	0.59	0.67	0.75	0.84
8000	0.089	0.179	0.268	0.36	0.45	0.54	0.63	0.71	0.80	0.89
9000	0.095	0.190	0.285	0.38	0.47	0.57	0.66	0.76	0.85	0.95

Determine from the upper part of the table the coefficient to be used. Enter the column in the lower part of the table headed by that coefficient. Run the eye down the column until opposite the length of the line chained. The result is the discrepancy in feet which may reasonably be expected. For 0.002, 0.005, etc., use columns 0.02, 0.05, etc., adding one decimal place.

84(*b*). ILLUSTRATIONS. Two measurements made with a steel tape under ordinary conditions are 651.2 and 651.15. The discrepancy is 0.05. Nearest value in Table II for 600 ft. for given conditions is 0.048, or ratio is 1/12,500, which is considered *good*. In very accurate work this would not be considered as even passable.

TABLE II

Allowable Discrepancies in Chaining

Discrepancy of Two Measurements of the Same Line, Given for Different Lengths					Ratio of the Discrepancy to Length of Measured Line	Ratio of the Probable Error of Either Measurement *	Ratio of the Probable Error of Arithmetical Mean *	Results
300	**400**	**600**	**800**	**1000**				
Measurements made with *chain*, plumb bob, chain pins, and average speed								
0.06	0.08	0.12 ft.	0.16	0.2	1/5000	1/10,000	1/15,000	Excellent
0.12	0.16	0.24 "	0.32	0.4	1/2500	1/5000	1/7500	Good
Measurements made with *steel tape*, plumb bob, chain pins, and average speed								
0.012	0.016	0.024 ft.	0.032	0.04	1/25,000	1/50,000	1/75,000	Excellent
0.024	0.032	0.048 "	0.064	0.08	1/12,500	1/25,000	1/37,500	Good
0.06	0.08	0.12 "	0.16	0.20	1/5000	1/10,000	1/15,000	Fair
Measurements made with steel tape, spring balance, thermometer, sag eliminated, accurate alignment								
0.003	0.004	0.006 ft.	0.008	0.01	1/100,000	1/200,000	1/300,000	Excellent
0.006	0.008	0.012 "	0.016	0.02	1/50,000	1/100,000	1/150,000	Good
0.012	0.016	0.024 "	0.032	0.04	1/25,000	1/50,000	1/75,000	Fairly good
0.018	0.024	0.036 "	0.048	0.06	1/16,000	1/32,000	1/48,000	Passable

*** Too much importance must not be attached to probable errors of only two observations.**

84(*c*). Comparison of Tables I and II. If discrepancies for 300, 600, 1000, and 5000 ft. taken from Table I are compared with values taken from Table II, it will be seen that for lengths up to, say, 1000 ft. there is not much difference, but beyond that point the discrepancies from Table II increase much more rapidly than do those from Table I. This is of course to be expected, since in Table II discrepancies vary as the length of the line, whereas in Table I they vary as the square root of the length. For example, in good chaining under ordinary conditions the apparent error for 300 ft. should not exceed 0.035 to 0.07 (Table I) or 0.024 (Table II); for 600 ft. the values are closer, 0.049 to 0.1 (Table I) and 0.048 (Table II); for 1000 ft., 0.063 to 0.13 (Table I) and 0.08 (Table II). For 5000 ft., or a little less than a mile, the value from Table II has increased to 0.4 ft., which is 0.12 ft. greater than 0.28, the maximum value from Table I. The rougher the measurements the more striking is the difference between the two tables for lengths over 1000 ft., as shown by the first, second, and fifth lines in the above table for length 5000.

85. Requirements Corresponding to Different Degrees of Precision. (*a*). In ascertaining the requirements for any given limit of precision the sources of error summarized in Art. 77 must be considered. It is necessary to know what errors may be neglected and to what extent the others must be eliminated or corrected. This is the only way in which the surveyor can be reasonably sure that the *discrepancy* or the *probable error,* as the case may be, really indicates the precision of his work. Some of the questions he must settle are how accurately to read the tape, to mark the tape lengths, to keep the chain horizontal; to what extent sag must be eliminated, and within what limits the temperature must be known. In a general way the answers to these questions are indicated in Tables I and II. As a further aid, Table III on page 64 has been prepared. This table shows for each source of error the *approximate* errors per 100 ft. which correspond to different ratios of precision.

85(*b*). The chief use for Table III is to indicate the precautions which are necessary to attain certain degrees of precision. Thus, for example, it is seen that a variation of 15° in temperature will cause an error of ⅛ in. per 100 ft. corresponding to 1/10,000, and a sag of 7⅜ in. will cause the same error. If one end of a 100-ft. tape is 12 in. out of line, it will cause an error of 0.005 ft., or 1/16 in. (or 1/20,000), whereas it will take a variation in pull of 7½ lb. to cause the same error. An error of ¼ in. in the length of the tape, in marking tape lengths, or in reading the tape, is equivalent to 1/5000. An idea of the extreme precautions necessary for a precision of 1/100,000 is gained from the last column, where a sag of 2¼ in. will in itself cause an error equal to this limit. A variation of 1½° in temperature or 1½ lb. of pull will cause the same error; hence the need for a thermometer and spring balance.

85(*c*). Combining errors. It is comparatively easy to estimate the approximate error for a single tape length which may be ex-

TABLE III

	Errors per 100 Ft. from Different Sources Corresponding to Different Ratios of Precision				
Ratio of precision	1/5000	1/10,000	1/20,000	1/50,000	1/100,000
Error in 100 ft.	0.02	0.01	0.005	0.002	0.001
	ft. or in.	ft. or in.	ft. or in.	ft. or in.	ft. or in.
1. Error in *length* of the tape	0.02 ¼	0.01 ⅛	0.005 1/16	0.002 1/40	0.001 1/80
2. Dif. in *elevation* of ends of tape	2.0 24	1.4 16¾	1.0 12	0.63 7 9/16	0.44 5⅜
3. *Center* of tape out of line	1.0 12	0.7 8⅜	0.5 6	0.3 3⅝	0.24 2⅞
4. Change in *temperature*	30°	15°	7.5°	3°	1.5°
5. One *end* of tape out of line	2.0 24	1.4 16¾	1.0 12	0.63 7 9/16	0.44 5¼
6. *Sag* at the middle of tape	0.86 10⅜	0.61 7⅜	0.43 5⅛	0.27 3¼	0.19 2¼
7. Variation in the *pull*	30 lb.	15 lb.	7.5 lb.	3 lb.	1.5 lb.
8. Error in *marking* tape lengths	0.02 ¼	0.01 ⅛	0.005 1/16	0.002 1/40	0.001 1/80
9. Error in *reading* the tape	0.02 ¼	0.01 ⅛	0.005 1/16	0.002 1/40	0.001 1/80

pected from each source of error under given conditions, but it is not a simple problem to combine these errors to determine the total error which may be expected in a line a number of tape lengths long. Thus, for example, a surveyor may know: (1) that the error in the length of his tape does not exceed, say, 1/16 in.; (2) that the maximum error in keeping his tape horizontal will not exceed 1.4 ft.; (3) that the center of his tape will never be out of line more than 4 in.; (4) that the variation in temperature will not exceed 15°; (5) that one end of the tape lined in by eye will never be more than 12 in. out of line; (6) that sag will not exceed 7 in.; (7) that the variation in pull will not exceed 3 lb.; (8) that the error in marking each tape length will not exceed 1/16 in.; and (9) that the tape will be read to the nearest 0.01 ft. From the table the errors in feet per 100 ft. are: (1) = 0.005, (2) = 0.01, (3) = 0.002, (4) = 0.01, (5) = 0.005, (6) = 0.01, (7) = 0.002, (8) = 0.005, (9) = 0.01.

Assuming that these values correspond to the same degree of care in chaining and that all have the same algebraic sign, their sum (0.059) is the maximum possible error that could occur in 100 ft., a precision of about 1/1700. It is evident, however, that to add the errors from all nine sources, and then multiply the result by the number of tape lengths in the length of a line, will not give the total error for the following reasons: (1) Constant errors from

some sources may be plus, whereas constant errors from other sources may be minus, and hence even constant errors may tend to balance each other. (2) Whereas that part of the total error due to constant or cumulative errors varies as the number of tape lengths, that part due to accidental or compensating errors varies as the *square root* of the number of tape lengths. The term error as used here means the *true* or unknowable error as distinguished from the discrepancy between duplicate measurements. As already pointed out, the discrepancy is not affected by constant errors.

The problem of how far the constant error from each source must be eliminated in order that the discrepancy between two measurements of a line shall indicate the true error within prescribed limits (assuming no mistakes are made) is a complicated one. Even when proper algebraic signs can be given to constant errors from different sources, it is still difficult to assign values which would represent the same care in the corresponding operations of chaining. It is difficult also to assign consistent allowable values for accidental errors from different sources. In practice, time and expense usually limit the precautions to be taken in eliminating constant errors, and where the work is important enough to warrant it, the aim is to eliminate or correct the error from each source as far as it is possible for human skill to do it under the prevailing conditions. In ordinary work, however, the practical question is not how to attain the *greatest* precision but how to attain the *required* precision, and this means that the chainmen should know what errors to slight and what not to slight. Chainmen too often waste time and effort in eliminating errors which are relatively unimportant, and ignore at the same time errors which are much more serious. A careful study of this chapter should help to correct such a tendency; it should also furnish the fundamental knowledge that enables a surveyor to decide what equipment and what methods of measurement should be used to attain a desired degree of precision.

The following examples are intended to show how such knowledge may be applied in practice. In each it is assumed that a 100-ft. steel tape is used and that its length at a temperature of 68° F. and a tension of 10 lb. is known within the limits of error indicated in the example. The assumptions for maximum and minimum errors are those considered reasonable for the conditions given, but obviously the values are not to be taken as exact for they are, at best, estimates. Any one of them should be changed when under the conditions of chaining it is not in accord with good judgment.

85(*d*). ILLUSTRATION. *Conditions.* Measurements made by experienced surveyors with a good tape, but without the use of thermometer, spring balance, or instruments for measuring difference in elevation between the two ends of the tape. Points marked with pins. Character of the ground such that constant plumbing down is necessary. These are the conditions likely to prevail in surveys of the usual type. The degree of precision may vary from about 1/1500 to 1/6500. These results were arrived at as follows.

Assumptions. (1) Length of tape can easily be ascertained to the nearest 0.01 ft. by comparing with a standard length; hence the error is ±0.005. (2) Difference in elevation between the two ends of the tape averages 2 ft. and, from Table III, Art. 85, the corresponding error is +0.02 ft. (3) The normal tension for a 100-ft. tape weighing 1½ lb. and supported only at its two ends would be about 27 lb., but the chainmen would probably exert, on the average, not more than 15 lb.; the error due to sag under this pull would be, from Art. 74(*a*), about +0.04 ft. This error is likely to be reduced to about +0.02 ft. since some of the tape lengths would be over ground which would afford more or less support and, moreover, on the steeper slopes, tape lengths of 50 ft. or less might be used, thus tending to reduce the error due to sag. (4) Since no thermometer is used, a variation of 68° to ±15° might easily occur, equivalent to an error of ±0.01 ft. (5) Error due to plumbing down and marking is difficult to determine except by trial; it is likely to be at least ±0.02. (6) Chainmen accustomed to a pull of 15 lb. or more to compensate for sag are likely to maintain the same pull when the tape is stretched on the ground and hence supported; an excess of 5 lb. above the standard tension of 10 lb. is equivalent to −0.004 ft. This excess in pull tends to reduce error due to sag. (7) The error that occurs when the tape is not stretched straight may be considered negligible; if it is large enough to take into account, it is due to careless chaining. (8) The error that occurs when an end of the tape is out of line is not likely to be large; it is always plus and should not exceed +0.005 ft.—it may be negligible.

SUMMARY OF ERRORS PER TAPE LENGTH OF 100 FT.

Sources of error	*Maximum* (Signs alike)	*Minimum* (Signs + or −)
1. Probable error in length of tape	+0.005	−0.005
2. Tape not horizontal	+0.02	+0.02
3. Residual sag	+0.02	+0.02
4. Variations in temperature	+0.01	−0.01
5. Plumbing down and marking	+0.02	−0.02
6. Excess pull (always minus)	0.0	−0.004
7. Tape not stretched straight	0.0	0.0
8. Ends of tape off line	+0.005	0.0
Algebraic sum	+0.080	+0.001

Note that the corrections would have opposite algebraic signs.

It is to be noted that under the assumptions made, all sources of error are constant except that in plumbing down, which is accidental. Moreover, the largest errors are not only positive but also occur with greatest frequency, with the possible exception of the error in the length of tape. Consequently, the resultant error is nearly always positive. It follows that the length of a line, obtained by measuring under the conditions assumed, is likely to be longer than the true length.

The resultant error which may be expected in a line 1000 ft. long (ten tape lengths) ranges from +0.66 ft. to +0.15 ft. These values are obtained by multiplying each value in the table by 10 except that for the error in plumbing (accidental) which is multiplied by $\sqrt{10}$ (Art. 81).

$$\text{Maximum: } 10(+0.005 + 0.02 + 0.02 + 0.01 + 0.005) + 0.02\sqrt{10} = +0.66$$

$$\text{Minimum: } 10(-0.005 + 0.02 + 0.02 - 0.01 - 0.004) - 0.02\sqrt{10} = +0.15$$

The error of +0.66 is equivalent to about 1/1500, and of +0.15 to about 1/6500. This range of degrees of precision is about what may be expected in ordinary chaining under the conditions assumed at the beginning of this illustrative example.

Anything that can be done to reduce the errors due to the more important sources, namely, tape not horizontal, sag, temperature, and plumbing, will result in a marked increase in the degree of precision. If, for example, the chaining is done on supporting stakes set on a slope slightly less than 100 ft. apart, so that the tape ends always come on stakes, and if the correction for slope is made (Art. 60(*a*)), the error due to the inclination of the tape is practically eliminated; and, since no plumbing down is necessary, the error due to plumbing and marking is reduced to that made in marking. Under these conditions the range of degrees of precision may be from about 1/2400 to about 1/22,000. The latter high degree of precision could result only from a favorable balancing of errors, as when an error due to a temperature below the normal is balanced by excessive pull. If the surface is smooth like that of a pavement, the error due to sag is practically eliminated. Variation in temperature is the only important source of error that remains, and the range may then be from about 1/4500 to about 1/6500. To be reasonably sure of attaining a precision of 1/10,000 or even 1/6500, it is usually necessary to take into account the temperature of the tape.

85(*e*). ILLUSTRATION. *Conditions.* Experienced surveyors; length of 100-ft. tape at 68° and 10 lb. is known within ±0.001 ft.; constant use of hand level, thermometer, and spring balance; every reading of the tape to 0.001 ft. or less; plumb bob used on slopes, no supporting stakes; ends of tape not out of level more than 0.4 ft.; sag compensated by normal tension for 50-ft. lengths; tape kept on line by eye but with more than ordinary care. Character of the ground fairly favorable for measurements. Under these conditions the errors from different sources assumed in the table below are considered reasonable and a corresponding degree of precision of 1/8500 to 1/17,000 may be attained.

SUMMARY OF ERRORS PER TAPE LENGTH OF 100 FT.

Sources of error	*Maximum* (Signs alike)	*Minimum* (Signs + or −)
1. Probable error in length of tape	+0.001	+0.001
2. Tape not horizontal 0.4 ft.	+0.001	+0.001
3. Residual sag	0.0	0.0
4. Variations in temperature ±8°	+0.005	−0.005
5. Plumbing down and marking	+0.02	−0.02
6. Variation in pull 2 lb.	+0.001	−0.001
7. Tape not stretched straight	+0.001	0.0
8. Ends of tape off line 0.5 ft.	+0.001	0.0
	+0.030	−0.026

It is to be noted that sources of error have been eliminated to such an extent that plumbing down is the only source of a relatively large error. It is necessary for the chainman at one end of the tape to do three things simultaneously, namely, level the tape to within 0.4 ft., apply normal tension, and plumb down. It is obviously difficult for one man to do this alone.

In a line 1000 ft. long the resultant error may range from −0.118 ft. to 0.058 ft., found from the tabulated values as follows:

$$10(+0.001 + 0.001 + 0.001 + 0.001) + (+0.02 + 0.001 + 0.005)\sqrt{10} = -0.118 \text{ ft.}$$

$$10(+0.001 + 0.001) + (-0.02 - 0.001 - 0.005)\sqrt{10} = +0.058 \text{ ft.}$$

These errors are equivalent respectively to 1/8500 and 1/17,000. The latter is the highest precision to be expected; ordinarily it probably would be considerably less. To obtain a precision greater than 1/10,000 it would be advisable to set stakes or other supports such as portable "bucks" at intervals of not more than 100 ft. This would permit slope chaining with the possibility of a precision of 1/20,000, or higher, all other conditions remaining the same. If in such a case, occasional plumbing down is necessary, it should be done with extraordinary care by at least two men, preferably with the use of some device for holding the suspended tape steady when the pull is applied. When the surface is smooth like that of a pavement, a degree of precision as high as 1/20,000 may readily be attained.

The method of procedure in measuring lines with a high degree of precision is given in detail in Chapter 8.

85(*f*). Requirements for different degrees of precision summarized. For a precision of 1/1500 to 1/5000 the length of the tape should be known to the nearest 0.01 ft.; no correction for temperature, sag, or pull; in slope chaining, tape held level by eye and plumb bob used. If the error in holding the tape level and the error in plumbing down are eliminated by measuring along slopes on supports or on a smooth surface the degree of precision may be increased to 1/2500 to 1/13,000, depending upon how favorable conditions are for chaining. To be reasonably sure, however, of attaining a precision as great as 1/5000, the temperature of the tape should be known to be within the range, say, of 60° to 75°, assuming the tape to be very nearly its true length at about 68°.

For a precision of 1/5000 to 1/17,000 the length of the tape should be known to within ±0.001 at standard temperature and pull; correction for temperature and pull should be made, and this means the constant use of thermometer and spring balance, both attached to the tape; error due to one end of the tape held lower than the other, kept small by use of hand level; plumb bob used, no supports for tape; sag compensated by normal tension; tape kept on line by eye. The source of greatest error not eliminated is plumbing down. If this error is avoided by measuring slopes on supports or on a smooth surface the precision may be 1/20,000 or greater.

CHAPTER 7

BASIC PROBLEMS IN CHAINING

CERTAIN problems that occasionally arise in chaining are solved by the application of geometric or trignometric principles. In working out such problems in the field, it will be helpful to recall how the scale and compasses are used in similar constructions in the drafting room. In the latter part of this chapter are given methods for special work in chaining such as referencing a point, locating details, and making a chain survey.

86. To Construct a Right Angle with a Tape by the 3–4–5 Method. If the lengths of the sides of a triangle are in the ratios of 3 : 4 : 5, one angle of the triangle must be 90°. Hence to construct a right angle, form a triangle with the tape, using, for example, any one of the following combinations: [3–4–5], [6–8–10], [9–12–15], [12–16–20], [15–20–25], [18–24–30], [21–28–35], and so on, with multiples of 3, 4, and 5.

86(*a*). ILLUSTRATION. In Fig. 86(*a*) one chainman holds the 0 and the 48-ft. mark of a 50-ft. cloth tape together, a second man holds the tape at the 36-ft. mark, and a third holds it at the 20-ft. mark. When stretched the tape will form a right-angled triangle. (Why?)

In practice it may be necessary to modify the method of using the tape just illustrated. If it is a *steel* tape, it may be undesirable to bend it sharply enough to form the vertices of the triangle. Moreover, there may not be three chainmen available. Modified methods are given in the next article.

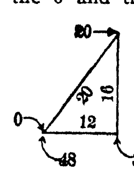


FIG. 86(*a*).

87. Perpendiculars. The methods of establishing perpendiculars and parallels with the tape are based upon familiar geometric constructions. It is to be remembered, however, that straight lines and arcs of circles cannot be drawn upon the ground with the same ease that they can be drawn on paper. The ground is often so rough or sloping that they cannot be drawn at all. In many cases, therefore, chainmen must exercise ingenuity in applying geometric principles. The difference between methods of the drafting room and methods of field work in making the same geometric construction is brought out in the following illustrations. *It is important not only to know geometric constructions, but also to know how the tape should be used in making them.*

87(*a*). TO ERECT A PERPENDICULAR TO A LINE AT ANY POINT IN IT. The 3–4–5 method of the preceding article may be used. Two other methods are illustrated in Figs. 87(*a*) and 87(*b*).

The 3–4–5 Method. Let it be required to erect a perpendicular to the line *ab* at the point *c* (Fig. 87). Establish the point *e* 15 ft. from *c*. The 20-ft. mark is held at *c*, the 35-ft. mark at *e*, and the 60-ft. mark and the zero mark are brought together at a point on the perpendicular. If only two chainmen are available, pins may be stuck at *e* and *c*, and the tape looped around these pins. One chainman keeps the 20-ft. and the 35-ft. marks at *c* and *e*, respectively, while the other brings the 60-ft. mark and the zero point together. A tape must be flexible if it is to be stretched around three sides of the triangle.

The method of applying the 3–4–5 principle given in the following illustration makes it possible to use longer sides for the triangle with any fixed

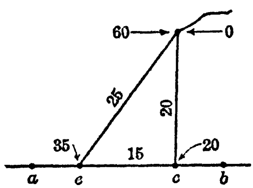


FIG. 87.

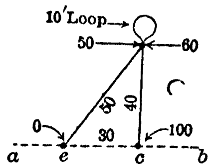


FIG. 87(*a*).

length of tape and avoids sharp bends in the tape. In Fig. 87(*a*) it is desired to construct a right angle at a point *c* in the line *ab*, one side of which shall coincide with *ac*. Establish a point *e* 30 ft. from *c*. The 0 end of a 100-ft. tape is held at *e*, the 100-ft. end at *c*, and the 50-ft. and 60-ft. marks are brought together, the portion of the tape not used forming a loop. The tape stretched tight forms a right-angled triangle with sides 30, 40, and 50 ft. long. By fastening the ends of the tape at *e* and *c* one man can do all the work.

Question. If a 50-ft. tape is used and the 25-ft. and 30-ft. marks are brought together, what should be the distance from *c* to *e?*

Note. The use of *two* 100-ft. tapes (Fig. 87(*a*)) would make possible still longer sides for the triangle. This holds true for some of the other geometric constructions that follow. Moreover, when there is a considerable amount of geometric construction involved in chaining, the work may be facilitated by the use of two tapes.

87(*b*). ILLUSTRATION. *Geometric Construction.* Required to erect a perpendicular at *b* to the line *ab*. From *any* point *c* describe an arc of a circle of radius *cb*, intersecting *ab* at *e*. Find *d* in line with *ec*.

Field Work. The field work with a 100-ft. tape could be done as follows. Establish *c* 50 ft. from *b* by holding 0 at *b*. Keeping the 50-ft. mark at *c*,

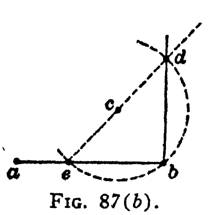


FIG. 87(*b*).

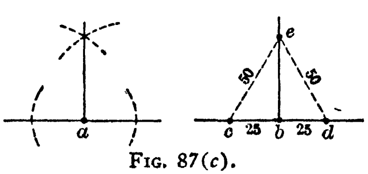


FIG. 87(*c*).

find *e* by swinging the 0 end of the tape and, with 0 at *e*, 50-ft. mark still at *c*, stretch the tape until the 100-ft. mark determines *d*. The point *d* may be checked by the use of another method, for example, the 3–4–5 method.

87(*c*). ILLUSTRATION. The difference between using the compasses and the tape is illustrated in Fig. 87(*c*). With the compasses everything is done with intersecting arcs as indicated in the figure. With the tape a better way is as follows. With the 25-ft. mark at *b*, establish *c* opposite the 0 mark and *d* opposite the 50-ft. mark. Keeping 0 at *c*, hold the 100-ft. mark at *d* and stretch the tape, establishing *e* at the middle of the tape (50-ft. mark). Or if it is a steel tape, bring the 45-ft. mark and 55-ft. mark together to avoid bending, thus establishing a point on the perpendicular 45 ft. from both *c* and *d*. If a check is desired, find another point on the opposite side of the line in

exactly the same manner, or if this is impossible, another point on the same side, using different distances.

87(*d*). REMARK. In each of the above illustrations notice that *one* point of the tape was fixed during the entire work in order to save unnecessary settings of the tape. In the geometric constructions which follow it is not enough simply to recognize the method, but consider carefully how each one could be carried out in the field, setting and stretching the tape the smallest number of times.

87(*e*). IF TWO SIDES OF A RIGHT TRIANGLE ARE OF EQUAL LENGTH, THE HYPOTENUSE IS EQUAL IN LENGTH TO 1.414 MULTIPLIED BY THE LENGTH OF A SHORT SIDE. This principle suggests another method of constructing a right angle similar to the 3–4–5 method. For example, if two sides are each 25 ft. long, the hypotenuse will be 35.35 ft. The principle is useful also in checking a right angle that has been constructed by some other method.

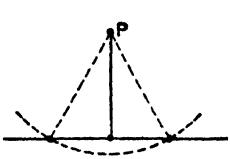


FIG. 87(*f*).

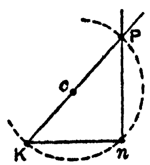


FIG. 87(*g*).

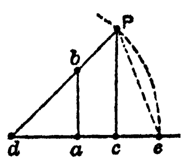


FIG. 87(*h*).

87(*f*). TO LET FALL A PERPENDICULAR TO A GIVEN LINE FROM ANY GIVEN POINT. In Fig. 87(*f*) from the given point *P* swing an arc intersecting the given line in two points. The required perpendicular through *P* passes halfway between the two points of intersection.

Question. If an arc could not be drawn on the ground, how would the tape be used?

In Fig. 87(*g*) the construction is that of Fig. 87(*b*) worked backward. Choose any convenient point on the line as *K*, find the point *c* halfway between *P* and *K*, and with *c* as a center and *cK* as a radius cut the given line at *n*, the foot of the required perpendicular.

In Fig. 87(*h*) a perpendicular is first erected at any convenient point as *a*, and a line through *P* constructed parallel to it. (See Art. 88.) Or any two points as *d* and *b* may be assumed in line with *P*, and *c*, the foot of the required perpendicular, found from similar triangles $dc = (da \times dP) \div db$.

A Second Method. With any point on the given line as *d* swing an arc through *P*, intersecting the given line at *e*. To find *c*, since $dP = de$, then $ce = \overline{Pe}^2/2de$.

Question. How could the methods of Fig. 87(*h*) be carried out best with a tape? Which methods of erecting or letting fall a perpendicular are best to use when the perpendicular is near or at the end of a line and the line itself cannot be produced?

87(*g*). A PERPENDICULAR OF GREATER LENGTH than can readily be constructed as one side of a triangle may be obtained as follows. (1) Lay off an auxiliary line parallel to the given line (Art. 88), and nearer to *P*. (2) Construct a perpendicular from *P* to this auxiliary line. (3) Extend the perpendicular to the given line. For example, assume that in Fig. 87(*g*) the distance from the given point *P* to the given line is longer than the length of the tape. A line parallel to the given line *Kn* and nearer to *P* may be constructed by one of the methods explained in the next article. A perpendicular to this line from *P* is first constructed, and then it is extended to the given line by "sighting in." The right angle at the intersection of the perpendicular and the given line should be checked. Other methods of constructing long perpendiculars may readily be devised. It is obvious that the longer the perpendicular the more difficult it will be to obtain a true perpendicular and the more necessary it will be to check results. The method used should be selected with this in mind.

88. Parallels. *To run a line through a given point parallel to a given line.*

88(*a*). In Fig. 88(*a*) a perpendicular *Pp* is let fall from the given point to the given line, and at some other point in the line as *r* another perpendicular *rs* of equal length is erected.

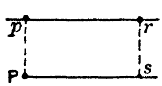


FIG. 88(*a*).

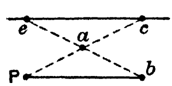


FIG. 88(*b*).

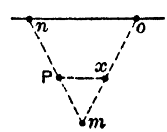


FIG. 88(*c*).

88(*b*). In Fig. 88(*b*) choose any convenient point on the given line as *c* and find *a* halfway between *P* and *c*. Assume a point *e* and make *ab* equal *ea*.

88(*c*). In Fig. 88(*c*) assume any convenient line as *mn* through *P*, and make *mo* equal to *mn*. Find *x* when $mx = mP$.

Remark. Other methods will suggest themselves to the student. The method of Fig. 88(*b*) is quick and accurate and easily checked by the method of Fig. 88 (*a*). A check on the method of Fig. 88 (*b*) is the equality of the diagonals.

89. To Lay Off or to Measure an Angle. Angles may be laid off by means of a steel tape with considerable accuracy, the error often not exceeding 5′ of arc. The trigonometric methods of the chapter, "Methods of Plotting Angles," Part II, are used. Since measurements are in feet instead of in inches and since the tape is used instead of drawing instruments, the operations in the field are somewhat different from those in the drawing room, as, for exam-

ple, when erecting perpendiculars. The trigonometric methods are of course the same.

89(*a*). Tangent method. (See Art. 625.) The base by which the tangent of the angle is to be multiplied may be any convenient number of feet, say 50 or 100. Perpendiculars are erected by one of the methods of Art. 87.

Illustration. To lay off an angle *ABd* of 148° 24′ clockwise from *AB*, using a 100-ft. tape. Lay off from *AB* produced 31° 36′, the supplement of the required angle, as follows. Find *e* 50 ft. from *B* on *AB* produced. At *e* erect a perpendicular (Art. 87(*a*)). The natural tangent for 31° 36′ is 0.6152, and this multiplied by 50 = 30.76. Find *d* on the perpendicular, 30.76 ft. from *e*.

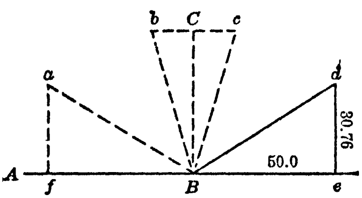


Fig. 89(*a*).

An angle somewhat greater than 90°, such as *ABc*, would be laid off by first erecting a perpendicular *BC* and then by constructing from this perpendicular as a base, the angle *CBc*. Similarly, an angle somewhat less than 90°, such as *ABb*, would be laid off by constructing the angle *CBb* from *BC* as a base. A relatively small angle such as *ABa* would be laid off from a base *Bf*.

Remark. Since the precision with which an angle may be laid off by the tangent method depends to a considerable extent upon the accuracy with which a perpendicular such as *de* is erected, right angles should be checked. The method of Art. 87(*c*) may be used for this purpose.

89(*b*). Chord method. (See Art. 627.) The radius may be any convenient number of feet, say 50 or 100. As circles cannot often be drawn on the ground, many of the practical suggestions for using the compasses do not hold good for the tape. Use the tape in a manner similar to that shown in Fig. 89(*b*), one portion of the tape stretched to form the chord of known length, the other portion forming the required side of the angle.

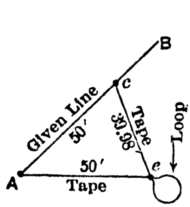


Fig. 89(*b*).

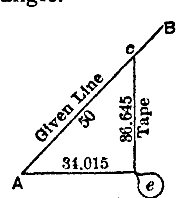


Fig. 89(*c*).

Illustration. To lay off an angle of 47° 8′ clockwise from *AB*, using 100-ft. tape. Chord for unit radius is 0.7996. For 50 ft. it is 39.98 ft. Find *c* 50 ft. from *A* on *AB*. Holding 0 mark at *c* and 100 mark at *A*, bring 50-ft. mark and 39.98 together. This will give *e*, a point on the required side.

Remark. The chord method is usually the most satisfactory one to use in the field.

89(*c*). Sine-cosine method. For any unit radius the sine and cosine distances can be found and used in the same way that the two distances were used in the preceding example.

Illustration. To lay off an angle of 47° 8′. Sine = 0.7329, cosine = 0.6803. Find *c* 50 ft. from *A* as before. Sine × 50 = 36.645, cos × 50 = 34.015. When the tape is stretched as shown in the figure it will form a *right angle* at *e*, a point on the required side.

89(*d*). MEASURING ANGLES. Angles between two given lines may be measured with the tape by reversing the methods outlined in Arts. 89(*a*), (*b*), and (*c*).

89(*e*). SUGGESTIONS. When the angle to be laid off or to be measured is large, it is often best to measure or lay off its complement or supplement. (See Art. 627(*d*).)

In the three illustrations just given a 50-ft. base or radius was chosen. Longer units, 100 ft. or more, may often be used to obtain greater precision. In such cases the use of two tapes may be advantageous.

90. Obstacles to Chaining which prevent chainmen from measuring directly between two points give rise to a set of problems in which distances are found by indirect measurements. In this article all these problems are solved by geometric methods.*

Obstacles to chaining are of two kinds: (1) *obstacles to measurement* and (2) *obstacles to alignment.* For any problem likely to occur in practice there are usually several geometric solutions; it will suffice to give two or three for each problem. The student with a good knowledge of geometry should be able to devise other methods.

90(*a*). TO MEASURE BETWEEN TWO POINTS WHEN BOTH ARE ACCESSIBLE.

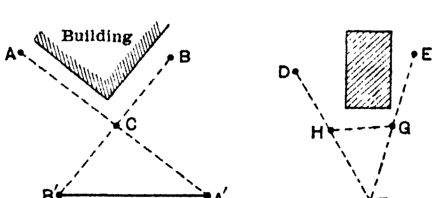


FIG. 90(*a*).

First Method. Assume *C*, make $CB' = BC$ and $CA' = AC$; then $A'B' = AB$.

Second Method. Establish *F* at any point equal distances from *D* and *E*. Make $FH = FG$ any convenient distance. Then $DE = (HG \times DF) \div HF$.

When the two points are intervisible, methods illustrated in Fig. 90(*b*) may also be used.

Third Method. When the obstacle is small, the following method is recommended as simple and accurate. The points *A*, *B*, and *D*

* Many of the same problems can be more easily solved by trigonometric methods, provided a transit is used for measuring angles. (See Chapter 25.)

in Fig. 90(*a*),3 have been established in a straight line, and the distance AD is required. On a perpendicular, judged by eye, through B assume C any short distance d from B, preferably one in *whole* number of feet. Chain AC and CD. Then, when the ratio of d to AB is small, say not more than 1 : 10, the following approximate method may be used for computing the distance AD to the nearest 1/100 ft.:

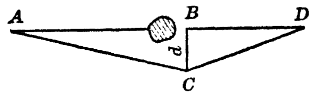


FIG. 90(*a*),3.

$$AB = AC - \frac{d^2}{2AC} \quad \text{and} \quad BD = CD - \frac{d^2}{2CD}$$

$$AD = AB + BD \quad \text{(See approximate method in Art. 60(}a\text{).)}$$

When d is some whole number of feet the solution is simple. A partial check may be obtained by chaining BD. This method is particularly useful when a transit line passes through the trunk of a tree or other small obstacle.

90(*b*). TO MEASURE BETWEEN TWO POINTS WHEN ONE IS INACCESSIBLE.

First Method. Lay off AC any convenient length perpendicular to the required distance AB, and lay off DC perpendicular to BC. Then $AB = AC^2 \div AD$.

Second Method. Assume I anywhere in line with the required distance FG. Assume H anywhere. Make $HJ = HI$ and $HK = HG$. Establish L in the line FH and at the same time in the line JK produced. Then $KL = FG$.

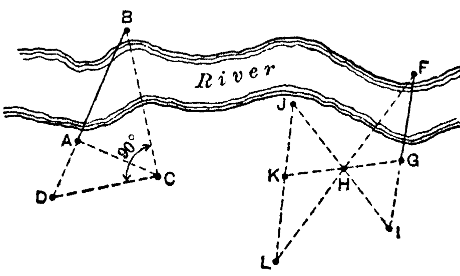


FIG. 90(*b*).

90(*c*). OTHER METHODS. Assuming that the point B in Fig. 90(*b*) is not a fixed point but any point in line on the opposite bank of the river, how could an equilateral triangle be constructed and used for finding the distance AB? How could a right triangle with a 60° angle at C be used for finding the same distance?

90(*d*). TO MEASURE TO AN INACCESSIBLE POINT HIGH ABOVE GROUND, i.e., to measure the height of an object. Set a transit pole in the ground some distance from the object and with the eye some feet away from the pole sight across the top of the pole to the inaccessible point. This line of sight is the hypotenuse of two similar triangles. The unknown height is the vertical side of one triangle, and the transit pole the vertical side of the other. The two bases of the triangles can be measured and the proportion formed. This method will of course give only approximate results. For accurate results, use a trigonometric method carried out with transit and tape. (See Art. 382(*d*).)

91. Special Cases of Obstacles to Chaining. (*a*). TO CHAIN OVER A HIGH WALL OR OTHER SIMILAR OBSTACLE. Drive two nails through a board, the heads projecting on one side and the points on the other. Lay the board across the wall with the points of the nails up, and bring both nails in line. Suspend a plumb line from the head of each nail and thus establish two points on the ground, one each side of the wall, a known distance apart, i.e., distance of nails apart. A transit pole with plumb lines tied to it can be used in place of the board and nails.

91(*b*). TO CHAIN BETWEEN TWO POINTS *A* AND *B* WHEN WOODS OR UNDERBRUSH INTERVENE SO THAT NEITHER POINT CAN BE SEEN FROM THE OTHER. Starting at one of the points *A*, chain along a straight line *AC* as nearly as can be guessed at in the direction of the other point *B*. This line is often called a *random line*. Measure the distance *CB* that this line strikes to one side or the other of *B*. Assume any distance on *AC* as *AD*, and from similar triangles $DE = (AD \times CB)/AC$. Lay off *DE*, thus establishing the point *E*. If this is done with reasonable care, the line *AE*, when produced, should pass through *B*, or near enough, at least, to show where a line should be cleared for the final chaining.

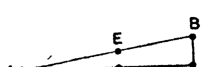


FIG. 91(*b*).

91(*c*). TO CHAIN BETWEEN TWO POINTS WHEN THE BROW OF A HILL INTERVENES. (Fig. 91(*c*).) Let *A* and *D* be two points with the brow of a hill between them. One chainman, *B*, goes as far back as he can and still see *D*. The other chainman, *C*, goes as far down hill as he can and still see *A*. *B* and *C* are both out of line, but they proceed to line each other in alternately. *B* lines *C* in with *D*. *C* then lines *B* in with *A*. *B* again lines *C* in with *D*, and *C* again lines *B* in with *A*. Thus the process is carried on until both *B* and *C* are in line. Poles are set at *B* and *C*, and these enable the chainmen to measure in a straight line between *A* and *D*. The method of Art. 91(*b*) can also be used.

Practical Suggestions. Poles may be set at *A* and *D* to begin with. The chainmen themselves can hold either poles or plumb lines. Time will be saved if when one chainman is lining the other in he will send him a little farther in the direction in which he is moving than appears necessary. Thus in Fig. 91(*c*) *B* moves *C* to *C″* instead of to *C′*, and *C″* then moves *B* to *B″* instead of to *B′*, and so on.

Remark. When a valley intervenes between two points a similar method can be employed to establish points on line in the valley. In most cases, however, since one point can be seen from the other, a pole is set on line with the transit. When a transit is not at hand still another method is used. One chainman holds a plumb line over one point and, placing his eye so that the string covers a rod held at the other point (on the opposite hill), lines in a second rod held in the valley until it too is covered by the string.

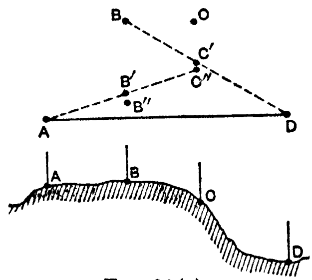


FIG. 91(*c*).

92. To Reference a Point.

(*a*). TO FIX ITS POSITION WITH RESPECT TO TWO OR MORE POINTS, so that if it is lost or destroyed it may be easily restored. There are three ways of doing this with the tape, corresponding to three of the methods of locating a point in Art. 10: (1) by two distances; (2) by offset; (3) by two intersecting lines.

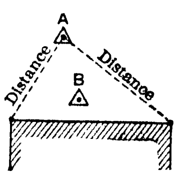


FIG. 92(*b*).

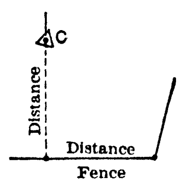


FIG. 92(*c*).

92(*b*). ILLUSTRATION. Let *A* in Fig. 92(*b*) be a tack in a stake. Two distances to the corners of a building near by were measured, thus referencing *A*. If the stake is destroyed, another may be reset easily, *provided the intersection of the two reference lines is sharp and definite.* The point *B*, for example, cannot be referenced accurately from the same two corners because it is too nearly in a straight line between them for the intersection to be definite.

92(*c*). ILLUSTRATION. In Fig. 92(*c*) the point *C* was referenced by dropping a perpendicular from *C* to a fence line, measuring this perpendicular and also the distance from the point where it intersects the fence line to a fence corner. This method is more difficult and less accurate in field work than that of the preceding illustration.

92(*d*). ILLUSTRATION. A third method of referencing a point is to establish two other points in the same straight line, one each side of the given point, and measure the distance from the given point to each of these reference points. Or, better still, to establish four reference points so that the given point may be determined without measurement by the intersection of two straight lines. (See Art. 10, Fig. 10(*a*).)

Other methods of referencing a point corresponding to methods 3, 4, and 5 (Art. 10) could be used, but each of them involves the measurement of one or more angles and is, therefore, more suited to work in which both tape and transit are employed.

93. Locating Houses, Fences, Streets, Streams, and Other Details with the Tape.

Not only in chain surveys, but also in

transit surveys, the chainmen are often called upon to locate details of the survey with reference to transit lines by the use of the tape alone. The general method is to locate enough limiting points of any detail to determine the position of the entire detail. Each of the limiting points is usually located with respect to at least two points of reference by one of the three methods just given for referencing a point. The points of reference and the lines between them form a system of horizontal control. A simple illustration is given in the next article. Location of details by the tape alone is best learned, however, in connection with transit surveys, and the methods are therefore more fully discussed in Chapter 22.

94. To Make a Chain Survey. It is sometimes necessary to make a complete survey using the tape only. Frequently this can be done with as great accuracy as the purpose of the survey requires. In Fig. 94 is shown the survey of a house and lot, in which no angles were measured. A station S was chosen from which all four corners of the lot could be seen. The four sides of the lot and the four broken lines were measured. From these eight measurements the boundaries of the lot were plotted. Had it not been for the house it would have been simpler to divide the quadrilateral by a diagonal into two triangles instead of four.

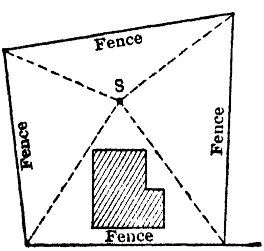


FIG. 94.

When the four fence lines have been located, the only detail left is the house. The limiting points are the six corners, each of which may be located with respect to two points of reference. These two points may be chosen on any known line—a broken line, a fence line, or a side of the house. For example, each of the two front corners of the house may be located with respect to two points on the front fence line, and each of the other four corners may be located, directly or indirectly, with respect to the two front corners. (What two points on the front fence line would it be best to take for reference points?) Check measurements should be taken. For example, each of the four points in which the front and rear lines of the house intersect the side fence lines might be located with respect to the nearest fence corner. Also, the distance from each of these four points to the nearest corner of the house might be measured. Finally, the dimensions of the house should always be measured.

A second method of making a chain survey of the boundary lines is to measure the lengths of the four sides of the lot and in addition

to measure with the tape the four angles at the fence corners (chord method, Art. 89(*b*)). This method could be used when there is no point such as *S* from which all corners of the lot are visible.

94(*a*). General method for a chain survey. Establish a system of triangles to serve as reference lines. This network of triangles should be as simple as possible and planned with an eye to locating details. Measure the three sides of each triangle. Locate details with respect to the most convenient reference lines by any of the methods for linear measurement given in Chapter 22.

94(*b*). Remark. The transit is used so generally that students are likely to think that a survey cannot be made with a tape alone. For short lines and small areas, however, the tape will often answer every purpose. Even horizontal angles can be measured with it (Art. 89), though this is not often necessary. The reference lines are laid out in triangles to save measuring angles, and this is the chief difference between a chain survey and a transit survey. When it is impracticable to divide the area surveyed into triangles, it may be necessary to measure angles with the tape, as in the second method of Art. 94. Even in such a case, it may be possible to establish a system of triangles for a portion of the area. Check measurements should be taken to be sure that the reference lines have been correctly surveyed, particularly when the area, or any portion of it, has not been divided into triangles. The location of important details should also be checked. (See check measurements for locating the house in Art. 94.)

CHAPTER 8

PRECISE LINEAR MEASUREMENT

In order to measure distances with a high degree of precision, errors in linear measurement must be reduced to a far greater degree than in ordinary chaining. This chapter is devoted largely to a discussion of how this reduction of errors is accomplished.

95. To Chain a Line with a High Degree of Precision. The chainmen should have a thorough knowledge of: (1) the various sources of error; (2) how to correct or to eliminate the error from each source; (3) to what extent the error from each source should be eliminated to attain a desired degree of precision. A knowledge of the precautions that must be taken in order that the error from a given source should not exceed a permissible limit, and a knowledge of the relative importance of errors from different sources may be gained from a study of Chapter 6; but the practical application of this knowledge is not easy. The chief difficulty is that the degree of precision with which a line is chained depends, not upon the amount of error from a single source, but upon the combined effect of errors from all sources (Art. 85(*c*)). The effect of some of these errors is to increase the length of the tape; on the contrary, the effect of others is to decrease the length. To make sure that a desired degree of precision will be attained, chainmen frequently carry the elimination of errors much further than they should. The art of chaining within a specified degree of precision and without waste of time is gained mainly from field experience in applying intelligently the fundamentals of accurate linear measurement.

As a rough guide, it may be stated that to attain a degree of precision greater than $1/4000$, precautions must be taken to eliminate or reduce errors from the more important sources (Art. 85(*d*)). For a degree of precision of $1/10{,}000$ or higher, precautions should be taken to eliminate or reduce errors from *all* sources (Art. 85(*e*)). The methods of measurement explained in that article should result in a precision of $1/15{,}000$ to $1/50{,}000$, depending upon the accuracy with which they are applied. This range is adequate for most purposes of plane surveying. For first-order precision, however, such as that required in the measurement of base lines for geodetic

triangulation, still more accurate methods must be used, such as those described in Art. 100.

95(*a*). METHODS OF SUPPORTING THE TAPE. (1) Uniformly supported on an approximately plane surface; (2) supported at two points of the tape only ("two-point" method); (3) supported at three or more points, generally at equidistant points. Each method has its disadvantages. In the first method it may be difficult to determine the true temperature and the true tension throughout the length of the tape because of the contact between the tape and the supporting surface. Incidentally, in stretching the tape repeatedly along a rough surface, such as a concrete pavement, or in dragging it over that surface, the cross section of the tape may become reduced to such an extent that the length of the tape under tension will be increased. In the second method more field work is involved and a larger field party is required. Nevertheless, this is often the best method; it is seldom used if the length of the tape exceeds 100 to 150 ft. In the third method, even more field work is involved, but the method is extensively used in base-line measurement, particularly when the measurement is made with long tapes.

Portable tape supports or "bucks," such as those described in Art. 99, are generally used for supports in the two-point method. Portable bucks and some form of intermediate supports, such as those described in Art. 100, may be used when a tape is supported at three or more points. In some cases, more permanent supports may be set in the ground along the line. The length of interval between intermediate supports depends on the slope of the ground, the length of the tape, and other practical considerations. For a 100-ft. tape, a 25-ft. interval is often used; for a 50-meter tape, a 12.5-meter interval.

When the ground is nearly level or when it slopes uniformly, the supports may be placed so that they keep the tape at the same level, or at the same grade, throughout the length of the measured line. When, however, there are changes in slope, it is generally necessary to break down the line into sections of different slopes; the supports for each section are then set to the slope of the tape for that section. It is desirable to avoid slopes greater than 10 per cent. This often may be done by erecting in hollows and gullies relatively high supports, such as small rigid towers with platforms on top.

95(*b*). SUPPORTS FOR A SHORT BASE LINE such as a base line in a triangulation survey of an ordinary bridge site. Supporting stakes or posts driven or set *firmly* in the ground may be used, say 4 in. by 4 in. for the two end hubs, and 2 in. by 4 in. for intermediate points. At any point where an end of the tape will fall during the various measurements of the line, the supporting post should be set on line in such a position that the end of the tape will be near the center. The post may then be sawed off to the level or to the

slope on which the line will be measured. The top may be capped with a zinc, copper, or brass plate upon which a fine line, perpendicular to the measured line, can be scratched to mark the point where the end of the tape falls. Each of the other posts may be set off line but with one side nearly on line so that a nail to support the tape may be driven into it. Much of the lining in of posts and nails may be done by eye. To keep the tape in position on the tops of those posts that have been sawed off to grade, two small nails may be driven part way into each post in such a way that there will be one on each side of the tape when it is stretched. To keep the two end hubs from being disturbed during construction a fence may be built around each one. If it is desired that these hubs should be more or less permanent, they may be made of concrete. It is usually desirable to reference carefully each end of the base line.

95(*c*). STANDARDIZATION OF TAPES. A tape is standardized under specified conditions of temperature, method of support, and tension. The standard temperature adopted by the U.S. Bureau of Standards is 68° F. or 20° C. The standard tension *for a tape supported horizontally throughout its length* is 10 lb. when the length of the tape does not exceed 100 ft., 20 lb. when the tape is longer than 100 ft. There is no standard tension for use in testing a tape that is not supported throughout its length.

In standardizing a tape, the character of the test depends primarily upon how the tape is supported and upon the tension desired. To reduce to a minimum the calculated corrections of errors, a tape should be tested under the same conditions of support and of pull as those under which the tape will be used in the field. For example, in standardizing a 100-ft. steel tape, one or more of the following tests may be made, according to the field requirements.

1. To determine the length from the zero graduation mark to the 100-ft. mark when the tape is uniformly supported throughout its length on a horizontal plane surface, at a standard temperature, and under a standard tension. This is a basic test required by the U.S. Bureau of Standards as a prerequisite for all other tests of a tape.
2. Same as (1) except that the length of tape tested is from the zero mark to a specified intermediate graduation mark. This is an important test because the basic test in (1) may show no appreciable error in the distance from the zero mark to the 100-ft. mark and yet there may be an error in the distance from the zero mark to an important intermediate graduation. Or, if there is an error in the distance from the zero mark to the 100-ft. mark, it may not be uniformly distributed throughout the length of the tape, but may occur in a short section. For example, the entire error may occur in a section from the 10-ft. mark to the 25-ft. mark. (See Art. 717(*d*) for an actual case of this kind.) It may be desirable, therefore, to test the interval from the zero mark to each of the most important intermediate graduations.
3. To determine the distance from the zero mark to the 100-ft. mark when the tape is supported at these two points only and the points are in the same horizontal plane. The tension and temperature for this test should be specified.
4. Same as (3) except that the points of support are at the zero mark and at a specified intermediate graduation mark, and the tested distance is between these two points.
5. To determine the tension (or pull) required to bring the 100-ft. mark exactly 100 ft. from the zero mark when the tape is supported at these two points only and the points are in the same horizontal plane. The temperature should be specified.
6. Same as (5) except that the zero point and a specified intermediate graduation mark are the points of support.

7. To determine the distance from the zero mark to the 100-ft. mark when the tape is supported at these two points and also at one or more specified intermediate points. Temperature and tension must be specified.

8. To determine the length from the zero point to a specified intermediate graduation under the same conditions as those in (7).

The U.S. Bureau of Standards is prepared to standardize a tape for practically any combination of conditions that commonly occurs in the use of tapes. A schedule of these conditions, specifications for a standard tape, and the standard values for temperature and tension are given in Art. 715.

A graduation mark must have an appreciable width even on the best of tapes, and in precision chaining it may make a difference in reading the tape whether a graduation point is assumed to be at the center of the corresponding graduation mark or at some other point of the mark. In standardizing tapes, the U.S. Bureau of Standards fixes the positions of graduation points as follows. "Unless otherwise stated, the comparisons of tapes with the Bench Standard are made at the *centers* of the lines on the edge to which the shortest graduations are ruled. If all graduations extend entirely across the tape, the ends of the graduation lines away from the observer when the zero of the tape is at his left hand are used."

The use of a tape in the field tends to change its length; consequently a tape should be tested whenever there is any likelihood that its length has been altered. "When in doubt standardize the tape." In work of a high order, it may be desirable to test the tape almost daily. The surveyor should be able to do this for himself by comparing the tape either with a standardized tape which is kept for that purpose only, or with some form of permanent standard. Methods of standardizing tapes are explained and different types of permanent standards are described in Arts. 715–717.

95(*d*). THE INCLINATION OF THE TAPE TO THE HORIZONTAL should be determined and recorded for each section of the line that is measured with the tape inclined. The inclination may be measured directly with a clinometer, or indirectly by determining the difference in elevation between the two ends of the slope. For the latter method, a hand level, a surveyor's level, or a transit with telescope level may be used. Both speed and precision are increased by the use of the level or the transit.

When a line is relatively long with many changes in the inclination of the tape, it is often best to determine the elevations of the points of change in succession as if running a profile through those points. A light level is useful for this purpose. In judging the accuracy with which the inclination of the tape should be determined, it is well to remember that if one end of a 100-ft. tape is 1.4 ft. higher than the other, the horizontal projection of the tape is 0.01 ft. less than 100 ft. In reducing slope distances to corresponding horizontal distances, the approximate formula in Art.

60(*a*) may be used for grades of 5 per cent, or under. Time may be saved by the use of reduction tables (Table III, p. 1104).

95(*c*). THE TENSION IN THE TAPE during a measurement should, in general, be the same as that in the tape when it was standardized. For a 100-ft. tape uniformly supported throughout its length, the standard tension is 10 lb.; but for the same tape supported at the zero mark and the 100-ft. mark only, the tension may be more than twice as much, particularly if it is caused by a pull sufficient to bring the 100-ft. mark exactly 100 ft., measured horizontally, from the zero mark. When it is necessary to calculate a correction for tension, the method explained in Art. 73 may be used.

The pull on the tape may be measured by a spring balance such as that in a spring tape handle. A special spring balance, often used in first-order chaining, consists of a heavy spring with dial and hand pointer (Art. 100(*a*)). A spring balance should be calibrated by testing it in a horizontal working position and comparing results with those obtained with a standardized balance or by some special device. One device for such a test is a standard weight supported by a cord over a frictionless pulley, as explained in Art. 716(*d*).

95(*f*). TAPE STRETCHERS. The usual procedure in chaining is to keep the pull on the tape at a constant amount equal to that for which the tape was standardized. There are a number of devices for keeping the pull steady. The most common tape stretcher is a staff, the lower end of which is held stationary on the ground while the upper end is moved to stretch the tape. Two staffs are generally used, one at each end of the tape. A loose-fitting leather loop, with an attachment to receive the steel loop on the end of the tape, may be slipped over the staff at the rear end of the tape and moved up or down according to the height of the supports; the pull of the tape will hold it in place. On the forward staff a small frame or collar may be used for holding the spring balance which acts as a connecting link between the end of the tape and the staff. The frame slides on the staff and is held at any desired elevation by a set screw or some form of friction grip. Lacking any staff, such as described, the chainman at either end can brace his hands by means of a stake or pole held obliquely with one end in the ground or braced against his foot to resist the pull (Art. 99(*c*)). Whatever happens, the supports should not be disturbed.

In chaining a relatively short line, such as a short base line, a portable pulley-and-weight device or a bell-crank device (Art. 716(*d*)) can be used to exert the desired pull on the tape. Either of these devices automatically keeps the pull constant, thus leaving the chainmen free to concentrate on the other operations involved in accurate chaining. For this reason either device could be used to advantage more than it is, even when it has to be carried along a line of considerable length.

95(*g*). THE TEMPERATURE OF THE TAPE should be measured and recorded each time that a linear measurement is read. To avoid serious errors due to variation in temperature, several precautions should be taken. Two tape thermometers of the same weight should be used, one about 3 ft. from each end of the tape. These thermometers should be calibrated either by the U.S. Bureau of

Standards or by frequent comparison with thermometers at a local weather bureau. The *bulb* of each thermometer should be in contact with the tape; otherwise an error of as much as 5° F. may occur. If a thermometer has no provision for attaching it to the tape, it may be held by adhesive tape. Tapes should not be used until they have attained approximately the temperature of the air. Measurements should be made when the atmospheric temperature is uniform, i.e., during cloudy or foggy days or during night hours. This requirement is less essential when invar tapes are used. Errors due to thermal expansion may be greatly reduced, even under unfavorable atmospheric conditions, by the use of invar tapes; but such tapes are easily distorted and must be handled with care (Art. 714(*b*)). It is well to have spare tape thermometers as part of the field equipment. Corrections for temperature may be made by computation or by means of tables (Art. 71).

95(*h*). Repetition of measurements. Several measurements of a line are generally desirable—sometimes as many as six. For long lines, measured with extreme care, two measurements may be sufficient. Half of the measurements may be in one direction along a line and half in the opposite direction. It is good practice to use two or more standardized tapes as a check and to measure a line at least once with each tape. When corrections for errors from all sources have been made, the measurements of a line by means of different tapes should agree within the permissible limit of error. When long tapes are employed, a shorter tape is often used for measuring odd distances shorter than the long tape length. The graduations on the long tape may then be few in number. When a line is long, it may be best to divide it into two or more sections, and then to obtain the length of each section by repeated measurements.

95(*i*). Reduction to sea level. Since all geodetic tables for computing geographical positions are based on an adopted spheroid at sea level, and since all published data of the federal and state triangulation surveys are generally reduced to this spheroid, it is sometimes desirable or necessary to reduce the measured length of a line which is not at sea level to its equivalent length at sea level. This reduction to a sea level spheroid is given by the series

$$c = L\frac{H}{R} + L\frac{H^2}{R^2} + L\frac{H^3}{R^3}$$

For relatively short lines at relatively low elevations only the first term need be computed or

$$c = L\frac{H}{R}$$

L is the measured length of the line and H is its mean height above sea level. R is the approximate value of the radius of curvature of the earth's surface at sea level; it may be taken as 6,367,000 meters or 20,889,000 ft. (log 20,889,000 = 7.31992). c is the correction to be *subtracted* from L for lines above the sea level.

SEA LEVEL REDUCTION *

Elevation (in Feet)	*Sea Level Factor*	*Elevation (in Feet)*	*Sea Level Factor*
Sea level	1.0000000	3000	0.9998565
500	0.9999761	3500	0.9998326
1000	0.9999522	4000	0.9998087
1500	0.9999283	4500	0.9997848
2000	0.9999043	5000	0.9997609
2500	0.9998804	5500	0.9997370

* From *Special Publication 194*, U.S. Coast and Geodetic Survey, Table 1, p. 8.

96. Field Parties and Equipment. The number of men in a party that can work to the best advantage and the equipment for the party depend on such considerations as the character of the ground, level or sloping, open or obstructed, and upon the degree of precision required. This will be evident from a study of the two illustrative examples of precision chaining in Arts. 99 and 100.

97. To Find the Probable Error. Five measurements of a short base line resulted as follows: 419.862, 419.863, 419.875, 419.861, and 419.858 ft. Assuming that all constant errors have been eliminated as far as practicable, it is required to find the probable error of a single measurement and the coefficient of precision for chaining. This is done by the method of least squares, using the formulas in Art. 45. It will be noticed that the algebraic sum of the residuals is not zero as it should be. This is because the values of v were taken only to the nearest unit in the third decimal place, which is sufficient for all practical purposes. The work is given in detail below.

Measurement	v	v^2
419.862	0.002	0.000004
419.863	0.003	0.000009
419.857	0.003	0.000009
419.861	0.001	0.000001
419.858	0.002	0.000004
5)4.301		$\Sigma v^2 = 0.000027$
419.8602 = mean		

$$E_0 = 0.6745\sqrt{\frac{\Sigma v^2}{n(n-1)}}$$

$$= 0.6745 \times \sqrt{\frac{0.000027}{20}}$$

$$= 0.00078$$

The probable error of the mean is 0.00078 or about $\frac{1}{540{,}000}$.

The probable error of a single observation is $E = 0.6745\sqrt{\frac{\Sigma v^2}{n-1}}$,

or in this case $0.6745\sqrt{\frac{0.000027}{4}} = 0.00175$ or about $1/240{,}000$.

97(*a*). TO DETERMINE COEFFICIENT OF PRECISION (Art. 82). The probable error of the total observed length $R_s = r\sqrt{N}$ (Art. 81) or $0.00175 = r\sqrt{N} = r\sqrt{4}+$ or $r = 0.0008+$. Hence the probable error of a line, say, 900 ft. measured under exactly the same conditions may be expected to be about $0.0008\sqrt{9} = 0.0024$ ft.

The probable error of the mean of $1/540{,}000$ indicates a degree of precision greater than that usually attained except in measurements of first-order accuracy.

98. Illustrative Examples of Precise Linear Measurement. Two examples of precise linear measurement will be given. In the first example, second-order accuracy is required in the measurement of traverse lines; in the second example, first-order accuracy is required in the measurement of a base line. In each example are given the equipment required, the party organization, the duties of each member of the party, and the general field procedure. From a study of these examples there may be gained (1) a knowledge of the practical application of the methods used in eliminating sources of error in chaining and (2) a comparison of the extent to which these methods are applied in second-order chaining with the extent to which they are applied in first-order chaining.

99. Second-Order Chaining.* *Method.* Length measurements should be made with 100-ft. tapes of invar or of steel, supported either at the 0-ft. and 100-ft. marks only, or throughout the entire tape. The two-point-support method can be adapted to all ground conditions and, therefore, is used almost exclusively. The supported-throughout method should be used chiefly for measurements on railroad rails. It can be used on concrete road surfaces, but even when great care is taken, the wear on the tape is excessive. Reduction in cross sections due to wear increases the length of the tape under tension because of the increased stretch and decreased sag.

If possible, measurement should be made on hazy days, unless an invar tape is used. Measurement over bridges or other structures should always be made on cloudy days, or at night, and should be repeated several times to overcome errors due to the expansion of the structure. Taping procedure is demonstrated in Figs. 99(*c*) and 99(*d*).

99(*a*). EQUIPMENT. The equipment for one taping party should consist of the following:

One tape, as specified; five to ten chaining tripods; one spring balance, as specified; two standardized thermometers; two tape stretchers, designed to be used for both types of tape support; two rawhide thongs; five to ten banker's pins for marking; two plumb bobs; adhesive tape, ½-in. and 1-in. widths; one keel; fifty stakes, 2 in. by 2 in. by 30 in.; one transit, preferably with attached level; one self-supporting target; one level (if no transit level is available); one clinometer, or hand level, if neither transit nor surveyor's

* Taken from *Manual 20* entitled *Horizontal Control Surveys to Supplement the Fundamental Net,* published by the American Society of Civil Engineers.

level is available; one level rod, graduated to hundredths of a foot; two folding rules graduated to tenths of feet; one brush hook; one hatchet; one machete; one 6-lb. or 8-lb. hammer, or wooden maul; one or two round-end shovels; record books and pencils.

Many types of chaining tripods are used successfully. A standard type of portable tape support is shown in Fig. 99(*c*). The *type* recommended in the manual, *Horizontal Control Surveys*, is similar except that it is made of wood, has a flat hexagonal top, and stands considerably higher above the ground. It is described in the manual as inexpensive and light in weight, which makes it easy to transport and easy to handle in the field. It is steady on most surfaces, except turf, which cannot be cleared quickly from beneath the tripod feet. Heavier tripods eliminate this difficulty. All tripods should be painted red or red and white so that they can be seen easily by automobile drivers.

99(*b*). PERSONNEL. The minimum taping party consists of the chief (who acts as marker), recorder, tension man, rear tapeman, and instrument man. When taping across a thoroughfare, another man must be added to direct traffic. Increased speed will be obtained if a rodman is added to relieve the recorder of handling the rod, and a laborer or two to handle tripods, and so forth. A levelman must be added if the transit is not equipped with a level, or if a hand level is used.

99(*c*). FIELD PROCEDURE. TAPE SUPPORTED AT TWO POINTS. A target is set at the point toward which measurement is to be made, and the tripods are distributed roughly in position. The transit is set up at the point of beginning and sighted on the target. Although alignment by transit is not necessary, it increases the speed of the party greatly, and is well worth the extra man. If the beginning point is not readily accessible to the tape, a taping tripod is placed under the instrument, carefully in order not to disturb it, and the starting point is transferred to the edge of the top of the taping tripod by the instrument plumb bob. This tripod is not removed until the taping of the section is completed.

The tape is stretched out in the line of progress with the 100-ft. mark forward, and a thermometer is attached at the 2-ft. mark with adhesive tape so that the bulb is in contact with the measuring tape, but free from adhesive tape. A loop of rawhide or string is passed through the eye of the tape at the zero end, and tied 6 to 18 in. from the tape. The tape end is laid on the starting tripod. The rear tapeman passes his stretcher through the loop and places the lower end of the stretcher on the ground against the outside of his right foot. The upper end is under his right arm and behind his shoulder. (See Fig. 99(*c*).) In this position he leans over the tape to see that the zero graduation is held exactly on the mark. This is readily controlled by adjusting his stance. However, he may find it helpful to grasp the tape near its end and behind the mark, applying a slight kinking force, just sufficient to control the position of the zero graduation.

The tension man passes his stretcher through a 6-in. loop of rawhide attached to the spring balance, snaps the spring balance to the tape, and, using the same position employed by the rear tapeman, applies a 20-lb. tension. (See Fig. 99(*d*).)

The chief of party who acts as marker places a tripod in line (as directed by the instrument man) and under the 100-ft. graduation. The tension man slides his rawhide thong until the tape just clears the top of the tripod. The marker must see that the tape is dry, clean, and free from all obstructions, and may run a light rag along its entire length at this time to remove any moisture or dirt. The marker gently depresses the tape to touch the marking surface of the forward tripod and, on a signal from the tension man that he has exactly 20 lb., and from the rear tapeman that the mark is right, he marks the tripod at the 100-ft. mark. When the tripod has a wooden top, the mark may be made with a hard pencil or with a T-shaped banker's pin which is forced into the wood to mark the point, and is always left sticking in the tripod. Bristol board of the thickness of the tape may be secured to the top of the tripod with Scotch masking tape, so that the edge of the tape butts against the edge of the Bristol board. The terminal mark of the tape can then be transferred to the board with a marking awl or a sharp, hard pencil. The Bristol board can be renewed at any time. On the heavier type of tripod, the mark may be made on a strip of white adhesive tape attached temporarily to the top of the tripod as shown in Figs. 99(*c*) and 99(*d*). Tension is released slowly, then reapplied for a check on the marking, signals from the tension man and rear tapeman being repeated. The recorder obtains the



Fig. 99(*c*). The rear chainman, by adjusting his stance, keeps the zero graduation mark of the tape exactly opposite the mark on the top of the tripod.



Fig. 99(*d*). The tension man, by adjusting his stance, keeps the tension at the required amount. The chief of party gently depresses the tape, marks the point on the tripod opposite the 100-ft. graduation mark of the tape, and takes the thermometer reading. When there is only one thermometer and it is attached to the tape near the zero end, the rear chainman takes the temperature reading.

temperature from the rear tapeman, holds the rod on the tops of the chaining tripods for the instrument man, and records the rod readings.

A record is made for each individual tape length or partial tape length, which includes the length used, the temperature, and the inclination.

The marker moves back to support the center of the tape, and it is then carried forward, the tape being held clear of all contacts by the marker, the tension man, and the rear tapeman. After the second tape length is measured, the recorder may begin picking up the tripods. He can carry about five of these, to be distributed later by the entire party. When it is necessary to bring the transit up, one of the tripods is placed accurately on line and the instrument is set up over it. For distances of less than a tape length, the tape is read independently by both the chief of party and the recorder. If the reverse side of the tape is graduated in meters the metric reading should be recorded as well. The head of the tape is carried beyond the end point, the zero mark being at the back tripod as usual. If the setup is more than 50 ft., a 20-lb. tension is used; otherwise a pull of 10 lb. is used. This affords a close approximation for proportional application of the standard tape correction.

When measuring where the tripods are unsteady, as on turf, stakes 2 in. by 2 in. by 30 in. should be used. They can be handled and driven by the marker and recorder, but, if possible, an extra man should be added for this work.

99(*d*). Field procedure. Tape supported throughout. When the tape is supported throughout, the procedure is much the same as in the foregoing description, except that no transit alignment is necessary on railroad rails. The rails themselves are sufficiently accurate. Stretchers are placed in front of the foot, which is placed on the base of the fulcrum. The recorder must aid the rear tapeman in making contact. On railroad rails or asphalt roads marks can be made with a sharp awl, but on concrete surfaces a piece of adhesive tape should be stuck to the pavement and marked with a hard pencil.

99(*e*). Backward measurement. It is best to measure each section in two directions. Although this is not demanded by the accuracy required, it provides the only proper check against blunders and bad practices. The results, reduced for temperature and inclination, should agree within one part in 30,000. The method for entering traverse measurement records is shown in the sample form.

Sample Form for Recording Traverse Measurement

From Station: *Mon 138* To Station: *Mon 139*
Date: *November 4, 1940* Time: *9:30 A.M.* Tape No.: *36*

Section		Temperature in ° F.	Setup in Feet	Tape Support		Inclination Difference in Feet	Remarks
From	To						
Mon 138	1	22	100	2	20	1.7	
1	2	22	100	2	20	2.5	
2	3	23	100	2	20	9.68	
3	4	25	100	2	20	4.1	
4	Mon 139	25	42.75	T	10	0.0	Flat on Pavement
			442.75				

99(*f*). Leveling. Leveling may be done with a surveyor's level, the attached level on a transit, a hand level, or a clinometer. All have been used successfully, but the first two increase both speed and precision. When a surveyor's level, or a transit level is used, rod readings are taken to hundredths of a foot on the tops of the tripods. A reading is taken on the same tripod from each of the two instrument positions, when the instrument is moved, and care taken to denote which reading was obtained from each position.

An extra man should be available when the hand level is used. He should carry a light notched stick to support the level, and, standing near the 50-ft.

mark, should take a reading to tenths of a foot on both tripods for each tape length, recording the difference in elevation. Collapsible foot rules, graduated to tenths of a foot, should be carried by the tension man and the rear tapeman for the levelman to sight on.

The clinometer is most successfully employed when 4-ft. taping tripods are used. It is placed on one tripod and sighted at a small target on the next tripod. The angle of inclination or the percentage grade is recorded.

99(*g*). Field computations. The measurements should be reduced as soon as possible, either in the field or in the field office. This is especially true when short triangles form part of the measurements. A form for computation is given below.

Form of Computation for Reducing Measurements

Section	Date	Dir. of Meas.	Tape No.	Tape Support	Uncorrected Length		Temp.	Corrections					Reduced Length	Adopted Geodetic Length
					Tape Lengths	Feet		Temp.	Incli-nation	Tape and Catenary	Setup (+) (= +Cor.)	Sea Level		
							Fahr.	Feet	Feet	Feet	Feet	Feet	Feet	Feet
(*a*) TRAVERSE LINE														

References: Temperature correction, Art. 71(*a*); inclination correction, Art. 60; reduction to sea level, Art. 95(*i*). The tape and catenary correction is determined when the tape is standardized. The length of the tape as determined for the proper type of support should be chosen, and the following formula applied:

$$C = L - 100.000$$

in which C = the decimal part of a foot to be added, and L = the length of the tape in feet for the proper support. Temperature and inclination corrections should be applied while the chaining is in progress, or as soon thereafter as possible, in order to determine whether the work is acceptable.

99(*h*). Rejection limits. Backward and forward measurements should agree within one part in 30,000, after reduction for temperature and inclination.

100. To Measure a Base Line with a Precision of the First Order, that is, with a resultant probable error that does not exceed, for example, about 1/600,000. Such precision requires first-class equipment and a certain procedure in field work that has become more or less standard practice. An excellent treatment of both equipment and procedure is given in *Manual of Engineering Practice 10,* entitled *Technical Procedure for City Surveys,* issued by the American Society of Civil Engineers.* The following epitome is composed of selected portions of the manual, quoted almost verbatim.

* Even more explicit directions for base-line measurement may be found in *Manual of First-Order Triangulation,* prepared by the U.S. Coast and Ge-detic Survey.

100(*a*). EQUIPMENT. Three 50-meter tapes, two for use in the field and one kept as a local standard, each graduated at the 0-, 25-, and 50-meter points, sometimes also at the 12.5- and 37.5-meter points; a 25- or 30-meter tape, graduated in meters on one side and in feet on the other, for measuring the odd lengths. The meter unit is recommended because the National Bureau of Standards is equipped to standardize meter tapes more accurately than foot tapes. Each tape should be tested for absolute length (Art. 95(*c*)) under the same conditions that will be encountered in the field; for example, supported at three points, five points, or throughout its length. Tests should be made both before and after field measurements. Inequality in the graduations of a tape is often a source of serious error; hence the short (25- or 30-meter) tape should be standardized for each meter while supported throughout. An invar tape is preferable to a steel tape because its coefficient of expansion (about 0.0000001 per degree centigrade) is so low that the tape can be used during the day under varying temperatures with no appreciable loss in accuracy, whereas, on the other hand, a steel tape (coefficient about 0.000011 per degree centigrade, or one hundred times as great) should be used only at night or possibly on a cloudy day (Art. 71).

Spring balance consisting of a heavy spring with dial and hand pointer, having a minimum capacity of 20 kg. or more, tested in a horizontal position for the tension to be used in the field. In addition to this test by the Bureau of Standards, a local test should be made before and after each day's work. This may be done by rigging up a frictionless pulley such as a bicycle wheel, and hanging a standard weight by a cord extending over the wheel and attached to the spring balance in a *horizontal* position (Art. 716(*d*)).

Tape thermometers, preferably graduated in degrees centigrade, calibrated by the Bureau of Standards or by frequent comparisons with the local weather bureau thermometers.

Portable tape supports for the ends of the tape, each consisting of a small rigid tripod, weighing 15 to 20 lb. and standing about 15 to 18 in. high, with an iron-plate head surmounted by a segment of a wooden cylinder covered with a sheet of copper or brass. Intermediate tape supports consisting of an upright iron pipe about 3 ft. in height, supported at its bottom through a ball-and-socket joint on a small iron tripod, with a projection for supporting the tape that slides on the pipe and may be adjusted at any desired height by means of a set screw. It may sometimes be more convenient to drive wooden posts at 25-meter intervals for tape supports, with the posts at the 50-meter points capped with sheets of copper. Intermediate posts should be set slightly off line so that nails driven on the proper grade will hold the tape vertically on a straight line.

Miscellaneous equipment includes suitable devices for keeping the tape stretched and steady; level and level rod for determining end elevations of the tape and intermediate changes in its grade.

100(*b*). FIELD PROCEDURE. A minimum of six men is required for the most efficient operation. Of the two men making the forward contact, one stretches the tape and maintains the proper tension, and the other makes and checks the forward mark. One man attends to the intermediate supports and acts as level rodman; two men give tension and contact to the rear end of the tape; and the sixth man acts as levelman and keeps a set of check notes. When the base is located on a city street and traffic is encountered, additional helpers are needed to guard the tape against traffic interference. More helpers are required during night work to hold lights for illumination.

On flat grades, intermediate tape supports may be placed on grade closely enough by eye. Their elevations should be taken, however, for the purpose of checking the grades. On steep grades, the intermediate supports should be "lined in" for grade as closely as possible by eye, their elevations taken, and inclinations computed for each fractional tape length.

Each base should be divided into sections ½ to ¾ mile (800 to 1200 meters) in length. Each section of the base should be measured at least twice, in opposite directions and with different tapes. Two such measurements are sufficient, provided the discrepancy between the two does not exceed 5 mm. per km. The final criterion for acceptability, of course, is the probable error in the total length of base. If this is within allowable limits, and the probable errors for the component sections of the base are permissible, no additional measurements need be made. If it is necessary to use steel tapes, the safest and most economical practice is to use four or more tapes, and to measure each section four or more times. In this way, compensating errors due to

erroneous determinations of the temperature of the tape are more likely to be disclosed than when only two tapes are used, and the section is measured twice.

Accurate determination of the temperature of the tape is important, especially when steel tapes are used. Two small thermometers, of the same weight as those used by the Bureau of Standards in laboratory tests, should be lashed to the tape about one meter from each end. Steel tape measurements should be made only while the temperature is uniform, such as during night hours. Cloudy and foggy days may also sometimes be utilized to advantage. The tape should be stretched flat, supported so that it does not touch the ground or pavement, and should be allowed to rest about one-half hour before using in order to attain the temperature of the outside air.

Base-line tapes should never be supported throughout their length on a wet smooth surface, such as a steel rail or a paved street, because under these conditions a suction is created between the tape and the supporting surface, causing uneven tension in the tape. When it is advisable to locate a base along a railroad tangent, the tape should be raised from the rail by supporting it on small wooden or glass rollers of such diameter as to prevent the tape from touching the rail between supports, which are usually placed at the 12.5-, 25-, and 37.5-meter points of the tape.

100(*c*). COMPUTATIONS. The various steps in the computation of a base line are: (1) correction for slope, i.e., reduction to the horizontal—for grades of 5 per cent or less, the approximate method may be used (Art. 60); (2) temperature correction, or reduction to a standard temperature (Art. 71); (3) absolute length correction (Art. 68); (4) reduction to sea level and to rectangular coordinate datum plane when the purpose of the survey renders such reductions desirable (95(*i*)); (5) computation of probable error of base (Art. 97). (In the *Manual on Engineering Practice 10* will also be found forms for field notes and specific directions for computations.)

CHAPTER 9

USE OF THE LEVEL AND THE LEVELING ROD

In this chapter directions and suggestions are given for setting up the level, for the use of its telescope, and for reading the leveling rod. Different types of levels and leveling rods are described in Chapter 52.

101. Care of the Level. First of all, treat the level kindly. Remember, many of the parts are of soft metal. Never force a screw. All screws should work easily; if they do not, there is some trouble which brute force will not help. The diameter of the head of a thumb screw often gives a leverage out of all proportion to the strength of the screw, and it is comparatively easy to strip the thread, or even to twist off the head of the screw. Beginners too often feel that clamps should be set as tight as possible, with the result that each time they have to be set a little tighter and soon become useless. The condition of the clamps on comparatively new school instruments should be a lasting object lesson. If the object-glass slide in the telescope moves hard, probably dust or grit is inside, and it will not take a very strong man either to destroy the rack and pinion, or to cut deep scratches and grooves along the surface of the slide. Likewise, do not force the eyepiece or any other movable part of the level. If the instrument is kept in good condition all such parts should move easily. Rubbing the object glass hard to clean it may get the dust or dirt off, but the glass may be scratched. Treat it gently.

In taking the level out of its box, do not lift it by the telescope or by any other delicate part, but take hold of the lower part of the instrument. Make sure that the level is securely screwed to the tripod head. It it drops off, it costs money, as more than one levelman can testify. Make the wing nuts in the upper part of the tripod just tight enough for friction to hold each leg in any position. Do not let a level stand with the tripod legs so close together that it is easily upset. On the other hand, if the legs are too far apart they may open out and let the level down, particularly if the wing nuts are a little too loose; this is most likely to happen on a smooth surface such as a floor or pavement. Do not place one of the tripod shoes so nearly on a line with the other two that the center of gravity of the instrument does not fall well within the triangle be-

tween the three shoes. The more nearly this triangle is an equilateral triangle, the more stable the instrument. Do not leave a level unguarded, particularly when it is where it might be knocked over by people, moving vehicles, or animals; or where it might be upset by wind or blasting; or where something might fall on it—the dead limb of a tree, for example.

Before picking up the level to carry it, clamp the spindle tight enough so that the telescope will not swing back and forth, but loose enough so that if bumped it will yield readily. The focusing screw on the objective slide should not be so loose as to allow the objective slide to run out. In carrying the level keep it in front when passing through a low or narrow place such as a door, or through brush and trees, especially when there are overhanging limbs; but do not carry the instrument with head down and legs up. Be careful to have a firm footing before passing an instrument over a fence or a wall, or when taking it from someone on the other side. To prevent the loss in the field of small parts of the level, keep the pin clips attached with strings, do not remove the sunshades from the telescope until the day's work is done, and then replace the telescope cap over the lens.

When the level has been exposed to moisture, wipe it dry before putting it away. Clean mud and dirt from the tripod legs. Do not put a dry tripod in a damp place as swelling may break a wing nut. Since loose tripod shoes are a source of error, keep them tight. Keep all moving parts of an instrument *clean* and slightly lubricated, but do not allow oil or grease to remain on exposed parts for it will gather dust. (See Chapter 55 for instructions pertaining to the care of instruments, including cleaning and lubricating.)

102. Setting Up the Level. In setting up the level, the main object is to get the whole instrument into such a position that the line of sight will revolve in a horizontal plane and remain in that plane. The first step is to plant the tripod legs *firmly* in the ground with the head of the tripod approximately level. The second step is to adjust the upper part of the instrument by means of leveling screws so that the line of sight will be horizontal no matter in what direction the telescope may be pointed. To do this, turn the telescope until it is above two diagonally opposite leveling screws, and bring the bubble *approximately* to the center of the tube by turning the screws. Turn the telescope until it is over the other pair of diagonally opposite screws, and bring the bubble to the center. Turn it back until it is above the first two screws and repeat the process, first over one pair of screws and then over the other, until the bubble stays in the center of the tube when the telescope is

swung around a horizontal circle. If the instrument is in adjustment, the line of sight will then revolve in a *horizontal plane of sight.*

102(*a*). SUGGESTIONS FOR MANIPULATING THE TRIPOD.

1. Spread the tripod legs just far enough to bring the telescope at a convenient height since it saves stretching up or stooping down.

2. Manipulate the tripod so that at the start its head is approximately level. The less the thumb screws are used the better—saves wear, saves time. On a slope, place two legs on the downhill side. Aim to keep diagonally opposite screws parallel to the general line in which the leveling is being done.

3. The tripod must stand firmly. On soft ground push the legs well in, but not as if they were to stay for all time. Pushing against a leg crosswise is a waste of strength and may break the leg by bending; push lengthwise. If there are lugs on the shoes of the tripod, the shoe of each leg may be forced into the ground by pressing on its lug with one foot. On hard, smooth surfaces, such as floors and pavements, put the shoe points, if practicable, in cracks or holes, to prevent slipping. It is especially important that the level should not settle after it is once set up.

102(*b*). SUGGESTIONS FOR MANIPULATING THE LEVELING SCREWS.

1. In turning the two diagonally opposite leveling screws, the thumbs always move toward or away from each other. The knowledge of which way to turn the screws should soon become instinctive. It may be helpful, however, to remember that *the bubble will move in the direction in which the left thumb is turned.* When the thumbs turn toward each other the bubble will move toward the right; away from each other, left.

2. When starting to level up, time may be wasted by being too particular about bringing the bubble exactly to the center. Only toward the end of the process is great care necessary. When the instrument becomes so nearly level that the bubble needs to be moved merely a part of one division, the final centering is best done with *one* screw; when both screws are a little loose turn the screw that will move the bubble toward the center and at the same time make both screws tight; similarly, when both screws are a little tight, loosen the one that, in turning, will move the bubble toward the center.

3. Keep all four leveling screws bearing evenly. If one of a diagonally opposite pair of screws is turned faster than the other, it will either bind or be raised off the bearing plate. Stop turning this screw until the other catches up. If both screws of a pair bind, loosen one of them slightly. If all four screws are turning hard, the tripod head may be too much out of level, or one pair may have been forced, or possibly the screw threads need cleaning. If the tripod head is obviously badly out of level, it may be necessary to reset the tripod. If this is not the case, try loosening two screws. Never force screws. Keep them quite loose as long as the bubble is considerably off center; they should turn so easily that they may be *rolled* between the hands, a more speedy process than turning with the fingers. After leveling up, take hold of the upper part of the level to see if it is firm. If it rocks, one or more of the leveling screws is not bearing properly.

103. Use of the Telescope. In order to use a telescope intelligently one should be familiar with the theory of lenses (Art. 706) and should know how the different parts of a telescope are constructed (Art. 707). (1) To make the cross hairs clearly visible, move the eyepiece in or out. On some instruments the eyepiece is moved by screwing or twisting it, but on others a focusing screw with rack and pinion is used. The proper focus depends solely upon the eye of the observer. On many modern instruments the eyepiece is finely graduated on the outside to enable one to set it quickly at a position to suit the eye once that position has been

determined. Lacking this graduation a fine mark scratched on the surface will serve the purpose. (2) To focus on an object, move the objective slide in or out, remembering that the nearer the object the farther out the object glass must be run. (Why?) (Art. 706.) "Near-far and far-near." This rule, however, does not apply to an internal focusing telescope. (3) To get the line of sight on the rod quickly, sight along the top of the telescope before sighting through it. (4) The eyepiece and the objective slide should move in and out freely. Never force either; instead, see if a slide needs cleaning or if a rack and pinion is out of order.

103(*a*). PARALLAX. If in sighting on a leveling rod the horizontal cross hair appears to move up or down on the rod when the eye moves slightly up or down, this movement is due to parallax, as explained in Art. 737. It is of the greatest importance to eliminate this parallax. It is a source of error too often neglected and may render accurate work impossible. To eliminate it, two steps are necessary.

1. Point the telescope toward the sky or any light surface, and move the eyepiece in or out until the cross hairs appear sharp and black, i.e., until they are in perfect focus.

2. Focus the objective on a leveling rod about 100 yd. distant and test for parallax by moving the eye slowly up or down across the eyepiece. If the cross hair appears to move over the image of the graduations on the rod, parallax is present. This parallax may be due to poor focusing of either the eyepiece or the object glass. To eliminate this parallax, refocus the object glass until parallax disappears. Then, if the graduations on the rod are not distinct, move the eyepiece the small amount necessary to make them appear so. The image of the graduations is now in the plane of the cross hair, and both are in focus of the eyepiece.

The greater the precision required, the more necessary it is to eliminate all parallax, whereas the higher the power of the telescope, the more difficult it is to do this. In leveling of a high order this adjustment should be tested frequently during the field work, for, as the eye of the observer tires, the focal distance of the lens of the eye changes, causing a blurring of the cross hairs. This should be corrected by refocusing the eyepiece. (See Art. 737 for additional information concerning parallax.)

104. The Position of the Leveling Rod. The leveling rod is used to measure the *vertical* distance from the horizontal plane of sight to a given point, usually, but not always, *below* the plane. With the base (zero end) of the rod on the point, the rod is held

vertical in all directions, like a plumb line, with its graduated face toward the level.

105. Sighting on the Rod. (1) If the rod is a self-reading rod, and a target is not used, the levelman sights to ascertain what horizontal line on the graduated surface appears to be covered by the horizontal cross hair. (2) If a target on the rod is used, the levelman sights in order to bring the center of the target into the plane of sight. This he does by directing the rodman to move the target up or down until the horizontal cross hair appears to cover the horizontal center line of the target.

106. Reading the Rod. In explaining how to read a leveling rod it will be assumed that a target is used. Once having learned to take target readings correctly, it is an easy matter to read a rod directly through the telescope without the use of a target. Assuming that a target has been set in such a position on the leveling rod that the horizontal center line of the target is in the plane of sight, it becomes necessary to determine the distance of that center line from the zero point of the rod. The method of obtaining this target reading will depend not only upon the way in which the face of the rod is graduated but also upon whether the target is provided with an ordinary *scale* or with a *vernier.* Both cases are explained in succeeding pages.

107. Targets with Ordinary Scales. The targets on page 100 illustrate two forms of the *Philadelphia rod.* (For general purposes, this type of leveling rod is used by many engineers in preference to any other type.) The scales on all six targets are exactly alike, i.e., each is a tenth of a foot, subdivided into hundredths and half-hundredths (0.005 ft.). In the three upper figures the smallest division on the rod is *one-tenth* of a foot; in the three lower figures each tenth is further subdivided into ten parts, or hundredths of a foot. This latter subdivision is for reading without a target. In target readings *no attention is paid to hundredths of a foot on the rod.* In the three lower figures readings are taken to the *upper edge of a black mark* instead of to a fine line, as in the upper figures; this is the only difference between the two forms of rods so far as target readings are concerned. Note that the target scales *are not verniers.*

107(*a*). TO READ THE ROD. (1) Note on the rod the nearest *foot below* the zero or target center. (2) Note on the rod the nearest *tenth of a foot* below the target center. (3) Note the division on the scale opposite the tenth of a foot observed in (2).

When no division on the scale is exactly opposite the tenth of a foot observed in (2), the thousandths of a foot may be estimated.

For example, the reading for the target in the upper right-hand corner of Fig. 107 was obtained as follows. (1) The foot division on the rod that is nearest to and below the target center is 5. (2) The tenth-of-a-foot division on the rod that is nearest to and below the target center is 3. (3) The nearest hundredth-of-a-foot division on the scale that is *above* the 3 on the rod is 1. The reading thus far is, therefore, 5.31 ft. (4) The 3 division on the rod is somewhat less than halfway between 1 and 2 on the scale, and the thousandth-of-a-foot reading is, therefore, somewhat less than 0.005, estimated as 0.003. Thus the final reading is 5.313 ft.

The correct reading for each of the six rods in Fig. 107 is printed on the target. The methods of making these readings are indicated below.

Feet	+	Tenths	+	Hundredths	+	Thousandths	=	Reading
5	+	2	+	0	+	5	=	5.205
5	+	2	+	6	+	0	=	5.260
5	+	3	+	1	+	3	=	5.313
5	+	9	+	4	+	0	=	5.940
6	+	0	+	0	+	5	=	6.005
6	+	0	+	6	+	5	=	6.065

107(*b*). COMMON MISTAKES. (1) To omit a zero from a reading; thus, 5.25 instead of 5.205, or 6.05 instead of 6.005, or 6.65 instead of 6.065. (2) To omit two zeros from the reading; thus, 6.5 instead of 6.005. (3) To take the nearest numbered division *above* the target center instead of *below* it; thus, 6.94 instead of 5.94, or 6.105 instead of 6.005, or 6.165 instead of 6.065.

Remark. A study of Fig. 107 will show that the above mistakes are natural ones for the beginner to make.

108. Targets with Verniers. The underlying principle of the vernier and the method of reading verniers are explained in Art. 703.

When a target has a vernier, hundredths of a foot are read on the rod itself instead of on the target, as is done on an ordinary scale, and thousandths are read from the vernier, not estimated. The graduations on the *New York rod* (three upper figures, Fig. 108) do not differ from those on the second form of *Philadelphia rod,* except that hundredths of a foot are marked with fine lines. On the target, however, ten divisions correspond to nine on the rod (see middle figure), or each division is 1/1000 ft. shorter than a division on the rod.

To Read the Rod. (1) Read feet, tenths, and hundredths by observing the position of the vernier zero. (2) Read *thousandths* by

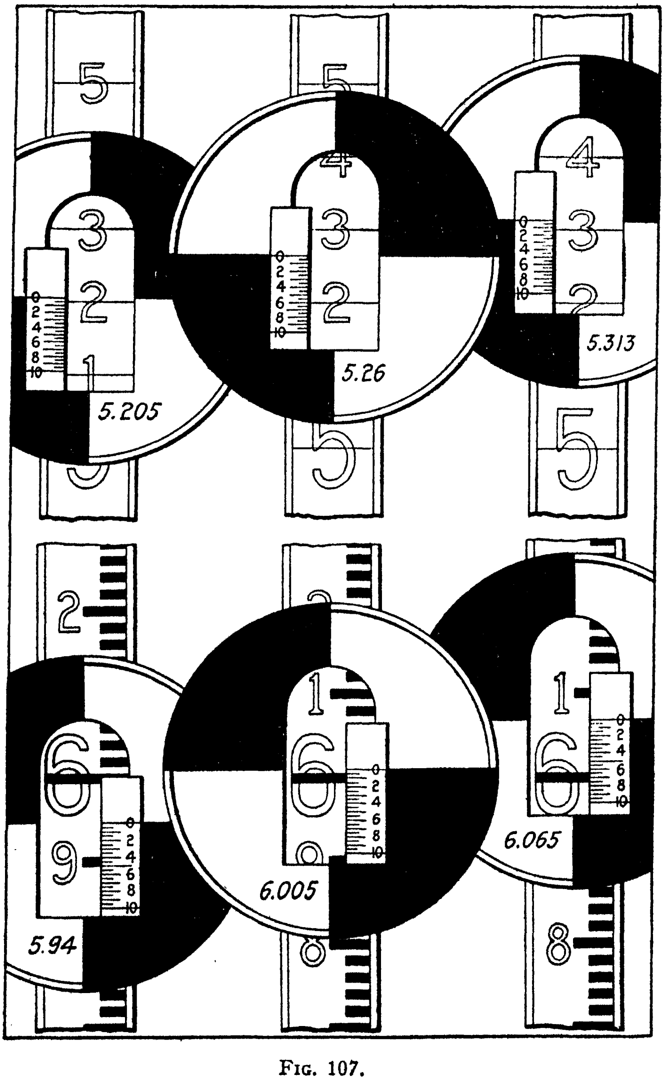


FIG. 107.

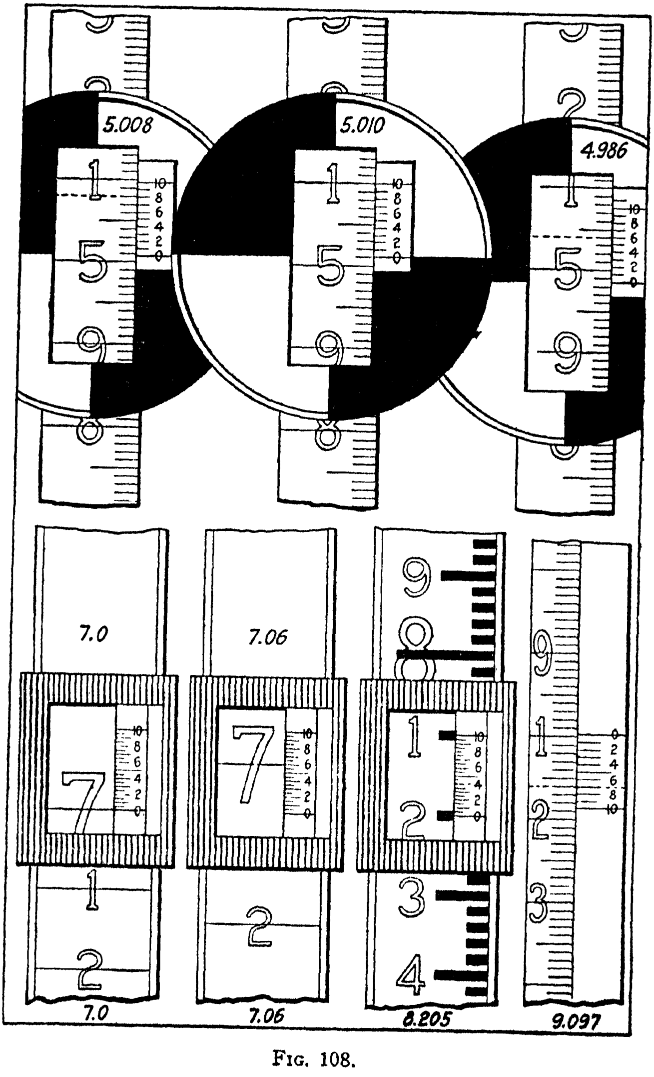


Fig. 108.

observing what mark on the vernier coincides with *any* mark on the rod.

Feet	+	Tenths	+	Hundredths	+	Vernier	=	Reading	Mistaken readings
5	+	0	+	0	+	8	=	5.008	5.08 or 5.8
5	+	0	+	1	+	0	=	5.010	5.10
4	+	9	+	8	+	6	=	4.986	5.986 or 5.16

109. Extended Rods. The method of reading an extended rod will be illustrated by rods having two sections, so graduated that in measuring any vertical distances between 7 and 14 ft., the front half of the rod must be extended upward, and the reading taken from the *back* of the other half. Each of the first three lower figures in Fig. 108 represents the back of a Philadelphia rod with an *ordinary scale*, whereas the fourth figure represents a New York rod with a *vernier*. The graduations on the back of the rod are numbered from the top *downwards*, or in just the reverse direction from the numbering on the front of the rod. Consequently, in reading the back of the rod, observe the nearest foot and tenth *above* the zero of the scale instead of below it, i.e., read *downward* instead of upward; this is the only difference between reading the back and the front of a leveling rod.

109(*a*). To USE A HIGH ROD. The first step is to set the center of the target exactly opposite the 7-ft. mark on the front of the rod. Before extending the rod the reading on the back should be exactly 7 ft. (see first figure), and the target should be exactly 7.0 ft. from the base of the rod. (See Art. 111 for precaution.) If the front half of the rod with the target fixed at 7 ft. is then raised, say, 0.06 ft., the reading on the back should be 0.06. (See second figure.) The reading for the third rod is 8.205. (Mistaken readings 8.25, 7.25.) For the vernier rod the correct reading is 9.097. (Mistaken readings 9.97, 9.107, 8.107.) One of the common mistakes in reading a high rod is to read the rod upward instead of downward, as illustrated by the mistaken readings 7.25 and 8.107. The omission of one or more zeros is another common mistake in high-rod readings as well as in ordinary readings, as illustrated by the mistaken readings 8.25 and 9.97.

Not all leveling rods are graduated from 0 to 7 ft. on the front and from 7 ft. to 14 ft. on the back. Many rods, for example, are designed to be set at 6.50 ft. for a high rod reading. The best rule to follow, therefore, is to set the target on the front at the *closed rod rear reading*.

109(*b*). WHEN A RETROGRADE VERNIER is used on a target, the entire vernier is *below* the center of the target instead of above. For an unextended rod it is read downward instead of upward, i.e., in the opposite direction to

rod graduations; for an extended rod, the retrograde vernier is read upward instead of downward. (See Art. 703(*e*) for an explanation of the retrograde vernier.)

110. Methods of Reading Rods Compared.

Target with a Scale	*Target with a Vernier*
1. It is not necessary to observe the *hundredth* of a foot on the rod; take the nearest tenth *below* the zero of the scale (or *above* it for extended rod).	1. The hundredth of a foot mark on the rod just *below* the zero of the vernier (or *above* it for extended rod) must be observed.
2. Thousandths are estimated.	2. Thousandths are read from the vernier.

111. To Test a High-Rod Reading. A leveling rod should be tested from time to time for high-rod readings. Clamp the target at the highest mark on the front of the rod and set the zero of the scale or vernier on the back of the rod at the same reading (7.0 ft. for the rods illustrated in Fig. 108). Measure with a steel tape the distance from the base of the rod to the center of the target. If this measurement does not check the reading (7.0 ft.), make a fine scratch on the face of the rod to indicate where the target should be set for high-rod readings. Set the target at this scratch, extend the rod to any reading, measure with a tape, and compare readings as before. Repeat this last step two or three times. It not only shows exactly where to set the target, but it settles all doubt as to how the rod is read when extended. If an error is found, attach a tag to the rod or mark the rod in some other way to indicate where the target should be set; for example, "The target on this rod should be set at 6.995 instead of at 7.0 ft. for all high-rod readings."

Remark. The error described above is very often found in leveling rods, especially if they have been used for some time. It is often caused by letting a high rod down "on the run," thus jamming the small piece at the upper end of the rod, and gradually knocking it upward. Out of a large number of school rods examined by the author over 75 per cent contained this error.

A Second Test. Ascertain, by measuring with a tape, the reading at which the scale or vernier on the *back* of the rod should be set in order that readings taken on the front of the rod *above* the joint will be correct. It is not safe to assume that when the joint is closed in perfect contact the readings above it will be correct. (Why is it better to have this joint near the top of the rod than near the bottom?)

112. The Boston Rod is another common form of target leveling rod. Like those already described, it has two sections; but the methods of taking readings differ somewhat from those already xplained. For low readings (less than 5.8 ft.), the target is clamped near the bottom of one section, and this section is then raised until the center of the target is in the plane of sight. For high readings, the rod is inverted end for end, thus bringing the target to the top of the rod. By extending the rod, the target can then be moved to any reading between 5.8 and 11.4 ft. This is an illustration of how methods of setting targets and reading rods may differ for different types of rods.

113. The Tape Rod. In this special form of self-reading rod, the graduations are on a movable metal tape which may be clamped in such a position on the face of the wooden portion of the rod that the elevation of any point on which the rod is held may be obtained directly from the rod reading, with little or no calculation. (See Art. 172(*a*).)

114. Use of the Hand Level. Hold the level to the eye and raise or lower the end until the reflection of the bubble is exactly bisected by the cross hair. The horizontal line of sight thus established is at the elevation of the eye. There are two ways of leveling with a hand level.

1. *To stand erect.* In this case the height of the line of sight is found by adding to the elevation of the ground upon which the observer is standing the height of the eye above the ground.

2. *To rest the level on a rod* of known height, say 5 ft. Hand level rods made especially for this purpose are generally used. This is more accurate than the preceding method, as the height of line of sight is always exactly 5 ft. above the ground upon which the rod rests. In either case, the rod upon which sights are taken may be any convenient type of self-reading rod, such, for example, as the Philadelphia rod. The hand level will, of course, give only approximate results; but it is useful, nevertheless, in certain types of surveying, such as the location of contours in topographic surveys.

115. Special Equipment. For special forms of leveling rods, such as precision leveling rods, automatic leveling rods, and flexible rods, see Art. 718(*b*). For special forms of targets, such as the angle target and the micrometer target, see Art. 718(*a*). For a rod level to aid in keeping the rod plumb, see Art. 718(*c*).

CHAPTER 10

LEVELING

THEORY

In this chapter the *theory* of leveling is explained without entering upon the practical details of the field work. Leveling terms are defined and forms given for level notes. Practical directions and suggestions for field work will be found in the next chapter.

116. Datum. In any given survey all elevations are referred to some common *datum.* This datum is usually an imaginary *level* surface, all points of which are assumed to have an elevation of *zero.* The *elevation* of any other point is its vertical distance above or below this surface or datum.

The mean sea level affords a convenient datum the world over, and elevations are commonly given as so much above or below sea level. It is often more convenient, however, to assume some other datum, especially if only the *relative* elevations of points are required.

116(*a*). DATUM LINE AND DATUM PLANE are two terms whose meanings, though obvious, are less general and comprehensive than the simple term datum. In fact, as pointed out in Art. 3, a level surface is not a plane surface but a curved surface, hence a level line is not a straight line but a curved line. This distinction is not often made, however, in plane surveying, since curvature of the earth's surface is ignored.

117. Leveling. A branch of surveying the object of which is: (1) to ascertain the elevations of given points with respect to a given or assumed datum; (2) to establish points at a given elevation or at different elevations with respect to a given or assumed datum. For example, to establish points on a level or on a slope for grading, or to fix the heights for different parts of a structure.

118. Methods of Leveling. (1) *Direct* or *spirit* leveling by means of a spirit level; (2) *barometric* leveling in which elevations are determined by means of a barometer; (3) *trigonometric or indirect* leveling in which the difference in elevation between two points is found by measuring at least one distance and one angle and then calculating the required difference in elevation by methods of trigonometry.

119. Differential Leveling. Direct leveling the object of which is solely to determine elevations of points regardless of the horizontal position of the points with respect to each other.

120. Profile Leveling. Direct leveling to determine the elevations of points at measured intervals along a given line in order to obtain a profile of the surface along that line. *Example:* A profile along the center line of a highway.

121. Bench or Bench Mark. A relatively permanent point of reference whose elevation with respect to some assumed datum is known. It is used either as a starting point for leveling, or as a point upon which to close as a check. In a survey of any extent there are usually a number of such bench marks, established with respect to the same datum and at points where they are most needed.

Illustration. A city establishes an elaborate system of bench marks so that by starting at any one of them a surveyor can determine elevations with respect to the same datum that the city uses in its engineering work, and if it is more convenient to end at a different bench from that at which he started, he will still have a check upon his work.

Remark. The elevation of the *first* bench mark is frequently arbitrarily assumed, as, for example, an even hundred feet, but this is equivalent to assuming arbitrarily a datum 100 ft. below that particular bench mark instead of assuming the sea level as the datum. This is done only when the *relative* elevations are required regardless of the elevations above sea level.

122. Stations. Points whose elevations are to be ascertained or points that are to be established at a given elevation. A station is where the leveling rod is held and *not* where the level is set up.

Remark. In transit surveying the transit is usually set up directly over a station. In leveling, on the contrary, the level is practically never set up over a station, but is placed at any convenient point from which two or more stations can be seen.

123. Plane of Sight. If a level is in adjustment and properly set up, the *line of sight* will revolve in a *horizontal* plane, i.e., it will always be in a horizontal *plane of sight* no matter in what direction the telescope is pointed. (See Fig. 124(*a*).)

Remark. Since this plane is perpendicular to a plumb line, it is tangent to a level surface at the point where the instrument stands. As already pointed out, however, the distinction between a level surface and a horizontal plane is not an important one in plane surveying.

124. The Two Essential Steps in Leveling. After setting up a level the first step is to ascertain the elevation of its *plane of sight* by measuring *up* to this plane from some point *whose elevation with respect to the assumed datum is known.* Then the elevation of any other point may be found by measuring *down* from the plane of sight. These two steps, (1) *measuring up to* and (2) *down from* the horizontal plane of sight, should be the first conception of spirit

leveling, for it is one most helpful in practice and comprises practically the whole theory.

124(*a*). ILLUSTRATION. Let the elevation of the bench mark in Fig. 124(*a*) be 100 ft. above some assumed datum. It is required to ascertain the elevations of the ground at *A*, *B*, and *C* with respect to the same datum.

The first step after setting up the level is to ascertain the elevation of its horizontal plane of sight. The line of sight strikes a leveling rod held on the bench 6 ft. *up* from the base of the rod. The elevation of the plane of sight is therefore $100 + 6 = 106$ ft.

If the telescope is then directed to *A* and it is 4 ft. *down* from where the line of sight strikes the rod to the base of the rod, then the ground at *A* must be 4 ft. below the plane of sight, or the elevation at *A* is $106 - 4 = 102$ ft. Likewise if the ground at *B* is 8 ft. below the plane of sight, its elevation

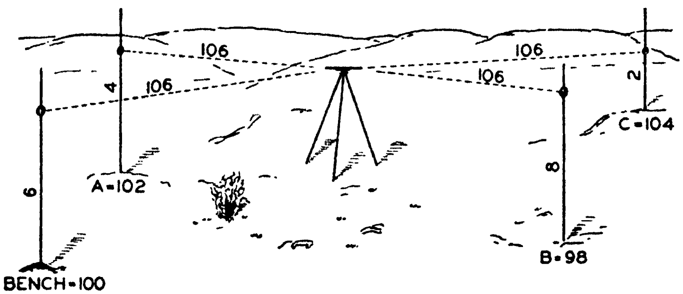


FIG. 124(*a*).

at *B* is $106 - 8 = 98$ ft. If at *C* the ground is 2 ft. below the plane of sight, its elevation at *C* is $106 - 2 = 104$ ft. Thus the elevations of any number of points may be obtained.

Remark. The leveling rod is graduated in only one direction, from the bottom up, and so there is really no difference between measuring *up* and measuring *down*. It is well, however, to think of measuring *up* to ascertain the elevation of the plane of sight, and measuring *down* to ascertain the elevation of points on the ground.

125. Height of Instrument (H.I.). For any particular setup this is the elevation of the plane of sight with respect to the assumed datum. Thus, for example, 106 ft. is the height of instrument in Fig. 124(*a*).

Remark. Notice that height of instrument does *not* mean the height of the telescope above the ground where the level stands, nor does it mean its height above the last station backsighted upon unless, as is sometimes the case, the elevation of this last station is the same as the assumed datum.

126. Backsight (B.S.). A sight taken on a rod held at a point of *known* elevation to ascertain how far the plane of sight is *above* that point, thus establishing the *height of instrument* with respect to the assumed datum. For example, the sight on the bench, Fig. 124(*a*), is a backsight.

Remark. Backsighting is equivalent to measuring *up* from the point of known elevation to the plane of sight.

127. Foresight (F.S.). A sight taken on the rod held at a point of *unknown* elevation to ascertain how far that point is *below* the

plane of sight, thus determining the elevation of that point with respect to the assumed datum. For example, in Fig. 124(*a*) sights on *A, B*, and *C* are foresights.

Remark. Foresighting is equivalent to measuring down from the plane of sight to the point whose elevation is required.

Confusion of Terms. The terms backsight and foresight are unfortunate terms because they imply direction. They have nothing whatever to do with direction. A backsight is not necessarily on a point behind the level, and a foresight may be anywhere in front, to one side, or behind, as shown in Fig. 124(*a*), where all the sights except that on the bench are foresights.

An effort has been made to avoid this confusion of terms by adopting the term *plus sight* in place of backsight, since the backsight reading is always *added* to a known elevation, and *minus sight* in place of foresight, because the foresight reading is always *subtracted* from the height of instrument. The chief objections to this change is that a minus foresight already has one meaning, i.e., a foresight taken to some point *above* the plane of sight, as occasionally happens, for example, in tunnel work or in leveling for a line of overhead shafting. The terms plus sight and minus sight would be preferable, nevertheless, if they could be universally adopted.

It will not be necessary for the beginner to ask the usual question, "How can you tell which is a backsight and which is a foresight?" if he will keep in mind that the object of backsighting is to establish the height of instrument (plane of sight), whereas in foresighting the object is to ascertain the elevation of some point.

128. Theory of Leveling. (*a*). STEPS IN LEVELING. From what has preceded, it is seen that the two steps in leveling for any one setup are: (1) To find how far the plane of sight (height of instrument) is *above* a point of known elevation by adding to that elevation the backsight rod reading, i.e.,

$$(1)\ \text{El.} + \text{B.S.} = \text{H.I.}$$

(2) To find how far *below* this plane of sight points are whose elevations are required, by subtracting foresights from the elevation of the plane of sight (height of instrument), i.e.,

$$(2)\ \text{H.I.} - \text{F.S.} = \text{El.}$$

128(*b*). NOTE. When the points are above the plane of sight the above rules still hold good provided minus signs are given to the sights. Thus, for example, a point on the ceiling of a room is used as a bench mark, and its elevation above some assumed datum is known to be, say, 112 ft. The leveling rod, held bottom end up for convenience, gives a backsight reading of 6 ft., or the plane of sight is 6 ft. below the ceiling, or its elevation is obviously 106 ft., i.e., $112 + (-6) = 106$. Suppose that the foresight reading to some point just below the ceiling is 5 ft. (rod bottom end up), then the elevation of this point is $106 - (-5) = 111$ ft.

Minus sights seldom occur in leveling except in special kinds of work.

128(*c*). NEW HEIGHT OF INSTRUMENT. If all the elevations required could be found from one setup of the level, no further explanation of fundamental principles would be required. Even when the level is frequently moved, however, the work consists simply in repeated applications of the two steps in a sort of endless chain. When it becomes necessary to move the level it may be set

up in a new place, the *new* height of instrument ascertained by backsighting on a point whose elevation was found from the preceding setup, and from this new height of instrument the elevation of new points may be found. Thus the process may be continued indefinitely. The point to be emphasized is *that for each new setup a new height of instrument must be ascertained.*

128(*d*). ILLUSTRATION. In running a line of levels, the stations on which the leveling rod is held are not necessarily in a straight line—they may be anywhere to one side or the other of the level as shown in Fig. 128(*d*). Let the horizontal broken lines represent *planes* of sight. Let the target readings at each station be indicated by the number at the point where the line of sight (dotted line) strikes the rod. Either the elevation of the first station must

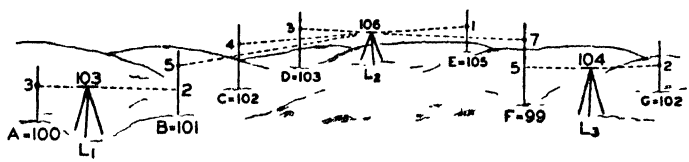


FIG. 128(*d*).

have been previously determined, or, if simply *relative* elevations are required as is often the case, it may be *assumed* as any convenient number, say 100. Then,

100 + 3 = 103 = H.I. when level is at L_1 103 − 2 = 101 = El. of *B*.
101 + 5 = 106 = H.I. " " " " L_2 106 − 4 = 102 = El. " *C*.
106 = H.I. " " " " L_2 106 − 3 = 103 = El. " *D*.
106 = H.I. " " " " L_2 106 − 1 = 105 = El. " *E*.
106 = H.I. " " " " L_2 106 − 7 = 99 = El. " *F*.
99 + 5 = 104 = H.I. " " " " L_3 104 − 2 = 102 = El. " *G*.

128(*e*). REMARK. The two steps in leveling are not likely to be forgotten if the student will form a mental picture similar to Fig. 124(*a*) or Fig. 128(*d*). This is much better than depending upon the memory to retain the formulas given at the beginning of this article. Always fall back on the original conception of leveling (Art. 124): (1) getting the distance of a plane of sight *above* some point of known elevation, and (2) getting the distances of points of unknown elevation *below* that plane.

129. Turning Point (T.P.). A point on which both a minus sight (foresight) and a plus sight (backsight) are taken on a line of direct levels, the minus sight from one setup in order to determine the elevation of the turning point, and the plus sight from the next setup in order to determine the new height of instrument. For example, in Fig. 128(*d*), *B* and *F* are turning points.

130. Intermediate Station (Int.). A station, intermediate between two turning points, on which only *one* sight, a minus sight (foresight), is taken, the sole purpose being to determine the elevation of the station. For example, in Fig. 128(*d*), *C*, *D*, and *E* are intermediate stations.

131. Effect of an Error in the Elevation of a Turning Point. An error in the elevation of a turning point causes the same error in the elevation of each of the succeeding turning points and inter-

mediate stations (why?), but an error in the elevation of an intermediate station affects only the elevation of that station.

132. Abbreviations.

Sta.	Station	El. or Elev.	Elevation
Int.	Intermediate station	T.P.	Turning Point
B.S.	Backsight	+S	Plus sight
F.S.	Foresight	−S	Minus sight
H.I.	Height of Instrument	B.M.	Bench Mark

133. Forms of Notes. The forms for level notes differ somewhat for different kinds of work. Standard forms are given in this article. Study these forms in connection with Fig. 128(*d*). The six columns of Form A correspond to the six columns on the left-hand page of the field book, the right-hand page being used for descriptions of stations and for other notes.

In all these forms, backsights and foresights are *observed* values which must be entered in the field, but heights of instrument and elevations are *calculated* values which may be left to be worked out in the office unless certain elevations are needed in the field for checking the work or for other purposes. Even then, it is not necessary to work up the notes for intermediate stations, but only for turning points.

Form A for Level Notes

(Left-hand page) (Right-hand page)

Sta.	B.S.	H.I.	Foresight T.P.	Foresight Int.	El.	
A	3.0				100.	
		103.0				
B	5.0		2.0		101.0	
		106.0				
C				4.0	102.0	Descriptions of stations go on this page. All bench marks and important stations should be so carefully described that they can be found at any time without difficulty.
D				3.0	103.0	
E				1.0	105.0	
F	5.0		7.0		99.0	
		104				
G			2.0		102.0	
	13.0		11.0			
	11.0	Check				
	2.0					

133(*a*). Comments on Form A. Each H.I. is entered on a separate line by itself. There is no great advantage in this except that, in looking over the notes, it is more evident, at least to the beginner, which H.I. was used in calculating any elevation. For example, it is obvious that the elevations of all four stations, *C*, *D*, *E*, and *F*, were calculated from a height of instrument

of 106.0 ft. (by subtracting the corresponding foresights). Thus each H.I. is used for all stations below it until the next H.I. below is reached. Notice also that the foresights on turning points are kept in a separate column. This is done in order that the foresights on turning points *only* may be readily added up. This sum is used in checking the notes by the method explained in Art. 134. The methods of calculating the heights of instrument and the elevations entered in Form A have already been indicated in Art. 128(*d*).

FORM B FOR LEVEL NOTES

(Left-hand page)					(Right-hand page)
Sta.	B.S.	H.I.	F.S.	El.	
A	3.0	103.0		100.0	Descriptions of stations go on this page. All bench marks and important stations should be so carefully described that they can be found at any time without difficulty.
B	5.0	106.0	2.0	101.0	
C		"	4.0	102.0	
D		"	3.0	103.0	
E		"	1.0	105.0	
F	5.0	104.0	7.0	99.0	
G		"	2.0	102.0	

133(*b*). COMMENTS ON FORM B. This form differs from Form A in two respects: (1) Each H.I. is not on a line by itself, but is entered on the same line with the B.S. and El. from which it was calculated. (2) Foresights on turning points are not in a separate column, but are entered in the same column with foresights on intermediate stations. Turning points may be easily distinguished, nevertheless, since for each turning point there is an entry in the backsight column, whereas for intermediate stations there are no such entries.

By comparing these notes with the arithmetical work in Art. 128(*d*), the method of calculating the different H.I.'s in the third column and the elevations in the last column will become clear. Thus each B.S. is added to the elevation recorded on the same horizontal line to get the H.I. from which the elevations are figured for all succeeding stations up to and including the next turning point.

This form is more compact than the preceding form and is, perhaps, the one most often used. It is frequently desirable, however, to skip every other line in entering notes, thus affording more room for corrections and for the descriptions of stations on the right-hand page.

FORM C FOR LEVEL NOTES

B.S.	T.P.	H.I.	F.S.	El.	Sta.
3.0	100.0	103.0			*A*
5.0	101.0	106.0	2.0		*B*
			4.0	102.0	*C*
			3.0	103.0	*D*
			1.0	105.0	*E*
5.0	99.0	104.0	7.0		*F*
			2.0	102.0	*G*

FORM D FOR LEVEL NOTES

Sta.	B.S.	F.S.	Dif.	El.	Check
A	3.0			100.0	3.0 2.0
B	5.0	2.0	+1.0	101.0	+1.0 100.0 101.0
C		4.0	+1.0	102.0	
D		3.0	+1.0	103.0	5.0 7.0
E		1.0	+2.0	105.0	−2.0 101.0
F	5.0	7.0	−6.0	99.0	99.0
G		2.0	+3.0	102.0	

133(*c*). Comments on Form C. This form differs from the preceding form in the order of arrangement of the columns, and in having a separate column for elevations of turning points. It is not so common a form as the others, but in some respects it is more convenient. Its chief advantage is that quantities that are to be added or subtracted are adjacent. Thus corresponding quantities in the first two columns are added to obtain a corresponding quantity in the third column, and from quantities in the third column are subtracted quantities in the fourth column to obtain quantities in the fifth column. It is also convenient to have stations and their elevations in adjacent columns, particularly in profile leveling.

133(*d*). Comments on Form D. This form is entirely different from the other three forms. The first three columns are filled out in the field in the usual manner. The column of differences is found by subtracting each F.S. from the preceding F.S., except in the case of the first F.S. after each turning point, and this is subtracted from the B.S. on that point. The elevations in the fourth column are found by adding *algebraically* each difference to the preceding elevation. The checks in the last column are found by subtracting from the B.S. on one turning point the F.S. on the next turning point, and adding this difference algebraically to the elevation of the first in order to check the elevation of the second.

This form is not much used. Its chief advantage is the check afforded for the elevation of each *intermediate* station. There is no such check in the other three forms. For this reason this *method of differences* is especially useful in working up notes for profile leveling, since the computation for every point on the profile is checked once and turning points are checked twice.

The four forms of level notes just explained are by no means all those in common use, but they illustrate basic methods of keeping level notes. These methods may be modified or used in different combinations in devising special forms of notes to meet special needs. Some of these forms are shown on pp. 132 and 133, where methods of keeping level notes are illustrated by examples from practice.

134. Rule for Checking Calculations. The calculations involved in working up any portion of level notes, as, for example, the portion between two bench marks or between any other two stations, may be checked as follows: Add all backsights together; add all *turning point* foresights together; the difference between these two sums should equal the *difference* between the elevations of the first and last stations in the portion of the notes checked. The foresight on the last station should be included in the sum of foresights, but the backsight on that station, if there is any, should not be included in the sum of backsights. Note that foresights on intermediate stations are omitted from the calculations. (Why?)

134(*a*). Illustration. In Form A, sum of backsights = 13; sum of turning point foresights (including that of *G*) = 11; thus, 13 − 11 = 102 − 100 = 2 (check).

134(*b*). Note. Another method of applying the same rule: (1) Add the sum of all the backsights to the elevation of first station. (2) Subtract from this result the sum of the *turning point* foresights. The result should be the elevation of the last station. Thus, (100 + 13) − 11 = 102 (check). See also the level notes on p. 133 for another illustration.

134(*c*). Note. In leveling around a closed figure, the first and the last stations coincide and, therefore, the difference between the two sums should equal zero, provided there has been no error in the field work. Usually, however, because of errors in field work, there is a slight difference between the original elevation of the first station and its elevation as obtained after

leveling around the closed figure. In such a case this difference and the difference between the sums of the backsights and foresights should be equal.

The difference in the sums of backsights and foresights is a check on calculations but not on field work. It enables one to detect numerical mistakes made in working out heights of instrument and elevations of turning points, but it does not disclose mistakes in using the level, reading the rod, or entering the rod readings. Moreover, it is not a check on the computed elevations of intermediate stations, because no backsights are taken on such stations.

135. Special Methods of Spirit Leveling. (a). FLYING LEVELS. A line of levels run with more rapidity and less precision than usual. Useful when only approximate results are desired; also as a rough check on more accurate leveling.

135(b). CROSS LEVELING. Taking levels on each side of a main line at right angles to that line in order to determine a vertical cross section of the surface of the ground, or of underlying strata, or of both. For example, taking cross levels at known intervals along the center line of a highway or of a railroad. (See Art. 176.) Cross levels are also used for determining contours of the surface of the ground. (See Art. 453(c).)

135(c). RECIPROCAL LEVELING. A method of determining the difference in elevation between two points A and B by *two* sets of sights. (1) Set up near A, backsight on A and foresight on B. (2) Set up near B, backsight on B and foresight on A. Add the difference in elevation determined from the first setup to that determined by the second setup and divide the sum by two. The resulting mean difference in elevation should be more accurate than the difference in elevation as determined from either setup alone. (See Art. 179 for practical application.)

135(d). DOUBLE-RODDED OR DOUBLE-LINE LEVELING. Leveling in which two independent lines of *turning points* are used. At each setup of the level, a backsight is taken first upon a turning point in line A and then upon one in line B. A foresight is then taken on another turning point in line A and one in line B. Turning points in each line may be chosen wherever convenient. Thus two independent lines of levels are run with only one line of setups. Heights of instrument determined from one line may be checked by corresponding heights of instrument determined from the other line. (See Art. 174 for practical application.)

135(e). PRECISE LEVELING. Leveling in which the degree of precision required is too great to be attained by ordinary methods, and in which, therefore, special equipment, or special precautions, or both, are necessary to eliminate, as far as possible, all sources of error. (See Chapter 15 for practical application.)

136. Establishing Points at Given Elevations is the reverse of the process of leveling to ascertain elevations. To establish any particular point, the difference between its required elevation and the height of instrument is calculated, and this difference is measured *down* from the height of the instrument by means of a foresight. The rod reading used for the foresight is called the *grade rod.*

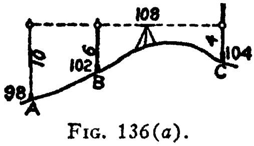


FIG. 136(a).

$$\text{Foresight} = \text{H.I.} - \text{Elev.} = \text{Grade rod.}$$

136(a). ILLUSTRATION. In Fig. 136(a) it is required to drive a stake at each of the points A, B, and C so that the tops of the stakes will be respectively 98 ft., 102 ft., and 104 ft. above some given datum. The height of instrument, found by backsighting on some point of known elevation, is 108 ft. The *grade rod* for A is therefore 10 ft., for B 6 ft., and for C 4 ft.

136(*b*). Grade line. The line to which a surface is brought in grading—it may be level, or on a slope, or on a vertical curve.

136(*c*). Grade. (1) The elevation of the finished surface of an engineering project. (2) The elevation of a particular point in a grade line, as the "crown of the road at grade 59.50 ft."; or, "sewer line, grade 21.19 ft. at station 1 + 50.00." (3) The rate of slope or degree of inclination from the horizontal as, for example, a rise or fall of 1 ft. measured *vertically* for each 100 ft. measured horizontally (a *1 per cent* grade); or of 5 ft. per 100 ft. (5 per cent grade); or with no rise or fall (a level grade). *Gradient* is a synonymous term for the rate of rise or fall as a "5 per cent gradient," or a "0.05 gradient." It is customary to use the sign + to indicate a rising grade line and the sign − to indicate a falling grade line. A large part of the leveler's work consists in establishing points of reference from which elevations required in construction may be determined. For a detailed explanation of methods used in such work, see Chapter 14.

CHAPTER 11

LEVELING

FIELD WORK

THE theory of leveling, explained in the preceding chapter, is exceedingly simple, but no branch of surveying requires greater care in the field work. Accurate results can be obtained only by observing certain precautions which tend to eliminate mistakes and errors. These precautions are so well understood that the methods of field work are more uniform in leveling than in almost any other branch of surveying. It is, therefore, somewhat easier to give definite directions and suggestions, but, as in all field work, it is not desirable to lay down hard-and-fast rules, or to follow blindly such directions as are given.

137. A Leveling Party usually consists of a levelman and a rodman. In rough country a third man is sometimes needed, to act as axeman. In very accurate work two rodmen may be used to advantage, and, in some cases, two levelmen as well.

138. Equipment for a Leveling Party. (Things most likely to be forgotten are put first.)

Rodman. Tacks or brads, crayon, stakes, hatchet, tape, notebook, and leveling rod.

Levelman. Necessary notes from transit party, datum, etc., notebook, adjusting pin, level.

Both men should provide themselves with hard pencils, pencil erasers, and scribbling pads.

Note. The equipment listed is for an ordinary party of two men. Additional equipment for larger parties will be described later. It is often necessary or expedient to use a transit for leveling instead of a level. Accurate work may be done with a transit provided it has a good level attached to the telescope and the whole instrument is kept in good adjustment, a fact which students are slow to realize. Of course the methods of work are practically the same as for the level.

139. Duties of Levelman. To run the level, to keep it in adjustment, to care for it, and to *protect it from injury* (Art. 101). To keep a general run of the work, seeing that no stations are omitted, that turning points are properly chosen, bench marks established, and all work done as expeditiously as possible. To take rod readings when a self-reading rod is used instead of a target. To keep field notes and to "work up" these notes.

140. Duties of the Rodman. To run the rod, to care for it, and to protect it from injury. To hold the rod plumb and still, and exactly on the point where it should be held. To choose turning

points and to establish bench marks. To be alert and attentive to signals from the levelman. To move the target and clamp it in exact position when a target reading is desired; to observe the target reading and see that it is entered correctly in the field notes. To cooperate with the levelman in working up the field notes, either in the field or after the field work is completed.

Note. Either the levelman or the rodman may record the field notes, depending upon which can do it the more efficiently. For example, if there are many target readings, it is usually best for the rodman to keep the notes. If, however, readings are taken by the levelman (from a self-reading rod), or if there are only a few, occasional target readings, it is usually best for the levelman to keep the notes. In any event, both men should cooperate in taking precautions to insure adequate and correct notes (an example of the teamwork desirable in all field work). In some cases it may be advisable for the levelman and the rodman each to keep independent field notes, particularly if they are beginners just learning the process of leveling.

141. Some Things to Know Before Beginning Work. (1) How to set up a level (Art. 102). (2) How to use the telescope to get accurate results (Art. 103). (3) How to read the rod (Arts. 106–110). (4) What points of the rod are danger points, i.e., points where mistakes in reading are likely to occur (Arts. 107(*b*) and 109(*a*)). (5) How to test the rod for a high-rod reading (Art. 111). (6) What is meant by *datum, bench mark, station,* and *plane of sight* (Arts. 116, 121, 122, and 123). (7) What the two essential steps in leveling are (Art. 124). (8) What is meant by *height of instrument, backsight, foresight,* and *turning point* (Arts. 125–129). (9) Why it is important to take readings with greater care on turning points than on intermediate points (Art. 131). (10) How to keep notes (Art. 133), and how to check them (Art. 134). (11) How to care for the level (Art. 101).

Remark. As soon as the student has become familiar with the fundamental principles and operations in leveling, he should begin a study of the sources of errors and the methods of eliminating or correcting them. (See the next chapter.) This is the quickest and surest way of learning to do accurate and reliable work. Eventually, when the student has become well accustomed to the use of the level, he should learn how to adjust it (Chapter 54).

142. Bench Marks for a Sea Level Datum. When it is desired to use the mean sea level as a datum, it is necessary to start with the first backsight on a bench mark, the elevation of which is known with respect to that datum. Permanent bench marks of this character have been established throughout the United States by the U.S. Geological Survey and by the U.S. Coast and Geodetic Survey, particularly in cities and thickly populated areas. These bench marks are usually bronze plates, set in concrete or stone, on which is inscribed the elevation of the top of the plate with respect to mean sea level. Similar bench marks have been established by state and municipal governments, particularly along highways and

streets, and by private companies such as railroad companies and public utility companies. It is not often necessary, therefore, to go a great distance to find a bench mark that will serve as a starting point.

142(*a*). Note. In many cases a city or a railroad or some other agency has a system of bench marks based on a datum other than mean sea level. Generally, however, the difference in elevation is known, and from this difference the elevation of any bench mark with respect to mean sea level can be calculated.

142(*b*). Temporary datum. When, for any reason, it is not feasible or essential to start from a bench mark of known elevation with respect to mean sea level, a bench mark may be established and its elevation arbitrarily assumed. Later, if desired, it may be possible to obtain the elevation of this bench mark with respect to mean sea level, and then all elevations may be changed to conform to a sea level datum. In a great many instances, only the *relative* elevations of points are desired, in which case it is immaterial what elevation is assumed for the first bench mark. For example, elevations of points on the surface of the ground are needed temporarily for grading. A temporary bench mark may be established and its elevation assumed at an even hundred feet or any other convenient number, regardless of sea level or any other datum.

142(*c*). To establish a mean sea level datum from tidal observations.
1. Set a tidal gauge in such a position that both high- and low-water elevations may be read. The gauge may be "homemade," but the best results will be obtained from such gauges as are described in the U.S. Coast and Geodetic Survey Special Publications 26 and 104.
2. Choose a place for the gauge where it may be protected from bad weather and where it is as free as possible from variations in the tides due to local conditions, preferably in a place near the open sea. The water should be deep enough to permit the bottom of the gauge to be set well below the lowest tide.
3. When the gauge has been fixed firmly in position, determine, by leveling, the elevation of the zero point (or some other point of the graduation) with respect to near-by bench marks, established for the purpose if none exists. Test this elevation from time to time in order to detect any movement of the gauge and to restore the gauge to its true position should any displacement occur.
4. Take observations daily for at least one lunar month; for accurate results, they may well extend over a year or more. The total number of observations for low tide should, of course, equal the total number for high tide.
5. Add the values of all observations, both for low tide and high tide, and divide the sum by the total number of observations. The mean value, thus obtained, should be the value of mean sea level. However, even when precautions have been taken to insure accurate observations, the value finally obtained may be only approximately correct; it may, nevertheless, be correct enough for all practical purposes.

143. Field Work Routine. In the explanation of the theory of leveling (Art. 128), it was shown that in the procedure for each setup of the level there are two steps, namely, (1) to backsight in order to determine the elevation of the plane of sight and (2) to foresight in order to determine required elevations. In the fol-

lowing explanation of field routine it is assumed that a rather high degree of precision is required and that all rod readings are made by means of a target.

143(*a*). DIFFERENTIAL LEVELING. Let it be assumed that the sole purpose of the leveling is to determine the difference in elevation between two points, as, for example, two bench marks, and that the distance between them is great enough to necessitate several setups of the level. The levelman sets his level up at any convenient point where he can sight on a rod held on the first bench mark as a starting point. The rodman holds his rod on that point, moves the target up or down according to the signal from the levelman, and clamps the target at the signal all right. *He then holds the rod a second time,* and if the levelman again signals "All right" he reads the rod, and, as he passes the level on his way to the next turning point, the levelman checks the reading. Both men enter the reading, each in his own notebook. A turning point is then chosen at such a distance from the level that the length of the foresight will be about equal to the length of the backsight just taken. The rodman holds the rod on the turning point, and the target is set as before, a second sight being taken after clamping. The levelman takes up the level and moves on. As he passes the rodman on the way to the next setup, he checks the foresight reading, and each enters it in his notebook. The levelman then sets up his level where he can backsight on the rod held on the turning point, and the process is repeated. Thus for any number of setups the routine is the same, first a backsight then a foresight.

Note. It will be shown later that the general method of procedure just explained may be modified in many ways to meet different requirements. For example, it is not generally customary in practice for both men to keep the notes or to read the rod, but it is desirable that students should adopt this precaution, particularly when the work is new to them.

143(*b*). INTERMEDIATE STATIONS. When in addition to turning points and bench marks there are other points whose elevations are required, the routine procedure is the same except that foresights (and only foresights) must be taken on these intermediate stations. (See illustration, Art. 128(*d*).) The stations are likely to be scattered on both sides of the lines between turning points, and often there are quite a number of them. At each setup, foresights are taken on all those intermediate stations whose elevations can best be determined from that setup. After backsighting on one turning point, it is best to foresight first and as soon as practicable on the next turning point before the level has time to "get out"; afterward the foresights on near-by, intermediate stations may be taken.

143(*c*). IN LEVELING WITHOUT THE USE OF A TARGET the levelman not the rodman reads the rod, but in other respects the routine field work does not differ materially from that already described.

144. Signals. When the levelman wishes the rodman to raise the target he holds his hand *above* his shoulder. The hand held *below* the shoulder or the waist means to lower the target. In either case the nearer the target approaches to the line of sight the nearer the hand should approach the level of the shoulder. Dropping the hand to the side means to stop moving the target. A gentle motion of the hand up or down from the shoulder means to move the target *slowly* in the direction indicated. When the center line of the target coincides with the horizontal cross hair, the levelman signals "All right," either by bringing his hands together above his head or by holding his arms horizontal and moving his hands slowly up and down. The signal for plumbing the rod is to hold one arm straight up above the head and then to move it slowly sidewise by inclining the body in the direction in which the rod should be plumbed. An arm swung slowly in a circle above the head means "take a turning point."

Remark. Any other set of signals may be agreed upon, but they should be clear and unmistakable at several hundred feet away, and should enable the rodman to get the target into position in the shortest possible time. It is essential that the signals to move the target up or down should indicate to the rodman *about how much the target should be moved.* A common fault with levelmen is to hold the arm horizontal and move the fingers in such a way that the rodman cannot tell whether the target is to be moved up or down. *Avoid all shouting.*

145. Two Methods of Reading the Rod. (*a*). IN THE DIRECT METHOD, the levelman must observe what subdivision of the graduation of a self-reading rod appears to coincide or to coincide most nearly with the horizontal cross hair. The procedure is: (1) Observe the number of the nearest foot mark below the cross hair. (2) Observe the number of the nearest tenth of a foot just below the cross hair. (3) Count the number of hundredths between the tenth just observed and the graduation mark nearest to and just below the cross hair. (4) If it is desired to read to thousandths, it is done by estimation. The procedure just outlined may be modified to suit the requirements. Frequently it is sufficiently accurate to read to the nearest tenth of a foot, or to read to the nearest tenth and then estimate the hundredths. (How would the procedure be modified in obtaining the elevation of a point *above* the plane of sight, as, for example, a point on the ceiling of a tunnel?)

145(*b*). IN THE TARGET METHOD of reading the rod, the rodman, after clamping the target as directed by the levelman, should hold the rod again in order that the levelman may sight a second time

to make sure that the target has not slipped during the process of clamping. However, when reading only to hundredths of a foot, an experienced rodman often does not clamp the target at all. Instead, he merely holds it in position long enough to take the reading. The methods of reading targets have been explained in Arts. 106–110.

145(*c*). COMMON MISTAKES IN READING THE ROD.

1. To omit a zero or even two zeros from a reading as illustrated in Arts. 107(*b*), 108, and 109(*a*). It is well to remember that the mistake of omitting a zero is likely to occur when the cross hair (in direct reading) or the center line of the target is in the first tenth or the first hundredth of a foot after any foot mark; or in the first hundredth after any tenth of a foot mark. (For example, 3.21 instead of 3.021; or 3.02 instead of 3.002; or 3.42 instead of 3.402.)

2. To read the wrong foot mark, particularly when using a high rod (Art. 109(*a*)), or when, in a direct reading, the levelman can see through the telescope only *one* foot mark. To avoid a mistake of an even foot in target as well as in direct readings, it is well to observe what foot mark is next above or below the foot mark read. When a transit or level with stadia hairs is used in leveling, a mistake all too common in a direct reading and sometimes in a target setting is for the instrument man to mistake a stadia hair for the center (horizontal) cross hair.

Common mistakes indicate "danger points" in rod graduations which should always be kept in mind. Every rodman is likely to have his own danger points which he must guard against by extra care when a reading is near one of them.

145(*d*). METHODS COMPARED. Under ordinary conditions of visibility, direct readings are practically as accurate as target readings. Obviously speed is gained by direct readings. Indeed, a target may be more of an encumbrance than a help in a large part of ordinary leveling. Consequently the method of direct reading is widely used, even in precise leveling. For accuracy and convenience in direct readings, it is best to use a one-piece rod on which graduations have been made sharp and distinct expressly for such readings. In a great deal of ordinary leveling, however, the self-reading graduations on a target rod, such as those on a Philadelphia rod, are satisfactory.

146. Lengths of Backsights and Foresights. (*a*). THE MAXIMUM LENGTH of a backsight or foresight will depend upon the character of the surface of the ground, the atmospheric conditions, and the optical qualities of the telescope. On a steep slope, the

length of a sight uphill may be limited by the distance from the level to the point where the line of sight meets the ground; downhill, by the distance at which the line of sight passes over the top of the rod. On more nearly level ground, the maximum length of sight that is advisable may be limited by atmospheric conditions, such as "boiling" of the air, or by the optical qualities of the telescope, or by both. In ordinary work, for the best results the length of sight should not be less than 100 ft. or much greater than 300 ft. since beyond that distance errors are likely to increase rapidly.

146(*b*). BALANCING BACKSIGHTS AND FORESIGHTS. When the difference in elevation between any two points is determined from a single setup by backsighting on one point and foresighting on the other, certain errors may be eliminated if the lengths of the two sights can be made equal. (The errors in backsighting are balanced by the errors in foresighting as explained in the next chapter.) Hence in differential leveling, it is desirable at each setup to make the backsight on one turning point and the foresight on the next turning point equal in length (called *balancing* backsight and foresight). The greater the degree of precision required the more desirable this becomes. It should be noted, however, that when there are two or more setups between two stations, certain errors in determining the difference in their elevations may be practically eliminated by making *the sum of the lengths of foresights equal to the sum of the lengths of backsights.* When this is done, no special effort need be made to balance the backsight and foresight at any setup.

In a great deal of ordinary leveling, the degree of precision desired is not high, and satisfactory results may be attained without balancing sights at setups or without making the sums of backsights and foresights equal. Nevertheless, even in such work, if the two stations are far apart, a series of consistently *long* backsights and consistently *short* foresights, or vice versa, may result in too great an error. The most certain method of procedure, therefore, is to keep the lengths of backsight and foresight approximately equal at each setup. This procedure involves little extra work and should be followed whenever it is desired to determine elevations with a relatively high degree of precision, as, for example, elevations of turning points and bench marks. On the other hand, it is usually unnecessary as well as impracticable to balance backsight and foresight in determining the elevation of an intermediate station. (Why?) In ordinary work backsight and foresight may be made approximately equal by estimating the distances; in more accurate work, by pacing or by stadia measurements.

147. Where to Set Up a Level. Where a backsight can be taken on the starting point, a bench mark, or the last turning point, and where the length of backsight will not be too great. Where, if possible, there is solid footing for the tripod, but not where the surface is so smooth that the tripod shoes will slip. Where the level will not be disturbed or jarred. Where foresights can be taken on near-by intermediate stations if there are such stations.

147(*a*). WHERE TO SET UP ON SLOPES. In leveling *down* a steep slope, set up the level far enough downhill to bring the line of sight just above the bottom of the rod when held on the last uphill turning point, but *not* so far that the line of sight strikes the ground below the bottom of the rod. In leveling *up* a slope, set the level far enough uphill to bring the line of sight just below the top of the rod when held on the last downhill turning point, but *not* so far that the line of sight passes over the top of the rod. In judging how far downhill or uphill to go, choose what seems to be the right place for the level and then make a quick preliminary test as follows. Set one leg of the tripod on the ground; take the other two legs, one in each hand, move the telescope rapidly into nearly the same horizontal position that it will occupy when the level is set up, and sight along the outside of the telescope to see if the line of sight will strike the bottom of the rod, or the top, as may be desired. Instead of the procedure just outlined, a hand level can often be used to advantage in finding the position of setup of the level.

Note. On steep slopes the beginner almost invariably tries to take too long sights. Going uphill he finds that the line of sight is too high to strike the downhill rod; going downhill he finds the line of sight too low to strike the uphill rod. Result, loss of time in setting up the level again. On the other hand, unnecessarily short sights may result in more setups than are really needed. The procedure just outlined should help to avoid unnecessary setups caused by an attempt to take either too long or too short a sight. In following the procedure it is helpful to remember that since the height of the telescope above the ground is usually between 4 and 5 ft. and since the length of the leveling rod is usually 12 to 14 ft., *the length of a sight downhill may be approximately twice that of a sight uphill,* provided the slope is uniform.

It is sometimes necessary to set up a level in such a position that sights can be taken on two points both of which have been established as, for example, a backsight on a turning point and a foresight on a permanent bench mark. The level must be set far enough uphill for a sight on one point and far enough downhill for a sight on the other. In finding this position by trial, do not keep setting the level up at random, but use the telescope to test roughly where lines of sight will come until the right place for the setup becomes evident.

147(*b*). Balancing sights on a slope. In balancing a backsight on one point and a foresight on another when both points lie in a fairly *level* surface, it usually makes little difference how near the level is to the line between the two points. When those points lie in a *sloping* surface, however, the level should be set off to one side far enough to equalize, as nearly as practicable, the uphill and the downhill sights.

147(*c*). Illustration. In Fig. 147(*c*) a level was set up at *A* far enough uphill to bring the line of sight just below the top of an extended rod when held on the turning point, T.P. 5. A turning point T.P. 6 was then established far enough uphill to bring the line of sight just above the bottom of the rod when held on the turning point T.P. 6. The level is nearly on line between the two turning points. The distance from the level to T.P. 5 is 48 ft. whereas the distance to T.P. 6 is only 25 ft. These sights could be balanced by moving both the level and T.P. 6 downhill, but this might require an extra setup. Assume that the level is set up at *B* instead of at *A*, off to one side but at nearly the same elevation as that at *A*, so that sights on T.P. 5 and T.P. 6 can still be taken. It is obvious that these two sights are more nearly equal in length than were the two corresponding sights from the level at *A*. From *B* to T.P. 6 is 150 ft. and from *B* to T.P. 5 is 156 ft.

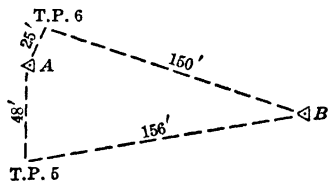


Fig. 147(*c*).

Assume that, owing to imperfect adjustment of the level, the line of sight is not horizontal, but that for each 100 ft. in the length of sight it rises 0.004 ft. Wherever the rod may be held, the line of sight will then strike it higher up than it should. When the level is at *A* the errors in rod readings due to imperfect adjustment are:

$$\begin{aligned} &\text{Error for T.P. 5: } 0.48 \times 0.004 = 0.00192 \text{ ft.}\\ &\text{Error for T.P. 6: } 0.25 \times 0.004 = 0.00100 \text{ ft.}\\ &\text{Error in leveling} \qquad\qquad\quad = 0.00092 \text{ ft.}\end{aligned}$$

The difference 0.00092 between the errors in the two rod readings is the error in the difference in elevation between T.P. 6 and T.P. 5. (Why?) When the level is at *B*, the errors in the two readings are:

$$\begin{aligned} &\text{For T.P. 5: } 1.56 \times 0.004 = 0.00624 \text{ ft.}\\ &\text{For T.P. 6: } 1.50 \times 0.004 = 0.00600 \text{ ft.}\\ &\text{Error in leveling} \qquad\quad = 0.00024 \text{ ft.}\end{aligned}$$

The error in the difference in elevation will be 0.00024 ft. It is obvious that, although the error for each sight taken from *B* is much greater than the error for the corresponding sight from *A* (since the lengths of sights from *B* are much longer), the resultant error of 0.00024 ft., which occurs when the level is at *B*, is only about one-fourth as great as the resultant error of 0.00092 ft. when the level is at *A*. By moving *B* farther away, the error may be reduced until it approaches zero, as the lengths of the two sights from *B* become nearly equal. The errors in this illustration are too small to be of consequence in ordinary leveling, but if the level is in poor adjustment, the error due to the failure to balance sights may be too great to be permissible.

148. Choosing Turning Points. (*a*). Locations. In choosing a location for a turning point, keep in mind (1) that it is desirable to balance backsight and foresight, and (2) that the length of sight should not be too great, depending upon the atmospheric con-

ditions, the optical qualities of the telescope, and the character of the surface of the ground (Art. 146). A rodman often wastes time by choosing a turning point too far up or downhill from the level. This may be avoided by making rough preliminary tests with a hand level, similar to those made in setting up a level on a slope (Art. 147(*a*)). Whenever a bench mark is established, it is well to use it as a turning point in order that its elevation may be checked at the same time that other elevations in the line of levels are checked. For if there is any error in the elevation of the bench mark, it will then be carried through in succeeding elevations and become apparent in the final check. This would not be true if only a foresight on the bench mark were taken.

148(*b*). SATISFACTORY TURNING POINTS. A turning point to be satisfactory must be stable; hence look for a point on something firm and solid, such as a solid rock, or a pavement or a curb. If none is easily found, drive a stake solidly into the ground, or a nail into the root of a tree, or in some other way establish a stable point. More trouble comes from choosing unsuitable turning points than from almost any other cause. Why choose, for example, a small stone in loose soil which may settle half an inch while the rod is being held on it and then read the rod to a thousandth of a foot?

A turning point should be definite and unmistakable so that no matter how many times it is used, *the rod will always rest on the same point.* It should be something easily described and easily found, so that should a mistake occur, it will not be necessary to go back to the last bench mark, but only to the last turning point. It is frequently advisable to mark the exact point with surveyor's crayon in such a way that it may quickly be found and identified. Metal turning points that can be forced into the ground are frequently used. The objection to such points is that if they are removed as the line of levels progresses and a mistake is made, it may be necessary to go back to the nearest bench mark since turning points have been destroyed.

149. Establishing Bench Marks. In most leveling operations, it is necessary to establish new bench marks in addition to any permanent bench marks that may exist. Indeed, it frequently happens that there are no permanent bench marks conveniently near. When possible it is desirable to start from a bench mark of known elevation, but when this is not feasible, a bench mark based on an assumed datum may be established. The location and character of new bench marks will depend upon the purpose for which they are to be used.

149(*a*). LOCATION. Bench marks should be established at such frequent intervals that wherever in subsequent work it may be necessary to set up a level, there will be a bench mark conveniently near. It is better to have too many than too few. They should be placed where they are not likely to be disturbed, and in some cases it may be advisable to establish *two* near enough together so that if one is displaced the other may be used.

Special kinds of work call for special locations of bench marks. For example, in leveling for the construction of a building, bench

marks may be required in such locations as at the nearest sewer manhole, on a curb near the building, or on some portion of an adjacent building.

The following examples will suggest where, in general, bench marks should be established: near the beginning and end of a line of levels; wherever in running levels work is temporarily stopped, as at noon or at night; at intervals of several hundred feet; wherever a branch line of levels is likely to be started; at the bottom and top of any high hill passed over; at the crossing of a railway or highway; on each bank of a stream crossed; in a city, near the street corners.

149(*b*). THE CHARACTERISTICS OF A SATISFACTORY BENCH MARK. If bench marks are to be used only temporarily, as for example, during the progress of a survey or during a relatively short period of construction, definite, solid, and comparatively permanent objects, easily described and found, similar to those already described for turning points (Art. 148(*b*)) may be used. Common examples are a protruding nail or spike in a blaze on the trunk of a tree, with the letters B.M. and the elevation marked directly over it in the blaze; or a slightly raised *rounded* surface on a rock, similarly marked with a point; or a cross chiseled on stone or concrete. Contrary to an impression more or less common, the elevation of a bench mark on the trunk of a tree is not changed by the growth of the tree. Sometimes, however, the soil is of such a character that even a large tree may be moved as a whole from the action of the frost or from other natural causes. Whatever is chosen as a bench mark, it should be such that *the bottom of the rod will always rest on exactly the same point.*

149(*c*). FOR A PERMANENT BENCH MARK, which may be used for years, an inscribed cast bronze plate or cap set in the top of a concrete or a stone monument is commonly used. In a cold climate, the monument should be set deep enough to withstand the action of frost, generally 3 to 4 ft. The top is usually flush with the surface of the ground, but in some cases it is set a foot or more below to protect it from injury. In lieu of a monument, an existing concrete or stone structure or a solid rock often provides a surface in which to set the bronze plate. The plate may be cast with a shaft which can be set in the concrete or leaded into the stone. The inscription may include the name or initials of the survey organization, elevation above sea level or other datum, references, and such other data as may be desirable. The lettering is usually raised although in some cases it may be countersunk. Among other bench marks sometimes used are: rods or bolts of non-corrosive metal set in concrete sidewalks; sections of steel rails or steel pipes driven into the ground. There are many other standard forms of permanent bench marks, but it may be necessary, nevertheless, to devise some special form to meet local requirements.

In selecting the type of bench mark and the method of placing it, consideration should be given to the following requirements: (1) accuracy in setting; (2) accessibility; (3) convenience; (4) permanency. Bench marks should be carefully referenced and a complete record made of them.

150. Routine Work of the Levelman. (*a*). SETTING UP THE LEVEL. The levelman should know *where* to set up the level (Art. 147) and *how* to set it up (Art. 102). Once it is in position with the plane of sight truly horizontal, the level may settle, thus lowering the plane of sight, or it may move slightly in such a way that the plane of sight ceases to be horizontal. Errors due to settlement or to some other displacement may be serious, as explained in Art.

166(*a*), and the levelman should take precautions to avoid them (Art. 102(*a*), 3).

150(*b*). SIGHTING. The levelman should be thoroughly familiar with the theory of lenses (Art. 706) and the optical principles upon which the construction of the telescope is based. He should also know how to focus, particularly in eliminating parallax (Art. 103(*a*), and how to sight quickly on the rod (Art. 103(3)). In addition, there are routine methods of procedure in sighting which it is well to follow in order to avoid mistakes and to reduce errors. Among these methods of procedure are: While reading a rod or setting a target, keep the horizontal cross hair truly horizontal and the bubble exactly centered. No matter how carefully a level may be set up, the bubble will not remain indefinitely in the center of the tube. *One of the most important things to remember in all accurate leveling is that the bubble should be in the center of its tube at the instant of sighting.* The levelman should bring it into position, if necessary, just before sighting; after sighting, he should check to see if it is still centered. Meanwhile, the whole instrument, and especially the telescope, should be handled with a delicacy of touch that will preclude any serious displacements such as those revealed in the following experiment.

Experiment. The results of the following simple experiment may surprise even experienced levelmen who have never tried it, and will impress upon the beginner the great sensitiveness of a good level. Carefully set up and level the instrument. (1) Rest the hands rather heavily upon two legs of the tripod and watch the bubble "move out," returning to place when the hands are removed. (It will often move out three or four divisions, which might make an error of 0.15 ft. in a rod reading if the rod is as much as 300 ft. from the instrument.) (2) The same results will follow if the weight of the hand rests heavily on the focusing screw of the object glass or on the eyepiece at the rear of the telescope. *Moral. Keep the hands off the instrument* after it is focused on the leveling rod. (3) Place several fingers on one end of the glass bubble tube and observe how the warmth of the fingers will "draw" the bubble. *Moral.* Very accurate work cannot be done when the sun shines on one end of the tube and not on the other.

Note. When only a relatively low degree of precision is required, a slight displacement of the bubble may be of no consequence. One can judge from the table in Art. 163(*b*) how great a deviation is permissible.

It will be easier to keep the level in the center of the tube when sighting, if, in setting up, two diagonally opposite leveling screws have been placed parallel, or nearly so, to the line between the two points on which the backsight and foresight are taken.

It is desirable when practicable to set up the level in such a position that the sun will be behind the instrument. This not only keeps direct sunlight out of the telescope, but also brings the face of the leveling rod toward the sun for better illumination.

150(*c*). READING THE ROD. (1) Sight on the rod and focus clearly. (2) Cooperate with the rodman in plumbing the rod.

(3) If the *direct* method of reading is used, follow the procedure outlined in Art. 145(*a*); if a target is used, cooperate with the rodman in setting the target (Art. 145(*b*)). In either case, guard against the errors in sighting described in Art. 165.

150(*d*). KEEPING LEVELING NOTES. (1) Be familiar with different forms of leveling notes (Art. 133) and use the form best suited to the work in hand. (2) Follow a general systematic method of procedure in recording notes (Art. 153). (3) Check computations (Art. 134).

150(*e*). CHECKING FIELD WORK. The methods are: (1) to check on established bench marks of known elevation, or (2) to run a closed circuit, or (3) to rerun the line of levels, or to run a special line of check levels (Art. 155).

150(*f*). KEEPING WITHIN THE REQUIRED LIMITS OF ERROR. A good levelman is not one who continually works for results of high precision, but rather one who determines for each part of the work the best limits of error and then chooses a method of field work with these limits in mind. When the required degree of precision is very high, it is hardly possible to take too great precautions; when it is low, such precautions may be not only wasteful but also a reflection on the intelligence of the levelman. In order to carry on the field work intelligently, the levelman must have a thorough knowledge of the sources of error in leveling, the relative importance of different kinds of error, and how to avoid or eliminate errors. Nothing will contribute more to success in leveling than such knowledge. (See Chapter 12.)

150(*g*). CARE OF THE LEVEL. (1) Keep it in good running order (Chapter 55); (2) take precautions to reduce wear and to avoid accidental injury (Art. 101); (3) keep it in adjustment (Chapter 54).

Note. By taking proper precautions as outlined in Art. 101, the wear and tear on the level may be greatly reduced, and frequently permanent injury to some part may be avoided. It must be remembered that there are parts which, if once impaired, cannot be restored to their original efficiency.

150(*h*). TEAMWORK between the levelman and the rodman is essential, both in field work and in keeping field notes. Each within reasonable limits should help to check the work of the other. In leveling as in all surveying, an important part of the "game" is to avoid mistakes and errors; enough will occur even when there is good team work.

151. Routine Work of the Rodman. (*a*). HOLDING THE LEVELING ROD. (1) Hold the rod *exactly on the point* and make sure that it is the right point, not one slightly higher or lower, par-

ticularly if it is a turning point or bench mark. Keep the graduated side squarely toward the level. (2) Hold the rod *plumb*—failure to do so may result in an error as great as 0.1 ft. in a single sight. It is not easy to hold a rod plumb and still at the same time, particularly if wind is blowing. If one stands squarely behind the rod and holds it with both hands, he is in the best position to hold it still. Some surveyors, however, prefer to have a rodman stand facing the *side* of the rod, since from this position he can see better whether or not the rod leans toward or away from the instrument, while the levelman can tell by means of the vertical cross hair when it is out of plumb in the other direction. The vertical edge of a building or some other vertical line may often be used by the rodman as a guide. In addition it may be desirable to "wave" the rod. To do this keep the base on the point, but move the top slowly toward the level and then away from it, or vice versa. As the rod is moved *away* from a vertical position, the *direct* reading, as observed by the levelman, will become greater and greater; but as it is moved *toward* a vertical position it will become smaller and smaller. The rod will be plumb when the direct reading is a minimum. When this occurs, the levelman signals to the rodman that the rod is plumb, and the latter should then hold it steady in that position. When a rod is waved, its *graduated face* should be held directly over the B.M. If the *back* of the base is on the point, waving may actually lift the base of graduations above the B.M. and give a smaller, incorrect reading.

When a target is used, the rod is held as nearly plumb as possible, judged by the eye, until after the target has been set. The rod is then waved. If the target center, as observed by the levelman, does not appear to move *above* the horizontal cross hair, and if it coincides in its highest position with that cross hair, the rod was plumb when the target was set; otherwise the target must be reset. The higher a reading is from the base of a rod the greater the error from failure to plumb the rod. (Why?) When a reading is less than 2 ft. it is usually more accurate to plumb the rod by eye than by waving it, since the whole graduated face may be *raised* when the rod is inclined away from the level. (Why?) This error is inappreciable for the higher rod readings. In precise leveling, it is best to use a *rod level* for plumbing the rod; even in work of less precision its use will save time.

Do not allow dirt to accumulate on the base of the rod or in either abutting joint of a two-piece rod, obviously a source of error. When a high-rod reading has been taken with a two-piece rod, do not let the upper portion down "on the run"; doing this

may not only injure the rod, but also drive farther into the ground the point on which the rod is held.

151(*b*). Setting the target. The signals from the levelman should indicate whether the target is to be moved much or little. When it is nearly correct the rodman will save time by moving it slowly but steadily—not jumping first too high and then too low. After clamping the target, hold the rod again in order that the levelman may sight a second time to make sure that the target has not slipped. It is well also to wave the rod to see if it is plumb. When reading only to hundredths, it is generally not necessary to clamp the target. The rodman should know *exactly* where to set the target in order to take a reading with a high rod, and where to clamp together the upper and lower parts of a rod in order that readings on the lower part above the joint may be correct. (See tests for high-rod readings, Art. 111.)

151(*c*). Reading the rod. (1) Follow a careful, systematic method of reading the rod (Arts. 106–110). (2) Be familiar with the "danger points" where common mistakes occur (Art. 145(*c*)). (3) Acquire the habit of quickly checking each reading by a final inspection. Remember that a single mistake in a reading on a turning point affects the elevations of succeeding stations, and that later it may be difficult to determine at which turning point the mistake was made. It may be necessary to rerun the line of levels until it is found.

151(*d*). Choosing turning points. (1) Choose turning points that are definite and stable (Art. 148(*a*)). (2) Keep in mind the desirability of balancing foresight and backsight. (3) Use bench marks for turning points when practicable.

151(*e*). Establishing bench marks. (1) Know where in general bench marks are commonly located (Art. 149(*a*)). (2) Keep in mind the characteristics of a good bench mark (Art. 149(*b*)).

151(*f*). The care of the rod, though simple, is too often neglected, with resulting errors in reading and injury to the rod. The rod graduations should be tested from time to time by comparing them with the graduations on a good steel tape. The test described in Art. 111 should be made frequently.

151(*g*). Teamwork. A rodman can "speed up" the field work by choosing the right turning points and by holding the rod plumb and still as soon as the levelman is ready to sight. He should cooperate with the latter in recording leveling notes, in checking field work, and in keeping within the required limits of error. (See paragraphs (*d*), (*e*), (*f*), and (*h*) of Art. 150.) It is particularly important for him to keep in mind the sources of error in the use

of the rod and to know what precautions are necessary and what are not in order to attain any required degree of precision. There are, of course, many other ways in which an alert, capable rodman can contribute to teamwork.

152. Numbering Stations. The method of numbering stations depends upon the purpose for which the leveling is done. If the purpose is to establish bench marks, only turning points and bench marks are stations, and one method of numbering may be illustrated as follows: B.M. 1, T.P. 1, T.P. 2, T.P. 3, B.M. 2, B.M. 3, T.P. 4, T.P. 5, B.M. 4, etc.

In profile leveling, it is generally customary, especially in surveys for highways and railways, to make the number of the station indicate its distance from some starting point. Thus 0 + 50, 1 + 0, 1 + 50, 2 + 75, and 140 + 67.8 mean 50, 100, 150, 275, and 14,067.8 ft., respectively, from 0 + 0, the first station. Points at multiples of 100 ft. from the starting point are called *full stations* and other points *plus stations*. The bench marks and turning points may be numbered as in the preceding illustration. There is no advantage, however, in numbering a turning point if it is certain that it will not be used again.

152(*a*). NOTE. When the line along which a profile is taken is a broken line, the vertex of the angle formed by two successive straight lines is likely to be a plus station, and is so numbered. The next *full* station is determined just as if there were no angle, and the numbering is thus continuous. When two straight lines are connected by a horizontal curve, the numbering is carried around the curve.

152(*b*). SPECIAL SYSTEMS OF NUMBERING OR OF LETTERING INTERMEDIATE STATIONS may be devised for special purposes. Assume, for example, that in planning for the foundation of a building, the elevations of certain points are required, such as points on the street pavement, sidewalk, sewer line, manholes, and neighboring buildings. Frequently in order adequately to describe the points a sketch is made in the field notes. Some simple system of numbering, or of lettering, or of a combination of the two that can be used on the sketch should be adopted.

153. Field Notes. It is important to keep in mind the general directions and suggestions, already given in Chapter 3, which hold good for all kinds of field notes and to be familiar with the standard forms of field notes for leveling which have already been explained and compared in Art. 133.

In general, the left-hand page of the notebook is reserved for numerical data such as the observed foresights and backsights and the calculated elevations. When it is important to balance backsights and foresights, the lengths of these sights are sometimes recorded in an additional column. The right-hand page is used mainly for descriptions of bench marks, intermediate stations, and oftentimes turning points, together with such notes and sketches as

may be necessary. Special forms of field notes may be used for special kinds of work, such as profile leveling and double-rodded leveling. (See Arts. 175(*c*) and 174(*a*).)

153(*a*). MISTAKES IN RECORDING will sometimes render useless the otherwise careful field work of a whole day. Entering the reading with figures interchanged, as 3.45 instead of 3.54, entering backsights and foresights in the wrong columns, or neglecting to enter a backsight at all, and mistaking the numerical value of a reading called out by another member of the party are some of the common mistakes. Mistakes in a numerical value may arise also from an illegible figure, or from one figure written on top of another or from one figure changed into another, or from a figure erased. (See suggestions in Art. 25.) Mistakes in recording are not easily detected by any method of checking.

153(*b*). MISTAKES IN COMPUTING which occur most frequently are adding a foresight instead of subtracting it, or subtracting a backsight reading instead of adding it; adding a backsight to, or subtracting a foresight from, the wrong height of instrument; ordinary arithmetical mistakes.

153(*c*). CHECKING THE NOTES. Computations on one page should be checked before beginning computations on the next. There are two methods of checking: (1) by the difference in the sums of the backsights and the foresights as explained in Art. 134, and (2) by the check described in "Comments on Form D." (Art. 133(*d*).) These checks do not disclose mistakes or errors in field work or in recording notes.

153(*d*). ILLUSTRATIVE EXAMPLES OF FIELD NOTES. Forms E and F shown in Fig. 153(*d*), correspond to the forms B and C in Art. 133. Form G shows the notes for three independent lines of leveling that were run to establish a series of bench marks for a sewer. The notes for each of the three lines are kept entirely in one column except that the difference between the reading for the backsight on one turning point and the reading for the foresight on the next turning point is entered in an adjacent column. This form, which is similar to that in Art. 174(*b*), is self-explanatory. It affords an excellent comparison of the three elevations obtained for each B.M., since these elevations are side by side on the same horizontal line, and it suggests how special forms of notes may be devised to suit special kinds of work. It will be noticed that the final elevations used are 0.01 ft. less than the mean values obtained. This is the result of correcting the elevation of the first B.M. to accord with elevations previously used on p. 69 of the field book as indicated by a note on the right-hand page.

FORM E

21 Sta.	B.S.	H.I.	F.S.	El.	
B.M.16	6.102	124.318		118.216	
31			5.2	119.1	
+50			6.4	117.9	
T.P.	5.813	126.315	3.816	120.502	
32			8.9	117.4	
B.M.17			7.235	119.080	

FORM F

21 +S.	B.M. and T.P.		−S.	El.	Sta.
6.102	118.216	124.318			B.M.16
			5.2	119.1	31
			6.4	117.9	+50
5.813	120.502	126.315	3.816		T.P.
			8.9	117.4	32
			7.235	119.080	B.M.17

FORM G FOR DIFFERENTIAL LEVELING

(1) 159 59.253		59.253		59.253	
+10.250		5.700		5.825	
69.503	+3.745	64.953	+3.747	65.078	+3.747
−6.505		1.953		2.078	
(2) 62.998		63.000		63.000	
+6.872		7.089		7.164	
69.870	+3.818	70.089	+3.817	70.164	+3.817
−3.054		3.272		3.347	
(3) 66.816		66.817		66.817	

Fig. 153(d).

FORM E

22

PROFILE NOTES, STA. 31 TO STA. 40

Cross on corner of lower stone step of court house

Ground

" — B.M. 16 = 118.216
Σ B.S = 11.915
130.131

T.P. — Σ F.S. = 11.051

Ground — B.M. 17 = 119.080 (check)

Nail in root of oak tree, east bank, Eagle Stream

FORM F

22

PROFILE NOTES, STA. 31 TO STA. 40

Cross on corner of lower stone step of court house

Ground

" — B.M. 16 = 118.216
Σ B.S. = 11.915
130.131

T.P. — Σ F.S. = 11.051

Ground — B.M. 17 = 119.080 (check)

Nail in root of oak tree, east bank, Eagle Stream

FORM G FOR DIFFERENTIAL LEVELING

Use 59.243 160

Pt. on E end of a curb, N. curb, Tilton St. 60' W. of W line Mansfield St.

B.S. — *Note: Levels adjusted from*

⊼ — *Bench 18 on page 69*

F.S.

Use 62.990
N.E. cor. catch basin, N.W. cor. Mansfield and Tilton Sts.

B.S.

⊼

F.S.

Use 66.807
Pt. on N. end of a curb, W curb Mansfield St. side driveway into #194

Fig. 153(*d*).

154. Common Mistakes and Sources of Error. (*a*). LEVELMAN. Level out of adjustment; unstable setup of the level; bubble out of center; parallax.

154(*b*). RODMAN. Rod not tested for length, particularly high rod; use of unstable or indefinite turning point; rod not held on exactly the *same* point for foresight and backsight on a turning point; rod not held plumb; target not set accurately when a high rod is used; rod not fully extended for a high rod.

154(*c*). LEVELMAN AND RODMAN. Mistakes in reading the rod; failure to balance foresight and backsight, particularly in accurate leveling; mistakes in recording notes, such as entering F.S. in B.S. column or vice versa or entering an incorrect numerical value; mistakes in working up notes, such as subtracting a B.S. or adding a F.S.

155. Checking the Field Work in Leveling. When a line of levels does not form a closed circuit, it may be possible to check the field work by starting from a permanent bench mark and closing on another. When this is not practicable, the only remaining method of checking the field work is by running a second line of levels. One method of doing this is by the double-rodded method explained in Art. 174. A better method, one which should be used in work of greater precision, is to rerun the levels in the opposite direction. For example, it is desired to establish a bench mark B starting from another bench mark A of known elevation. Levels are first run from A to B, and the difference, d, in elevation between A and B is determined; levels are then run from B to A, and a second difference in elevation d', determined. There will usually be a discrepancy between these two differences in elevation. If this discrepancy is too great to come within the permissible limits of error, levels in both directions are repeated until the discrepancy between d and d' does not exceed the permissible limits of error. The mean value $\frac{1}{2}(d + d')$ is then used in calculating the elevation of B. Rerunning a line in the opposite direction, as just explained, will usually result in a greater discrepancy than if it were rerun in the same direction, but the mean result should be more nearly correct because certain cumulative errors are thus eliminated.

When a high degree of precision is not required as, for example, in a great deal of profile leveling, the leveling may be checked near the close of a day's work by running a set of flying levels back over the line to the staring point, checking on intermediate bench marks, new or old, and finally on the one at the starting point. In class work, where there are several parties, checks may be on common

bench marks established by the instructor at intervals of several hundred feet.

155(*a*). How CLOSE TO CHECK. Whichever check is used, the question arises as to what discrepancy or error is permissible. This depends partly upon the conditions under which the leveling is carried on, but still more upon the purpose for which the elevations are needed. Results are affected by the instruments used, by the nature of the ground covered, by the time in which the work must be done, by the personal equations of the rodman and the levelman, and by the atmospheric condition.

The error depends to a large extent on the number of setups and the distance covered. In class work a party, at first, ought to run half a mile and back with a difference of less than 0.10 ft. between the assumed elevation of the starting point and its elevation as found upon returning to it. This is assuming eight or ten setups each way. A difference of 0.03 ft. would indicate excellent work for the twenty setups, ten each way, but with good instruments and under favorable conditions students should soon be able to come within this limit using ordinary precautions. This does not mean that the absolute error is only 0.03 ft., for some of the errors are compensating, whereas the cumulative errors are to a large extent eliminated by running in opposite directions.

As a rough check it may be assumed that under ordinary conditions the total error E should not exceed $0.02\sqrt{s}$, where s is the total number of setups. If the number of setups is not known, it may be obtained by dividing the total distance by *twice* the average length of sight.

When a line of levels between two points is checked by rerunning in the opposite direction the maximum discrepancy permissible between the first difference in elevation d and the second difference in elevation d' is usually specified by means of a formula, $E = C\sqrt{M}$, in which E is the maximum discrepancy permissible, C, a specified constant, and M, the distance in miles between the two points.

Customary limits of error and methods of distributing errors are treated more in detail in Arts. 170 and 171.

156. Classification of Levels. The standard classification of levels, based on the degree of precision required, is *first order, second order, third order,* and *fourth order.* The degree of precision is specified by means of a formula, $E = C\sqrt{M}$, in which the value of the quantity C is specified for each order of leveling. The application of this formula is explained in Art. 170, and the com-

mon values for *C*, corresponding to the different orders of leveling, are given in the table in Art. 170(*b*).

156(*a*). First-order leveling is frequently called *precise leveling*. These synonymous terms are reserved for leveling in which extraordinary precautions are taken to eliminate errors. The method of procedure in such work, based on the theory of errors in leveling given in Chapter 12, is outlined in detail in Chapter 13.

156(*b*). Second-order leveling. Although the degree of precision generally specified for second-order leveling is only about one-half as great as that specified for first-order leveling, it is nevertheless sufficiently high to require a method of procedure designed to eliminate certain sources of error.

157. Specifications and Method of Procedure for Second-Order Leveling. The following specifications and method of procedure for second-order leveling are taken from the manual entitled *Technical Procedure for City Surveys*, published by the American Society of Civil Engineers. It is assumed that permanent bench marks have been established throughout the city by first-order leveling.

Second-Order Levels

157(*a*). Specifications.

1. All second-order level lines shall originate with, and close upon, first-order level bench marks, or shall be run in closed circuits.

2. Criteria for acceptable final results shall be: (*a*) The maximum closing error of any line or circuit shall not exceed 0.04 ft. $\sqrt{M}$, *M* being the length of the line or circuit in miles. (*b*) Final adjustments shall be made by the method of weighted junction points.

157(*b*). Preparation. The field reconnaissance for second-order levels is usually just a matter of locating and establishing bench marks along the routes selected. Bench marks for this class of leveling may be of a type less permanent than those described for first-order levels. When the second-order system of levels parallels the second-order traverse, all permanent traverse monuments should be utilized as bench marks. Additional bench marks may be established on fire hydrants, catch-basins, concrete sidewalks, steps, etc. For future recovery, all bench marks should be completely described and referenced by steel tape measurements.

157(*c*). Equipment. The equipment required for second-order leveling includes a well-constructed and accurate engineers' level, and a direct-reading level rod. The extension type of rod may be used provided that its setting, when extended, is carefully checked at frequent intervals. It is also well to provide a portable turning point.

157(*d*). Personnel and field procedure. A minimum of two men is required for the field work of second-order leveling. The party chief acts as level observer, and the other man serves as rodman. By using two rodmen instead of one man, the rate of progress may be increased, but the cost per mile will probably not be any lower.

The principal rules of field procedure in second-order leveling are:

1. The level shall be tested for error of bubble once each day, and shall be adjusted if found to be in error by more than 0.01 ft. per 100 ft. of distance

2. The difference in length between a foresight and a corresponding backsight for any setup shall not exceed approximately 50 ft. (Distances in second-order leveling are usually paced, and the number of paces on each sight is noted in the record book.)

3. The cumulative backsight distances shall never differ from the cumulative foresight distances on any section by more than about 100 ft. (A continuous total of the number of paces for both backsight and foresight distances should be kept in the record book.)

Only one wire is read in this class of leveling, and to avoid the possibility of extensive rerunning due to reading the bottom or top wire instead of the middle wire, it is advisable to use a level having only one wire.

158. Establishing Points at Given Elevations is merely reversing the method used in determining the unknown elevations of given points. The applications of this kind of leveling, however, are many and varied, especially in construction work. Some of the most common of these applications, particularly methods used in establishing points for grades, are explained in Chapter 14.

159. Errors in Leveling. The best field work does not consist in always leveling with the utmost precision for this is needlessly expensive, but in leveling with just enough precision to meet the requirements. To do this, some errors must be eliminated while others may be ignored, and this is more difficult than to try indiscriminately to eliminate all errors. If it is desired, for example, to obtain elevations to the nearest hundredth of a foot, how close must be the rod reading? Is it necessary to keep the bubble exactly in the center? Must the rod always be exactly plumb? May the effect of the sun shining on the level be ignored? Must the curvature of the earth and atmospheric refraction be taken into account? To answer these and similar questions, it is necessary to know such fundamentals as: Which errors are accidental and which are cumulative; what are likely to be the magnitude of various kinds of errors; what can be done to eliminate these errors? No one can do really good leveling until he has acquired a thorough understanding of the whole subject of errors in leveling. This subject is treated in the next chapter.

160. Speed. Speed in leveling is gained not so much by hurrying as by the skilful manipulation of the level, by a knowledge of the best method of procedure, and by not wasting time. Time may be wasted in setting up in the wrong place, in setting up too high or too low to strike the rod, in slow manipulation of the level, in getting the target into position, in reading the target, and in waiting for a rodman to saunter slowly from station to station. The rodman should guess about where to move the target or the rod, and should have it there by the time the levelman is ready to sight. He should watch signals closely, and these signals should show whether the target must be moved much or little. Avoid moving the target in jumps first too high, then too low. Often a levelman can call out the approximate reading, thus helping the rodman to set the target quickly.

Speed and accuracy may often be gained by increasing the number of men in a party and the number of instruments used. *Examples:* The use of an extra level as explained in Art. 173 and Art. 179; the use of an extra rod as explained in the article "Reciprocal Leveling" (Art. 179), and in the chapter on precise leveling (Art. 199(*d*), 4). Time may also be saved in some cases by special methods of procedure such as double-rodded leveling (Art. 174).

Remark. At first, speed is not so important as accuracy. The student should not forget, however, that in practice the two should be combined. So much depends on the requirements of the work, on the nature of the ground covered, and on the instruments used, that it is difficult to give an idea of what the speed should be. It may be said in a general way that in class work a level party working under favorable conditions should soon be able to set up and take the backsight and foresight in 10 min., or, on an average, at the rate of six stations an hour. If the average length of sight is 200 ft., i.e., 400 ft. between turning points, this means about half a mile an hour.

CHAPTER 12

ERRORS IN LEVELING

IN this chapter sources of errors in leveling are explained, the relative importance of errors from different sources is discussed, and methods of correction or elimination are given. Limits of error are suggested, and the precautions which are necessary in accurate leveling are outlined. The general discussion of errors in Chapter 4 should be reviewed before beginning this chapter. The construction of the level is described in Chapter 53.

161. Sources of Errors in Leveling.

1. Instrumental errors, (*a*) errors of adjustments and (*b*) defects in the construction of non-adjustable parts.
2. Mistakes in manipulating the level and leveling rod.
3. Mistakes in reading the leveling rod and in keeping field notes.
4. Errors in sighting.
5. Errors due to changes in the position of the level or rod.
6. Errors due to natural sources.
7. Mistakes in recording and computing.
8. Personal errors.

162. Instrumental Errors. (*a*). ERRORS DUE TO IMPERFECT ADJUSTMENT. Errors are *compensating;* they are practically eliminated if the lengths of the foresight and the backsight are made exactly equal. The adjustments of the level (explained in Chapter 54) are for the purpose of making the line of sight generate a *horizontal* plane of sight when the level is properly set up. If the line of sight is not parallel to the bubble axis (the principal error of adjustment), it will generate the surface of a cone the axis of which is vertical. Hence rods held equally distant from the instrument will be intersected at the *same* level by an imaginary base of the cone, and the error due to the fact that the line of sight does not move in a plane is eliminated.

Note. Errors in leveling between two stations or bench marks due to imperfect adjustment may also be eliminated by making the sum of the lengths of foresights between the two stations equal to the sum of the lengths of backsights. It is not so easy to do this, however, as it is to balance backsight and foresight at each setup.

162(*b*). SLUGGISH BUBBLE. If the bubble is sluggish in its movement, an error may result from the bubble's coming to rest in the wrong position, even though it may creep back to the correct position while the sight is being taken. The error is *compensating* and

partially avoided by observing the bubble *after* the target is set to see if it is still in the center of its tube, but such a bubble is a constant source of annoyance and delay.

162(*c*). ERROR IN THE MOVEMENT OF THE OBJECTIVE SLIDE. Error is *compensating*, and is eliminated by making the backsight and foresight equal, since in that case the focus is not changed and hence the slide is not moved. If the objective slide is slightly worn, it may not move in a truly horizontal direction. This is one reason why extremely short sights should be avoided since, for such sights, the slide must be moved out nearly its entire length.

162(*d*). ERROR IN ROD GRADUATION. Not likely to be appreciable in ordinary work if the rod is graduated by a reputable maker, nevertheless every new rod should be tested. In a test of five new rods, four were found to be 0.006 to 0.008 ft. too long in the first 7 ft., and the fifth was 0.016 ft. too long in 13 ft. These errors would not, of course, be permissible in accurate leveling, such as first-order or second-order leveling.

162(*e*). ERROR DUE TO A DEFECTIVE JOINT. An extension rod with an abutting joint should be thoroughly tested at frequent intervals by the method explained in Art. 111. The wear and tear from letting the rod down "on the run" and from other sources may seriously affect this joint, and failure to test the rod may result in a large *cumulative* error.

163. Mistakes in Manipulation. (*a*). THE LEVEL. Mistakes in setting up, imperfect focusing, unnecessary clamping of the spindle, errors in centering the bubble, failure to watch the bubble, resting the hands on the tripod or telescope. (See "Experiment," Art. 150(*b*).) Precautions should be taken to keep the bubble in the center of its tube. If it moves out slightly, bring it back with the leveling screws. Often in work of great precision mirrors are used, or a bubble tender brings the bubble to position with a slight pressure of the fingers. The longer the sight the more necessary it is to keep the bubble centered at the instant of sighting.

163(*b*). NOTE. The levelman should know the error caused by a displacement of the bubble judged by the division marks on the glass tube. The value of a single division varies greatly for different instruments, but it may be ascertained easily for a given tube. (See Art. 705(*a*).) The value is commonly given in units of angular measure, and in 18-in. levels it is usually 20″ to 30″; in some levels it is 10″ or even less, whereas in others it will be found to exceed 60″ or 1′. On page 141 is a table which shows the errors in feet to be expected with instruments of different sensitiveness and for sights of different lengths. *D* in the table stands for one division on the tube.

163(*c*). THE LEVELING ROD HELD SO THAT IT LEANS TOWARDS OR AWAY FROM THE INSTRUMENT will make a reading too large. Error is compensating on turning points. (Why?) Avoided by swaying

ERRORS IN FEET CORRESPONDING TO DIFFERENT DISPLACEMENTS OF THE BUBBLE

Sensitiveness of Level		$D = 10''$			$D = 30''$			$D = 1'$		
Deviation of Bubble		$\frac{1}{2}D$	D	$\frac{3}{2}D$	$\frac{1}{2}D$	D	$\frac{3}{2}D$	$\frac{1}{2}D$	D	$\frac{3}{2}D$
Distance from Rod to Instrument	50	.001	.002	.003	.003	.006	.009	.006	.012	.018
	100	.002	.004	.006	.006	.012	.018	.012	.024	.036
	150	.003	.006	.009	.009	.018	.027	.018	.036	.054
	200	.004	.008	.012	.012	.024	.036	.024	.048	.072
	250	.005	.010	.015	.015	.030	.045	.030	.060	.090
	300	.006	.012	.018	.018	.036	.054	.036	.072	.108

the rod slowly backward and forward as directed in Art. 151, or, better still, by using a rod level, a special attachment devised for plumbing the rod.

Question. Why is the error not likely to be compensating when leveling entirely uphill or downhill?

163(*d*). ALLOWING DIRT OR GRIT TO ACCUMULATE on the base of the rod or in either abutting joint is another source of error. (On what part of the rod would readings be affected by dirt in the joint?)

164. Mistakes in Reading the Rod and in Keeping Field Notes. Common mistakes in reading the rod are discussed in Art. 145(*c*). Common mistakes in recording and in working up field notes, and methods of avoiding and detecting such mistakes are given in Art. 153(*a*), (*b*), and (*c*).

165. Errors in Sighting. (*a*). The difficulty of telling when the cross hair coincides with the center of the target, or, in the case of a self-reading rod, of determining the exact reading which the cross hair appears to cover, leads to an error which is hard to avoid; it is, however, compensating. The coarseness of the cross hair, the type of rod, the form of the target, the length of sight, atmospheric conditions, and the eyesight of the levelman, all have to do with such an error. It may help the levelman after looking steadily at a target until he thinks it is in a correct position to close the eye momentarily and then look again before giving the signal to clamp; or in the case of a self-reading rod, after the first sight, to close the eye momentarily, and then to take a second or check reading. In either case, he should see to it that the horizontal cross

hair of the level is kept truly horizontal. (See adjustments of the level.) If the length of sight exceeds 300 ft., the cross hair is likely to cover so great a space as to make accurate reading difficult. On the other hand, it is well to avoid sights much less than 100 ft. in length. When there are stadia hairs, a common blunder is to mistake one of these hairs for the horizontal cross hair.

An error in sighting too often ignored, but one which may be serious, is due to parallax. Eliminated by correctly focusing the eyepiece and the object glass (Art. 103(*a*)).

In precise leveling, errors in sighting are reduced by the use of three cross hairs in the telescope. For each backsight or foresight three rod readings are taken, one for each hair. The mean of these three readings, it may be assumed, will probably be more accurate than a single reading obtained when only one cross hair is used. For a specified method of taking three readings, see Art. 199(*d*), 4.

When there is only one cross hair, errors in sighting may be reduced by repeating an observation several times and by taking the mean value of these observations.

165(*b*). EXPERIMENT. The following experiment was tried to ascertain the probable error of setting a target at different distances. A level was set up with two leveling screws in the line of sight. An instructor held an umbrella over the instrument and kept the bubble in the center of its tube, while eighteen students who had used the level just enough to become accustomed to its use set the target, each in turn, at three different distances, the instructor reading the rod for all settings. The probable errors for a single observation were as follows:

Conditions	*Instrument*	*Length of Sights* 100	200	300
Sun shining	Level 1	0.00196	0.00196	0.00370
Level in shade	" 2	0.00147	0.00229	0.00327
Air slightly boiling at 200 and 300 ft.	" 3	0.00130	0.00157	0.00213

Levels 1 and 2 were 18-in. wye-levels, 1¾-in. objective, magnifying power about 35 diameters. Level 3 was a 15-in. wye-level, 1⅛-in. objective, magnifying power about 30 diameters. The results for levels 1 and 2 were obtained by one group of students; the results for level 3 were obtained by another group of the same number of men (18). The above results indicate that, for beginners at least, under favorable conditions the chances are even that the error of sighting and of setting the target will not exceed, say, 0.002 ft. for a 100-ft. sight, 0.0023 for 200 ft., and 0.0037 for 300 ft.

The error of reading a rod directly without the use of a target is not so great as is commonly supposed. From a large number of readings taken by a class of over 100 beginners, using "speaking" rods of various styles with graduations more or less dim, the probable errors for a single observation at different distances were found to be (in feet):

Length of sight	50	100	150	200	250	300
Probable error	0.0024	0.0035	0.0038	0.0048	0.0062	0.0065

By comparing these results obtained under unfavorable conditions with those above obtained under favorable conditions, it is seen that target readings are but slightly more accurate than direct readings. Under certain conditions they have been found to be less accurate.

166. Errors Due to Changes in Position of the Level or the Rod. (*a*). SETTLING OF THE INSTRUMENT between the backsight on one turning point and the foresight on the next. Cumulative,

tending to make all elevations too high because the foresight reading is decreased, and the calculated elevation of the second turning point will therefore be too high. Avoided by setting up the instrument solidly on firm ground, and taking the foresight on the second turning point immediately after backsighting on the first. For this reason two rods and two rodmen are sometimes used. Rerunning in the opposite direction tends to eliminate this error also. On warm winter days when the ground is thawing the error due to settling may be very large.

166(*b*). THE LEVEL DISTURBED OR JARRED by passing trains, vehicles, or by the levelman walking around it. Error most likely to occur on loose soil, or on a smooth surface where the tripod shoes do not catch. On some instruments mirrors are placed so that the levelman can watch the bubble without leaving his position behind the instrument, and in precise leveling he can observe the cross hair and the bubble at the same time. As a precaution in ordinary work, set two legs of the tripod in a line parallel to the general direction of the stations, and see that all three legs are reasonably firm before leveling up.

166(*c*). SETTLING OF A TURNING POINT between the foresight and the backsight on it. Cumulative, tending to make elevations too high because the backsight reading will be increased, thus increasing the calculated height of instrument. Easily avoided by choosing proper turning points. Rerunning in the opposite direction tends to eliminate this error.

166(*d*). NOTE. Errors due the settling of the instrument or of a turning point may be reduced by alternation of the order of B.S. and F.S. in successive setups.

167. Errors Due to Natural Sources. (*a*). CHANGES IN THE INSTRUMENT DUE TO THE EFFECT OF SUN AND WIND. The sun shining on a level may cause its parts to expand unequally. For example, in running towards the sun the front wye will expand, making the foresight too high; hence the elevation will be too low. On backsighting the other wye will expand, making the backsight too small; hence the height of instrument too low. The error is therefore likely to be cumulative, making all elevations too low. Again, if the sun heats one end of a bubble tube more than the other the bubble will move *towards* the heated end. (Try breathing on one end of a tube which is level, or warm one end by holding the fingers on it, watching the bubble to see if it is drawn over.) Errors due to the sun may be avoided by shading the instrument with an umbrella, or working on a cloudy day. The effect of wind and heated air currents is to make the target seem to tremble or dance, and

often renders accurate work out of the question. In precise leveling a tent or some combination of umbrella and windbreak is used to shelter the level from both sun and wind. There is usually less wind at daybreak than at any other period of the day, a period also when the atmosphere is in the best condition for long sights.

The error due to the expansion of a wye, and certain other errors as well, may be avoided by keeping the line of sight and the bubble axis parallel (by adjustment) and by taking precautions to have the bubble centered at the instant of sighting.

167(*b*). CHANGE OF LENGTH OF LEVELING ROD FROM VARIATIONS IN TEMPERATURE AND HUMIDITY. Inappreciable in ordinary work, but in more precise leveling it is quite possible that errors from this source may exceed the errors arising from the leveling itself. It is difficult to correct for the expansion and contraction in an ordinary leveling rod, though it can be done to some extent by marking gauge points on the rod and by measuring between these points to ascertain any changes in lengths which may occur. It is preferable, however, to use a better type of rod, such, for example, as one with a graduated strip of invar metal. (See Art. 718.)

167(*c*). ERRORS FROM CURVATURE AND REFRACTION.

1. *Curvature.* The difference between a horizontal line of sight and a level line is about 8 in. in 1 mile, and varies as the square of the distance, i.e., the curvature of a level surface is 0.66725 ft. multiplied by the square of the distance in *miles*. (See Art. 6(*b*).) The effect of curvature is to *increase* the rod reading.

2. *Refraction.* The effect of refraction is the same as if the line of sight were curved downward, or concave towards the earth's surface, and hence the rod reading is *decreased.* The curve is irregular because of varying atmospheric conditions, but for average conditions it is assumed to have a diameter about seven times that of the earth. The correction for refraction is $\left[\frac{(\text{Distance in feet})^2}{\text{Mean radius of earth in feet}}\right] \times 0.07$, where 0.07 is the mean value of the coefficient of refraction.

3. *Combined curvature and refraction.* The effect of refraction is to reduce the correction for curvature. The table on p. 145 gives approximate values for the correction for combined curvature and refraction.

The distance D and the correction c are both in feet except in the last column, where the distance is given in miles.

An empirical formula giving approximate results is: Curvature and refraction, in feet, = 0.574 (distance in miles)^2.

The error due to curvature and refraction is of little consequence in ordinary leveling. In precise leveling it may be eliminated to a large extent by making the backsight and the foresight equal for each setup, and by reducing to a minimum the interval of time between these two sights. The error due to refraction may be further

CORRECTION FOR COMBINED CURVATURE AND REFRACTION

D	*c*	*D*	*c*	*D*	*c*	*D*	*c*	*D*	*c*	Miles	*c*
300	.002	1300	.035	2300	.108	3300	.223	4300	.379	1	.571
400	.003	1400	.040	2400	.118	3400	.237	4400	.397	2	2.285
500	.005	1500	.046	2500	.128	3500	.251	4500	.415	3	5.142
600	.007	1600	.052	2600	.139	3600	.266	4600	.434	4	9.141
700	.010	1700	.059	2700	.149	3700	.281	4700	.453	5	14.282
800	.013	1800	.066	2800	.161	3800	.296	4800	.472	6	20.567
900	.017	1900	.074	2900	.172	3900	.312	4900	.492	7	27.994
1000	.020	2000	.082	3000	.184	4000	.328	5000	.512	8	36.563
1100	.025	2100	.090	3100	.197	4100	.345	5100	.533	9	46.275
1200	.030	2200	.099	3200	.210	4200	.362	5200	.554	10	57.130

reduced by keeping the line of sight at least 3 ft. above the ground, and by shortening the lengths of sight when heat waves make longer sights uncertain. It may even be advisable to discontinue the leveling during the middle of the day, or even to do all the leveling at night.

167(*d*). NOTE. In warm climates, accurate leveling is often difficult in the daytime. In Panama, for example, the most important single difficulty is the excessive heat waves which make accurate sights of over 150 ft. practically impossible between 9 o'clock in the morning and late in the afternoon. An added difficulty during the dry season is that the velocity of the wind is likely to be high during the daytime. It has been found that both difficulties can be largely eliminated by leveling at night with the bubbles, cross hairs, and leveling rod properly illuminated. Under the given conditions both speed and accuracy were higher than could have been obtained working in the daytime. In places where the length of sight is limited by conditions other than heat waves, no material advantage is obtained by night work. (See "Precise Levels Run at Night," *Civil Engineering*, December 1938).

168. Personal Errors. *The personal equation.* Errors which arise from causes peculiar to the individual are not easily avoided. They may not amount to much in ordinary work, but they are taken into account in precise leveling, especially in comparing results. A man may habitually read the rod too high or too low, or he may manipulate the level in some peculiar manner so that slight errors of the same sign (cumulative) occur. Running levels in opposite directions, the rodman and the levelman changing places before going over the ground a second time, or, in running a single line of levels, changing places at every other setup, helps to eliminate errors of this kind.

169. Elimination of Errors—Recapitulation. (*a*). MISTAKES. Keep in mind the "danger points" in reading the rod (Art. 145(*c*)). Mistakes in reading the rod and in recording the readings are best avoided by having two men keep independent notes, reading the rod independently, and comparing notes *after* both have read the rod. The levelman should avoid mistaking a stadia hair for the middle horizontal cross hair.

169(*b*). Constant errors. By making backsight and foresight equal for each setup several constant errors are eliminated, such, for example, as instrumental errors due to imperfect adjustment and the error due to curvature and refraction. The level, however, should be frequently tested for adjustment. When it is impossible to make backsight and foresight equal, reciprocal leveling may be employed (Art. 179). Rerunning a line in the opposite direction tends to eliminate certain other constant errors. The error due to the settling of the turning point or to holding the rod on slightly different points for foresight and backsight should be avoided—it is usually inexcusable. In many cases steel turning points are used to eliminate such errors. The error due to settlement of the instrument is reduced by taking the foresight as soon as possible after the backsight, and in precise leveling two rodmen are used for this purpose. The error due to the bubble's being out of center is avoided by instinctively glancing at the bubble just before and after sighting, by the use of mirrors on levels of precision, and sometimes by employing a bubble tender. Errors due to the sun's shining on the instrument are avoided by shading the level or working on a cloudy day. The error due to the boiling of the atmosphere is avoided by suspending work during midday hours. Rod errors are avoided by testing the rod graduations, by keeping the base of the rod and the abutting joints free from dirt and grit, by examining the setting of the target as well as the vernier reading for an extended rod, and when the rod is closed by examining the vernier reading to see that it reads 6.5 ft. or 7.0 ft., not depending upon the abutting end to bring it back to place. These last errors are eliminated altogether by using a one-piece rod; many surveyors will use no other in important work. The error due to not plumbing the rod is avoided by waving the rod, or by using a rod level. For precise leveling in which many of the foregoing precautions are adopted see Chapter 15.

169(*c*). Accidental errors, such as small errors in sighting and errors resulting from small disturbances of the instrument, can be reduced only by adjusting independent or related observations, such as repeating an observation two or more times and then calculating a mean value.

170. Customary Limits of Error. Probably in no branch of surveying does the law of accidental errors (that the total error will vary as the square root of the number of observations (Art. 81)) accord more closely with experience than in leveling. This is because constant errors are more easily eliminated and hence accidental errors are likely to predominate, even in ordinary work.

There are two ways in which the precision of leveling may be judged: (1) by comparing the two elevations of the same point as found from independent lines of leveling—the difference is not an error but a discrepancy; (2) by the difference between the *assumed* elevation of some starting point and the elevation found by completing a circuit, i.e., returning to the same starting point. This is a *true* error, called the closing error of leveling.

In either case the total error of a line of levels may be expected to vary about as the square root of the number of setups. When, however, the number of setups per mile is practically constant, it is more convenient to express the error in terms of the distance, thus:

$$\text{Error in feet} = \text{Constant}\sqrt{\text{Distance in miles}}, \text{ or } E = C\sqrt{M}$$

In this formula, E is the *upper* limit of error, that is, the greatest error permissible. The degree of precision is indicated by the value of the constant C. Thus, for example, a constant of 0.01 would indicate an exceptionally high degree of precision whereas a coefficient of 0.40 would indicate a very low degree. The value C to be adopted in any given case will depend upon the purpose of the work, the period of time in which it must be done, the expense warranted, and other local conditions.

170(*a*). ILLUSTRATION. The general use of the formula may be illustrated as follows. Assume that a coefficient of 0.02, corresponding to a high degree of precision, has been adopted; that a line of levels has been run between two bench marks, that M (in miles) is the distance between these bench marks, and that from these levels the difference in elevation between the two bench marks has been obtained; that a second line of levels has been run in the opposite direction between the two bench marks, and that a second difference of elevation has been obtained. The first and second differences in elevation are not likely to be the same. The *discrepancy* E between the two should not exceed an amount in feet obtained from $0.02\sqrt{M}$. (The *custom* of calling the value represented by E an "*error*" is firmly fixed, but no great harm is done provided it is clearly understood that it is a discrepancy and not a true error.)

170(*b*). COMMON VALUES FOR C. In the following table are given some of the common values for the coefficient C in the expression $E = C\sqrt{M}$. In the first column is indicated the order or character of leveling; in the second column, the maximum length of sight that should be allowed; in the third column, the decimal of a foot to which readings should be taken; in the fourth column, whether or not backsights and foresights should be carefully balanced.

Order	*Sight*	*Readings*	*Sights Balanced*	C
First	300	Thousandths	Carefully	±0.017
Second	300	Thousandths	Carefully	±0.040
Third	300	Thousandths	Approximately	±0.050
Ordinary	500	Hundredths	Roughly	±0.100
Rough	1000	Tenths		±0.400

170(*c*). THE DEGREE OF PRECISION THAT MAY BE EXPECTED will depend not only on the lengths of sight, the accuracy of rod readings, and the care with which sights are balanced, but also on the method of procedure followed in order to eliminate other sources of error, the equipment used, the personnel of the leveling party, and the character of the ground over which levels are run. A general idea of the equipment, personnel, and method of procedure for leveling of the first order may be gained from a study of Chapter 15; for leveling of the second order, see Art. 157.

Since the total error for a given distance increases as the square root of the number of setups increases, it is more difficult to attain a high degree of precision in a hilly country, where a comparatively large number of setups is necessary, than in a flat country where a smaller number is required. For example, the degree of precision obtained in precise leveling, as indicated by the coefficient C, varied in one instance from about 0.013 ft. for nearly level ground to about 0.033 ft. for gradients entirely between 1 in 20 and 1 in 10.

Beyond a certain limit a small increase in precision requires a large increase of effort, and the difference in precision between ordinary leveling and precise leveling is not at all in proportion to the difference in the skill and care required.

170(*d*). PERMISSIBLE ERRORS IN LEVELING. The following table, computed from the formula $E = C\sqrt{M}$, gives the permissible error in *feet* for different distances in miles for different coefficients of precision.

PERMISSIBLE ERRORS IN LEVELING FOR DIFFERENT COEFFICIENTS AND DIFFERENT DISTANCES

Miles	0.05	0.04	0.03	0.02	0.01
0.5	0.035	0.028	0.021	0.014	0.007
1	.050	.040	.030	.020	.010
2	.070	.056	.042	.028	.014
3	.086	.069	.052	.035	.017
4	.100	.080	.060	.040	.020
5	.112	.089	.067	.045	.022
6	.122	.098	.073	.049	.024
7	.132	.106	.079	.053	.026
8	.141	.113	.085	.057	.028
9	.150	.120	.090	.060	.030
10	.158	.126	.095	.063	.032
15	.194	.155	.116	.077	.039
20	.224	.179	.134	.089	.045

Illustration of the use of table. What discrepancy between two lines of levels is permissible at the end of 4 miles if 0.05 is the adopted value of C? Answer 0.1 ft. What is the permissible error of closure of a circuit 6 miles long if 0.02 is the value adopted for C? Answer 0.049 ft.

171. Distribution of Errors.

In this discussion of the distribution of errors in leveling, it is assumed that proper precautions have been taken to eliminate sources of constant error so that the remaining errors are largely accidental and tend to offset each other. Only simple cases will be considered.

171(*a*). CASE 1. LEVELS RERUN IN THE OPPOSITE DIRECTION. Starting from a bench mark A of known elevation, a line of levels is run to establish the elevations of new bench marks B, C, and D. Assume that levels have been run forward and backward between each pair of successive bench marks as explained in Art. 155. Let d be the difference in elevation between A and B that was determined by running from A to B, and d', a second difference

in elevation that was determined by running from B to A. Assuming that the levels have been run with the same care in both directions, the mean difference of elevation $\frac{1}{2}(d + d')$ is then used in calculating the elevation of B, or, what amounts to the same thing, one-half of the numerical difference between d and d' may be applied as a correction to the elevation of B that was determined by running from A to B.

The same process of adjustment may be repeated for levels that have been run forward and backward between B and C. Starting with the adjusted value for the elevation of B, the adjusted value for the elevation of C is calculated. Thus adjustments may be made from bench mark to bench mark throughout the whole line of levels.

Note. It is necessary to observe in the field whether or not the discrepancy between the two differences of elevation obtained by running forward and backward between two successive bench marks exceeds the maximum permissible error, for if it does, the levels should be rerun. But the adjustment of the values for elevations may be left until the field work is completed. In working up the field notes some systematic method of procedure should be followed that will help to avoid the common mistake of adding a correction to an elevation when it should be subtracted or vice versa.*

171(*b*.) CASE 2. A CLOSED LEVEL CIRCUIT. Starting from a bench mark A of known elevation, a line of levels is run over a closed circuit ending on A. Let E represent the error of closure. It is required to distribute this error in adjusting the elevations of new bench marks established on the circuit. It is customary to distribute the closing error uniformly around the circuit in proportion to the distances of the bench marks from the starting point. For example, if the distance of a bench mark from A is approximately one-fourth the total length of the circuit, its elevation is adjusted by applying a correction equal to $\frac{1}{4}E$; if its distance from A is approximately five-eighths the length of the circuit, the correction is $\frac{5}{8}E$, and so on. If the elevation of A, found from running levels around the circuit, is *higher* than its elevation used at the start, i.e., if E is positive, then the correction applied to each bench mark will decrease the elevation of the bench mark, i.e., the correction will be negative. If E is negative, the correction for each bench mark will be positive, thus increasing its elevation.

As stated in the preceding article, errors do not increase in direct proportion to the distance run but as the square root of that distance. Even this statement is based on the assumption that the number of setups is practically constant per unit of length. But suppose that one section of the circuit is run over comparatively level ground, and another section, of about the same length, over steep slopes. Then the number of setups in the second section would be greater than in the first. The method of adjustment just described could, therefore, be questioned. This suggests that no method of adjustment can be applied indiscriminately, but local conditions must be taken into account, not merely those pertaining to the number of setups, but also all other conditions that tend to make accurate leveling more difficult over one section than over another. However, under ordinary conditions where, with the same care in leveling, nearly the same degree of precision may be attained over all sections of the circuit, this method of adjustment is sufficiently accurate for the purpose of distributing errors that do not exceed permissible limits of error, since such errors are small in proportion to the lengths of the distances run.

171(*c*). CASE 3. A JUNCTION OF TWO OR MORE LINES OF LEVELS. Assume that two or more lines of levels have been run from a common bench mark A of known elevation over different routes to a new bench mark B. It is probable that the result will be two or more different elevations for bench mark B. It is required to determine an adjusted value.

If the different lines have been run with nearly the same degree of precision, a greater error may be expected over a long line than over a short one. Consequently, in adjusting values, greater weight should be given to the elevation of B determined by the levels over a short line than to an elevation determined by levels over a long line. It is customary, therefore, to assign the weights of the lines as inversely proportional to their lengths. The "weighted mean" is then calculated, and the result is taken as the adjusted value of the elevation of B.

* See *Engineering News-Record,* March 3, 1938, p. 345, for such a method of procedure.

The same method of adjustment is used at the junction of several lines of levels which start from different bench marks whose elevations have been previously determined with respect to the same *datum*.

Illustrative Example. Assume that a bench mark B is at the junction of three lines of levels whose lengths are nearly in the proportion of 2, 4, and 5, and that the corresponding elevations determined for B are 30.23 ft., 30.31 ft., and 30.26 ft. With 20 assumed as a convenient multiple and divided by 2, 4, and 5, the weights for the three lines are respectively 10, 5, and 4. The weighted mean or adjusted value for the elevation of B is:

$$\frac{10 \times 30.23 + 5 \times 30.31 + 4 \times 30.26}{10 + 5 + 4} = 30.25 + \text{ft.}$$

Since the whole number of feet is the same for each reading, the same result may be obtained by omitting 30 ft. from each elevation: $(10 \times 0.23 + 5 \times 0.31 + 4 \times 0.26) \div 19 = 0.25+$ ft. If the whole number of feet is not the same, omit only those digits before the decimal points that are alike. For example, if the third elevation had been 31.26 instead of 30.26, the value would be: $(10 \times 0.23 + 5 \times 0.31 + 4 \times 1.26) \div 19 = 30.46+$ ft.

171(*d*). ADJUSTMENT OF A LEVEL NET. In leveling of the highest degree of precision, the discrepancies due to the errors in a level net are usually adjusted by methods based on the theory of least squares. However, unless the net is quite extensive an approximate method may be used. The method of successive distributions devised by Professor George H. Dell and explained in this article is easily understood, is mathematically correct, and involves only simple arithmetical computations for its application. It will give results which agree very closely with those obtained by the method of least squares and with considerably less computation. For example, in the problem to be explained later, the discrepancy in results obtained by the two methods applied to level circuits of five to thirty miles in length differs by less than 0.05 ft.

The method of successive distributions can be used to advantage in connection with work done with an ordinary engineer's level since in such work it is doubtful whether the precision obtainable in the field warrants an attempt to deal with elevations or readings in units of less than 0.01 ft., or 0.005 ft. Nevertheless, with a little added effort, the precision of the calculations is capable of further extension by increasing the number of successive cycles of distribution. Consequently the method may be considered sufficiently precise to meet the requirements of the majority of level nets, except those of large geodetic surveys.

The method of successive distributions has the great advantage that the computations can be made with a slide rule. For this purpose a "polyphase" type of rule is preferable since an entire series of values can be read from a single setting of its slide.

The explanation of the method which follows, including the illustrative problem, is a condensation of the paper entitled "The Adjustment of a Level Net" by George H. Dell, published in the

Transactions of the American Society of Civil Engineers, Volume 101.

DEFINITIONS AND PRELIMINARY EXPLANATIONS

To facilitate description, the following terms will be used:

Control Bench Mark: A bench mark with an elevation which is to remain fixed, and to which the elevations of other bench marks are to be adjusted.

Junction Point: A bench mark at which three or more level lines meet.

Section: The length of level line connecting consecutive junction points or control bench marks.

Intermediate Bench Mark: A bench mark located at some intermediate point of a section.

Sub-section: A part of a section terminating at an intermediate bench mark.

Circuit: A polygon or closed figure bounded by sections (but containing none within its interior).

Exterior Section: A section situated along the boundary of the net, and hence belonging to only one circuit.

Interior Section: A section situated within the interior of the net, and hence common to two circuits.

Positive Direction of a Section: The direction in which the correction is to be computed (regardless of whether the difference in elevation is positive or negative).

Direction Arrows. "Main direction arrows" are those used to designate the positive directions of the sections. In general, these direction arrows serve also to indicate the direction in which the closing error of a circuit is calculated. In certain interior sections, however, the latter direction may be contrary to the former, and is then distinguished by means of an "auxiliary direction arrow," which is written beside the section and within the circuit in question, as shown in Fig. 171(*d*),1.

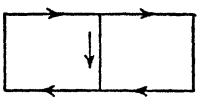


FIG. 171(*d*),1.

Weights. The weight, p, of an observed difference in elevation in a given section depends upon the length of the section, the number of setups it contains, the number of independent measurements entering into the data, and the quality of instruments and survey methods used. On ordinary slopes, the number of setups is approximately proportional to the length of the line, and, therefore, it is customary to weight the observations in inverse proportion to the distance (Art. 171(*c*)). (The distance may be determined by stadia measurements, by pacing, or by scaling from a map.) Moreover, the weight is directly proportional to the number of times the levels were run; that is, a difference in elevation which is determined from the mean of two separate runnings of a section is given twice as much weight as a difference in elevation obtained from a single running.

Effective Lengths. It is convenient, in the problem of the level net, to deal with weights in terms of the *effective lengths* of sections. The effective length of a section is defined as the reciprocal of the weight; that is, $1/p$; consequently, it is directly proportional to the length of the section and inversely proportional to the number of measurements.

In the illustrative problem which follows, the effective length of each section is obtained by dividing the *relative length* by the weight for that section. The relative length is the actual length of the section divided by some number arbitrarily selected (the same for all sections). The smaller units thus obtained facilitate the computations. In the illustrative problem, the length of each section is divided by 800. In that problem all sections were run forward and return, except that from C to E, which was run only one way (Fig. 171(*d*),3). All sections except the one from C to E were given weights of 1, whereas the section from C to E was given a weight of ½. Consequently the effective lengths of all sections except that from C to E are identical with the corresponding relative lengths. (See in this article, Table II, columns 4 and 5.) The effective length of the section from C to E is the relative length divided by ½, or $5 \div \frac{1}{2} = 10$. Note that the effective length of a section which has been leveled once is double the effective length of that section when leveled twice.

Cycle. One complete distribution of the closing errors in the sections of a level net. There are as many cycles as there are successive distributions and the number depends upon how many are required to attain the desired degree of precision.

Illustrative problem

The level net chosen to illustrate the method of adjustment is shown in Fig. 171(*d*),2. It consists of four circuits and nine bench marks. The bench marks, *A, B, C, D,* and *E* are at junction points; intermediate bench marks are at *F, G, H,* and *K*. The bench mark at *A* is assumed to be a *control* bench mark whose correct elevation, therefore, is known (416.15 ft.)

The elevations of all bench marks are given in Table I. These elevations were taken from field notes for lines of levels run over six different routes as follows (Fig. 171(*d*),2): route 1: *A* to *B* to *C* and return; route 2: *C* to *D* to *G* to *E* and return; route 3: *D* to *F* to *B* and return; route 4: *C* to *E* (one way); route 5: *A* to *H* to *E* and return; and route 6: *A* to *K* to *E* and return.

The approximate distances between bench marks are shown in Fig. 171(*d*),2.

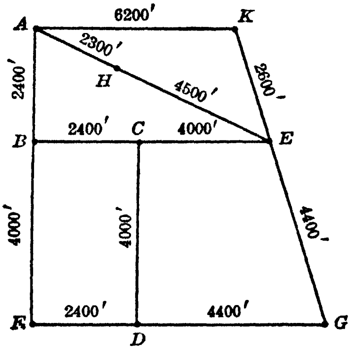


Fig. 171(*d*),2.

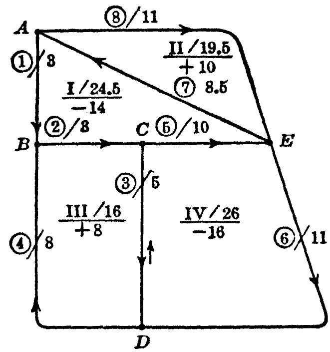


Fig. 171(*d*),3.

A numeral in a circle is the number of the corresponding section; the numeral beside the circle is the effective length of that section. Each Roman numeral designates the circuit within which it is placed; the number beside the Roman numeral is the total effective length of the circuit; and the number below the horizontal bar is the closing error of the circuit expressed in hundredths of a foot. For example: circuit IV, effective length 26.0, closing error − 0.16 ft.

Direction arrows are assigned arbitrarily in such a way as to secure continuous routing through all the circuits with a minimum number of auxiliary direction arrows. A rounded corner in Fig. 171(*d*),3 indicates that the corresponding section continues beyond that corner and that the corresponding bench mark is an intermediate bench mark.

TABLE I

Elevations from Field Notes

Route 1		Route 2		Route 3		Route 4		Route 5		Route 6	
Point	Elevation	Point	Elevation	Point	Elevation	Point	Elevation	Point	Elevation	Point	Elevation
A	416.15	*C*	424.06	*D*	421.66	*C*	424.06	*A*	416.15	*A*	416.15
B	422.59	*D*	421.70	*F*	424.11	*E*	425.16	*H*	418.22	*K*	410.02
C	424.06	*G*	428.65	*B*	422.59	..	......	*E*	425.35	*E*	425.34
B	422.65	*E*	425.25	*F*	424.05	..	......	*H*	418.32	*K*	409.94
A	416.17	*G*	428.58	*D*	421.58	..	......	*A*	416.28	*A*	416.04
..	......	*D*	421.62	..	......	..	......	..	......	..	......
..	......	*C*	423.92	..	......	..	......	..	......	..	......

Note that the elevation of control bench mark *A* has been fixed at 416.15 ft.

Adjustment Procedure

1. *Sketch the level net* as in Figs. 171(*d*),2 and 171(*d*),3. Values for effective lengths and closing errors are to be added later.
2. *Tabulate elevations* from field notes of the various routes in a form similar to that in Table I.
3. *Assign a system of direction arrows* as previously explained in connection with Fig. 171(*d*),3. Note that these direction arrows are not dependent in any way either on the order or the direction of routes as run in the field.
4. *Number the sections, determine their effective lengths* as previously explained, and record these values on the plan as in Fig. 171(*d*),3. Determine the effective length of each *circuit* by adding the effective lengths of all sections in that circuit. Record the results beside the respective roman numerals and above the horizontal lines as in Fig. 171(*d*),3.
5. *From the field data in Table I, calculate the observed difference in elevation over each section in the positive direction* (the direction shown by the main direction arrow in Fig. 171(*d*),3). Arrange these values together with lengths of sections, relative lengths, and effective lengths as in Table II. Also determine the observed differences in elevation over the subsections. Pay particular attention to the algebraic signs of the observed differences in elevation.

TABLE II

Computation of Effective Lengths and Corresponding Observed Differences in Elevation

(*a*) Relating to Sections

Section (1)	Between Points (2)	Distance, l, in feet (3)	Relative length (4)	Effective length L' (5)	Field Results * Forward (6)	Field Results * Back (7)	Observed difference in elevation † (8)
1	*A* and *B*	2 400	3	3	+6.44	−6.48	+6.46
2	*B* and *C*	2 400	3	3	+1.47	−1.41	+1.44
3	*C* and *D*	4 000	5	5	−2.36	+2.30	−2.33
4	*D* and *B* (via *F*)	6 400	8	8	+0.93	−1.01	+0.97
5	*C* and *E*	4 000	5	10	+1.10	......	+1.10
6	*E* and *D* (via *G*)	8 800	11	11	+3.55	−3.63	−3.59
7	*E* and *A* (via *H*)	6 800	8.5	8.5	+9.20	−9.07	−9.14
8	*A* and *E* (via *K*)	8 800	11	11	+9.19	−9.30	+9.24

(*b*) Relating to Subsections

Section (1)	Subsection: between points: (9)	Ratio, length of subsection to length of section (10)	Field Results * Forward (11)	Field Results * Back (12)	Observed difference in elevation † (13)
4	*D* and *F*	24 : 64	+2.45	−2.47	+2.46
6	*E* and *G*	44 : 88	−3.40	+3.33	+3.36
7	*E* and *H*	45 : 68	+7.13	−7.03	−7.08
8	*A* and *K*	62 : 88	−6.13	+6.10	−6.12

* In the direction actually run. † In the positive direction.

6. *Prepare a table similar to Table III to show the closing error (observed difference in elevation) for each section in each circuit* shown in Fig. 171(*d*),3. Find the closing error for each *circuit* by adding algebraically the closing errors for the sections in the circuit, and record these errors on the plan as in Fig. 171(*d*),3, i.e., below the horizontal lines, thus forming fractions the numerators of which are effective lengths of circuits.

TABLE III

COMPUTATION OF CLOSING ERRORS (OBSERVED DIFFERENCES IN ELEVATION)

Section No.	Circuit No. I	Circuit No. II	Circuit No. III	Circuit No. IV
1	+6.46	......	......	......
2	+1.44	......	+1.44	......
3	......	......	−2.33	+2.33 *
4	......	......	+0.97	......
5	+1.10	......	......	+1.10
6	. ....	......	......	−3.59
7	−9.14	−9.14	......	......
8	. . ..	+9.24	....	......
	+9.00	+9.24	+2.41	+3.43
	−9.14	−9.14	−2.33	−3.59
Summation	−0.14	+0.10	+0.08	−0.16

* This value taken in the negative direction of section 3; that is, as shown by the auxiliary direction arrow in Fig. 171(*d*), 3.

The steps in the adjustment procedure which remain to be explained are: (7) To distribute closing errors. (8) To tabulate closing errors which remain after adjustment. (9) To determine the algebraic sum of the corrections for each section. (10) To verify the circuit closures. (11) To apply the values of the circuit corrections to the observed differences of elevation to obtain adjusted elevations.

7. *Distribute the closing errors of the four circuits* found in Table III, i.e., −0.14, +0.10, +0.08, and −0.16 ft. A convenient form for use in computation is shown in Table IV. The closing errors for circuits, i.e., −14.0, +10.00, +8.00, and −16.00 in column 3, are taken from Table 3, but each value in that table has been multiplied by 100 for convenience in expressing units of closing errors.

Each fraction in columns 4, 6, 8, and 10, Table IV, represents the effective length of the corresponding section divided by the effective length of the corresponding circuit. The data for the fractions may be taken from the plan in Fig. 171(*d*),3. The relations expressed in column 4 are based on the assumption that the closing error in any section is to the closing error of the corresponding circuit as the effective length of the section is to the effective length of the circuit. Note that in column 4 the algebraic sign of the closing error for circuit I is plus instead of minus. This makes 1.71, 1.71, 5.71, and 4.85 *corrections* instead of closing errors, and the total correction for the circuit is +13.98, indicating a negligible error in the computations. Likewise the corresponding results in column 4 for circuits II, III, and IV are corrections.

In circuit II, the correction for section 7, i.e., +4.85 is carried across from circuit I and added to the original closing error for circuit II, giving +4.85 + 10 = +14.85. In column 4 this is changed to −14.85. (Sum of the section *corrections.*)

In circuit III, the correction for section 2, i.e., +1.71 is carried across from circuit I, making the total +8.00 + 1.71 = +9.71.

In circuit IV, the correction for section 3, i.e., +3.04, is carried across (with reversed sign) from circuit III and the correction for section 5, i.e., +5.71, is carried across from circuit I. The reason that the algebraic sign of −3.04 was reversed is because in circuit IV the closing error was calculated

in the direction indicated by the auxiliary direction arrow in Fig. 171(*d*), 3. The closing error is $-16.00 + 3.04 + 5.71 = -7.25$.

It should be noted that in dealing with sections marked by the auxiliary arrow, the sign of the correction is always changed upon being carried over into the adjoining circuit.

The adjustment of the circuit errors is continued from cycle to cycle of computations as far as may be desired—in this problem through four cycles. The procedure in each cycle is the same as that explained for the first cycle. However, it is necessary at the beginning of each cycle to determine the closing error for each circuit. This is done by transferring and combining corrections for sections determined in the preceding cycle *which have not previously been transferred* in that cycle. For example, for circuit I in the second cycle, the values transferred from the first cycle are for sections 2, 5 and 7, respectively, -1.82, $+2.79$, and -6.48. Note that in the first cycle there are two values for section 2, namely $+1.71$ and -1.82, but since the value $+1.71$ had been transferred previously (from circuit I to circuit III) and the value -1.82 had not been transferred previously, only the latter was carried over to column 5, circuit I, in the second cycle. Likewise there are two corrections in the first cycle for section 5, namely $+5.71$ and $+2.79$; the latter value is transferred to circuit I in the second cycle. The algebraic sum of the three closing errors $(-1.82 + 2.79 - 6.48 = -5.51)$, the closing error for circuit I in the second cycle, is distributed among sections 1, 2, 5, and 7 by the same ratios (3/24.5, 3/24.5, 10/24.5, 8.5/24.5) as those used in the first cycle. The procedures for circuits II, III, and IV in the second cycle, similar to that just explained for circuit I, will become evident from a study of Table IV

The procedures for the third and fourth cycles, similar to that just explained for the second cycle, are indicated in Table IV.

8. *Tabulate the closing errors which remain at the close of each cycle of distribution as in Table V.* (Values obtained from Table IV.)

This Table V may be used to determine the number of cycles required to obtain a satisfactory adjustment of the level net. The rapid decrease in values effected by the second and third cycles and the increased uniformity in the closing errors of the four circuits indicate that in the adjustment of level nets of ordinary precision, three cycles often would be sufficient. Note that the values in Table V are 100 times the true values so that the errors appear to be 100 times larger than they really are. For example, the closing error for circuit I at the end of the fourth cycle is $0.63 \div 100 = 0.0063$ ft.

9. *Determine the algebraic sum of the corrections of each section which were calculated in successive cycles.* Record close to the line of each section line as in Fig. 171(*d*),4 the corrections for that section calculated in Table IV. For example, for section 4 in circuit III, the values, found by reading across

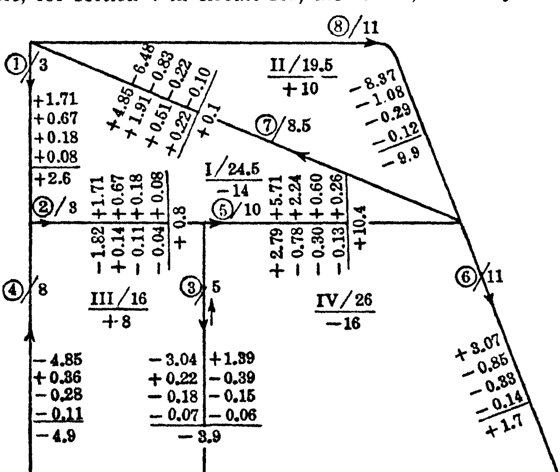


FIG. 171(*d*),4.

TABLE

DISTRIBUTION OF

Circuit No. (1)	Section No. (2)	First Cycle: Closing error (3)	First Cycle: Distribution (4)	Second Cycle: Closing error (5)	Second Cycle: Distribution (6)
I	..	−14.0			
	1	....	$\frac{3}{24.5} \times +14.00 = +1.71$	....	$\frac{3}{24.5} \times +5.51 = +0.67$
	2	....	$\frac{3}{24.5} \times +14.00 = +1.71$	−1.82	$\frac{3}{24.5} \times +5.51 = +0.67$
	5	....	$\frac{10}{24.5} \times +14.00 = +5.71$	+2.79	$\frac{10}{24.5} \times +5.51 = +2.24$
	7	....	$\frac{8.5}{24.5} \times +14.00 = +4.85$	−6.48	$\frac{8.5}{24.5} \times +5.51 = +1.91$
		−14.00	+13.98	−5.51	+5.49
II	..	+10.00			
	7	+4.85	$\frac{8.5}{19.5} \times -14.85 = -6.48$	+1.91	$\frac{8.5}{19.5} \times -1.91 = -0.83$
	8	....	$\frac{11}{19.5} \times -14.85 = -8.37$	....	$\frac{11}{19.5} \times -1.91 = -1.08$
		+14.85	−14.85	+1.91	−1.91
III	..	+8.00			
	2	+1.71	$\frac{3}{16} \times -9.71 = -1.82$	+0.67	$\frac{3}{16} \times +0.72 = +0.14$
	3	....	$\frac{5}{16} \times -9.71 = -3.04$	−1.39 *	$\frac{5}{16} \times +0.72 = +0.22$ *
	4	....	$\frac{8}{16} \times -9.71 = -4.85$	....	$\frac{8}{16} \times +0.72 = +0.36$
		+9.71	−9.71	−0.72	+0.72
IV	..	−16.00			
	3	+3.04	$\frac{5}{26} \times +7.25 = +1.39$	−0.22 *	$\frac{5}{26} \times -2.02 = -0.39$ *
	5	+5.71	$\frac{10}{26} \times +7.25 = +2.79$	+2.24	$\frac{10}{26} \times -2.02 = -0.78$
	6	....	$\frac{11}{26} \times +7.25 = +3.07$	....	$\frac{11}{26} \times -2.02 = -0.85$
		−7.25	+7.25	+2.02	−2.02

* Note the reversal of the algebraic sign when the closing error or the correction for indicated by the auxiliary arrow

IV

CLOSING ERRORS

Third Cycle		Fourth Cycle		Section No.	Circuit No.
Closing error (7)	Distribution (8)	Closing error (9)	Distribution (10)	(2)	(1)
				..	I
....	$\frac{3}{24.5} \times +1.47 = +0.18$	....	$\frac{3}{24.5} \times +0.63 = +0.08$	1	
+0.14	$\frac{3}{24.5} \times +1.47 = +0.18$	−0.11	$\frac{3}{24.5} \times +0.63 = +0.08$	2	
−0.78	$\frac{10}{24.5} \times +1.47 = +0.60$	−0.30	$\frac{10}{24.5} \times +0.63 = +0.26$	5	
−0.83	$\frac{8.5}{24.5} \times +1.47 = +0.51$	−0.22	$\frac{8.5}{24.5} \times +0.63 = +0.22$	7	
−1.47	+1.47	−0.63	+0.64		
				..	II
+0.51	$\frac{8.5}{19.5} \times -0.51 = -0.22$	+0.22	$\frac{8.5}{19.5} \times -0.22 = -0.10$	7	
....	$\frac{11}{19.5} \times -0.51 = -0.29$	....	$\frac{11}{19.5} \times -0.22 = -0.12$	8	
+0.51	−0.51	+0.22	−0.22		
				..	III
+0.18	$\frac{3}{16} \times -0.57 = -0.11$	+0.08	$\frac{3}{16} \times -0.23 = -0.04$	2	
+0.39 *	$\frac{5}{16} \times -0.57 = -0.18$ *	+0.15 *	$\frac{5}{16} \times -0.23 = -0.07$ *	3	
....	$\frac{8}{16} \times -0.57 = -0.28$	....	$\frac{8}{16} \times -0.23 = -0.11$	4	
+0.57	−0.57	+0.23	−0.22		
				..	IV
+0.18 *	$\frac{5}{26} \times -0.78 = -0.15$ *	+0.07 *	$\frac{5}{26} \times -0.33 = -0.06$	3	
+0.60	$\frac{10}{26} \times -0.78 = -0.30$	+0.26	$\frac{10}{26} \times -0.33 = -0.13$	5	
....	$\frac{11}{26} \times -0.78 = -0.33$	....	$\frac{11}{26} \times -0.33 = -0.14$	6	
+0.78	−0.78	+0.33	−0.33		

section 3 is carried across from circuit III to circuit IV, or vice versa. This reversal is in circuit IV, Fig. 171(*d*),3.

TABLE V

ERRORS REQUIRING DISTRIBUTION

Cycle	Circuits			
	(I)	(II)	(III)	(IV)
1	−14.00	+14.85	+9.71	−7.25
2	−5.51	+ 1.91	−0.72	+2.02
3	−1.47	+ 0.51	+0.57	+0.78
4	−0.63	+ 0.22	+0.23	+0.33

from cycle to cycle, are −4.85, to +0.36, −0.28, and −0.11. For each of the interior sections 2, 3, 5, and 7 there are two sets of values in Table IV. This is because each of these sections is common to two circuits. For example, the values for section 2 are obtained by reading across from cycle to cycle, first in circuit I and then in circuit III.

Find the algebraic sum (to the nearest 0.1) of the individual corrections which have been recorded in Fig. 171(*d*),4 for each section. For each of the interior sections, this will be the algebraic sum of the quantities on both sides of the line. The sum for any section is 100 times the actual correction for that section. Thus for section 1, the actual correction is 2.7 ÷ 100 = 0.03 to the nearest hundredth.

Note that when a section is marked by an auxiliary arrow, the signs of all quantities on that side of the section which bears the arrow are reversed when the individual corrections of the section are totaled (section 3, for example).

10. *Verify the circuit closures.* If the closing error of a circuit (as found in Table III) is correctly distributed among the sections in that circuit, the sum of the corrections for the sections should equal the closing error for the circuit (Table III) but with opposite sign. Tabulate the totals of the corrections for sections, circuit by circuit, as in Table VI. Totals for sections may be taken directly from Fig. 171(*d*),4 divided by 100, and then entered in the table to the nearest hundredth of a foot. Note that the algebraic sign for the total correction for section 3 in circuit III is reversed for circuit IV since in circuit IV the corrections for section 3 are taken in the negative direction as indicated by the auxiliary direction arrow in Fig. 171(*d*), 3.

TABLE VI

VERIFICATION OF CIRCUIT CLOSURES

SECTION CORRECTIONS

Section No.	Circuit I	Circuit II	Circuit III	Circuit IV
1	+0.03	......	......	......
2	+0.01	......	+0.01	......
3	......	......	−0.04	+0.04
4	......	......	−0.05	......
5	+0.10	......	......	+0.10
6	......	......	......	+0.02
7	0.00	0.00	......	......
8	......	−0.10	......	......
Summation	+0.14	−0.10	−0.08	+0.16

11. *Compute the adjusted elevations.* Apply the values of the circuit corrections (in Table VI) to the observed differences of elevation (columns 8

and 13, Table II) to obtain (*a*) the adjusted differences in elevation between bench marks and (*b*) the adjusted elevations of the bench marks themselves.

TABLE VII

COMPUTATION OF ADJUSTED ELEVATIONS OF BENCH MARKS

Section or Subsection (1)	Observed Difference in Elevation (2)	Correction (3)	Adjusted Difference in Elevation (4)	Initial Point (5)	Adjusted Elevation (6)	Terminal Point (7)	Adjusted Elevation (8)
1. *A* to *B*	+6.46	+0.03	+6.49	*A*	416.15	*B*	422.64
2. *B* to *C*	+1.44	+0.01	+1.45	*B*	422.64	*C*	424.09
3. *C* to *D*	−2.33	−0.04	−2.37	*C*	424.09	*D*	421.72
5. *C* to *E*	+1.10	+0.10	+1.20	*C*	424.09	*E*	425.29
D to *F*	+2.46	−0.02	+2.44	*D*	421.72	*F*	424.16
E to *G*	+3.36	+0.01	+3.37	*E*	425.29	*G*	428.66
E to *H*	−7.08	0.00	−7.08	*E*	425.29	*H*	418.21
A to *K*	−6.12	−0.07	−6.19	*A*	416.15	*K*	409.96

The distance *D* to *F* (Fig. 171(*d*),2) is three-eighths the distance *D* to *B* (section 4); hence the correction is in the same proportion or three-eighths of the correction for section 4 in Table VI, i.e., $\frac{3}{8} \times (-0.05) = -0.02$. Corrections to differences in elevations of bench marks on other subsections are computed in a similar manner. Thus the correction for *E* to *G* is $\frac{1}{2}\ (+0.02) = +0.01$; the correction for *E* to *H* is $\frac{1}{2} \times (0.0) = 0.0$; the correction for *A* to *K* is about $\frac{3}{4}\ (-0.10) = -0.07$.

Note. Another method of adjusting a level net, which gives results identical with those obtained by the method of least squares, is explained in a paper, "An Improved Method for Adjusting Level and Traverse Surveys," by Clarence Norris and Julius L. Speert in *Transactions of the American Society of Civil Engineers,* Volume 101, 1940.

The adjustment of a level net by the method of least squares is explained in the *Manual of First-Order Leveling,* U.S. Coast and Geodetic Survey, Special Publication 140, and also in the manual, *Technical Procedure for City Surveys,* published by the American Society of Civil Engineers.

For the application of the Dell method, just explained, to the adjustment of a traverse net, see Art. 572.

CHAPTER 13

LEVELING

SPECIAL METHODS

THIS chapter deals with special methods and applications of spirit leveling such as double-rodded leveling, profile leveling, preliminary leveling for grading, and reciprocal leveling; also with trigonometric leveling, and barometric leveling.

172. Use of Automatic or Self-Computing Leveling Rods. (*a*). AN AUTOMATIC TAPE ROD is a special form of self-reading rod which may be set in such a manner that elevations of points may be read directly without any arithmetical work such as subtracting the foresight from the height of the instrument. It consists of an endless or continuous metal ribbon, so graduated that the numbers increase downward from the top of the rod instead of upward from the bottom as in an ordinary leveling rod. The ribbon, passing over rollers at the top and bottom of the rod, may be moved up or down and set in any position.

When the rod is held on the starting point or first bench mark, the ribbon is adjusted until the reading taken from the level corresponds to the elevation of the starting point. If now the rod is held on any other point, the line of sight will strike the tape at a graduation mark that corresponds to the elevation of that point, provided the tape has not been disturbed between the backsight and foresight. For example, assume that the elevation of the first bench mark is 594.83; the ribbon is moved up or down until the line of sight is exactly at the 4.83 mark. Assume now that the reading at the first turning point is 3.29; the elevation at that point is, therefore, 593.29, the figures 59 having been prefixed to the rod reading. At the second setup the rod is held on the first turning point for a backsight, and the ribbon moved until the reading is 3.29. In this way, the work may be carried on from turning point to turning point as far as may be necessary. All elevations on turning points, and on intermediate stations, if there are any, are read directly, and the columns for backsight, foresight, and height of instrument are entirely eliminated from the level notes.

172(*b*). AN AUTOMATIC LEVEL ROD WITH REMOVABLE FOOT NUMBERS is one in which the foot numbers may be snapped on or off to change them from one foot mark of the rod to another. One form, when extended, has a 10-ft. work-

ing face with graduations similar to those of a Philadelphia rod except that the tenths for each foot are numbered downward instead of upward. The rod may be used either as a self-reading rod or as a target rod.

The method of using the rod is similar to that just described for the tape rod except that at each backsight foot numbers are reset. For example, assume that the elevation of the first bench mark is 594.83. When the rod is held for a backsight on the bench mark, it is raised slightly on the short sliding base until the line of sight strikes at a 0.83 mark, where the rod is firmly clamped. At the first foot mark above this reading, snap on the figure 4. Above that is figure 3 and so on for all foot marks to the top of the rod. At the foot marks below the reading, snap on the figures 5, 6, 7, and so on to the bottom of the rod. All elevations from foresights may then be read directly, including the elevation of the first turning point. At the second setup the process of starting away from the first turning point is the same as starting from the bench mark.

172(*c*). Use of the inverted leveling rod. An ordinary leveling rod, such as a two-piece Philadelphia rod, may be used within limits as an automatic rod by holding it with the top down and bottom up. There is little advantage in this unless there are a considerable number of elevations to be ascertained from a single setup.

Let it be required, for example, to determine the elevations of a large number of points in a small plot of land. The procedure is as follows. Establish a bench mark, preferably at the point of lowest elevation. For a backsight on this bench mark, hold the rod inverted, and as a rule fully extended. Assume the elevation of the bench mark to be the same as the rod reading. To find the elevation of any other point, hold the inverted rod on that point and take a direct reading.

Illustration. Suppose that the reading on an inverted rod, held on a bench mark, is 2.14; this is assumed as the elevation of the bench mark. Suppose that when the rod is moved to another point, the reading is 3.65; this is the elevation of that point with respect to the datum assumed for the bench mark.

When it is desired to ascertain elevations with respect to a bench mark already established, the backsight on the benchmark is taken as follows. Hold the inverted rod on the bench mark and slide the movable portion of the rod, up or down, until the reading from the instrument is the same as the elevation of the bench mark. For example, if that elevation is 302.15, slide the movable portion of the rod until the reading from the elevation is 2.15 and clamp it in this position. If the elevation of the bench mark is such that the reading does not fall upon the movable portion of the rod, or if it strikes too near the top or the bottom of the movable portion, set the rod at any whole number of feet greater than the elevation of the bench mark that will make the reading on the movable portion of the rod one that is convenient for use. Then subtract this whole number of feet from all foresight readings. For example, assume that the elevation of the bench mark is 308.55 and that it is desired to have the line of sight strike near the 3-ft. mark on the movable portion of the rod. Add 5 ft. to 208.55, making it 313.55, and set the rod so that the reading is 3.55. If at some other point the foresight reading is, for example, 5.32, the elevation of that point will then be 315.32 − 5 = 310.32.

If a second setup is necessary, the method of procedure for starting from the turning point is the same as that for starting from the first bench mark. Thus the line of levels can be extended indefinitely. It is obvious, however, that when, because of large variations in surface elevations, a considerable number of setups is necessary, the numerous settings of the rod required and the relatively small range on the rod within which sights may be taken make it doubtful if any advantage will be gained by the use of the inverted rod.

173. Leveling with Two Levels. By the use of two levels, speed in leveling may be increased without any sacrifice in precision. Precision, indeed, is often greater. Three students with two levels and one rod can run a line of levels in about one-half the time required by the usual party of two working with only one level. The method of procedure is as follows. (1) Levelman *A* sets up his level where he can backsight on the starting point and foresight

on the first turning point, T.P. 1. (2) By the time these two sights have been taken, levelman *B* has set up his level, has focused the telescope, and is ready to backsight on T.P. 1. (3) The rodman, who has just held the rod on T.P. 1 for a foresight by levelman *A*, now holds it for a backsight on the same turning point by levelman *B*. He then goes ahead to establish T.P. 2. (4) By the time levelman *B* has completed the foresight on T.P. 2, levelman *A* is ready to take a backsight on that turning point. Thus the levelmen alternate in setups throughout the line of levels.

In some cases, both speed and precision may be increased by using two rods as well as two levels.

174. Double-Rodded Leveling or Double-Line Leveling is a method of leveling in which *two* independent lines of turning points are used, that is, at each setup there are backsights on two turning points and foresights on two others. By beginning with a common backsight on the starting point and ending with a common foresight on the final point, two independent lines of levels between any two points may be run by this method with comparatively little extra work. The heights of instrument obtained from one line may be used, *as the work progresses*, to check those obtained from the other. This is particularly advantageous when there are no near-by permanent bench marks or other convenient means of checking the leveling.

Two corresponding turning points may be near together, but, in order to avoid making the same mistake of reading the wrong foot mark first for one point and then for the other, it is best to have one point a foot or more higher than the other. There is some advantage, particularly in recording notes, in considering all the higher points as one line of levels and all the lower points as another. When there are two rodmen, one holds his rod on all points of the "high line" and the other on all points of the "low line." Speed may be increased somewhat if two rodmen are employed, but nearly as satisfactory work can be done with one.

If there were no errors in leveling, the height of instrument at any setup would be the same whether determined from the low line or the high line of levels. Normally, however, there will be a small discrepancy between the height of instrument as determined from the low line and as determined from the high line; consequently there will be *two* heights of instrument for each setup. Let D represent the discrepancy between the two heights of instrument at any setup and D' the discrepancy between the two heights of instrument at the next setup. The variation ($D - D'$ or $D' - D$), between the two discrepancies should be nearly the same from setup to setup, and

in accurate leveling it should not exceed 0.003 ft. If, at any setup, this variation indicates too great an error, backsights on the two turning points are repeated. If no material error is found, it will then become necessary to start from the two turning points used for backsights at the preceding setup and to repeat the leveling from those points. This suggests that a record should be kept of the location and character of turning points in order that they may be found and identified easily. When metal turning points are used for the backsights from any setup, they should not be disturbed until the work has been checked at the next setup.

It is important to remember that the discrepancy between the two heights of instrument at any setup is the *cumulative* discrepancy up to that point and that, therefore, it is likely to increase as the length of line increases. For example, if at 1000 ft. from the starting point it is 0.005 ft., it may be twice as great at 2000 ft.

When a bench mark is established from any setup, and the discrepancy between the two heights of instrument at that setup is great enough to warrant an adjustment, the elevation of the bench mark is determined by subtracting the foresight reading from the *mean* of the two heights of instrument. The bench mark may then be used as a turning point for both lines of levels. The mean height of instrument is also used in determining any other required elevation, such as that needed for a check on an established bench mark. When practicable, both lines of levels should start from a common *permanent* bench mark and close on another.

Double-rodded leveling does not eliminate sources of constant error and, therefore, does not necessarily result in increased precision. It does, however, reveal most of the common mistakes in observing and recording.

174(*a*). THE FIELD NOTES may be recorded in any one of the forms used for differential leveling, modified to permit double the usual number of entries at each setup. In order that there may be no confusion between the two lines of levels, some systematic procedure should be followed, both in the field work and in recording notes. One method is to run a low line and a high line of levels, as already suggested, and then always to take and to record the reading on a low-line turning point before taking and recording one on the corresponding high-line turning point, or vice versa.

174(*b*). IN THE SPECIAL FORM OF NOTES on page 164, the values for the low line are in one column and the values for the high line are in another. By this arrangement the two lines are less likely to be confused, not only in recording observations, but also in computing heights of instrument and elevations. Starting from a common bench mark the first height of instrument is, of course, the same for both lines. For the second setup, however, there is a difference of 0.002 ft.; for the third, of 0.005 ft.; and for the fourth, of 0.006 ft.

SPECIAL FORM FOR DOUBLE-RODDED LEVELING

Sta.	*Low Line*		*High Line*	
B.M. 1	59.253		59.253	
	+10.250		+10.250	
H.I. 1	69.503		69.503	
	− 6.505	+3.745	− 6.231	+4.019
T.P.	62.998		63.272	
	+6.872		+ 6.600	
H.I. 2	69.870		69.872	
	− 3.054	+3.818	− 2.348	+4.252
T.P.	66.816		67.524	
	+ 5.320		+ 4.607	
H.I. 3	72.136		72.131	
	− 4.213	+1.107	− 3.619	+0.988
T.P.	67.923		68.512	
	+ 6.412		+ 5.817	
H.I. 4	74.335		74.329	
	− 3.461	+2.951	− 3.461	+2.356
B.M. 2	70.874	+11.621	70.868	+11.615
		59.253		59.253
	Check	70.874		70.868
B.M. 2	Mean elevation	70.871.		

Notice that the increase between two successive differences does not exceed 0.003 ft.

In the third and fifth columns are entered the values found by adding algebraically the backsight and the foresight for each height of instrument. These values are used to check the difference in elevation between B.M. 1 and B.M. 2 as indicated. The difference between the elevation of B.M. 2 as obtained from the low line and as obtained from the high line is only 0.006 ft. Often this discrepancy would be of no consequence and either value could be taken as sufficiently accurate, but in more precise work the mean value of 70.871 ft. would be used.

A COMMON FORM FOR DOUBLE-RODDED LEVELING

Sta.	+	*H.I.*	−	*Elev.*
B.M. 1	10.250	69.503		59.253
B.M. 1	10.250	69.503		59.253
T.P. 1 L.	6.872	69.870	6.505	62.998
T.P. 1 H.	6.600	69.872	6.231	63.272
T.P. 2 L.	5.320	72.136	3.054	66.816
T.P. 2 H.	4.607	72.131	2.348	67.524
T.P. 3 L.	6.412	74.335	4.213	67.923
T.P. 3 H.	5.817	74.329	3.619	68.512
B.M. 2 L.			3.461	70.874
B.M. 2 H.			3.461	70.868
	+56.128		32.892	
	32.892			Mean = 70.871

23.236 ÷ 2 = 11.618
59.253

Check 70.871

174(*c*). THE MORE COMMON FORM OF NOTES for double-rodded lines is somewhat more compact than the special form but also more confusing, particularly to beginners and sometimes even to more experienced surveyors.

In checking the computations, the difference between the sum of all the backsights and the sum of all the foresights is divided by 2 in order to obtain the mean difference in elevation between B.M. 1 and B.M. 2. Likewise, the elevation of B.M. 2 is taken as the mean of the two elevations obtained from the two lines of leveling.

175. Profile Leveling. The purpose of profile leveling is to determine the elevations of points at short measured intervals along a fixed line, such as the center line of a highway, or of a railway, or of a sewer. This fixed line may be a single straight line, or it may be composed of a succession of straight lines (broken line), or of a series of straight lines connected by curves. Sometimes profiles are required along two or more parallel or nearly parallel fixed lines, such as the center line and the two curb lines of a street.

175(*a*). INTERVALS. The length of the interval between two successive points will depend partly upon the degree of precision required and partly upon the changes in slope; the greater the precision, the shorter the intervals; the more uniform the surface, the longer the intervals. A common interval is 100 ft., or 50 ft., or 25 ft. Whatever interval is adopted in any given case, it may be advisable to make it shorter where the slope of the surface is changing rapidly, or longer where the grade is uniform. Elevations should be taken at the top and bottom of slopes, or at any abrupt change in the surface of the ground, and at all other critical or significant points of the profile, regardless of where these points may happen to be with respect to the adopted interval.

175(*b*). GENERAL PROCEDURE. The general method of field work is the same as that already described in Art. 143(*b*) for obtaining elevations of intermediate stations; in fact, points on the profile line *are* merely intermediate stations. It is usually necessary to take a considerable number of intermediate foresights from each setup, and these should be taken *after* the foresight on the next turning point. (Why?) It is generally best to set up the level to one side of the profile line to avoid too short sights on stations near the instrument. Unnecessary work may be saved by setting up where foresights to points on the profile line may be taken *backward* as well as forward. When practicable, levels should be started from a bench mark of known elevation. It is generally advisable to establish new bench marks at suitable intervals, a thousand feet more or less, as the leveling progresses. The customary method of *numbering stations* is explained in Art. 152.

175(*c*). THE DEGREE OF PRECISION required in profile leveling is usually considerably greater for turning points and bench marks

than for points on the profile, except when a profile point is used as a turning point. For ground profiles, readings on profile points are generally taken to the nearest tenth of a foot and on turning points to the nearest hundredth, with no special effort to balance backsight and foresight distances. For a smooth surface, such as the top of a railroad rail, it may be desirable to take readings on intermediate stations to the nearest hundredth of a foot and on turning points to the nearest thousandth, in which case backsights and foresights on turning points should be balanced.

175(*d*). CHECKING FIELD WORK IN PROFILE LEVELING. Unless it is practicable to start from one established bench mark and close on another, there is no easy method of checking the field work. Even a check on a bench mark does not reveal errors in the elevations of profile points. Such a check is valuable, however, because it shows the magnitude of the *cumulative* error in leveling between bench marks. When there are no permanent bench marks near the profile line, it is sometimes practicable and desirable to run short lines of differential levels to permanent bench marks some distance away. This is not only a partial check on the field work, but it also ties in the profile leveling with the system used in establishing the permanent bench marks. The only method of checking the elevations of intermediate stations is to rerun the profile levels over the whole line. Since an error here and there in the elevations of intermediate stations is generally of little consequence, a check on turning points only will usually answer every purpose.

175(*e*). FIELD NOTES for profile leveling are commonly kept in the standard form shown in Form B, Art. 133(*b*), or in some slight modification of that form as, for example, Form C, Art. 133(*c*). One modification is to record the *foresights* on bench marks and turning points in a separate column from those on intermediate stations. This is an advantage in checking computations by the method of Art. 134. Another modification is to record the *elevations* of bench marks and turning points in a separate column from those of intermediate stations. This makes it easy to distinguish the profile points from those not on the profile line. Sometimes both these modifications are adopted. All bench marks and intermediate stations should be adequately described and, in some cases, turning points as well. Examples of field notes for profile leveling are given in Forms E and F in Art. 153(*d*).

176. Preliminary Cross Leveling in Route Surveying. When the preliminary center line of such a project as a highway, railroad, canal, or ditch has been established and profile levels have been run along a portion or the whole of that line, preliminary

cross levels are usually taken to determine the shape of the surface of the ground for a relatively short distance on each side of the center line. From these levels, elevations are obtained which may be used in plotting a topographic map of a relatively narrow strip of land along which the route will be finally located.

As explained in the preceding article, elevations on the profile of the center line were determined at regular intervals or stations and at additional points on the profile wherever there was any abrupt change in the surface of the ground. Cross levels are usually taken at each of these regular stations and at the additional intermediate points. The cross levels, at any point, are merely profile levels on a line through that point perpendicular to the center line and extending a relatively short distance on either side of that line. Starting from the elevation of the point on the center line, already determined from the profile leveling, elevations may be determined at intervals along the cross line in such a way as to show the profile of the surface of the ground along that line, including all changes of any magnitude in that surface.

When a cross line is short, its direction may be judged by eye; when it is long, the right angle may be established by a more accurate method. Elevations along the line are usually determined either by means of an engineer's level and level rod, or by the use of a hand level and topographer's rod, depending upon the degree of precision required, and sometimes upon the length of the cross line.

The preliminary center line of a route usually consists of a series of straight lines. At the angle of intersection of two such lines, the cross line perpendicular to one line extends on either side of that line, but the cross line perpendicular to the other line need extend only into the larger angle of intersection, i.e., the angle greater than 180°.

From preliminary cross levels, contours of the surface of the ground may be determined as explained in more detail in Art. 467(*e*). A topographic map which shows these contours may be used in determining the location of the final route.

For cross-section leveling in route surveying, see Art. 195.

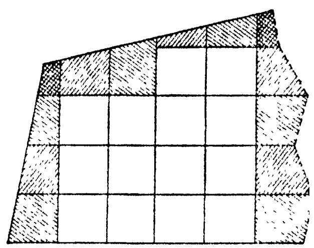
177. Preliminary Leveling for Grading a Plot of Land. The purpose of the leveling is to obtain surface elevations which may be used in determining where the graded surface will lie. These surface elevations in many cases must also be considered in deciding upon the character of the graded surface, i.e., whether it shall be one plane surface or a combination of two or more plane surfaces, or a curved surface, or partly plane and partly curved. After

the location and character of the graded surface have been determined, the original surface elevations may be used in estimating the cut and fill that will be necessary in grading. (See Art. 597.)

177(*a*). General method. When the plot of land is rectangular, divide it into either squares or rectangles and determine the elevations of the corners of these squares or rectangles. When the boundaries of the plot are irregular, divide as large a portion as practicable into squares or rectangles and extend the lines forming the squares or rectangles until they intersect the boundaries. Determine the elevations of these points of intersection as well as the elevations of the corners of the squares or rectangles.

177(*b*). Illustration. In Fig. 177(*b*), the unshaded portion has been divided into squares, and the division lines have been extended to intersect the boundaries. The elevation of the points of intersection and the corners formed by the boundary lines are determined as well as the elevations of the corners of the fourteen squares.

For large areas of land, it may be necessary to modify slightly the method just explained. The area may be divided into large squares or rectangles with each side several hundred feet in length, and each of these unit squares or rectangles may be subdivided into small squares or rectangles. In rough country, where only approximate results are required, the elevations of the corners of the large unit squares may be determined with an engineer's level and leveling rod, and these corners may then be used as points of vertical control. The elevations of the smaller squares may then be determined with a hand level and a topographer's rod, checking on points of vertical control at the corners of the large squares. In many cases, if the ground is quite uneven, it may be necessary to take elevations of intermediate points along the sides of the smaller squares wherever there is a decided change in the surface of the ground.


Fig. 177(*b*).

The general method of preliminary leveling by subdividing an area into squares or rectangles is often used in determining contours. (See Art. 453(*d*).)

177(*c*). Practical suggestions.

1. Make the squares as large as possible, and yet have the surface of the ground in any one square approximately a plane. The more uniform the surface the larger the squares; very rough ground, small squares.

2. A transit and a tape may be used for establishing the corners of the squares, particularly those on the boundary line or perimeter of an area. In small plots of land, it is often accurate enough to line in other points by eye.

For example, if, in Fig. 177(*b*), points are established on the perimeter of the unshaded area, the corners of the squares within that area and the points of intersection on the boundary lines may be lined in by eye. Distances may be measured with a tape, or by the stadia method, or sometimes by pacing.

3. Take extra rod readings to cover hollows and mounds within a square, which otherwise would cause appreciable errors in results. The location of such points should be carefully indicated on the sketch in the field notes.

4. Chain pins can be used to mark corners and intermediate points on the ground. In many cases, however, stakes are better, for the number or letter corresponding to the corner may then be marked on the stake.

5. Use some simple but adequate system of designating points where elevations are taken. For a small area where elevations are taken only at the corners of the squares or rectangles, corners may be numbered 1, 2, 3, etc., in the order in which elevations are taken. For large areas or when there is a considerable number of intermediate points, some such system may be used as that shown on p. 488 in which the division lines running in one direction are lettered *A, B, C,* etc., and those at right angles are numbered 0, 1, 2, 3, etc.

6. In leveling over large areas with a hand level, proceed systematically from point to point by a route which will require a minimum amount of work, and at the same time provide frequent checks on points of vertical control established with the engineer's level.

7. When the plot of land is relatively small, an automatic leveling rod or an inverted rod may be used to advantage. (Art. 172.)

8. Sketches in the field notes should show not only the subdivision of the area into squares and rectangles with corners properly designated, but also the location of intermediate points if there are any. An intermediate point on a division line is often designated as a plus station. For example, a point between corners 2 and 3, 18 ft. from 2, may be designated on a sketch as 2 + 18 or simply as *p* + 18. If the sketch is sufficiently large, the corresponding elevation of the points may also be entered. Frequently, the sketches are prepared before going into the field.

178. Leveling for a Borrow Pit. In order to determine the amount of material removed from a borrow pit, it is necessary to take levels before and after excavation. This may be done by the method just described or by a slight modification of it.

Instead of dividing the surface area into squares, it is better in most cases to proceed as follows. (1) Lay off a reference line along one side of the area of the borrow pit and far enough away so that the line will not be disturbed. (2) Set stakes along this line at regular intervals, the length of the interval depending upon whether the surface of the ground is even or uneven. (3) At each stake establish a line perpendicular to the reference line by setting a reference stake on the opposite side of the borrow pit and at any convenient distance back from the edge. (4) On each of these cross lines run profile levels in such a way that all irregularities of the surface will be measured. Only elevations of points within the area of the borrow pit need be taken. If there is any uncertainty as to the boundary of this area, however, enough elevations should be determined to make sure that the final area is covered. (5) After the material has been excavated, run profile levels over the same cross lines as before, taking elevations within the area of excavation.

From the cross sections determined by the preliminary levels and the final levels, the volume of material excavated can be calculated. (See Chapter 42.)

179. Reciprocal Leveling. When it is required to determine the difference in elevation between two points so situated that no place for the level can be found from which the lengths of foresight and backsight will be even approximately equal, certain errors which may occur when sights are unequal can be practically eliminated by *reciprocal* leveling.

Let A and B be points on opposite sides of a river, or of a deep ravine, or of some other obstacle. Set up at some convenient point C, and from rod readings on A and B, calculate the difference in elevation between these two points. Set up at some convenient point D, and from rod readings on A and B, calculate a second difference in elevation. The mean of these two results should be very nearly the true difference in elevation between A and B.

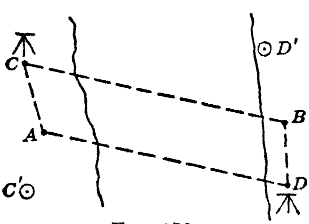


Fig. 179.

The distance from C to A and from B to D should be great enough to permit accurate rod readings. The level might have been set up at some such point as C' instead of at C, or at D' instead of at D. In order that as little time as possible may elapse between backsight and foresight, it is well to use one rod at A and another at B. When sights are long, targets are used, frequently of special design. The lengths of the longer sights CB and DA should be approximately equal in order to eliminate errors of the instrument and the effect of curvature and refraction. When a rather high degree of precision is desired, it is best to repeat each of the longer sights, CB and DA, several times and thus obtain a mean value for the reading at A and for the reading at B. When it seems desirable, the shorter sights, CA and DB, may be repeated for a similar purpose. In precise leveling the longer sights may be repeated as many as ten times, and the shorter, five times. When such a large number of sights is to be taken, the rods may be held in position by guy wires.

When the distance between A and B is large and a high degree of precision is desired, it is necessary to eliminate, as far as possible, errors due to changes in atmospheric refraction and to the effect of the sun shining on the level. For this reason, it is best to do the leveling on a cloudy day or to repeat the leveling on different days with different atmospheric conditions. The next best

precaution is to set up in a spot where the line of sight will be well above the ground or water and to protect the level from the sun's rays. In any event, the time that elapses between the observations taken at *C* and those taken at *D* should be reduced to a minimum. To do this, it may be desirable to use two levels, one at *C* and the other at *D*, thus making it possible to take nearly simultaneous readings from *C* and *D*. This, however, does not eliminate errors due to imperfect adjustment of either instrument. As an additional precaution, therefore, the level at *C* may be moved to *D*, the level at *D* moved to *C*, and a second set of simultaneous readings taken. From the two sets of observations, four values may be obtained for the difference in elevation between *A* and *B*. Unless there is obviously too great an error in one or more of these values, the mean of the four is taken as the final difference in elevation. To obtain the best results, the two levels should have telescopes of nearly the same magnifying power and bubble tubes of about the same sensitiveness.

179(*a*). PRECISE RECIPROCAL LEVELING. The following method of procedure for reciprocal leveling of the first order is taken from *Technical Procedure for City Surveys*, published by the American Society of Civil Engineers. (See also Chapter 15 for a discussion of first-order leveling.)

In first-order leveling it is frequently necessary to carry the line of levels across a river or a bay which is not spanned by any structure stable enough to support the instruments or rods. Under such circumstances it is necessary to adopt a special procedure of an accuracy comparable to that of regular leveling over the ground. The most common procedure which is most convenient both in the field and in the office is called the movable-target method.*

Two permanent bench marks or turning points are established on opposite sides of the river, at approximately the same elevation, as high above the water as possible without unduly increasing the length of sight. One of the rods, provided with a special target that may be bisected and can be clamped at any position upon the rod, is guyed with wires in a vertical position over the distant bench mark. The level is set up about 20 ft. from the near bench mark, and a backsight is taken on the near rod in the usual manner. Next, the telescope is focused on the distant rod, with the level bubble centered, and the target is made to coincide with the middle wire, by means of prearranged signals. Upon the completion of this operation the rodman records the target reading, signals the reading to the levelman, and loosens the target, while the levelman throws the bubble out of center. After a lapse of 5 to 10 min., in order to allow a change in atmospheric conditions, the operation is repeated again and again until a set of ten observations is secured. At no time during these distant rod observations may the focus of the telescope be changed.

The level is then moved carefully to the opposite side of the river, where it is set up about 20 ft. from the other bench mark, and a set of ten reciprocal observations is taken on the distant rod. The level should be transported as rapidly as possible, so that atmospheric conditions during the two sets of observations may be the same, or similar, and in such a manner that the focus of the telescope and the adjustment of the level bubble are not changed. To ascertain whether or not the focus of the telescope is the same for both sets of distant observations, the distant target upon the latter setup is read before the near rod is observed.

* For an alternative method, see *Special Publication 140*, U.S. Coast and Geodetic Survey.

The mean difference between two such sets will be free from the effects of level error, curvature, and refraction if the atmospheric conditions are the same for the two sets. In cloudy, moderately calm weather, the mean result is less likely to be affected by refraction varying with atmospheric conditions. The effect of refraction may also be minimized by making simultaneous target observations with two similar instruments and by having the observers, with their levels, change stations in the middle of the observations.

The mean value of the ten target observations is computed before the instrument is moved, so that doubtful observations may be rejected and replaced with new readings. The usual rule for doubtful observations is to reject all those having residuals greater than the probable error of a single observation, multiplied by five. (See Arts. 42 to 47 for probable error of a single observation and probable error of the mean of several observations.)

The difference in elevation for each of the pairs of setups is then computed and corrected for index error, absolute length of rod, temperature, curvature, and refraction, and for collimation error K (Art. 199(c),8). If the two corrected differences in elevation are in reasonable agreement, a mean value is computed, and the operation is complete.

Since the level error K and refraction are the principal sources of discrepancy in this operation, greater accuracy may be secured by repeating the entire procedure on another day, under atmospheric conditions as different as possible.

180. Trigonometric Leveling. Leveling by means of vertical angles and horizontal distances. The theory is based on the trigonometric properties of a right triangle. A transit is used for this work.

180(a). THE TANGENT METHOD. In Fig. 180(a) let T represent a point in a transit telescope at the intersection of three axes, namely, (1) through the line of sight, (2) through the center of the supporting axis, and (3) through the plumb-bob line. Let it be required to find the difference in elevation between points A and T. Measure the horizontal distance TB and the vertical angle ATB; find the distance BA by trigonometry from $BA = TB \times \tan ATB$. In like manner the difference in elevation between points T and C may be found from $CD = TD \times \tan DTC$. Thus the elevations of any number of points below or above T may be found.

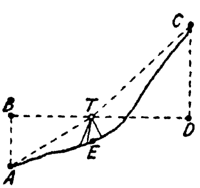


FIG. 180(a).

The difference in elevation often required is that between the point E on the ground vertically below T and some other point such as C. This difference may be obtained, of course, by measuring the distance TE and adding it to CD. Another method is this: Hold the sight pole vertically at E and mark on it the height ET to the center of the supporting axis of the telescope. (One method of marking is to wrap paper around the pole and use a rubber band for the mark. A level rod with the target set for the height ET is still better.) In measuring the vertical angle, hold the pole (or rod) vertically at C and sight on the mark on the pole (or the target on the rod). The line of sight will then be parallel to a line

from *E* to *C*. The calculated vertical distance will then be from the point *D* to a point on the pole (or the target on the rod) above *C* and not the distance *DC*. This calculated distance will be the required difference in elevation between *D* and *C*.

Illustration. A transit stands at station *E*. A sight pole is held so that the height above the ground of the supporting axis of the telescope may be marked upon it. The cross hairs are then brought on this mark when the pole is held at a second point *A*. The vertical angle as read at the transit is 5° 20′. The horizontal distance from *E* to *A*, equal to *TB*, is known to be 421.6 ft. Tangent of 5° 20′ is 0.09335. Then 421.6 × 0.09335 = 39.36 ft., the difference in elevation of the *ground* at *A* and at *E*.

The accuracy of the method may be judged by the following results. If the vernier on the vertical arc reads to minutes the error in reading the vertical angle should not exceed 1′. If, in the above illustration, the heights are computed for three readings, 5° 19′, 5° 20′, and 5° 21′, the three results are respectively 39.234, 39.356, and 39.483, or a difference for one minute of 0.12 for 421.6 ft. If the vertical angle had been, say, 20° larger, the error for one minute would still be approximately the same. Thus, for three readings, 25° 19′, 25° 20′, and 25° 21′, the three results would be respectively 199.438, 199.590, and 199.741, or a difference of about 0.15 ft. in 421.6 ft. It is thus seen that the error which arises from reading the vertical arc is comparatively small. By the use of the gradienter this error can be made much less (Art. 181). *It must be remembered, however, that there are other sources of error.* The level bulb on the telescope may not be sensitive, and it may also be out of adjustment, i.e., the line of sight may not be horizontal when the reading on the vertical arc is zero. There may be an error in the horizontal distance. Thus, if in the illustration the distance had been called 420.6 instead of 421.6, the error in height for 5° 20′ would be 0.09 ft., and for 25° 20′ it would have been 0.47 ft. It is seen, however, that the method is sufficiently accurate for most purposes, and can be used a great many times in place of a level and leveling rod. The method is also used for determining the height of an inaccessible point (Art. 382(*d*)).

Practical Suggestions.

1. The horizontal distance *TB* or *TD* may be measured (*a*) with the tape, (*b*) with the stadia, or (*c*) by triangulation. When for any reason the *horizontal* distance from *T* to *C*, i.e., *TD*, cannot be measured, *TD* may be found by triangulation, treating *C* as an inaccessible point (Art. 382(*a*)), or in some cases it may be possible to measure the hypotenuse *TC* and then proceed as explained in the "Sine Method."

2. If the elevations of many points are to be reduced from vertical angles, the use of a reduction table or a special form of slide rule will save much computation.

3. If the elevations of points are required with reference to some datum, first find the elevation of *T*, i.e., the height of instrument, by backsighting on some point of known elevation, as, for example, some bench mark.

180(*b*). THE SINE METHOD. *A second method* often used, especially in stadia work, is one in which the hypotenuse is measured instead of the base.

Illustrations. Let *T* in Fig. 180(*b*) represent the center of a transit, and *a* a point on the ground underneath *T*. Let *a′* represent a target set as far above the bottom of the rod as *T* is above *a*, i.e., $a'G = aT$. Then $Ta' \times \sin a'TD = a'D = Ge$, or the elevation of *G* above *a*.

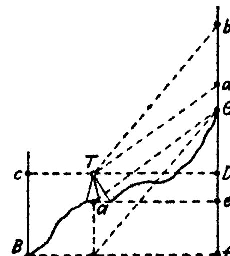


FIG. 180(*b*).

Let B represent some point of known elevation *not* under the instrument, as, for example, some bench mark. With the telescope horizontal, set the target at c. Let b be an imaginary point underneath T, of the same elevation as B, i.e., $Tb = cB$. Without changing the target hold the rod at G and let b' represent the target for this position of the rod. Then $b'G = Tb = cB$ and $Tb' \times \sin b'TD = b'D = Gf$, or the elevation of G above B.

Caution. Since all the methods illustrated are in common use, it is very necessary in keeping field notes to make clear what point is taken as a reference point, T, the center of the supporting axis; a, a point on the ground under the instrument; or B, some point not under the instrument; also whether the base or the hypotenuse is measured.

181. Leveling with the Gradienter. The gradienter is a tangent screw attached to the supporting axis of the telescope. By means of a micrometer head, small vertical angles may be measured, or large angles may be read with the vertical arc supplemented by the micrometer screw. As a rule the value of the screw thread is such that one complete revolution of the screw will move the horizontal hair over a space of 1 ft. on a vertical rod at a distance of 100 ft., so that angles are measured *in terms of the tangent.* If the distance to a rod at any point is known, and the vertical angle is found by the gradienter, the elevation of the point may be determined by trigonometry, as already explained. The distance from the transit to the rod may be found also with the gradienter by turning the screw through one complete revolution. If it passes over 1 ft. of the rod, the latter is 100 ft. away, 1½ ft., 150 ft. away, and so on in the same proportion. This method of obtaining distances is similar to the stadia method. Thus, if necessary, the entire work of finding the elevation of a point may be done with the gradienter alone, without using a tape. This is also true of the stadia. (For the use of the gradienter in establishing grade points, see Art. 188(g).)

182. Theory of Barometric Leveling. Since the barometer is an instrument for measuring atmospheric pressure and since this pressure is less for high altitudes than for low altitudes, differences in elevation may be estimated by differences in barometer readings. Roughly speaking, the average reading of the barometer at sea level is 30 in. and the barometer falls *about* 1 in. for every 900 ft. of ascent above the sea level.

If the decrease in the density of the air were *exactly* uniform and if the pressure at a given elevation were always the same, barometric leveling would give very satisfactory results.

182(a). Atmospheric sources of error. The pressure of the atmosphere at any point is constantly changing because of changes in meteorologic conditions. This change may amount to as much as 2 in. or more. In other words, the elevation of the same place as indicated by the barometer at two different times may change as much as 1800 ft. For the same reason barometer readings taken simultaneously at two different places may be exactly the same and yet the elevations of the two places be quite different. The error from this source may be largely eliminated by taking barometer readings over a long period of time at different seasons of the year and hours of the day, and

thus obtain a mean value. This method, of course, cannot be used in ordinary surveying.

182(*b*). TEMPERATURE also affects barometer readings in two ways: It affects (1) the barometer itself and (2) the weight of the air, for the warmer the air the lighter it is. In a mercurial barometer the height of the column of mercury is affected by the temperature. It is usually claimed that aneroid barometers are compensated for temperature, and though this is not strictly true, it is customary to make no correction for the temperature of the barometer itself, as for that of a mercurial barometer. In using either barometer the greatest uncertainty arises from the method of determining the average temperature of the air column. Thus, in finding the difference in altitude between two stations, the temperature of the air taken at each station is necessarily the temperature of a stratum of air near the earth. The common assumption that the average temperature of the air column, upon which the difference of pressure between the two stations depends, is the average of the observed temperatures at these stations is an erroneous one.

182(*c*). BAROMETRIC FORMULAS. Numerous formulas, differing chiefly in experimental constants, have been developed for establishing a relation between altitude and barometer readings. Various formulas and corresponding tables of corrections are given in *Smithsonian Meteorological Tables*, published by the Smithsonian Institution, Washington, D. C. In ordinary barometric leveling, those terms in the formulas which depend upon the temperature of the barometer, humidity, latitude, gradient, and other still smaller sources of error may be ignored, leaving only the mean temperature of the air to be taken into account. Simple correction coefficients may then be applied as explained in Art. 184(*a*).

183. The Aneroid Barometer. (See Art. 182 for the theory.) The aneroid barometer is less accurate than the mercurial barometer, but it is far more portable and convenient, and therefore it is used almost exclusively in surveying. It consists of a metallic box from which the air has been exhausted. The varying pressure of the atmosphere acting upon a thin corrugated top of the vacuum chamber moves it in or out, and this motion is transmitted by multiplying levers, chains, and springs to a pointer moving around a dial which is graduated so that atmospheric pressure is recorded in *inches* to correspond to the varying heights of a column of mercury. In addition the altitudes in *feet,* corresponding to various pressures, are usually given by a second set of graduations—less accurate than the first because dependent upon barometric formulas for reducing pressure to altitude.

In addition to the atmospheric sources of error discussed in the preceding article, the aneroid barometer is subject to many instrumental errors such as: (1) varying sensitiveness or lack of sensitiveness of the corrugated top; (2) incorrect graduations; (3) failure of chains and levers to respond quickly to movements of the corrugated top. The instrument is not intended for accurate work and is used chiefly for reconnaissance or exploration in hilly country, and for rough topographic surveys where it is desired to sketch in contours quickly without great regard for accuracy.

Aneroid barometers vary in diameter from 2 or 3 in. to 10 or 12 in. Those of the ordinary type, particularly the smaller models, are often not sufficiently sensitive and are slow in action; moreover, they do not stay in adjustment for any great length of time and should be continually tested.

183(*a*). AN ALTIMETER, similar to those used in airplanes, is now available for use in surveying. It is a special form of aneroid barometer, but more sturdy, stays in adjustment better, and is without the usual lag of the aneroid. It is so sensitive that differences of elevation of 2 or 3 ft. can be detected.

184. Leveling with the Barometer. There are three methods: (1) single observations; (2) simultaneous observations; (3) extended series of observations.

184(*a*). THE METHOD OF SINGLE OBSERVATIONS. The barometer is carried from point to point and a single reading taken at each station. The readings thus obtained involve all atmospheric errors, especially those due to the changes in the atmosphere which take place during the interval between observations.

The altitude scale should not be used except for very rough work, and only for the purpose of ascertaining *differences* of altitudes. If the scale is adjustable the barometer may be taken to some bench mark or point of known elevation and set at the altitude of that point; then the elevations of other points may be read directly from the altitude scale. This means, however, that tem-

perature and other sources of error are ignored, and only approximate results can be expected. A better method is to use the scale corresponding to inches and correct for temperature as follows:

1. Take the barometer reading (in inches) at the first station or bench mark, and note the temperature of the atmosphere.

2. Take the barometer reading and note the temperature at the second station.

3. Find from Table XI, p. 1122, the altitude corresponding to each of the barometer readings, and subtract one from the other. The result is the difference in altitude of the two stations *uncorrected for temperature*.

4. Add the two temperatures and from Table XII, p. 1123, find the correction coefficient C for that sum.

5. Multiply the difference obtained in (3) by $(1 + C)$, *giving to C its proper algebraic sign* ($-$ if $t + t' < 100°$ F. and $+$ if $t + t' > 100°$ F.). The resullt is the approximate difference in altitude of the two stations.

184(*b*). SUGGESTIONS. (1) Notice that the erroneous assumption in regard to the average temperature of the air column in Art. 182(*b*) is involved, as are all the other sources of error mentioned in Art. 182. (2) Readings for altitudes should not be taken just before or after or during a storm. (3) The shorter the time between the readings at two different stations the better. (4) At the beginning of a day's work compare the barometer reading at some station with the reading at the same station on the day before, and thus get an approximate correction for the atmospheric changes if any have taken place. It is well to do this at intervals during the day, as the pressure at a given point may change very much in a short time. (5) Hold the barometer in the same position when reading it, always vertical or always horizontal, preferably vertical. (6) The barometer is likely to "catch" or "hang"; tap it gently with the finger before reading it. (7) The barometer "drags," sometimes registering at one moment the elevation where it was twenty minutes before. Hence, in ascending a hill consecutive readings are likely to be too low; in descending, too high. This dragging is a most common source of error.

184(*c*). SIMULTANEOUS OBSERVATIONS, i.e., taking simultaneous readings at different places with barometers which have been previously compared. The aim of this method is to eliminate as far as possible error due to changes in atmospheric conditions during the time which elapses between the observations at the first and second stations. One barometer is kept at the base or starting point, and it is assumed that whatever change is indicated by this barometer, as read at intervals previously agreed upon, should also be allowed for in reading the other barometer read at the same intervals. This assumption may or may not be true, but the method is the most satisfactory one which the engineer can use when stations are widely separated, since he can seldom carry on a long series of observations at any one station to determine the mean reading.

The watches of the base and field operators should be synchronized, and there should be a clear understanding with regard to the intervals for simultaneous readings. The lengths of these intervals will depend upon the rapidity of change in air pressure due to varying atmospheric conditions, and upon the relative degree of precision required. It may be desirable sometimes to take readings as often as every five minutes or even every two minutes. By plotting base readings, a curve may be obtained by means of which field readings may be corrected.

184(*d*). EXTENDED SERIES OF OBSERVATIONS are usually taken with mercurial barometers and cover long periods of time. The mean readings thus obtained often give excellent results, but of course the method cannot be used to any extent in surveying.

184(*e*). LIMITS OF ERROR. Errors may be expected in barometric leveling of 10 to 25 ft. If temperature is ignored, that alone may introduce an error as great as 10 ft. (See correction in Table II for $t + t' = 180°$ F.) By simultaneous observations with two barometers and by taking other similar precautions, errors may often be corrected or reduced to as low as 5 or 6 ft. Under favorable conditions, expert observers may obtain even better results. In the ordinary topographic surveying where bench marks have been established, the best method is to start at one bench mark and close as frequently as possible on the other bench marks. Thus, by comparing with the elevations obtained by spirit leveling, one can get the general run of errors to be expected in the use of the barometer under the given conditions.

There is an impression more or less prevalent that aneroid barometers are of little use in surveying. When, however, only approximate elevations are required, the aneroid barometer may be used to advantage, and, in some cases, when the time for making a survey is limited, barometric leveling is the only method that can be used for determining elevations. Moreover, with the development of the modern altimeter, a fairly high degree of precision may be attained more easily than with the ordinary aneroid barometer.

184(*f*). ILLUSTRATION. The following is an example of the usefulness of barometric leveling. It was proposed to construct a large water-supply main about 15 miles in length. Only two days were available in which to determine a suitable route, locations for standpipes, and other hydraulic features. By means of two altimeters, one of which was kept as the base instrument at a point of known elevation, the necessary preliminary information was determined in one day. Probably there is no other way by which such accurate information could have been obtained in the limited time available.

CHAPTER 14

LEVELING TO ESTABLISH GRADE POINTS

THIS chapter deals with methods of establishing elevations for purposes of construction. It includes methods of establishing grade points for curbs, pavements, sewers, and roads; of setting stakes for grading land areas; and for setting slope stakes. The typical cases of special work selected for this chapter will suggest methods to be followed in any work of a similar character that is likely to arise in the experience of a young engineer. (See also Chapter 57, "Surveys for Construction," particularly Art. 788.)

185. Establishing Elevations for Purposes of Construction. A large part of leveling consists in establishing reference points of known elevations so distributed that all other elevations needed during construction can be readily obtained from these points. Such points are often called *grade points*, and a line through such points, a *grade line*. This kind of leveling, often referred to as "giving elevations," is used constantly in all kinds of engineering construction.

The elevations of grade points to be established and the amount of cut or fill at each point are usually determined before going into the field. For example, assume that the center line of a sewer has been located, profile levels run, and the profile plotted. Assume also that the elevations for the grade line of the sewer have been determined, and that this grade line has been plotted on the profile sheet; that on this sheet have been given the elevations of all important or critical points, such as points on the grade line where the rate of grade changes, together with the cut or fill at each point, and such other data and information as may be needed in establishing elevations for the construction of the sewer. This profile sheet, or a blueprint of it, can then be used in the field in determining where grade points should be established and in calculating grade rods for such points. In like manner profile sheets may be used in the field in establishing grade lines for various other types of engineering construction such as railroads, highways, and city pavements.

In many cases, elevations for grade points and values for cut and fill are worked out in the form of field notes, and these notes are used instead of a profile sheet in establishing elevations. (See notes for Albany Avenue sewer, Fig. 191(*e*).) In other cases the

information needed for establishing grade points is furnished on grade sheets. One form of grade sheet, for example, used when grade stakes have already been driven and the elevations of their tops have been determined, consists of four columns. In the first column are entered the designation of each grade point by number or letter, as 14 + 75 or B; in the second column is entered the vertical distance from the top of the stake to the grade point, as 0.74′ down or 0′–8⅞″ down; in the third column is entered the elevation of the top of the stake as, 543.74′; and in the fourth column, the elevation of the grade point as 543.00′. In many cases the last two columns may be omitted.

For grading plots of land, and for the construction of foundations, masonry structures, buildings, and other engineering structures, the elevations, distances, dimensions, and other data are usually given on maps, plans, and working drawings, instead of on profile sheets.

The general method of establishing any point at a given elevation is explained in Art. 136. The most essential thing to remember is that the point is determined by measuring down from the height of instrument a distance equal to the *grade rod*. This is simply reversing the method of ascertaining the unknown elevation of a point. Since the foresight reading is equal to the grade rod, the operation may be expressed as follows:

$$\text{Grade point} = \text{H.I.} - \text{F.S.} = \text{H.I.} - \text{Grade rod}$$

185(*a*). METHODS OF MARKING OR FIXING GRADE POINTS. In giving elevations for grading, points are generally fixed by means of grade stakes; but where these cannot be driven, it is necessary to use some other device such as spikes or chisel marks. In giving elevations for the construction of foundations, sewers, and various other types of engineering structures, points are usually fixed by means of straight boards or planks, called *batter boards*, supported in a horizontal position, and generally so set that their upper edges are at designated elevations. Grade stakes and batter boards should be set, if possible, where they will not be displaced during construction. In any case, one or more bench marks should be established near by so that if any grade stake or batter board is displaced, it may be re-established easily and quickly. When bench marks have been previously established, they may be used for checking elevations.

185(*b*). PRACTICAL SUGGESTIONS.

1. Frequently reference points of elevation may be established from similar points near by without the use of leveling. For example, if a straight line of stakes has been set 100 ft. apart with their tops at grade, intermediate

stakes can often be set with sufficient accuracy by stretching a cord across the tops of the original stakes. Likewise, if several batter boards have been set, say 50 ft. apart, over a straight trench, intermediate batter boards can often be set by means of a cord, or even by eye.

2. In construction work it is continually necessary to make vertical measurements from points of reference which have been established. For example, in constructing a sewer, batter boards are placed at intervals across the trench and vertical measurements are made downward from the batter boards, or from string stretched between batter boards, by means of a steel tape or other measuring device. In cases where the men on construction work are not accustomed to the decimal system, it may be desirable or even necessary to give vertical distances in feet and inches as, for example, in giving the height of the upper edge of a batter board above the top of a stake driven to grade.

3. When a considerable number of elevations are to be established from a single setup of the level, either an automatic leveling rod or an ordinary rod inverted can frequently be used to advantage. (See Art. 172.)

186. General Methods of Setting Grade Stakes. Grade stakes are usually set along lines that have already been located on the ground, such, for example, as the gutter line of a pavement or the center line of a railroad track. In order that the stakes shall not be displaced during construction, it may be necessary to set them to one side, usually on an offset line parallel to the line on which the grade points are really needed. For example, if grade stakes were set on a curb line of a street, it would be difficult to lay the curb without displacing them. It is better, therefore, to set them on an offset line parallel to and about 2 ft. from the curb line. It is often necessary to set grade stakes on two or more parallel lines, as, for example, on the lines for the two edges of a pavement; or on a considerable number of parallel lines such as those used in grading a plot of land. (See Art. 194.)

If it were always possible to set a stake with its top "at grade," this would be the ideal method of indicating a grade point. In many cases, however, this method cannot be used since the top of the stake would come either too far below the surface of the ground or too far above it. Moreover, it is frequently impracticable to drive the stake far enough into the ground. There are several ways in which stakes may be set.

1. Each stake may be driven so that its top is at grade—the best method when it can be used.

2. Stakes may be driven so that all their tops are the same distance above grade, preferably a whole number of feet, as 1 ft. or 2 ft.—a good method, but not always practicable.

3. Each stake may be driven with no special effort to have its top at grade, and then a horizontal line may be drawn across one face to mark the grade, or some whole number of feet above or below grade. This method may be used when stakes are not easily driven to the desired heights. In some cases, it may be desirable to saw the stakes off at the horizontal line.

4. Each stake may be driven without regard to grade, the elevation of its top afterward ascertained, and the amount of cut and fill, measured from the top, marked on the stake. This method makes necessary many measurements during construction, a different one for each stake; hence mistakes are more likely to occur. A modification of the method is to draw a horizontal line anywhere across one face of the stake, ascertain its elevation, and mark the cut or fill from this line instead of from the top of the stake.

5. Each stake may be driven and the cut and fill marked on it, with the understanding that the cut or fill is to be measured from the existing surface of the ground. This method could be used only for rough grading.

186(*a*). GENERAL METHOD OF PROCEDURE. The required elevation at each point where a stake is to be set must be determined. It is well to do this in the office before going into the field. These elevations may be entered in the level notes in the usual way, or they may be shown on profile sheets, maps, or other drawings. After the instrument is set up and the height of instrument found, the grade rod should be figured for each stake that is to be driven from that particular setup. If method 1 of setting stakes is used, set the target at the grade rod for any stake, and drive the stake until, when the rod is held on its top, the center of the target is covered by the horizontal hair. If method 2 of setting stakes is used, proceed in the same way, except use a "field" grade rod which will bring the top of the stake a convenient distance above ground, and a whole number of feet above or below grade.

186(*b*). ILLUSTRATION. The required elevation of a point is, say, 120 ft. The actual elevation of the ground at that point, given in the notebook or found at the time by taking a foresight, is 128.4 ft. The H.I. has been found to be 133.6 ft. The *true* grade rod is, therefore, 133.6 − 120 = 13.6 ft. As the surface of the ground is 128.4 − 120 = 8.4 ft. above grade, the top of the stake should be about 9 ft. above grade or about 0.6 ft. above the ground. Hence the *field* rod used in setting the stake is 13.6 − 9.0 = 4.6 ft. The stake is marked "Cut 9.0 ft.," and the contractor is told to measure from the tops of stakes.

A similar method of procedure may be used for method 3. In method 4 the cut or fill marked on the stake is simply the difference between the elevation of the top of the stake and the grade elevation at that point; in 5 it is the difference between the elevation of the ground and the grade elevation.

186(*c*). PRACTICAL SUGGESTIONS. (1) First drive a stake *approximately*, being careful to stop before it is too low. Hold the rod and let the levelman indicate how much more it should be driven. (*An alternative method.* Turn the stake upside down and stand it on the ground. Hold the rod alongside, have the levelman sight the target to proper grade, and mark the stake where the bottom of the rod is. Then the portion of the stake from the mark to the point of the stake is the portion to be driven. Axeman can then drive with confidence to within a slight amount of the correct position before checking with the rod for the final setting.) (2) Don't split hairs. Keep in mind how close the contractor can work, and drive stakes accordingly. It is folly, for example, to drive stakes to the nearest hundredths of a foot for ordinary grading; the nearest tenth will often do. On the other hand, stakes for curbs and pavements should be set with greater accuracy. (3) Time can often be

saved by not using a target at all, i.e., by reading the grade rod directly from the instrument. (4) Sometimes, as in the case of sewers, some other device than stakes must be used for giving grade. (5) Stakes should be close enough together to enable the contractor to do his work, but not so close as to be in his way. (6) There are two very common mistakes which occur in setting grade stakes. One is to observe the wrong horizontal cross hair, i.e., a stadia wire, the other is to set the target incorrectly for the grade rod. The only absolute check is to test the stakes by setting up the instrument in a different position, but errors may often be detected by the eye alone. If the tops of three or more stakes should be on the same grade line, and any one of them is enough above or below this line to affect construction, the error will be apparent to the eye in sighting across the tops of the stakes. Hence form the habit of *frequently sighting by eye, to make sure that stakes are properly set.* (7) An automatic grade rod can often be used to advantage in setting grade stakes. (8) A transit is sometimes better than a level for setting grade stakes, particularly for shooting in or for work in which stakes must be lined in as they are set.

186(*d*). THE FIELD NOTES are kept in the usual form, except that when they have been worked up, additional data will appear in the notes, such as a column of grade rod values, grade elevations, height of ground above grade, and height of grade mark above ground. A study of the notes for giving a grade for a sewer (Fig. 191(*c*)) will suggest forms for similar work in setting grade stakes.

187. Marking Grade Stakes. The main object in marking a grade stake is to indicate whether or not the stake is "at grade," and if it is not, to show how far the corresponding grade point is above or below the stake. In any case, the men on the work must know from what point on the stake measurements should be made, i.e., from the top of the stake, or from a horizontal line marked on the face of the stake, or from a nail driven horizontally part way in or from a point marked by some other method.

If the tops of all the stakes are at grade, and this is understood by the men on the work, there is no reason for indicating this fact on the stakes. If, however, some of the tops are at grade and some are not, those that are should be clearly marked with a G or with some other symbol. When a stake is not at grade, the required cut or fill should be marked on one face as, for example, C 10.5′, or F 8.6′. Instead of the letters C and F, some engineers use a minus sign for cut and a plus sign for fill; others use a plus sign for cut and a minus sign for fill. This system is confusing and a source of trouble particularly to the man who does the grading. The ambiguity can be avoided by always using C for cut and F for fill, and this should be the general practice.

An arrow is often used to indicate the point from which a measurement should be made. If the top of a stake is at grade, an arrow is drawn upward to the top and marked G. Some surveyors merely mark an X on the top of the stake to indicate that the top is at grade. If the grade is marked by a horizontal line on the stake, the arrow ends on the horizontal line. An arrow drawn downward from the top of the stake or from a horizontal line indicates that the cut should be measured from the top or from the line; similarly,

an arrow drawn upward to the top of the stake or from a horizontal line indicates the point from which the fill should be measured. It is often necessary to mark on another side of the stake the station number, as for example, 20 + 50 or A 23. (See Fig. 195(*k*).) Stakes should be marked with crayon which will not be washed out by rain or easily obliterated in any other way.

188. To Set Stakes on a Grade between Two Fixed Points. (*a*). THE MOST GENERAL CASE is when the stakes are at unequal intervals, and when the ground is too uneven to permit the use of the "shooting-in" method. The grade will be the hypotenuse of a right-angled triangle, the base of which is the horizontal distance between the two fixed points; the difference in their elevations will be the altitude. By similar triangles the elevation of any point on the grade will then be proportional to the distance of that point from one of the fixed points. The elevations of the points having been calculated, the stakes may be set as explained in Art. 186.

188(*b*). ILLUSTRATION. The elevations of two fixed points *A* and *B*, 360 ft. apart, are respectively 226.42 ft. and 233.12 ft. The finished grade is a uniform slope between the two points. It is required to set stakes 40 ft., 120 ft., and 300 ft. from *A*. The difference in elevation is 6.7 ft. Hence ($^{40}/_{360}$) × 6.7 ft. = 0.74, and 0.74 + 226.42 = 227.16 ft. = elevation at 40 ft. from *A*. In the same way 2.23 + 226.42 = 228.65 = elevation at *A* + 120, and 5.58 + 226.42 = 232 = elevation at *A* + 300. As a check the difference in elevation for the remaining 60 ft. is $\frac{1}{6}$ × 6.7 = 1.12, and 232.0 + 1.12 = 233.12, which agrees with the elevation of *B*.

188(*c*). PRACTICAL SUGGESTIONS. (1) See Art. 186(*c*) for suggestions for setting stakes. (2) When practicable, set stakes at equal intervals so that the difference between successive grade rods will be a constant.

188(*d*). TO SHOOT IN A GRADE BETWEEN TWO FIXED POINTS. This consists in establishing a line of sight *parallel* to the grade, so that one grade rod will answer for all the points. The first step is to establish two fixed points at grade. These are usually stakes, one at each end of the line. The grade elevations of each of these two points with respect to a bench mark or some other point of reference being known, each of the stakes is driven until its top is at grade, as determined by the calculated grade rod. This work may involve a preliminary line of differential levels from a bench mark or some other point of known elevation. The next step is to establish a line of sight at a known distance above grade *parallel to grade*. It is most easily done with a transit, although it may be done almost as easily with a level. If a level is used, it is best to set it up with one pair of leveling screws in line with the two given points, so that these screws can be used for inclining the telescope.

The method of procedure is: (1) Set up near one stake, a foot or so behind it. (2) Hold the rod on the stake and set the target

at the height of the center of the objective lens. (3) Hold the rod on the second stake; incline the telescope until the horizontal hair covers the center of the target. This brings the line of sight parallel to grade, provided the height of the objective lens has not been changed appreciably by inclining the telescope. (4) Hold the rod on the first stake, and set the target over again if the objective lens has been changed. (5) Hold the rod at the second stake, change the line of sight slightly if necessary, and *establish a permanent foresight* at the given distance above the stake. (6) Set the intermediate stakes without changing the target. This cannot be done if the surface of the ground is very irregular; hence this method is suitable only for comparatively uniform slopes.

188(*e*). NOTE. When the transit is used it is often better to set up directly over one of the stakes, and fix the target at the height of the supporting axis, thus avoiding the readjustment in inclining the telescope. For the same reason some surveyors prefer to set the level directly over a stake instead of a foot or so behind it.

188(*f*). PRACTICAL SUGGESTIONS. (1) Make sure that the line of sight is parallel to grade before setting intermediate stakes, no matter how many times the target must be reset—two trials are usually sufficient. (2) One way to establish a permanent foresight is to wrap a strip of paper about a pole, slip a rubber band over it, stick the pole behind the stake, move the rubber band until it coincides with the line of sight. Then any change in the line of sight which may occur during the setting of intermediate stakes may be detected readily and corrected by the levelman. (3) The error involved in setting the target opposite to the center of the objective lens, without looking through the telescope, is very small, often less than that which ordinarily occurs in setting the target at a distance of 300 ft. or 400 ft. (4) If the surface of the ground is too far above or below the required grade to drive the stakes to grade, the method can still be used, provided all the stakes can be driven some constant distance above or below grade. (5) When a grade line is uniform for a long distance so that the two end stakes are too far apart for accurate sighting from a single setup, stakes may be set at grade at intervals of, say, 300 ft. before the work of shooting in the grade is started. (6) See Art. 186(*c*) for other practical suggestions for driving stakes.

188(*g*). TO ESTABLISH A GRADE BY THE USE OF A GRADIENTER. If a transit or a level is equipped with a gradienter (Art. 181), grade stakes may be set by a method similar to that just explained, except that the line of sight is established as follows. (1) Make sure that the gradienter reading is 0 when the line of sight is level. (See "Adjustment of the Gradienter," Art. 747.) (2) Make the line of sight parallel to the grade by turning the micrometer screw of the gradienter until the reading corresponds to the gradient. For example, let it be required to lay off a 1.7 per cent grade. If one complete revolution of the screw moves the telescope until the inclination of the line of sight is 1 ft. per 100 ft., i.e., a 1 per cent grade, and if the circumference of the drum is divided into 100 equal parts, turn the screw one complete revolution plus seventy graduations on the drum. The line of sight is now parallel to the grade; hence it is set for shooting in.

189. To Set Grade Stakes for a Curb or Sidewalk. A transit in good adjustment for leveling is the best instrument for this work. The object is to set a line of stakes from which the contractor can determine not only the line of the edge of the curb toward the roadway, but also the elevation, at any point, of the top of the curb. This line should be parallel to the curb line and far enough away so that the stakes will not be displaced during the setting of the curb. (1) Assuming that the curb line is straight, establish two points known to be on the curb line. If the curb is for a city street, one of these points may be near one end of a block, and the other, near the other end. (2) From each of the two points thus established, set a stake about 2 ft. from the curb line on the sidewalk side, thus fixing the offset line. (3) Set stakes on this line at regular intervals, driving each stake far enough into the ground so that it will stay in place, but not so far that its top will be below grade. (4) Set each stake with its top at grade, using either the grade-rod method or the shooting-in method. The latter is generally the better method when it is practicable to use it. (5) After the stakes have been driven to grade and not before, establish the exact offset line by driving nails into the tops of the stakes.

If the grade is on a vertical curve, there is no fundamental difference in the field work except that the shooting-in method cannot be used.

189(*a*). GRADE ROD METHOD. *Illustration.* A curb 376.1 ft. long is to be set to a 2 per cent grade. The lower end begins at a point of 129.64 ft. elevation. Stakes are set on line every 50 ft. The grade rod will therefore be 1.0 ft. *less* for each succeeding stake as the grade ascends. The 3 + 50 stake is 26.1 ft. from the end, so that $0.02 \times 26.1 = 0.522$ must be subtracted from the grade rod used for that stake to get the grade rod for the end stake; i.e., if the grade rod for the first stake is, say, 8.94, for the 3 + 50 stake it is 1.94, and for the last stake it is 1.42.

189(*b*). PRACTICAL SUGGESTIONS. (1) When practicable, set the instrument midway between the ends, and high enough to control the last or highest stake. (2) If the offset line is not more than 2 ft. from the curb line, a carpenter's level can be used in setting the curb. In some cases, however, it may be necessary to make the offset greater. (3) The usual interval between the stakes is 25 or 50 ft. On a vertical curve, however, it may be necessary to make the interval as small as 10 ft. (4) If the grade line runs slightly below the surface of the ground at any point, a hole may be dug, and the stake driven to grade in the hole. (5) Where it is impracticable to drive stakes to grade, one of the other methods given in Art. 186 may be used. For example, in resetting old curbs or in setting new curbs in an old street, the offset line on the sidewalk may be marked by spikes driven flush in the sidewalk or, if this be impracticable, by light chisel marks on the surface. The curb setter can then be furnished with a memorandum giving the distance that each of these points on the sidewalk is above or below the grade of the curb opposite to it. (6) When a curb line strikes the side of a pole or tree, drive a nail, in line and at grade.

190. To Establish Grades for Pavements. Since curbs are usually set before the pavement is laid, the stakes set for the curbs may be used by the contractor in obtaining elevations for the gutter

line, provided that line is parallel (measured vertically) to the curb line. When it is not parallel, it becomes necessary to give additional grade points for the gutter. This may be done either by setting stakes or by marking, with a chisel, points on the face of the curb where the surface of the pavement will strike the curb. In some cases, it may be desirable or necessary to set a row of stakes on the center line of the street, especially if the street is wide, in order to establish the grade line for the top of the crown of the roadway. The engineer often marks out the templet used in laying a pavement. Unless the street is very wide, this templet corresponds to one-half the cross section. Its lower edge is a curve such as an arc of a vertical circle or a parabola, which starts from the gutter line and rises to the center of the street in order to give the proper crown to the pavement. When a templet is not used, it may be necessary to set intermediate stakes on the line between each center stake and the corresponding gutter point. In this case, an ordinate to the surface of the pavement must be computed for each intermediate stake; from this ordinate, the corresponding grade rod can be calculated after the height of the instrument has been determined.

191. To Give Grades for a Sewer. (*a*). First method. Assume that the profile has been run along the center line of the sewer and that the grade of the sewer has been established on the plotted profile. On an offset line, parallel to the center line, set grade stakes at such intervals as the contractor may desire. These stakes should be far enough away not to be displaced during construction and, preferably, on the side where excavated material will not be deposited. After the trench has been excavated, set batter boards by measuring up from these grade stakes, and on these batter boards mark points directly over the center line of the sewer.

In some cases, the profile levels are run along the offset line instead of the center line.

191(*b*). Second method. Assume that the center line of the sewer has previously been run and that the sewer trench has been excavated. Set batter boards either by the grade rod method or by shooting in. The center line of the sewer is then marked on the batter boards.

191(*c*). Third method. Same as the second method except that no attempt is made to set the batter boards exactly parallel to the grade of the sewer. Instead, upright strips of wood (grade slats) about 2 in. wide are nailed to the batter boards in such a way that a line along the edges of the strips will be directly over the center of the sewer. The grade is then established by marking points on

these uprights, either by driving a nail horizontally *part way* into the edge of each slat or by cutting a notch.

191(*d*). PRACTICAL SUGGESTIONS.

1. When in any one of the three methods just given, grade points are established from calculated grade rods, it is well, in order to avoid mistakes, to work in one direction, say uphill, so that the target will always be moved downward at each setting. In many cases, however, work may be saved and mistakes avoided by shooting in grade points, a method which is particularly adapted to giving grades for sewers.

2. The *lowest* line along the inside surface of a sewer, often called the flow line, is commonly used as the grade line. Points on this line are established by vertical measurements from the upper edges of the batter boards or from the marks on grade slats, or from string stretched between batter boards.

3. The upper edges of the batter boards or the marks on the grade slats should be in a line parallel to the grade line of the sewer and preferably a *whole* number of feet above it, in order that it will not be necessary for the contractor to measure to subdivisions of a foot.

4. Batter boards are set at such intervals as the contractor may desire. They may be nailed to wooden stakes, driven firmly into the ground, or fastened with screw clamps to heavy iron bars used in place of stakes. The upper edge of each batter board may be brought to a horizontal position by means of a carpenter's level.

5. When the ground is very uneven, it may be necessary to set the batter boards or the marks on the grade slats, as the case may be, at *unequal* distances above the grade line of the sewer. Each distance, nevertheless, should preferably be some *whole* number of feet above that line.

6. As sewers are usually laid in streets, it is generally easier to measure the intervals between batter boards along the surface of the ground and then to find the proportional rise or fall for each of these distances. (See (2) in Art. 191(*c*).) The *per cent* for grade thus found does not accord with the definition in Art. 136(*c*), but the difference, numerically, is small. On rough uneven ground or very steep slopes it may be more convenient to measure between grade slats by horizontal stretches, as in ordinary chaining, although a final O.K. measurement should be made parallel to the sewer, since the contractor will be paid for the length of sewer as thus measured, and not for its horizontal projection.

7. When it is left to the contractor to set batter boards from grade stakes, he should be furnished with a set of notes which gives the distance that he must measure up from each grade stake to the upper edge of the corresponding batter board. The contractor may prefer that these distances be given in feet and inches to avoid the use of decimals.

8. When the tops of three or more batter boards have been established parallel to the grade line of the sewer, they may be checked roughly by sighting along the tops or by stretching a cord. The protruding nails or notches in grade slats may be checked in the same manner.

9. A straight, stiff, grade pole may be made for the use of the contractor and inspector in measuring down from the batter boards, grade slats, or string stretched between batter boards to the grade line of the sewer. A common form is of wood, 2 in. by 2 in., with a metal bracket at the foot of the pole and at right angles to it. Graduations are marked on the pole 1 ft. apart, beginning at the *bottom* of the bracket so that when the pole is in position for measuring down from the batter boards or grade slats, the bottom of the bracket will coincide with the grade line of the sewer. It is often convenient to have on the pole an adjustable target which can be set at any foot mark.

10. Grade stakes and batter boards are likely to be knocked out or displaced, and for this reason, if for no other, bench marks should be established near the end of each straight strip, or at distances not exceeding, say, 400 ft.

11. Where construction is in earth, sewer "boards" may be used instead of batter boards. Such boards are about 2 in. × 8 in. × 12 in. They are set on edge in the ground, astride the proposed trench, with tops flush with the surface of the ground. Line nails are driven, and elevations are read on the tops of the boards at nails. The foreman is furnished with a sketch which shows the heights above boards to a certain string height. The foreman nails on the

verticals for line and grade and stretches a continuous string as he needs it. The method, of course, is not used for machine excavation.

191(*e*). Illustration. (See field notes for Albany Ave. sewer, Fig. 191(*e*).) *Data.* The blueprint called for the following flow line elevations:

31	Albany Ave. Sewer.			July 10	1944
Sta.	B. S.	H. I.	Rod	El.	Cut
B.M. = 0-01.30	55.610 3.207	58.817	13 3.203	.614	10 .7 9.3
-1.30 0	C.M.H			.006 45.62	
25 16 0+25.16			13 3.071	.126 .746	10 .5 9.5
30.01 +55.17			12 2.921	.150 .896	10 .6 9.4
29.99 +85.16			2.771	.150 46.046	10 .5 9.5
30.02 1+15.18	55.15 60.03		2.621	.150 .196	10 .4 9.6
29.78 +44.96	58.61 59.32		2.472	.149 .345	10 .1 9.9
28.86 +73.82	42.36		2.328	.144 .489	10 .2 9.8
29.63 2+03.45	275.47		2.180	.148 .637	10 .3 9.7
29.73 +33.18			2.032	.148 .785	10 .5 9.5
30.03 +63.21			11 1.882	.150 .935	10 .6 9.4
5.13 +68.34	C.M.H @ P.C.			.025 .960	
7.22 +75.56			1.821	.036 .996	10 .8 9.2
2+75.47	O.K. July	10 1944	11.15 A.M.	R.N.C.	W.T.T.

17	Newhall St.		18″ Tile	Sewer	
Sta.	Grade	H. a. G.	E.G.P.	Rod	H. I.
0	40.125				57.390
47.15	.118 40.243	12	52.243	5.147	
40.34 87.49	.101 40.344	12	52.344	5.046	
39.67 127.16	.099 40.443	12	52.443	4.947	
40.05 167.21	.100 40.543	12	52.543	4.847	
39.58 206.79	.099 40.642	12	52.642	4.748	
39.74 246.53	.099 40.741	12	52.741	4.719	57.460
39.90 286.43	.100 40.841	12	52.841	5.044	57.885

Fig. 191(*e*).

At the center of the manhole (or station 0), El. = 45.62 ft.; at 268 ft. upstream (or station 2 + 68), El. = 46.96 ft.; a straight grade between these two points or a rise of 46.96 − 45.62 = 1.34 ft. in 268 ft., or, approximately, a 0.5 per cent grade (for 1.34 ÷ 2.68 = 0.5). The different steps taken in giving the grade are outlined here.

1. The contractor set planks on edge across the trench, at intervals of about 30 ft. The engineer lined in the uprights or grade slats. As a slat could not be set over the manhole at station 0, one was set directly back of it (−1.3 ft.).

24" Tile, Grade	0.5 % 32
Inspector began	work July 8. 7 A.M.
C. Line of Sewer 8'	South of No. Curb.
On Intercepting S	ewer, Cor. Garden St.
Rock in Trench	
Sta. 0+57 From 10 10.0	Ties for M.H. of P.C.
+60 9.3	From pole #267 31.19
+63 8.5	" Hyd. base 57.31
+69 9.2	
+75 10.0	
12" Water Main Location	
	3.1 1+88
2" Gas Crosses 0.	8 1+82
at Sta. 2+05	1+73.5 Crosses ℄
3.'5 deep 0.6	1+67 Depth 4'
2.7	1+53 to top

on ℄ of B.S.	Street Bench	No.	Date 1944 June	Grade 1 : 400 18
4.315	53.075			M.H. So. Side Ivy St.
		1	10	18" Curve to Lilac St.
		2	10	At Sta. 85.5 drain in
		3	10	use for Ho. #29 No. side
		4	10	Depth on ℄ 5.8
		5	10	Conn. to T in Sewer
4.385	53.075	6	13	with 6" riser and
4.810	53.075	7	16	Qr. bend

FIG. 191(*e*).

2. Measured the partial distances between slats, 25.16, 30.01, 29.99, etc., and entered these distances in column 1, written in small figures just below the ruled lines. As the grade of the street was slight, these distances were measured along the surface.

3. Added each partial distance to the number of the preceding station to

get the number of the station at the succeeding grade slat, thus: (0) + 25.16 = station 0 + 25.16; (0 + 25.16) + 30.01 = station 0 + 55.17; (0 + 55.17) + 29.99 = station 0 + 85.16; and so on, thus completing column 1. Observed that the last station checked, very closely, with the total distance.

4. Multiplied each partial distance by 0.5 (per cent) and put the results in column 5, written in small figures just below the ruled lines, thus: 25.16 × 0.5 = 0.126; 30.01 × 0.5 = 0.150; and so on.

5. Starting with the known elevation of 45.62 for station 0, found the elevation of the flow line at each station by adding to the elevation of the preceding station the quantity in small figures just below, thus: 45.62 + 0.126 = 45.746; 45.746 + 0.150 = 45.896; and so on. Noticed that 46.96, the elevation thus obtained for station 2 + 68.34, checked with the elevation called for on the blueprint for that station. This completes column 5.

6. Backsighted on a bench mark to get H.I. For convenience, 55.610, the elevation of the B.M., and 3.207, the B.S., were both recorded in the second column. Recorded 58.817 (their sum) in the third column.

7. Subtracted each of the calculated elevations in column 5 from H.I. to get the grade rod and entered the results in the fourth column, the whole number of feet at the top of the space, the decimal part on the ruled line, thus: 58.817 − 45.746 = 13.071; 58.817 − 45.896 = 12.921; and so on.

8. Found, by trial, that it would be most convenient to cut the notches on the grade slats 10 ft. above the flow line; hence each grade rod was reduced by 10 ft., giving field rods of 3.071, 2.921, etc., thus completing column 4.

9. The notch on each grade slat was then cut, a "crow foot" marked on the slat with its point at the notch, and 10 marked between the lines of the crow foot. As the work progressed, the last column was filled out. Opposite station 0 + 25.16, for example, there are three quantities recorded, i.e., 10 ft., the difference between the grade rod and field rod, or the distance of the notch on the slat above flow line; 0.7 ft., the distance of the notch above the ground, and 9.3 ft. (or 10 − 0.7), the actual cut at that station. This column shows the actual depth of earth which the contractor has to remove; it is also useful in plotting the profile of the finished sewer.

10. Finally, measurements were taken between notches, skipping every other slat so as to make each measurement cover two intervals. These measurements of 55.15, 60.03, etc., in column 2, were added up, and their sum or the O.K. measurement was compared with the length obtained in the first column from the partial measurements. As might be expected, it was slightly shorter. The contractor is paid for the length of sewer as thus measured, i.e., as measured parallel to grade.

11. Explanatory notes and records of various underground structures and rock measurements were put on the right-hand page. Thus, for example, rock in the trench began at station 0 + 57, 10 ft. down measured from the cord stretched between notches on the grade slats; at station 0 + 60 the top of the rock was 9.3 ft. below the cord; at station 0 + 63, 8.5 ft. below; and so on. The direction of the 12" water main is given by offsets from the center line of sewer; thus at station 1 + 88 the offset is 3.1 ft., at 1 + 82 it is 0.8 ft., and at 1 + 73.5 it crosses the center line, with its top 4 ft. below the cord stretched between notches on the grade slats.

191(*f*). Illustration. (See notes for Newhall St. tile sewer.) The first and second columns correspond to the first and fourth columns, respectively, of the preceding illustration. The third column is the "height above grade" of the notches or nails in the slats and corresponds to the first part of the last column in the preceding illustration. The fourth column contains the "elevation of grade points" or actual elevations of the notches or nails. The fifth column gives the field rods. The third and fifth columns are combined in column 4 in the preceding illustration.

191(*g*). Suggestions. (1) The heading of the notes for sewers should give: name of street; size of sewer and material; rate of grade and date. (2) Manholes and other important points should be referenced, and notes made of curves, house connections, and bends, as illustrated in Fig. 191(*e*).

192. To Give Elevations for a Line of Shafting. The controlling points, say at each hanger, must be lined in first; hence the transit used for that purpose may also be used for leveling. The

elevations of the controlling points may be established by the grade rod method in the usual way. Since the points are usually above the instrument, *minus* grade rods must be used, and hence it will be more convenient, although by no means necessary, to hold the rod upside down. A bench mark should be established from which all elevations may be obtained.

Illustration. The elevation of the line of shafting at some hanger point is to be, say, 12 ft. The height of instrument is, say, 5 ft. Then $5 - 12 = -7$ ft. is the grade rod, i.e., the required point is 7 ft. *above* instead of below the line of sight. Set the target at 7 ft., and hold the rod upside down. The base of the rod will then be 7 ft. above the line of sight and hence will give the required point.

193. Elevations for Structures. In establishing elevations for the construction of the foundation of a building, batter boards are placed at the various corners and at such intermediate points as may be necessary. (See Arts. 782(*d*) and 783.) Similarly, batter boards are used in establishing elevations for foundations of such masonry structures as retaining walls, bridge abutments, and dams. (Leveling for the construction of structures is explained in Chapter 57, "Surveys for Construction.")

194. To Set Stakes for Grading a Plot of Land. In the article on preliminary leveling for grading a plot of land (Art. 177), a general method was explained which involved two steps: (1) dividing the plot into squares or rectangles, and (2) determining the elevations of the surface of the ground at the corners of these subdivisions. Setting stakes for grading the plot is the reverse of the second step, that is, grade elevations are established at the corners of the squares or rectangles. If these subdivisions have not been already laid out in the preliminary leveling, it will be necessary to do this. In that case the practical suggestions in Art. 177(*c*) hold good for laying out and designating the subdivisions. It may not be necessary to set a grade stake at every corner, particularly if the subdivisions are relatively small; they should be set only where the contractor is likely to need them. Since the stakes may be disturbed in grading, provision should be made for resetting them at any time. Convenient bench marks near by should be established, and the different rows of stakes should be referenced by points established in line but outside the area to be graded. When the finished grade has been decided upon, it remains to set each stake either by using the grade rod method or by shooting in the grade. Many of the data required for this field work may be prepared beforehand in the office and furnished on a grade sheet or in some other convenient form.

There are three general cases, according to whether the finished grade is (1) to be level, (2) to slope in one direction, or (3) to slope in two directions.

Remark. In each of the three cases it will be assumed for purposes of explanation that the surface of the ground is such that all stakes can be driven to grade. When, as is often true, this cannot be done, the customary method is to drive temporary stakes marked with the cut or fill *measured from the existing surface.* When the rough grading is completed, the final set of stakes may be driven with tops to grade, or with tops, say, one foot above grade, or with lines marked on the stakes. Occasionally it will be necessary to set stakes once for all before grading is begun, and to use different field rods for different stakes. (See illustration, Art. 186(*b*).)

194(*a*). CASE I. TO SET STAKES SO THAT THE FINISHED GRADE WILL BE LEVEL. It is required to grade to an elevation of 100 ft. (actual or assumed). (1) Set up the level and find H.I. Say it is 106.45. (2) Set the target at the grade rod, in this case 6.45. (3) If the ground is already nearly level it may be possible to drive all stakes to grade, using this grade rod, or if more than one setup is required, by figuring a new grade rod for each new setup of the level.

194(*b*). CASE II. TO SET STAKES SO THAT THE FINISHED GRADE WILL SLOPE IN ONE DIRECTION ONLY. In this case it is best to set the lines of stakes parallel to the direction of the slope. Then each line may be set in a manner similar to that explained for setting curb stakes (Art. 189). A quick method, when the surface of the ground is fairly even, is to shoot in one row of stakes in the direction of the slope, and then set the cross rows at right angles, with all the stakes in any one row at the same elevation. In rough ground it is usually necessary to set stakes by the use of field grade rods, as explained above.

194(*c*). CASE III. TO SET STAKES SO THAT THE FINISHED GRADE WILL SLOPE IN TWO DIRECTIONS. In this case, if the two directions of slopes are at right angles to each other, the work is similar to the preceding case, the stakes in one direction being set parallel to the slope in that direction. The cross lines in the other direction will then be parallel to the other slope. When the ground is nearly at the required grade, a quick way is to calculate the elevations of the four corner stakes, set these stakes, shoot in the outside rows, and then shoot in the intermediate stakes of each row from the two extreme stakes in that row. If, however, the ground is too uneven to set the stakes without using different grade rods, it then becomes necessary to calculate the elevation of the finished grade at each stake, and from these elevations find the grade rod to use for each stake, as in the preceding illustration.

When the two directions of slope are not at right angles to each

other, the problem is somewhat more complex. As a general rule, several points may be chosen to be used as controlling points. The elevations of these points having been determined, by calculation or by some other method, stakes may be set on a straight line between any two controlling points in the usual manner. Controlling points should be selected with a view of simplifying the work of calculation as well as the actual field work.

195. Cross-Section Leveling in Route Surveying. When the final center line of a route has been determined on a map, it is located on the ground by stakes set at stations usually 100 ft., or 50 ft., or 25 ft. apart. Profile levels are then run along this center line as explained in Art. 175, and the profile is plotted as explained in Chapter 50. The grade line is then determined in accordance with certain principles of design. For highways, the grade line is the center line along the finished surface; for railroads, it is the line along the base of the rail. For construction purposes, however, the grade line is usually the center line along the subgrade. The subgrade is the graded surface of the roadbed upon which rests the pavement of a highway or the track (ties, ballast, and rails) of a railroad. The grade line, the gradient of which will usually vary for different portions of the route, is laid out on the profile so that at any point the vertical distance between the original surface of the ground and the graded surface will be apparent.

Remark. The methods of cross-section leveling will be explained with special reference to highways and railroads, but these methods, as thus explained, can be easily modified to apply to other types of route surveying such as surveys for pipe lines and canals.

195(*a*). CROSS SECTIONS. A cross section of a railroad or of a highway at any point along the route is understood to be a cross section through that point at right angles to the center line, regardless of whether that line is straight or curved. Cross sections of railroads do not differ materially in their general outlines from those of highways except in the shape of the roadbed. The principal parts of a figure representing a cross section are: (1) a line representing the surface of the roadbed; (2) two inclined lines representing the side slopes; and (3) a line representing the surface of the ground which lies between the slope lines (Fig. 195(*f*)). A typical cross section of a highway is shown in Art. 601(*g*). Drainage ditches on each side of a railroad roadbed in cuts are considered as supplemental cross sections and are treated separately. Side ditches or gutters on each side of a highway roadbed, both in cut and in fill, are considered part of the cross section proper.

The roadbed of a railroad is usually a plane which is level transversely both on tangents and on curves. It can, therefore, be represented in a cross section by a horizontal straight line. The roadbed of a highway is generally crowned on tangents but becomes an inclined plane on curves. The width of a roadbed for either a railroad or a highway is usually greater in cut than in fill.

Side slopes of a cut or of a fill are usually plane surfaces represented in a cross section by straight inclined lines. The slope (inclination) is expressed as the number of units measured horizontally to one of the same units measured vertically. For example, 1½ to 1, is a slope that extends 1½ ft. measured horizontally to 1 ft. measured vertically. The slope may vary for different portions of the route according to the character of the material through which or of which the slopes are constructed. Among the more common slopes are the following.

	Cut		*Fill*	
	Maximum	*Minimum*	*Maximum*	*Minimum*
Earth and gravel	1 to 1	1½ to 1	1 to 1	1½ to 1
Sandy soil	1½ to 1	3 to 1	1½ to 1	2 to 1
Solid rock	Vertical	¼ to 1		
Loose rock	¼ to 1	1 to 1	1 to 1	

Common slopes range in each case from the maximum or steepest slope considered permissible to the minimum or most gradual slope that is likely to be necessary. Notice that only for a cut in solid rock is a vertical surface permissible. At the other extreme the slopes in sandy soil or clay may be as little as 2 to 1 or even 3 to 1.

It should be noted that in military usage slope is expressed as vertical: horizontal, i.e., 1 to 1½ means 1 ft. vertical to 1½ ft. horizontal.

195(*b*). NOTE. There is sometimes confusion as to whether the first quantity of a ratio represents a horizontal measurement or a vertical measurement. To avoid this confusion, it may be well always to let it represent a *horizontal* measurement, and remember that H comes before V in the alphabet. When there is any danger of ambiguity in the expression of a ratio, the letters H and V should be used, thus: 1½ H to 1 V.

195(*c*). THE PURPOSES OF CROSS SECTIONING are: (1) to set stakes for the guidance of those who do the grading; (2) to make such horizontal and vertical measurements as may be necessary to determine the area of each cross section. Areas of cross sections are used in computing earthwork volumes. (See Chapter 42.)

195(*d*). THE DATA NEEDED FOR CROSS SECTIONING are: (1) width of roadbed in both cut and fill for every portion of the line; (2) the subgrade elevation and the elevation of the ground surface at

every station on the entire line (obtained from the profile and usually entered in the cross-section notebook opposite its station); (3) the slopes to be used for different earth materials; (4) all information that may be needed in leveling such, for example, as the description and elevation of every bench mark that has been previously established along the line. Before going into the field, the surveyor should center these data in the cross-section notebook and check them carefully by comparing them with the corresponding data on the profile sheet.

195(*e*). A SLOPE STAKE at any cross section is one that is set at a point where the slope of the cut or of the fill intersects the surface of the ground. It is helpful to remember that slope stakes mark points of no cut or of no fill. Two slope stakes, one on each side of the center line, are commonly set for each cross section. Slope stakes at successive cross sections and on the same side of the center line will indicate where the corresponding upper edge of the cut or where the corresponding lower edge of the embankment will be after the grading is completed.

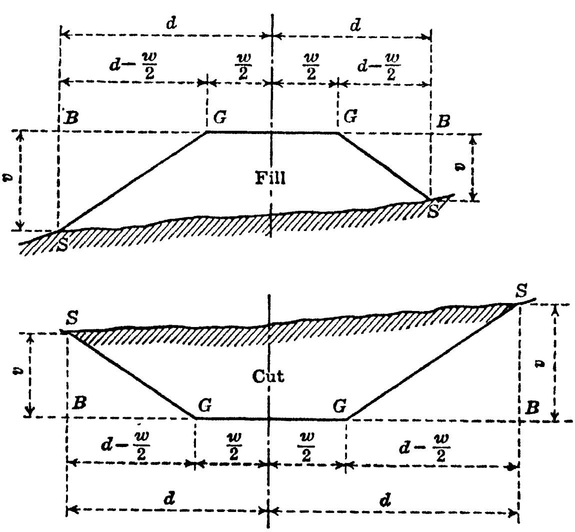


FIG. 195(*f*).

195(*f*). SETTING SLOPE STAKES. In each of the typical cross sections shown in Fig. 195(*f*), the points at which slope stakes should be driven are indicated by *S*. In cross sectioning, each of these points is found by trial. The test appplied in determining the correct position of a point *S* is as follows.

Test. The horizontal "distance out" (d) from the center line to the slope stake (S) must equal one-half width (w) of the roadbed (GG), plus the vertical distance (v) that the point S is above or below the subgrade (GG) multiplied by the ratio (r) of the slope.

$$d = \tfrac{1}{2}w + vr$$

Since the width (w) and the elevation of the roadbed are known, the test involves only two measurements, namely, the distance out (d), measured with a tape, and the vertical distance (v), found by leveling. It may be noted, incidentally, that the position of *any* point on a slope must fulfill the conditions of the test.

In setting a grade stake, a point is assumed on the surface of the ground as nearly on the slope as can be estimated and the test applied. If the measurements show that the point is not on the slope, another point is assumed, and the test repeated until some point is found that is on the surface of the ground and also on the slope.

195(*g*). VALUE OF THE VERTICAL DISTANCE v. For any particular cross section the *grade rod* is the vertical distance between the elevation of the subgrade at that section and the height of instrument or,

$$\text{Grade rod} = \text{H.I.} - \text{Grade elevation}$$

When the grade elevation is *above H.I.*, the grade rod is regarded as *negative*. For any setup *and single cross section* the grade rod is, of course, a constant quantity. If the leveling rod is held at any point on the ground surface, the rod reading is called the *ground rod* for that point.

The value of v may be found by subtracting algebraically the ground rod from the grade rod.

$$v = +\ (\text{or} -)\ \text{Grade rod} - \text{Ground rod}$$

A positive result indicates *cut;* a negative result, *fill.*

Note. In accord with the last statement, the sign + is quite generally used in field notes to indicate cut and the sign − to indicate fill. To avoid confusion, remember that "t" in the word "cut" is similar to +.

195(*h*). METHOD OF SETTING SLOPE STAKES ILLUSTRATED. The elevation of station 60 + 00 on the profile of the center line of a highway is 80.0 ft. It has been determined that the elevation of the subgrade at that station shall be 90.0 ft., that the roadbed shall be 24.0 ft. wide, and that the slopes shall be 1½ to 1. It is required to set the slope stakes S_l and S_r. (See Fig. 195(*h*).)

By backsighting on a point of known elevation, the H.I. of the level was found to be 86.0 ft. The *grade rod* for this setup is, therefore, 86.0 − 90 = −4.0 ft.

A ground-rod reading, taken at the center point C, was found to be 6.0 ft. The vertical distance v at that point is, therefore, −4.0 − 6.0 = −10 ft. (fill), which checks the fill obtained from the profile.

To set the left-hand slope stake, a trial ground-rod reading was taken at A, and found to be 10.6 ft. The value of v for this point is, therefore, −4.0 − 10.6 = −14.6 ft. (fill).

If A is the point sought, its distance out from the center line should be $12 + 14.6 \times 1\frac{1}{2} = 33.9$ ft., but, by measurement, it was found to be only 25.5. Consequently the point A is not far enough out to be on the slope.

The final trial ground-rod reading was taken at S_l and found to be 12.4 ft. The distance v at that point, is, therefore, $-4.0 - 12.4 = -16.4$ ft. (fill), and the distance out should be $12 + 16.4 \times 1\frac{1}{2} = 36.6$ ft. Actual measurement showed it to be 36.5 ft., or near enough for practical purposes.

After a preliminary trial or two, the right-hand slope stake was located at S_r. The ground-rod reading was 4.6 ft., the distance $v = -4 - 4.6 = -8.6$ ft. (fill). The distance out should be $12 + 8.6 \times 1\frac{1}{2} = 24.9$ ft.; it was found by measurement to be 24.8 ft.

Note. It is not necessary to know the elevation of a station in order to set slope stakes, provided the cut or fill at the center line is known. The elevation of the ground at the center may be assumed as 10 ft., 20 ft., or as any convenient quantity. The corresponding grade elevation can then be determined from the known center cut or fill,

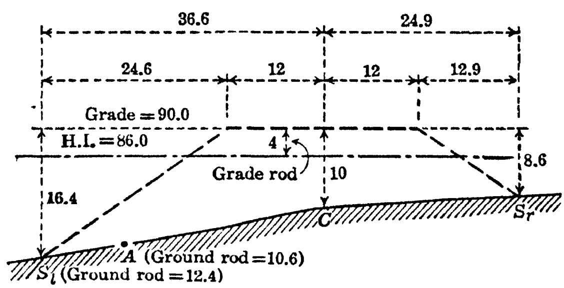


FIG. 195(*h*).

and the H.I., found by taking a backsight on the rod held at the center point. Or the height of instrument may be assumed, the corresponding elevation of the ground determined, and from this elevation and the known center cut or fill the grade elevation may be calculated.

195(*i*). SLOPE TABLES. In some handbooks and in some field books, there are slope tables from which may be read distances from the center of the roadbed for different values of the vertical distance v and for common slopes such as 1 to 1 or $1\frac{1}{2}$ to 1.

195(*j*). THE USE OF AN AUTOMATIC LEVELING ROD AND A SLOPE TAPE will eliminate most of the calculation in setting slope stakes and will speed up the work. The automatic rod has a movable, metallic tape which can be set to any position as explained in Art. 172. It can, therefore, be adjusted to the height of instrument in such a way that the rod reading at any point on the surface of the ground will be equal to the vertical distance v of that point above or below the plane of the subgrade.

Slope tapes may be obtained for different widths of roadbed and for different ratios of slope. For example, a common form is that graduated for roadbeds up to 26 ft. in width and for a slope of $1\frac{1}{2}$ to 1. The zero point on both the face and the back of the tape is 13 ft. from the end of the tape. The 13-ft. portion of the tape between the zero point and the end is graduated in feet and numbered from 13 at the end, backward to the zero mark. From this zero point onward, the main graduations on the *face* of the tape are 1 ft. apart, but on the back of the tape they are $1\frac{1}{2}$ times as great, i.e., $1\frac{1}{2}$ ft. apart. Each foot on the face of the tape is subdivided into tenths of a foot, and each of the $1\frac{1}{2}$-ft. spaces on the back of the tape is subdivided into $1\frac{1}{2}$ tenths of a foot.

Any reading on the back of the tape is obviously not the true distance from the zero mark but $1\frac{1}{2}$ times that distance. For example, the main graduation mark numbered "4" on the back of the tape is not 4 but 6 ft. from the zero mark, and this will become evident from the reading "6" on the face of the tape as indicated below.

Face	0	1	2	3	4	5	6, etc.
Back	0		1	2		3	4, etc.

The graduations on the back of the tape are merely a device for multiplying the vertical distance v by the slope ratio r in the formula $d = \frac{1}{2}w + vr$. When the tape is held with its zero point a distance $\frac{1}{2}w$ from the center line, the point for a slope stake, found by trial, will be that point at which the automatic rod reading (equal to v) and the reading

on the back of the tape are the same. The distance out may then be found by adding to $\frac{1}{2}w$ the reading on the face of the tape that is opposite to the reading on the back.

Illustration. The procedure for setting slope stakes by means of an automatic rod and a slope tape may be illustrated by the use of Fig. 195(*h*).

1. Hold the rod at the center point C, and adjust the movable tape until the reading is equal to the fill at C, i.e., until the 10-ft. mark is in the plane of sight.
2. If now the rod is held at any other point on the surface, the reading will then be equal to the fill at that point. For example, the reading at S_l will be 16.4, and at S_r, 8.6.
3. Hold the 12-ft. mark of the tape (which is 1 ft. from the end of the tape) at C, and, by repeated trials, find the point where the reading on the rod and the reading on the back of the slope tape are numerically the same, *i.e.*, at the point S_l, where the rod reading is 16.4 ft. and the reading on the back of the tape is also 16.4. The corresponding reading on the face of the tape will be 24.6, and the distance out from C to S_l will be, therefore, 24.6 + 12 = 36.6 ft. Similarly when the rod is at S_r, the readings will be: rod reading, 8.6 ft.; back of the tape, 8.6; face of the tape 12.9 ft.; distance out = 12.9 + 12 = 24.9 ft.

When a cross section is in cut, the procedure is the same as that just explained except that the rod is held with the numbers upside down for the reading at the center stake and for readings at the slope stakes.

When one-half the width of a roadbed exceeds 13 ft., the tape previously described can still be used. For example, if one-half the width of the roadbed is 15 ft., a stout cord or a wire 2 ft. long may be added to the end of the tape. In this case, in measuring distances out, 15 ft. instead of 13 ft. must be added to the reading on the face of the tape.

Note. The back of a tape can, of course, be graduated to correspond to any desired ratio of slope, and a full equipment would include several tapes graduated on the back to correspond, respectively, to the different slopes commonly used.

Question. How could an ordinary tape graduated only on one side (to feet and tenths of feet) be used for a slope of 1 : 1?

195(*k*). Marking slope stakes. The distance v from the plane of the subgrade to the surface of the ground at the point S where a slope stake is set (Fig. 195(*f*)) is commonly referred to as the cut or the fill at that slope stake, although in reality, there can be no cut nor fill at any slope stake. This vertical distance in feet is marked on one side of the stake, preceded by C for cut or F for fill, as C 6.3 or F 4.2 (Fig. 195(k)). On the opposite side of the stake is marked the number of the station at which the section is taken. The lettering extends lengthwise of the stake on both sides, and reads from the top downward.



Fig. 195(*k*).

A different method of marking stakes, illustrated in Fig. 195(*k*), is used to a considerable extent, particularly in highway cross sectioning. The numerator 10.5 is the cut, usually given to grade but sometimes to the bottom of the gutter, and the denominator is the distance out from the center line. The slope is sometimes given also; it is always well to indicate it where a change of slope occurs. The station number is marked on the opposite side of the stake.

Slope stakes are usually driven at a slant, with the sides upon which the cuts or fills are marked facing toward the center line.

195(*l*). The field procedure for cross sectioning at any given station, assuming that the height of instrument has been estab-

lished, is as follows. (1) Hold the rod at the center stake, determine the elevation of the surface of the ground at that point, and observe whether it checks within, say, 0.1 ft. or 0.2 ft., with the elevation as obtained from the profile. Mark the cut or fill on the back of the stake. (2) Set the two slope stakes as explained in Art. 195, marking the stakes as explained in Art. 195(*k*). (3) When the surface of the ground between the center stake and either of the two slope stakes does not slope uniformly, additional elevations should be obtained at points where the slope changes, i.e., at breaks in the surface. Each of these points should be located by its distance out from the center line. Enough elevations should be taken to furnish data for a fairly accurate profile of the surface of the ground on the cross line between the two slope stakes.

Suggestions. (1) The direction of the cross line on which the slope stakes are set can usually be estimated by eye, but if the cut or fill is great, it may be well to lay off the 90° angle with the tape or with the transit. (2) The leveling rod and the tape may be read to the nearest tenth of a foot. (3) When, at any station, the surface of the ground is level in a direction perpendicular to the center line, the only elevation required is that of the surface of the ground at the center stake, and the cut or fill from which the distance out to each slope stake is calculated is the same as the cut or fill at the center stake. (4) When part of a cross section is in fill and part in cut, the point on the surface of the ground at which the section changes from cut to fill should be determined, its distance out measured, and a stake set at that point. Note that at this point of zero cut or zero fill the ground rod is the same as the grade rod. (5) The cut or fill to grade and the distance out for every point of change in the profile of the ground surface between the two slope stakes should be recorded in the field notes in the form of a fraction as explained in Art. 195(*q*). Likewise, for each of the two points where slope stakes are driven, the distance *v* above or below grade and the distance out are usually recorded as a fraction.

195(*m*). NOTE. *The area of a cross section* needed in the calculation of earthwork may be determined by calculation directly from field measurements, or from distances scaled on the plotted cross section, or by means of the planimeter. All these methods are explained in Arts. 586–589 and 601. By keeping these methods in mind, and by continually asking whether sufficient measurements have been taken to determine the area, one is less likely to omit some necessary measurement of the cross section.

195(*n*). TYPES OF CROSS SECTIONS. The type of cross section at any particular station is determined mainly by the number of elevations it is necessary to take to secure a satisfactory profile of the surface line between slope stakes. A *level section* is one in which the surface line is level from slope stake to slope stake. A *three-level section* is one in which three elevations are taken, one at the center and one at each slope stake. A *five-level section* is one in which five elevations are taken, including those at the center and at the two slope stakes. An *irregular section* is one at which elevations are taken at irregular intervals, wherever there is a break in the ground. A *sidehill section* is one in which part of the section is in cut and part in fill. A large part of cross sectioning for highways involves irregular sections. An illustration of an irregular section which is also a sidehill section is shown on p. 711.

195(*o*). WHERE TO TAKE CROSS SECTIONS. For highway or railroad grading, stations are usually 50 or 100 ft. apart, and a cross section is taken at each station. Additional cross sections are taken wherever it may be necessary to secure the data required for satisfactory earthwork calculation. Where the cost of grading is high

and consequently greater precision is desired, sections may be taken nearer together. In rock excavation, for example, they may be only 10 to 20 ft. apart. Additional cross sections are also taken at the P.C. and P.T. of each curve.

The most complex cross sectioning occurs along portions of the line where cut changes to fill or vice versa, particularly if such portions are on a side hill. Under such conditions, a sidehill cross section may be wholly in cut, or wholly in fill, or partly in cut and partly in fill, depending upon where it is taken. It may be necessary, therefore, to take additional cross sections as follows. (1) At the point where the cut or fill runs out on the downhill edge of the road bed, i.e., where the elevation of the surface of the ground, $\frac{1}{2}w$ from the center, is the same as the grade elevation at the section; (2) where the cut or fill runs out on the center line, i.e., where the surface of the ground is at grade; (3) where the cut or fill runs out on the uphill edge of the roadbed.

Remark. A thorough understanding of the methods used in earthwork calculation is practically indispensable in deciding where to take cross sections.

195(*p*). A MODIFIED PROCEDURE FOR CROSS SECTIONING is commonly followed when the maximum cut or fill is, say, under 5 ft., particularly in setting grade stakes for a highway. Instead of setting slope stakes, vertical stakes are driven far enough from the center line so that they will be well outside the excavation or the fill, say, at a distance of 25 ft. from the center lines. Elevations of the tops of these stakes are then determined. These side stakes serve not only as grade stakes but also as reference stakes for the center line. When the grading is being carried on, taller stakes, if necessary, may be driven beside these side stakes with their tops at the elevation of the subgrade. The profile of the ground is taken between the two stakes, and, therefore, extends somewhat beyond the cut or the fill. The principal advantage of this method is that the cross sections may be taken before the grade line or the exact shape of the cross section of the highway has been determined. There is some disadvantage in construction, however, in not having slope stakes.

Note. When it is necessary to take cross sections by the method just described before final elevations have been determined, the elevation of the side stakes and of points on the cross profile can be taken with reference to some *assumed* convenient datum.

195(*q*). FIELD NOTES FOR CROSS-SECTION LEVELING. Field notes for leveling from station to station may be kept in accordance with one of the standard forms of leveling notes. The columns usually recorded on the left-hand pages of the field notebook are indicated by the following headings:

Sta.	B.S.	H.I.	F.S.	Elev. Ground	Elev. Grade

This form is like Form E, Fig. 153(*d*), except that there is a sixth column for recording the center-line grade elevations, which are usually determined in the office for all stations and then entered in the cross-section notebook before going into the field. The form is

also similar to that in Fig. 191(*e*), except for the last column. It is sometimes advantageous to have a column for grade rods as well as for grade elevations.

On the right-hand pages are entered the cross-section notes usually in a form similar to that shown below in which the cross-section notes for station 60 + 00 (Fig. 195(*h*)) are given.

Left	*Center*	*Right*
$\frac{F16.4}{36.6}$	F10	$\frac{F8.6}{24.9}$

In these notes, the cut or the fill and the distance out are entered as a fraction, with the distance out as the denominator. Some engineers prefer to record the distance out as the numerator and the cut or fill as the denominator. The distances out are to the slope stakes, i.e., to points of no cut or of no fill. The numerator in each case does not represent the cut or fill at a slope stake but the vertical distance v from the surface of the ground at the stake to the plane of the subgrade. The notes are obviously for a three-level section; for a five-level section there would be two additional fractions, and in each of these fractions the numerator would be the *actual* cut or fill to grade at the distance out indicated by the denominator.

The algebraic signs + for cut and − for fill are quite generally used in cross-section notes in place of C and F, but the reasons for using C and F in marking slope stakes (Art. 187) hold true to some extent in recording field notes.

If the cross-section notes are kept entirely within the left-hand half of the right-hand page, the other half of the page can be used for brief descriptive notes and comments. Field books in which right-hand pages have vertical as well as horizontal ruled lines are preferable.

Remark. The most complex cross-section notes are those for *irregular* cross sections of considerable width, such as the sidehill cross section of a highway shown in Fig. 601(*g*). Note in that cross section the relatively large number of surface elevations taken, as indicated by the fractions, and that the numerator of the fraction at the point where cut changes to fill is 0.0.

Question. How can the slopes be determined from the width of roadway (40 ft.) and the two fractions $\frac{4.0}{32.0}$ and $\frac{-4.5}{35.0}$?

CHAPTER 15

PRECISE DIFFERENTIAL LEVELING

IN precise differential leveling, errors in leveling must be reduced to a far greater degree than in ordinary leveling. This chapter is devoted largely to a discussion of how this reduction of errors is accomplished. (For precise reciprocal leveling see Art. 179(*a*).)

196. Introduction. For first-order leveling the permissible error is ± 0.017 ft.$\sqrt{M}$, and for second-order leveling it is $+0.040$ ft. $\sqrt{M}$ (Art. 170(*b*)). In first-order leveling it is necessary to eliminate as far as practicable every source of error, and even in second-order leveling, methods of precise leveling somewhat modified must be followed. Except in governmental surveys, the highest degree of precision required in leveling is generally greater than that of the second order but somewhat lower than that of the first order. Specifications and the field procedure for second-order leveling have been given in Art. 157. The term *precise leveling,* as used in this chapter, is to be understood, therefore, as applied to leveling of any degree of precision higher than that of the second order. Precise differential leveling involves the use of suitable instruments, the protection of the level from disturbing elements, careful manipulation of the level, and a program of procedures designed to reduce errors.

Every member of a leveling party should have a clear understanding of the errors from different sources and of their relative importance (Chapter 12). The value of a high degree of precision should be weighed against the cost of obtaining it; the cost of first-order leveling may be all out of proportion to the value derived from it. There is, perhaps, a tendency to specify first-order level- when second-order leveling would be sufficiently accurate and less costly.

197. Instruments and Equipment. (*a*). LEVELS. By following the methods and the program of procedures designed for precise leveling, a relatively high degree of precision may be attained with a good level of the ordinary type; but the field work may be more expensive than if a precise level is used. (See description of precision levels, Art. 729(*c*).) For example, in order to attain a specified degree of precision, it may be necessary to run levels over a given loop more times with an ordinary level than with a

precise level. Moreover, longer sights and longer loops are permissible with a precise level. In general, for work of the first order, a precise level should be used. In the selection of such a level, the characteristics of the telescope and the sensitiveness of the bubble tube are important considerations. (See "The Selection of a Level," Art. 730.) The level should be equipped with a reflecting mirror by means of which the level bubble may be observed while sighting at the level rod, and with a micrometer device which can be used as a gradienter. There should be two additional horizontal cross hairs in the telescope similar to stadia hairs; the value of the interval between hairs should be checked by a field test.

197(*b*). LEVELING RODS. A direct-reading, one-piece rod, similar to a stadia rod, is used in precise leveling. Rods are not equipped with targets. The metric system of units is used by the U.S. Coast and Geodetic Survey; the yard and decimals of a yard by the U.S. Geological Survey. To avoid fluctuations in length due to moisture and to changes in temperature, graduations should be on an invar strip attached to the rod. There should be a steel shoe on the rod of such a form that when the rod is held on a bench mark or turning point, the point of contact on the rod will be definite and always the same. The rod should be equipped with a plumbing level and a thermometer.

197(*c*). ADDITIONAL EQUIPMENT. Portable steel pins or plates for use as turning points; a large umbrella; a portable windbreak.

198. The Reduction of Errors. The sources of error in leveling and the methods of reducing or eliminating errors from different sources have been discussed in Chapter 12. Those which are of particular importance in precise leveling are summarized here.

198(*a*). SOURCES OF ERRORS. (1) Errors in the level itself and errors due to imperfect adjustment of the level (Art. 162); (2) error in the length or in the graduations of the level rod (Art. 162(*d*)); (3) manipulation errors (Art. 163); (4) errors in sighting (Art. 165); (5) errors due to changes in position of the level or of the rod (Art. 166); (6) errors due to natural sources—effect of sun, wind, variations in temperature, curvature of the earth, and refraction (Art. 167); (7) Personal errors (Art. 168).

198(*b*). REDUCTION OF ERRORS. (1) Use a good precise level and test its adjustments frequently. (2) Use a precise leveling rod that has been standardized for absolute length for a given temperature, for the coefficient of expansion, and for the index error (distance, if any, between the foot of the rod and the zero of the graduations). (3) Keep the weight of the hands off the level and its tripod. Make sure that the level bubble is at the center at the

instant of sighting. If the instrument is not equipped with a mirror, an assistant should keep the bubble centered. Plumb the leveling rod by means of a plumbing level. (4) Reduce the error in sighting by taking the reading for each of the three horizontal wires. If the mean of the three readings differs from the reading for the middle wire by more than the permissible limit of error (0.002 ft. for first-order leveling), the observation should be repeated. Each reading should be estimated to the nearest thousandth of a foot. (5) Take precautions to avoid settlement of the level or of the level rod (Art. 166). Use definite and stable turning points. (6) Protect the level from the rays of the sun not only during a setup but also when carrying it from one setup to the next. Use a windshield when required. Reduce the error due to refraction by keeping the line of sight well above the ground. The atmospheric conditions are most favorable for precise leveling on a cloudy day or during the night. In attaining a specified high degree of precision, it may be less expensive, especially in warm climates, to do the leveling under such conditions than it is to carry it on under a bright sun. (See note on night leveling, Art. 167(*d*).)

198(*c*). PROGRAM OF FIELD PROCEDURES FOR THE REDUCTION OF ERRORS. It is not sufficient merely to observe the precautions just summarized. On the contrary, it is equally important that there be a program of field procedures designed to reduce errors. The essentials of such a program are: (1) Lengths of sight should rarely be as great as 150 yd. (2) The difference in length between a foresight and a corresponding backsight should be as small as practicable and should not exceed 10 yd. (Distances are measured preferably by stadia or gradienter, but sometimes by pacing.) The sums of the backsight distances and the foresight distances should not differ by more than 20 yd. (3) The interval of time that elapses between a backsight and the succeeding foresight should be as short as possible. Shortening of the interval is best accomplished by employing two rodmen, one of whom holds his rod on even-numbered turning points, the other on odd-numbered turning points. (4) At one setup the backsight is taken before the foresight whereas at the next setup the foresight is taken before the backsight, alternating in this manner throughout the line leveled.

The field procedures just outlined tend to diminish errors from several sources such as those due to imperfect adjustment of the level, to the settling of the level, to poor visibility, to changes in refraction, to variations in temperature, and to curvature of the earth's surface. There is no such thing as the absolute elimination

of errors in leveling, but a surprisingly high degree of precision may be attained by following a good program of field procedure. *The field procedure in precise reciprocal leveling* is outlined in Art. 179(*a*).

199. Specifications and Procedures. It is impossible to outline any one program of field procedures that can be used under all conditions and for all purposes of precise leveling. The following specifications and field procedures for first-order leveling are taken from a manual entitled *Technical Procedure for City Surveys*, published by the American Society of Civil Engineers. This method, in turn, was adapted and revised for city conditions from *Instructions for First-Order Leveling* by the U.S. Coast and Geodetic Survey.

It is particularly important in studying these methods of procedure to note the precautions that are taken to eliminate errors, and to ask oneself just what errors are eliminated by each precaution.

First-Order Levels

199(*a*). Specifications.

1. The first-order levels shall constitute the basic control for all elevations. In general, the distribution of permanent monuments shall be at the rate of at least one pair of monuments (a pair to be considered as two intervisible monuments, spaced 500 to 1000 ft. apart) to each linear mile of traverse. It is recommended that the lines of first-order levels be parallel to and identified with the first-order traverse lines, and that the traverse monuments be used as bench marks.

2. All first-order level lines shall start from, and close upon, previously established first-order bench marks, if available. Spur lines should be avoided by running closed circuits.

3. Criteria for acceptable final results shall be as follows. (*a*) The rods used shall be tested at frequent intervals for absolute length and index error. The coefficient of expansion of the rods shall be determined by the National Bureau of Standards. (*b*) All observed differences of elevation shall be corrected for collimation error, absolute length, index error, and temperature. (*c*) All discrepancies between backward and forward runnings over the same section shall agree within 0.017 ft. $\sqrt{M}$, M being the length of the section, in miles. (*d*) Final results shall be adjusted by the method of least squares or by simultaneous weighted junction points.

199(*b*). Preparation. The first step in the level survey is the finding and checking of all federal and state first-order bench marks, within or near the city area, for inclusion in the city network of levels. These bench marks serve as a basis for the reduction of elevations to the mean sea level datum plane. The U.S. Board of Surveys and Maps will furnish information as to where descriptions and elevations of the bench marks, within the specified area, may be obtained.

When the level survey follows a first-order traverse survey, its lines will parallel the lines of the first-order traverse, and the traverse monuments will serve as bench marks. In this instance, the field reconnaissance is simply a matter of locating additional bench marks along the route (where such bench marks are required) and of selecting the shortest and most economical level routes between these bench marks.

When the levels are surveyed previous to the traverse, the reconnaissance should be completed, in every detail, as for a first-order traverse.

In addition to those required as station markers for the first-order traverse, permanent bench marks may, and in many cases should, be set along the first-order level routes.

Before leveling is begun, ample time should be allowed for the settlement of the bench mark monument and final setting of the concrete. For future recovery, all bench marks should be completely described and referenced by at least three horizontal steel-tape measurement.

199(*c*). Equipment. The major equipment used in first-order leveling includes a first-order level instrument and two first-order level rods. One of the most efficient types of leveling instruments is known as the U.S. Coast and Geodetic Survey precise level. Other level instruments may be used with satisfactory results, but the field work will probably cost more. Regardless of type, an instrument should be equipped with a very accurate level tube, a mirror or prism arrangement for viewing the level bubble almost simultaneously with rod reading, and a bubble-centering micrometer screw that can be operated while viewing the bubble.

An umbrella is needed for protecting the instrument from the direct rays of the sun, and a light cloth windshield is required as a protection against strong wind.

The first-order level rod, graduated in yards (or meters), is a direct-reading rod made of one piece of wood. In most recent models the graduations are marked on a strip of invar fastened to the rod at the bottom, and held under spring tension at its top. There is no uncertainty as to the absolute length of this type of rod which, unlike the wooden rod, is independent of temperature and humidity fluctuations. Both invar and wooden rods should be tested by the National Bureau of Standards for absolute length, index error, and coefficient of expansion. In testing a rod, a small thermometer is set in a groove on its rear face, and a universal level is attached to it to insure verticality.

When the older type of wooden rod is used, its length should be compared frequently with a standardized steel tape, for absolute length corrections to be computed later.

The smallest rod graduation is 0.01 yd., which is divided by estimation into ten parts, and the readings are recorded to the nearest 0.001 yd. Graduation in yards is used because in all first-order leveling the three wires of the diaphragm are read, and the mean of these readings, recorded in *feet*, corresponds to the sum of the three wire readings, in recorded *yards*. (See Art. 200(*a*).)

Portable turning points are required when suitable fixed objects are not available. These points are usually round iron pins, 12 to 18 in. long, pointed at the lower end. They are driven into the ground with a small hand axe or sledge, and are pulled out by means of a chain or ring attached at their lower end. When the leveling is carried across soft ground, it is preferable to use a turning point consisting of a circular iron plate, 6 or 8 in. in diameter, with a raised central point on the top surface and three or four prongs attached to the bottom.

199(*d*). Personnel and field procedure. The field party required for first-order leveling consists of four men. The party chief, acting as level observer, operates the instrument; the recorder keeps all notes, makes all field computations, and holds the umbrella or windshield when required; and the two rodmen carry the forward and rear level rods.

The general rules for field procedure in first-order leveling in cities are these.

1. All lines shall be leveled forward and backward independently and preferably under different atmospheric conditions, that is, on different days or in the forenoon and in the afternoon of the same day.

2. The forward and backward runnings along any line shall begin and end on permanent bench marks. No temporary bench marks may be used to hold the line; not even overnight.

3. Forward and backward levelings should be repeated again and again, if necessary, until the difference between the two is not greater than 0.017 ft. $\sqrt{M}$, in which M = length of section, in miles. (Judgment should be exercised in the application of this rule.) If any measure over a section gives a result differing by more than 0.020 ft. from the mean of the other measures over the section, this measure shall be rejected.

4. The observation program at each station should be as follows. Set up and level the instrument. Read the three wires of the diaphragm, as seen projected against one of the rods, to the nearest 0.001 yd., while the bubble is held continuously in the middle of the tube. Immediately after this, read the three lines of the diaphragm as seen projected against the other rod. The

abstract sum of the three wire readings (equaling the mean reading, in feet) may differ from the middle wire reading multiplied by three, by not more than 0.002 ft. It is desirable that the rod interval between the middle and upper wires shall not differ from the one read between the middle and lower wires by more than 0.002 yd. The recorder tests the results immediately after the observation, and the instrument should not be moved unless and until the tests satisfy these rules.

5. In the record, all instrument stations should be numbered consecutively, assigning the number 1 to the first station of the day. At odd stations the backsight is taken before the foresight; at even stations the foresight is taken before the backsight. Since the same rod is held on a station for both foresight and backsight, the effect of this practice is that at each setup the same rod, that is, the rod used for the backsight at the first instrument station, is read first.

6. At each rod station the thermometer should be read to the nearest degree and the temperature recorded.

7. The difference in length between a foresight and a corresponding backsight shall not exceed 10 yd.; in fact, it should be made as small as feasible without extra expenditure of time. The recorder should record the stadia distances, in yards, for both backsight and foresight, together with their continuous sums for each section. The sums of the backsight distances and the foresight distances should be kept as nearly equal as feasible without extra expenditure of time, and shall never differ by more than 20 yd.

8. One daily observation of the collimation error of the level should be made in the regular course of leveling and should be recorded in a separate opening of the record book. The procedure is as follows. Make the last foresight reading at an instrument station a part of the collimation check; have the back rod placed about 10 yd. back of the instrument and read the rod; move the instrument to a position about 10 yd. behind the foresight rod, and read the foresight and the backsight rods. (Each of the points at which the instrument stands is between the two points on which the rod is held.) The required constant, K, to be determined, namely, the ratio of the correction for any rod reading to the corresponding subtended rod interval, is

$$K = \frac{I_n - I_d}{i_d - i_n}$$

I_n = sum of the two near rod readings; I_d = sum of the two distant rod readings, one of which is the last foresight reading at the instrument station; i_n = sum of the two corresponding near subtended rod intervals; i_d = sum of the two corresponding distant rod intervals.* The total correction for curvature and refraction must be applied to the sum of the distant rod readings before substituting it in the equation. The level should not be adjusted if K is less than 0.005. If K is between 0.005 and 0.010, adjustment is not advisable, but if K is greater than 0.010, the level must be adjusted and should be redetermined at once. The adjustment for reducing K must be made by moving the level vial, not the reticule. (See Art. 742(*d*).)

9. The instrument should be shaded from the direct rays of the sun, both during observation and when moving from station to station.

10. The maximum length of sight should be 150 yd., and this maximum should be permitted only under the most favorable conditions.

200. Field Notes for Precise Leveling. The field notes shown and the accompanying explanation are taken from the manual entitled *Technical Procedure for City Surveys*, published by the American Society of Civil Engineers. In studying these notes, keep in mind that for each sight there are three readings, one for

* The final value of each rod reading is the mean of three readings, one for each wire. The corresponding *subtended rod interval* is the difference between the readings for the upper and lower wires, and, from the principle of the stadia, each interval is a measure of the corresponding length of sight. The correction K therefore is per unit of length.

The field work for determining the value of the numerator of the fraction is the same in principle as that for reciprocal leveling.

each of the three cross hairs. Keep in mind also that the leveling rod used is graduated in yards and that, therefore, the readings, entered in columns 2 and 7, are expressed in *yards*, but that the sum of the three readings for each sight, entered in column 3 or 8, is numerically equal to the mean of the three readings in *feet*.

200(*a*). FIRST-ORDER LEVEL NOTES. A sample page of first-order level notes is reproduced in the accompanying table. In this case, as in that of all specimen field records and computation forms, only the general arrangement and the essential details are indicated. Actually, such forms should provide for records of personnel, file numbers, miscellaneous remarks, etc.

Columns 2 and 7 contain the three thread readings (rule 4, "Personnel and Field Procedure," Art. 199(*d*)) in their proper order. The quantity below a set of three thread readings is the middle thread reading, multiplied by 3. It is used only as a check on the sum or mean of the three readings. If rods graduated in feet or in meters are used, this value would be the sum of three readings, and columns 3 and 8 would represent this quantity divided by 3.

FIRST-ORDER LEVEL NOTES: FORWARD RUN, B.M. 509 TO B.M. 510, CHERRY STREET

Instrument: *No. 7432* Time: *9:00 A.M.* Date: *October 6, 1925*
Weather: *Clear, wind moderate* Rods: *Nos. 897 and 898 yard units*

Number of Stations (1)	Backsight: Thread Readings (2)	Backsight: Sum (3)	Backsight: Thread Intervals (4)	Backsight: Sum of Intervals (5)	B.S. Rod and Temp. (6)	Foresight: Thread Readings (7)	Foresight: Sum (8)	Foresight: Thread Intervals (9)	Foresight: Sum of Intervals (10)	B.M. No. (11)	Descriptions and Remarks (12)
	1.113		*36.8*		*897*	*3.292*		*40.2*		*509*	
	0.745		*37.0*			*2.890*		*40.3*			
	0.375					*2.487*					
1	*2.235*	*2.233*	*73.8*	*73.8*	*72° F.*	*8.670*	*8.669*	*80.5*	*80.5*		
	1.093		*38.5*			*2.976*		*37.2*			
	0.708		*38.8*			*2.604*		*37.2*			
	0.320					*2.232*					
2	*2.124*	*2.121*	*77.3*	*151.1*		*7.812*	*7.812*	*74.4*	*154.9*		
	1.260		*33.4*			*1.760*		*34.3*			
	0.926		*33.6*			*1.417*		*34.5*			
	0.590					*1.072*					
3	*2.778*	*2.776*	*67.0*	*218.1*		*4.251*	*4.249*	*68.8*	*223.7*		
	2.642		*28.2*			*0.967*		*27.9*			
	2.360		*28.4*			*0.688*		*28.0*			
	2.076					*0.408*					
4	*7.080*	*7.078*	*56.6*	*274.7*		*2.064*	*2.063*	*55.9*	*279.6*	*510*	
		+14.208					*−22.793*				

Columns 3 and 8 contain sums of the three thread readings, each of which is numerically equal to a mean reading, in feet. If rods graduated in feet or in meters are used, each of these columns should be headed "Mean," and should contain sums of the three thread readings, divided by 3.

Columns 4 and 9 show the rod intervals intercepted on the rod between the middle and the upper threads, and between the lower and the middle threads. The two intervals, of course, should be to each other in the same proportion as the actual intervals in the diaphragm are to each other. For the sake of convenience, the thread intervals are multiplied by 100, the threads in this instrument being set at a ratio of 1 : 100. The sum of each pair of intervals is shown immediately below.

Columns 5 and 10 show the cumulative sum of the intervals. With this value shown for each instrument station, the total backsight and foresight distances may be kept nearly equal.

Column 6 shows the number and the temperature of the rod used in the backsight at each bench mark.

The algebraic sum of foresights and backsights is $-22.793 + 14.208 = -8.585$, and this is the difference in elevation between B.M. 509 and B.M. 510.

The space required for column 12 is generally much wider than that shown here. Frequently the entire right-hand page is used for descriptions and remarks. It is particularly important to describe the location of each bench mark. For example, opposite bench mark 509 the decription: *Bancroft and Cherry Streets at southeast corner.* And opposite bench mark 510 the description: *Cherry and Speilbush Streets at northeast corner.*

Note. The Adjustment of Elevations in first-order leveling by the method of least squares is not explained in this book. The subject is treated in the *Manual of First-Order Leveling,* issued by the U.S. Coast and Geodetic Survey, in the manual, *Technical Procedure for City Surveys,* and in various books on the adjustment of observations.

CHAPTER 16

BEARINGS AND AZIMUTHS

BEARING and *azimuth* mean so nearly the same thing in surveying that the two terms are often confused. The distinction is made clear in this chapter, as is also the distinction between *forward* bearing and *back* bearing, and between *forward* azimuth and *back* azimuth.

For the method of calculating the bearing of one line from the bearing of another line when the angle between the two is known, see Chapter 37, "Calculation of Bearings and Azimuths." For the method of surveying with the compass, see the next chapter. Before beginning this chapter, one should be familiar with the properties of the magnetic needle as explained in Art. 704.

201. Magnetic Declination. In Art. 704 it is explained that:

1. Except in a few localities the magnetic needle does not point to the *true* north, but to a *magnetic* north. The angle between a true north-south line and a magnetic north-south line as observed at any place is the *magnetic declination* for that place.

2. The magnetic declination differs for different places, and varies slightly from year to year in the same place.

3. The *agonic* line is a line passing through places where the declination is zero. In the United States it passes through South Carolina, North Carolina, Kentucky, Indiana, and Michigan.

4. For places *east* of this line the declination is *west*, and for places *west* of it the declination is *east*. This fact helps the memory in deciding whether the declination for any place is east or west.

5. *Local attraction* is a term used to denote any influence, in a given locality, which prevents the needle from pointing to the magnetic north. For additional information concerning the magnetic needle and changes in declination, see Art. 704.

202. Meridians. (*a*). A TRUE MERIDIAN is a great circle cut from the surface of the earth by a plane passing through the imaginary axis on which the earth rotates. There are an infinite number of such meridians. *Within the limits of a plane survey,* any portion of a meridian may be considered a straight line, and such a straight line through any point on the earth's surface is the *true meridian* or the *true north-south line* through that point. In a plane survey, true meridians through different points may be considered to be parallel lines in a horizontal plane.

202(*b*). NOTE. The *celestial sphere* may be conceived as a hollow sphere of infinite size with its center at the center of the earth. The *celestial poles* are the points in which the axis of the earth prolonged pierces the celestial sphere. A *celestial meridian* is a great circle of the celestial sphere passing through the celestial poles. The celestial meridian for any point on the earth's surface passes through the zenith for that point.

202(*c*). THE MAGNETIC MERIDIAN through any point may be considered in plane surveying to be a straight line whose direction is that indicated by a magnetic needle at the point, provided the needle is free to swing to its normal position with one end pointing toward the north magnetic pole. The true meridian and the magnetic meridian through a given point rarely coincide.

203. Meridians as Lines of Reference. Assume that the direction of each line of a survey is indicated by the angle it makes with any intersecting north-south line. Then the direction of every line is known with respect to any *single* north-south line that may be assumed as *a line of reference.* It is often desirable or necessary in a survey to indicate in a similar manner the directions of all important lines of a survey with respect to some one line of reference, though it is not always a true north-south line.

203(*a*). A true meridian through any point never changes direction. Moreover, such a meridian can be accurately established by solar or star observations and re-established, if necessary, by the same methods (Chapters 34 and 35). For these reasons, a true meridian is best for any survey for which a *permanent* line of reference is desired.

203(*b*). A MAGNETIC MERIDIAN has often been used in the past, particularly in compass surveying, as a reference meridian. Since its direction is determined by a magnetic needle, it cannot be established or re-established with a high degree of precision. Moreover, the magnetic meridian through a given point changes as the magnetic declination varies. Nevertheless, it is still useful as a line of reference in many cases; for example, in rough surveys made with a compass, or in checking directions of traverse lines and other lines of a transit survey.

203(*c*). AN ASSUMED MERIDIAN. When in a survey the directions of lines with respect to a north-south line, true or magnetic, is of no consequence, some other line of reference may be assumed in any convenient direction; frequently it is the first line of a traverse. Such a line can only be re-established by means of points that have been monumented or referenced.

204. The Bearing of a Line. The *bearing* of a line is the direction of that line with reference to a meridian. Let the point *C* in Fig. 204 be one end of a given line *CA*. A north-south meridian

through C forms an angle of 30° with CA. This *acute* angle is the *angle of bearing,* taken at C, of the line CA. In designating the bearing, it is necessary also to indicate the quadrant of bearing in which the line CA lies; in this case, N 30° E. Bearings are always measured from the north point or the south point so many degrees east or west, never from the east point or west point so many degrees north or south.

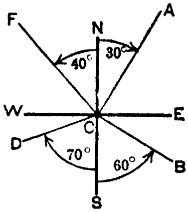


FIG. 204.

The four possible directions in which bearings are measured may be illustrated by the four lines in Fig. 204. In each case the bearing is taken at C, and the direction in which the angle is measured is indicated by the direction of the arrow. Thus the bearing of CA is N 30° E, *not* E 60° N; of CB, S 60° E, *not* E 30° S; of CD, S 70° W, *not* W 20° S; of CF, N 40° W, *not* W 50° N.

The angle of bearing never exceeds 90°. For example, a line 120° from north measured clockwise is *not* N 120° E, but S 60° E. It is well to remember that the *direction* (bearing) of a line with respect to a meridian involves two elements, namely, (1) *inclination,* measured by the angle of bearing, and (2) *sense* (the direction from the end of the line at which the bearing is taken toward the other end), indicated by two letters of the compass, the first of which must be N or S and the second E or W. This is analogous to the way in which a force in mechanics is indicated by inclination and sense.

204(a). TRUE BEARING. The bearing of a line with respect to a *true* north-south meridian. A true bearing never changes.

204(b). MAGNETIC BEARING. The bearing of a line with respect to a *magnetic* north-south meridian. This is the bearing that is read from a compass (unless the declination arc has been turned off), and differs from the true bearing at any place by the amount of the magnetic declination at that place. The magnetic bearing of a given line changes slightly from year to year.

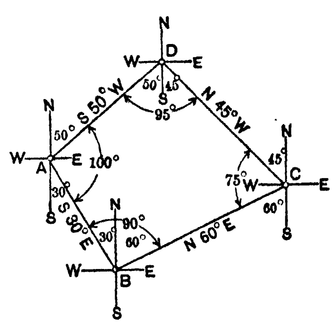


FIG. 205.

205. Forward Bearing and Back Bearing. Every line of definite length has two bearings, one for each end of the line. If the bearing of any line from one end is NE from the other end it will

be SW and vice versa; likewise, if from one end of a line the bearing is NW, from the other end it must be SE. This is an important distinction because it indicates direction northerly or southerly. Thus, for example, if the bearing of a line *AB* (Fig. 205) is given indiscriminately as either S 30° E, or N 30° W, one cannot tell whether *B* is southeast or northwest of *A;* but if from *A* to *B* (*AB*) it is S 30° E, and from *B* to *A* (*BA*) it is N 30° W, all doubt is removed. The *forward bearing* of a line is the bearing in the direction in which the survey is being run; the *back bearing* is that in the opposite direction. The two bearings of a line are numerically equal but with opposite letters.

Illustration. In Fig. 205, if the survey is made from *A* to *B* to *C* to *D* to *A,* the bearings as given on the figure are all *forward* bearings, but if the survey is made from *A* to *D* to *C* to *B* to *A,* the bearings as given are *back* bearings. Assuming the stations to have been occupied in the order *ABCD,* the bearing of *AB* is S 30° E, or back bearing is N 30° W; *BC* is N 60° E, or back bearing is S 60° W; *CD* is N 45° W, or back bearing is S 45° E; *DA* is S 50° W, or back bearing is N 50° E.

Note. In this book when referring to the bearing of any line, that end of the line at which the bearing was read will be given first. Thus if the bearing of a line *AB* read at *A* is S 30° E, then *AB* = S 30° E, and *BA* = N 30° W, i.e., the bearing of *BA* is the back bearing of *AB.*

Reading Bearings. The method of reading the needle in a compass is explained in Art. 279.

206. The Bearings Kept in a Survey are usually *all forward* bearings. It will not do to record forward bearings and back bearings indiscriminately. Though it is true that one can always determine from the needle whether the forward bearing is N or S, one can usually do this without consulting the needle, since after working in any locality a short time the compass points are well fixed in mind.

Note. When puzzled in the field as to whether the first letter of a bearing is N or S, imagine yourself walking along the line *in the direction in which the survey is being made,* and note whether you are going in a northerly or southerly direction. If the line should happen to be so nearly east and west that you cannot tell, then ask yourself if you are going easterly or westerly. If, for example, you know that the bearing of a line must be either NE or SW, you can usually determine in this way which of the two is the *forward* bearing.

207. Observed and Calculated Bearings. An *observed* bearing is one that has been obtained by direct observation in the field. A *calculated* bearing is one that has been derived by computation. For example, if the bearing of one of two intersecting lines is known and the angle between the two lines is also known, the bearing of the second line may be easily calculated from the known bearing of the first and the known angle. The method of calculating bearings is explained in Chapter 37. Calculated bearings and

calculated azimuths are used mainly in transit surveying, as explained later.

208. Things to Remember About Bearings. Summary. (1) The magnetic bearing differs from the true bearing by the amount of local magnetic declination. (2) Bearings are reckoned from either the north or the south points, never from the east or west. (3) Every bearing must be given by two letters. The first must be either N or S, the second either E or W. The omission of either letter renders the bearing of no value. (4) No bearing is over 90°. (5) Every line has two bearings, forward and back, equal numerically but with opposite letters. (6) The field notes should clearly indicate from which end of a line the forward bearing of the line is taken.

209. The Azimuth of a Line is the direction of that line with respect to a meridian, but it differs from bearing in that all azimuth angles are measured *clockwise* from the same starting point and hence may vary from 0° to 360°. When the meridian is a north-south line and angles are measured from the north as a starting point, the azimuth of a line in the NE quadrant is between 0° and 90°; for a line in the SE quadrant, the azimuth is between 90° and 180°; for a line in the SW quadrant, the azimuth is between 180° and 270°; for a line in the NW quadrant, the azimuth is between 270° and 360°. Thus in speaking of, or recording, azimuths, the compass points N, E, S, and W are not used, the *magnitude* of the azimuth alone indicating the direction of the line—a decided advantage.

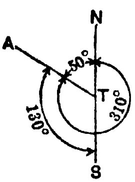


FIG. 209(*a*).

209(*a*) ILLUSTRATION. A transit standing at *T*, Fig. 209*a*, is pointed at *A*. The angle *NTA* measured *clockwise* is 310°, and if *NT* is a north-south line, the azimuth of *TA* is 310°. If the bearing of *TA* is required, it must be from the azimuth, 360 − 310 = N 50° W.

209(*b*). THREE TYPES OF AZIMUTHS. An azimuth is a ***true azimuth,*** a ***magnetic azimuth,*** or an ***assumed azimuth,*** according to whether the reference meridian is true, magnetic, or assumed. The starting point from which angles are measured may be at either end of the meridian. In this book, unless otherwise stated, azimuth angles are measured from the north of a north-south meridian. Many surveyors, however, measure azimuths from the south as a starting point, in which case the azimuth of *TA,* Fig. 209(*a*), would be 130° instead of 310°. In astronomical observations and in most governmental surveys the true south is taken as the starting point.

Note. In ordinary plane surveying, the chief advantage in measuring azimuths from the north instead of from the south lies in the fact that by so doing the algebraic signs will accord with the directions to which the surveyor is accustomed in trigonometric computations. (See Chapters 37 and 38.)

209(*c*). Assumed meridian. A meridian in any convenient direction may be assumed if desired. For example, in Fig. 209(*c*), let *BC* be any line of reference, *not* a N and S line. The azimuth of *TA with respect to BC* will be *BTA* if *B* is the end chosen as a starting-point, or *CTA* if *C* is the end from which azimuths are reckoned.

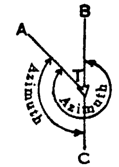


Fig. 209(*c*).

209(*d*). Back azimuth. Every line of definite length has two azimuths, one for each end of the line. Thus in Fig. 209(*d*), if the azimuth of *AB* is taken at *A* it is 40°, but if taken at *B* it is 220°. If the survey is being made in the general direction from *A* to *B*, then 40° is the azimuth and 220° the back azimuth.

The *back azimuth* of a line is the azimuth of that line measured at its *forward end.* Since azimuths are measured always in one direction (clockwise), the back azimuth is always equal to the *azimuth plus 180°.* If this sum is more than 360°, subtract 360°.

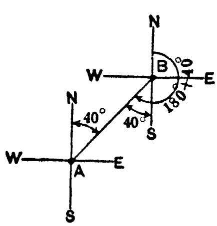


Fig. 209(*d*).

209(*e*). Choice of meridian. When, in any survey, there is nothing to be gained by either observing or calculating bearings, a line in any convenient direction may be *assumed* as a meridian for azimuths. Thus, for example, the first transit line in a survey is often assumed as a reference line or meridian, and then all azimuths are referred to this assumed meridian. When, however, it is desired to check azimuths roughly by means of the compass needle, or when, for any other reason, *magnetic* bearings are of importance, a magnetic meridian and magnetic azimuths are used. For surveys in which it is desired to have important lines permanently preserved or, if lost, accurately re-established, a true meridian and true azimuths are best.

209(*f*). Observed and calculated azimuths. An observed azimuth is one that has been obtained by direct observation in the field. A calculated azimuth is one which has been derived by computation in much the same way that bearings are calculated.

210. Facts to Remember about Azimuths.

1. Bearings are measured from either N or S, but azimuths are from N only (or from S only).

2. Bearings are measured either right or left, whereas azimuths are always measured to the right (clockwise).

3. Bearings are never over 90°, but azimuths are 0° to 360°.

4. In recording a bearing, *two* letters (compass points) are required, but in recording azimuths no letters are used.

5. The back azimuth is equal to the forward azimuth plus 180° and vice versa.

6. An *assumed* meridian may often be more convenient than either a true or a magnetic meridian. It should always be clear from the field notes what was used as the meridian.

7. There can be no ambiguity concerning the direction of an azimuth provided it is clearly stated in the field notes, once for all, whether azimuths are measured from the north or from the south, or, in the case of an assumed meridian, from which end of that meridian.

CHAPTER 17

COMPASS SURVEYING

SINCE the theory of transit surveying to be explained in subsequent chapters includes practically all the theory of compass surveying, it is necessary in this chapter to give only a few additional directions and suggestions for the use of the surveyor's compass. The first part of Chapter 16, which deals with the subject of bearings, should be studied in connection with this chapter.

211. Compass Surveying. The transit has taken the place of the surveyor's compass to such an extent that now the latter is used for rough work only, where speed and not accuracy is the first requirement. For such work the compass is a very useful instrument, and surprisingly good results may often be obtained with it. The positions of any two points with respect to each other are determined by the *bearing* and *the length* of the line between them, and this is the basis of compass surveying. The angle between two lines may be calculated easily if the bearing of each line is known. (See Chapter 37, "Calculation of Bearings and Azimuths.") The clumsy and inaccurate chain formerly used in compass surveying has practically been replaced by the steel chain tape.

The objections to the use of the compass are: (1) Under the most favorable conditions bearings cannot be read with any degree of certainty closer than 5′ or 10′. (2) The needle is unreliable, and under certain conditions, as for example in city surveying, local attraction may render it practically useless.

The advantages are these. (1) The compass is light and portable; and while it is less accurate than the transit, it requires much less time for setting up, sighting, and reading. (2) An error in the direction of one line does not, necessarily, affect other lines of the survey. (3) The compass is especially adapted to running straight lines through woods and other places where obstacles are likely to interfere with the line of sight. Thus, for example, when a foresight strikes a tree, the instrument may be set up beyond it at a point, found by trial, such that the bearing of the backsight on the tree will be the back bearing of the foresight. (4) Since the compass was the instrument in general use before the introduction of the theodolite and the transit, it is especially adapted to retracing old surveys.

212. Use of the Compass. Practical Suggestions.

1. Set up over the station and level up. If the instrument is fitted with a ball spindle only, it is made level by grasping the main plate *near the compass box*—not near the sight vanes—and moving it until both bubbles come to position, where it is held by friction

on the ball. If there are leveling screws, the instrument is made approximately level on the ball and the leveling perfected by the leveling screws.

2. The upright sights or sight vanes are usually unlike. For example, the one nearer the north point of the compass box may have graduations marked on it for vertical angles, whereas that at the south end is not graduated but has attached to it two little revolving disks or peepholes. If there is a counter for keeping tally in chaining, it is usually placed on the plate near the south sight. Fix in mind once for all which of the two uprights is the *south* sight, so as to tell it at a glance. Always keep this next to the eye in sighting, and always read the *north* end of the needle, as explained in Art. 279. Use a magnifying glass in reading, and look along the axis of the needle to avoid parallax. Read to the nearest half degree and estimate to the nearest 5′. It is good practice to glance at the *south* end of the needle as a check, especially if one moves around the instrument to take the reading, for in changing positions any source of local attraction about the person is likely to be detected by the corresponding change in the needle reading.

3. Protect the pivot in every way possible, for unless the pivot is sharp and perfect the needle may be sluggish and unreliable. Hence lift the needle off the pivot before starting to carry the compass to another point, and keep it off when not in use. Avoid excessive vibrations, after setting up, by turning the main plate before releasing the needle until the latter is nearly in the magnetic meridian. The same result will be obtained if the vibrations are checked by raising the needle slightly, but the longer the needle is in coming to rest the more accurate will be its final position.

4. Keep sources of attraction such as axes and chain pins away from the compass. Bits of steel about the person, such as pocket knives, wire in a hat brim, a steel suspender buckle, eyeglass cases, sometimes affect the needle. The glass of the compass box may become electrified from rubbing it—wet it with the finger or breathe upon it.

5. If the needle is sluggish it may need to be remagnetized by bringing it into the magnetic field of an electric light or a dynamo. In doing this the poles may be reversed. Test by putting the needle back on its pivot, and if reversal is found, put the needle back into the field the other end about. A more tedious method is to lay the needle down and rub each end from the center out with that end of a bar magnet which attracts the end of the needle that is being rubbed. In passing the magnet back from end to center, lift it some distance above the needle to avoid counteracting the rubbing.

6. For important lines take duplicate readings. After the first

reading, disturb the needle by lifting it or by tapping the box gently, and see if it comes to rest at the same reading as before. Another way is to turn the sights off the point sighted at and to bring them back again to check the original reading. This takes more time, but it is the surest check.

7. To guard against local attraction read the bearing of important lines at two different points—usually at the two ends.

8. When, because of obstructions, it is inconvenient to take the bearing of a line, take the bearing of a parallel line, established by offsets as near as practicable to the required line.

9. In ordinary work make sure that the declination arc is set at zero. When, however, it is desired to run lines by true bearings instead of by magnetic bearings, set off the declination arc (Art. 303(*i*)). In retracing old surveys the change in the declination of the needle may be ascertained and the declination arc adjusted so that the lines may be run from the bearings given by the original surveyor (Art. 218).

10. When the compass is put away, allow the needle to swing to the magnetic meridian and then lift it—it retains its magnetic strength longer in this position.

213. Compass Surveying—General Method. (*a*). REFERENCE LINES corresponding to transit lines, Chapter 21, are established, and the length and bearing of each ascertained. (See also Art. 304.)

213(*b*). WHERE TO RUN LINES. The suggestions in Art. 345 hold good. It is often possible, however, to make compass lines coincide with boundary lines, where in transit surveying it would be impossible. When this can be done it saves much unnecessary work (Art. 345(*a*)). Avoid choosing stations near sources of local attraction.

213(*c*). METHODS OF LOCATING DETAILS. The methods of Chapter 22 hold good, except that bearings of lines are read in place of angles in those methods which involve angular measurements. Because of the inaccuracy of the compass, methods involving linear measurements only are used even more than in transit surveying. The old rule that "when distances and direction to the same point disagree, distances govern" is more true in compass surveying than in transit work.

213(*d*). PRACTICAL SUGGESTIONS. (1) *Equipment* nearly the same as given in Art. 331, with the compass in place of the transit. (2) *Referencing stations* by marks on trees, etc. (See Art. 340.) (3) *Relation between office and field work.* (See Art. 343.)

214. Detection and Elimination of Local Attraction. In this article the term *local attraction* is used in its strict sense, and does

not include the deviation of the needle due to things about the person or to other avoidable sources.

214(*a*). Detection. When the bearings read at the two ends of a line do not agree, *local attraction* probably exists at one or possibly both ends, provided the compass is in good condition and no mistakes are made in observations. To ascertain the correct bearing, set up at a third point on the line, and see with which of the first two readings the third agrees. If it agrees with neither, try a fourth point, and so on until two readings do agree.

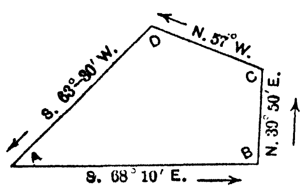


Fig. 214(*b*).

214(*b*). Elimination. The angle formed by two lines can be calculated from the bearings of the two lines regardless of local attraction, provided the bearings are both read at the same station, i.e., where the lines intersect. This fact suggests a method of eliminating local attraction from the bearings of a number of connected lines, as illustrated below.

Compass at	*Line*	*Bearing*	*Correction*	*Corrected Bearing*
A	*AD*	N 62° 40′ E	+50′	N 63° 30′ E
	AB	S 69° 0′ E	+50′	S 68° 10′ E
B	*BA*	N 67° 0′ W	−1° 10′	N 68° 10′ W
	BC	N 41° 0′ E	−1° 10′	N 39° 50′ E
C	*CB*	S 39° 50′ W	0	
	CD	N 57° 0′ W	0	
D	*DC*	S 57° 0′ E	0	
	DA	S 63° 30′ W	0	

Illustration. The bearing of each line in Fig. 214*b* was read at both ends of the line. The results in the third column above show that in only one case (the line *CD*) do the bearings agree. This indicates that there is no local attraction at either *C* or *D*. Hence the bearing of *DA* = S 63° 30′ W must be correct. But at *A* the reading is N 62° 40′ E, indicating an error of +50′ due to local attraction at *A* (+ = clockwise, − = counterclockwise). Applying this correction to the erroneous reading S 69° 0′ E of *AB*, the correct bearing of *AB* is S 68° 10′ E. But the reading at *B* is N 67° 10′ W, indicating an error of −1° 10′. Applying this correction to the other reading taken at *B*, i.e., *BC* = N 41° 0′ E, the correct bearing of *BC* is N 39° 50′ E. This checks with *CB* = S 39° 50′ W, taken at *C*, where there is no local attraction. Thus local attraction may be eliminated from any number of connected lines provided the correct bearing of one line is known.

Note. When the bearing of not even one line can be taken as correct, assume, for temporary purposes, that the correction at some one station, say the first station, is zero, and correct all the other bearings accordingly by the method just illustrated. It will usually be found that for a number of stations the correction is the same. It is therefore pretty safe to assume that at these stations no local attraction exists, and the bearings may again be corrected on this assumption.

215. Field Notes and Method of Procedure. (*a*). The field notes are usually kept by the sketch method (Art. 366). Thus, for example, if the compass had been used in the survey illustrated on p. 367, the bearing would be recorded along each line in place

of the angle. When simply the bearings of a few lines are taken and no sketch is required, the notes may be tabulated in a form similar to that in Fig. 215, a modification of Form 7, p. 356. Be careful to record whether the bearings are true or magnetic.

Station	Line	Bearing	Length		
D	DE	S.1°20'E	229.7	All bearings are Magnetic.	Declination = 10°58'W July 23, 1944
	DC	S.72°55'W			
C	CD	N.72°55'E	155.0		
	CB	N.8°45'E			
B	BC	S.8°45'W	87.3		
	BA	N.22°10'W			
A	AB	S.22°10'E	107.8		

Fig. 215.

215(*b*). Method of procedure. Set up at *A*, took the bearing of *AB*. Set up at *B*, took the back bearing of *AB*, i.e., the bearing *BA*, found that it checked with the forward bearing. Took the forward bearing of *BC*, and moved on to *C*. While this was being done the chainmen measured the lines between stations. Continued the work by taking the forward and back bearing at each station, as indicated in the notes.

Remark. The first column is unnecessary and may be omitted provided it is understood that the first letter of a line is the station at which the compass stands when the bearing is taken. If there are side shots for locating details, these may be entered in the tabulated form as in transit surveying, and an explanatory sketch made on the right-hand page, a method resorted to sometimes when there is not room on the sketch. In compass surveying, however, details are usually located by linear measurements only, and these should be put directly on the sketch itself.

If the methods of keeping notes for transit surveys have been mastered, it is an easy matter to adapt them to compass surveys.

216. Sources of Error. (1) Local attraction; (2) errors of observation; (3) instrumental.

The first source of error is annoying, but it can be eliminated by the method of the preceding article. The second includes errors in sighting and in reading, which can be avoided by following the suggestions in Art. 212. The chief source of error is the compass itself. Not only is it difficult to read the needle closely, but also different compasses will often give different bearings for the same line, even when the instruments seem to be in good adjustment.

217. Limits of Precision. The maximum error of bearing should not exceed 10′ if a good compass graduated to half degrees is used. Since 1 in 57.3 corresponds to 1°, this is equivalent to 1 in 344. If the bearing is correct to 5′, which is as close as the compass can be read with a reasonable degree of certainty, the error is 1 in 688. This is much less accurate than the results which may be reasonably expected in measuring distances with a chain tape. Of course, better results are often obtained with the compass, but a higher limit of error should not be required.

218. Rerunning Old Surveys. The chief use of the compass, in many communities, is in rerunning former surveys, especially in finding boundaries which have become more or less uncertain. The original surveys were made, say, 40 to 100 years ago, and the large tree or the pile of stones which once marked a corner has entirely disappeared. In such a case the surveyor can proceed only from such evidence as he can procure by examining town records, and by searching for existing bounds. If the owners who are interested will not accept his results, it is then a matter for the courts. The surveyor cannot, of course, settle a boundary dispute, though many persons seem to think so. He can, however, give expert testimony, and frequently he can persuade the disputants to accept a boundary line which he has run out without resorting to the courts. (See Chapter 58 for difficulties encountered in rerunning old surveys.)

Assuming that the lengths and bearings of the boundaries are known, it will be necessary to ascertain the change in the magnetic declination which has taken place since the original survey was made. If *two* points on any boundary line are still preserved, the most direct method is to sight along this line and to change the declination arc until the bearing agrees with the original. When, however, only one point on a boundary line remains, it will be necessary to find the change in magnetic declination in some other way. This change may often be calculated, provided the date of the original survey is known.* When tracing deeds in town records, however, it is not easy to know when one has gone far enough back, i.e., to be sure that one has reached the owner for whom the original survey was made. When the date has been ascertained and the change in the magnetic declination has been figured, the declination arc should be adjusted accordingly; or, if there is no provision for this adjustment, proper allowance should be made in the bearing of each line. The importance of dating a map and of giving the corresponding magnetic declination is obvious.

* When there is no local record of the change in magnetic declination from year to year, it may often be obtained from the *U.S. Declination Tables and Isogonic Charts*, published by the U.S. Coast and Geodetic Survey. (See also Art. 521 and Art. 801(*a*).)

CHAPTER 18

READING TRANSIT ANGLES

THIS chapter is divided into two parts to correspond to two distinct steps in reading a horizontal angle measured with a transit: (1) reading the *limb*, and (2) reading the *vernier*. In the first part are shown several different graduations of the limb, and in the second part several forms of verniers. To study all these graduations at the outset is likely to lead to confusion. Begin with the simplest forms as, for example, a limb graduated continuously (Fig. 225), and a vernier reading to minutes (Fig. 233).

Observe the common mistakes as illustrated by the incorrect readings in the book, and then practice with a transit graduated to correspond. When this much is thoroughly understood it is an easy matter to proceed with a transit more finely graduated. Before beginning this chapter the student should study Art. 722, in which the construction of the upper and lower plates of a transit is explained.

219. The Lower and Upper Plates of a Transit. In measuring horizontal angles with a transit, the magnitude of the angle is read from graduations on two circular plates. When the transit is set up properly, these two plates are level, and each may be revolved in a horizontal plane independently of the other. The lower plate or *limb* has on its outer edge a graduated circle. The upper plate carries either one or two verniers. The uprights (or standards) which support the telescope are attached to the vernier plate, and the zero of the vernier serves as an indicator. Hence the telescope cannot be revolved right or left without taking the indicator with it, and the *horizontal* angle through which it is turned is measured by the arc on the graduated limb passed over by the indicator. If the indicator is opposite the zero of the limb when the telescope points to the backsight, and stops exactly opposite a graduation mark when the telescope is turned to the foresight, the angle is read *without* the use of the vernier; when, as is more often the case, the indicator stops *between* graduation marks the vernier reading tells how far it has passed beyond the last mark.

220. Limb Graduations. The limb of the transit is commonly divided into 360°, and every *tenth* degree is usually numbered; but there are several systems of numbering and several methods of subdividing each degree.

220(*a*). COMMON SYSTEMS OF NUMBERING LIMB GRADUATIONS. (1) *Full circle,* 0° to 360° (Fig. 225); (2) *half circle,* 0° to 180°, then backward to 0° (Fig. 226); (3) *quadrant,* two zeros 180° apart, with numbers running up to 90° both ways from each zero (Art. 227).

220(*b*). COMMON METHODS OF SUBDIVIDING EACH DEGREE. (1) Into two parts, or 30′; (2) into three parts, or 20′; (3) into four parts, or 15′; (4) into six parts, or 10′.

220(*c*). REMARK. On most transits there are two rows of numbers, the outer row often corresponding to one system, the inner row to another. Each system has its advantages and is adapted to some particular kind of work.

On an ordinary transit each degree is divided into two or, at the most, three parts; it is only on instruments intended for more precise work that the limb is divided to 15′ or 10′.

221. General Method of Reading Angles. There are three distinct steps: (1) reading the limb; (2) reading the vernier; (3) combining the two readings.

Remark. The majority of mistakes made in reading angles are made in reading the limb, and not in reading the vernier. Moreover, a mistake in reading the limb usually amounts to at least 20′ or 30′ and often to several *degrees,* whereas a mistake in reading the vernier is seldom over a minute or two. Hence the working rule *"Make sure of the limb reading before looking at the vernier."* In learning to read angles the same rule holds good.

222. Reading the Limb. (*a*). No matter in which direction the plate carrying the indicator revolves there are always two readings for an angle: One is the value of the angle measured clockwise from the backsight, and the other is the value measured counterclockwise *from the same backsight.* Each value is the explement of the other, i.e., their sum is 360°.

If angles were always measured to the right (clockwise), the numbers of the graduations on the limb would always increase *clockwise,* and a common source of mistakes would be removed. In some kinds of work, however, as in deflection angles, readings are taken right and left, and this calls for numbers which increase counterclockwise as well as clockwise. These two rows of numbers must not be used indiscriminately, a constant source of mistakes with beginners.

222(*b*). LIMB READINGS FOR ANGLES TO THE RIGHT. Consider the indicator to have moved *clockwise.* (It is immaterial whether it did or not.) Observe which row of graduations increases *clockwise,* note the last number on that row passed by the indicator, and from this number determine the value of the graduation mark opposite the indicator or just behind it (Fig. 225).

222(*c*). LIMB READINGS FOR ANGLES TO THE LEFT. Proceed in a similar manner, but consider the indicator to have moved counterclockwise, and use the row in which the numbers increase counterclockwise (Fig. 225).

222(*d*). REMARK. For angles clockwise numbers increase clockwise, and for angles counterclockwise numbers increase counterclockwise. If one could stand *inside* the graduated circle, numbers for angles to the right would increase towards his right hand, and for angles to the left they would increase towards his left. Since, however, the observer stands *outside* the circle, the numbers for angles to the right increase towards his *left* hand, and vice versa. This leads to confusion which would be avoided if the terms clockwise and counterclockwise were always used in place of right and left respectively.

223. Common Mistakes in Reading the Limb. (1) Reading the number *ahead* of the indicator instead of behind it; (2) reading the inner circle of numbers instead of the outer circle, or vice versa; (3) reading *even* degrees instead of degrees and fractions of a degree as, for example, 43° 0′ instead of 43° 30′, or 11° 0′ instead of 11° 20′; (4) dropping a subdivision as, for example, 28° 20′ instead of 28° 40′.

224. Method of Reading the Limb Illustrated. On the succeeding pages are shown three different systems of numbering the graduations on the limb. The full circle as shown in each system is about half the true size. Below the circle is a portion of the limb greatly enlarged. The vernier is purposely omitted, but the indicators represent four different positions of the zero of the vernier. The readings noted *below* the arc are for angles taken to the right as indicated by the arrow, whereas those *above* the arc are for angles to the left. The first two common mistakes noted in the preceding article are illustrated by the incorrect readings in parenthesis. The third and fourth mistakes will be illustrated in the explanation of vernier readings (Arts. 233–236), since they are most often made when adding the limb reading and the vernier reading together.

Suggestions. In studying the illustrations keep in mind the fact that the limb of a transit remains stationary during the interval between the backsight and foresight while the indicator moves over an arc equal to the angle measured. Understand clearly the difference between measuring an angle to the right (clockwise) and to the left (counterclockwise). The following method of studying the illustrations is suggested.

1. Choose one of the four indicators, assume the angle to have been measured clockwise, and cover with a card the readings below the arc.
2. Notice which circle of numbers is increasing clockwise, find the number just behind the indicator, count up the intervening spaces, and read the angle.
3. Remove the card and see if the reading agrees with that given on the figure. Observe also the mistaken reading in parenthesis, and consider from what cause the mistake might be made.
4. Repeat the three steps given above for each of the other three indicators; also, for all four indicators assuming that angles were measured counterclockwise.

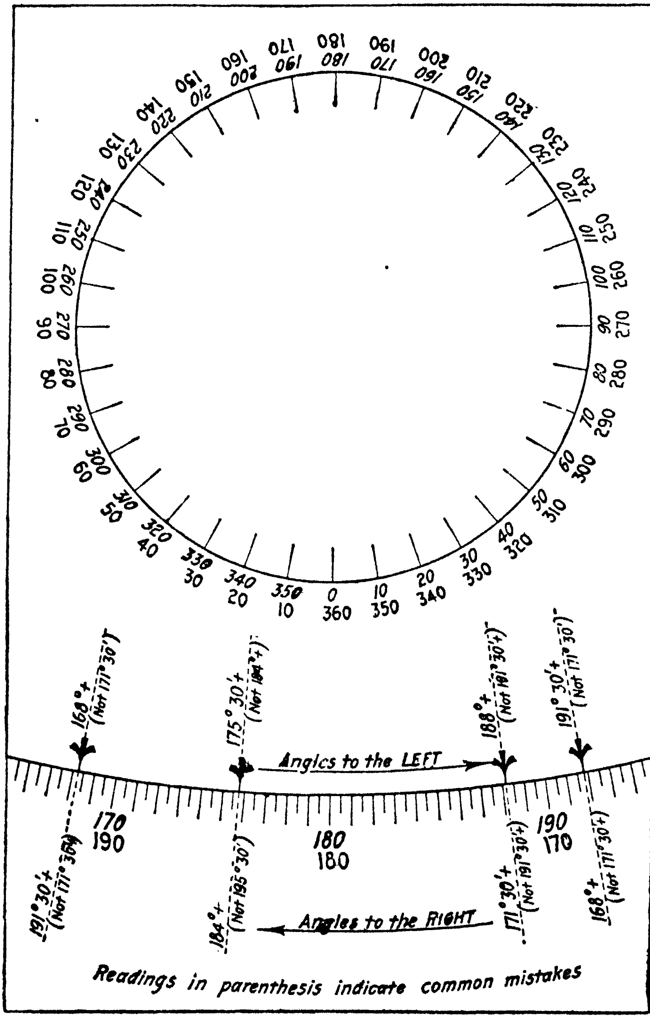

Angles to the LEFT
Angles to the RIGHT
Readings in parenthesis indicate common mistakes

FIG. 225.

225. Full-Circle System from 0° to 360° in Both Directions. (*a*) This is one of the best systems of numbers for all-round work. Angles may be measured in either direction, to the right or to the left, without confusion, provided simple precautions are taken. Thus for angles to the right always read the outside row if that is the row in which numbers increase *clockwise;* for angles to the left read the inside row. *When, as is often the case, all angles are taken to the right, ignore the inside row. Make it a point not to use it at all.* In some makes of instruments the numbers slant in the direction in which they should be read. (See full circle, Fig. 225.)

225(*b*). Reading the wrong circle. The portion of the limb shown enlarged is where the two circles have the number 180° in common, and this is where the mistake of reading the wrong circle of numbers is most likely to occur. Where the numbers are widely different it is usually sheer carelessness to read one for the other. If, for example, an angle of 100° is taken to the right, the indicator will stop opposite 100° and 260°, but an alert transitman would see that the angle could not be over 180° (the backsight produced), and hence would not read 260° in place of 100°. This illustrates, incidentally, the value of *habitually checking readings by estimating angles with the eye.*

225(*c*). Reading degrees ahead of the indicator. When the indicator stops just before it reaches a number it is a natural mistake to read as if it had gone by the number. This mistake is illustrated by the two incorrect readings of 171° 30′ and 195° 30′ for angles to the right, and by all of the incorrect readings to the left except 171° 30′ in place of 191° 30′. (What mistake does this illustrate?)

225(*d*). Reading a single circle counterclockwise. Notice that the sum of the outer number and the inner number at *any* point is always 360°. Hence when there is only one circle of numbers (0° to 360° clockwise) and it is desired to read an angle to the left, consider the indicator to have moved to the left (*counter*clockwise), but subtract the nearest number behind the indicator from 360°, and proceed as if the number were actually engraved upon the limb, as in fact it is when there are two circles. As the subtraction involves only *even degrees,* this method is better than to read to the right and subtract from 360°, for the latter subtraction involves *minutes* as well as degrees. Mountain transits and other transits intended for measuring angles to the right only, and hence having but one circle of numbers, may be used for measuring angles to the left by simply imagining the explementary number engraved on the limb as just suggested. (See Art. 226(*d*).)

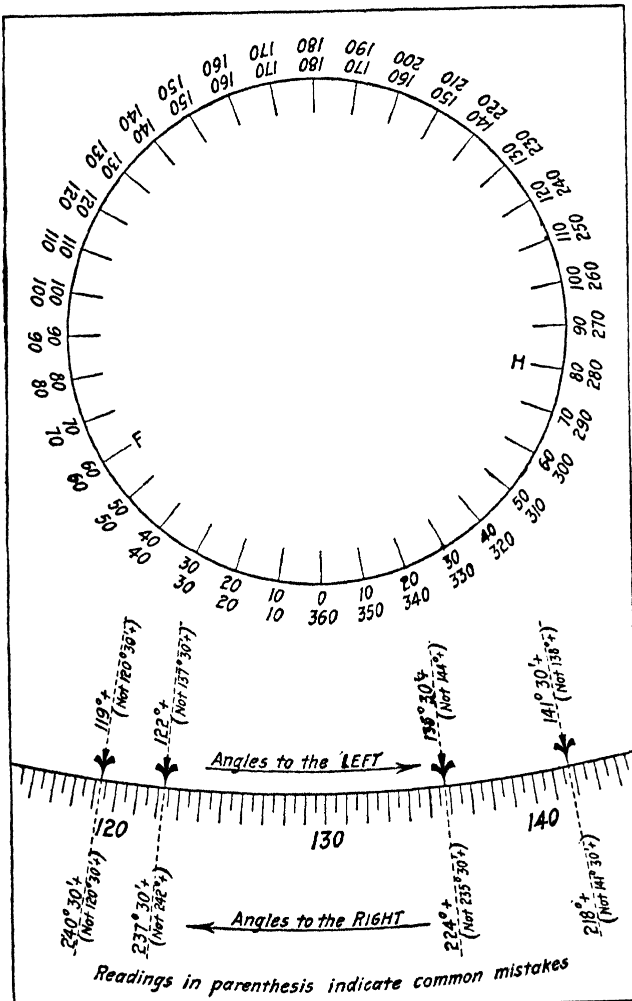

F
H
119°+
(Not 120°30'+)
122°+
(Not 137°30'+)
138°30'+
(Not 144°+)
141°30'+
(Not 138°+)
Angles to the LEFT
120
130
140
240°30'+
(Not 120°30'+)
237°30'+
(Not 242°+)
224°+
(Not 235°30'+)
218°+
(Not 141°30'+)
Angles to the RIGHT
Readings in parenthesis indicate common mistakes

FIG. 226.

226. Combination of Full-Circle and Half-Circle Systems. (*a*). In Fig. 226 the outside circle of numbers is a *full*-circle system, increasing clockwise exactly like the outside circle in the preceding illustration. No further explanation of its use, therefore, is needed.

The inner circle is a *half*-circle system and is used for measuring angles which do not exceed 180°, as, for example, deflection angles. Thus, if the indicator is at *F* in the inner circle, the deflection angle is 60° to the right, but if it is at *H*, the deflection angle is 80° to the left.

226(*b*). COMMON MISTAKES. The portion of the limb shown enlarged is near the 230° mark of the outer circle, but the outer numbers have purposely been omitted. The angles to the left are read in the usual manner. The mistaken readings for angles to the left all come from reading the number in front of the indicator instead of the number behind it. There is, of course, the same danger of reading the wrong row of numbers as in the illustration (Fig. 225).

226(*c*). SINGLE-CIRCLE GRADUATION. Notice that the sum of the corresponding numbers in the outer and inner rows is always 360° *after passing the 180° mark of the outer circle.* On some instruments, intended only for measuring deflection angles, the full-circle system is omitted. Suppose that it is desired to measure angles to the right *over* 180° on such an instrument. After the indicator passes 180° the numbers run backward or in the wrong direction for angles to the right. Subtract the number behind the indicator from 360°, and proceed as if the result were engraved on the limb as explained in the preceding article.

226(*d*). ILLUSTRATION. On the enlarged arc it is observed that the numbers are increasing in the wrong direction for angles to the right. If the indicator is between 120° and 130°, the number behind it, for angles to the right, is 130°. Hence 360 − 130 = 230°, and proceeding as if 230° were engraved opposite 130, as it really is when there are two rows of numbers (see complete circle), the final reading is 237° 30′+. Likewise, if the indicator is between 140° and 150°, the number behind it (outside the limits of the drawing) is 150, and hence 360 − 150 = 210°, from which 8° more is counted clockwise, making the final reading 218°+. The other readings, 224°+ and 240° 30′, are obtained in a similar manner.

226(*e*). REMARK. In taking deflection angles it is desirable to keep the zero of the limb nearly under the eyepiece of the telescope to save walking around the instrument when reading angles. If the half-circle system is used, it will be necessary, after setting the indicator at zero, ***to plunge the telescope before backsighting.*** In order to avoid this the ***quadrant*** system of numbering is often preferable, especially in railway work, where deflection angles are seldom more than 90° (Art. 227).

227. Combination of Full-Circle and Quadrant Systems. In this combination, one-half the *inner* circle is numbered from 0° to 90° in each direction from 0° exactly as in Fig. 226; but instead of continuing to increase to 180° as in that figure, they decrease to 0°. Thus there are *two* zero points on the inner circle, one directly across the circle from the other, and from each of these two zeros numbers increase to 90° both clockwise and counterclockwise. The chief advantage of this system over the full-circle and half-circle systems is that in measuring deflection angles it is not necessary to plunge the telescope before backsighting. However, in work which involves the measurement of deflection angles greater than 90° the combination of full-circle and half-circle is more convenient.

228. Verniers. (*a*). The principle of the vernier, the method of determining the *least count,* and methods of reading verniers are explained in Art. 703. It is easier to read a vernier correctly, once the least count has been ascertained, than it is to read the limb, and mistakes are less costly. Generally if a vernier is divided into ten equal spaces *between numbers* as shown in Fig. 233, it reads to minutes, i.e., each space is *called* a minute though it really covers nearly a degree. If the equal spaces are again subdivided into two parts, each part is *called* 30″ (Fig. 235); if subdivided into three parts, each of the smallest parts is *called* 20″ (Fig. 236); if into six parts, each part is *called* 10″.

228(*b*). REMARK. It helps, in reading the vernier, to think of the larger divisions as minutes, and the smaller subdivisions, if there are any, as 30″ or 20″ or 10″, as the case may be. It is well to remember, however, that each space on the vernier really covers a much greater arc than is indicated by what we call it.

228(*c*). THE LEAST COUNT as determined by inspection should be verified when using any vernier for the first time. From the principle of the vernier the least count, i.e., the difference between l, the smallest division on the limb, and v, the smallest division on the vernier, is $x = l - v = l/(n+1)$, where n represents the number of spaces on the limb corresponding to $n+1$ spaces on the vernier, or $nl = (n+1)v$, from which equation the value of x is derived. For verniers as usually constructed the following rule will also apply.

Divide the value of the smallest division on the limb by the number of parts on the vernier.

229. To Read a Vernier. Keeping in mind the value of the smallest division on the vernier it is an easy matter to get the vernier reading. If the zero or indicator exactly coincides with a mark on the limb, the angle is read without using the vernier; otherwise find some mark on the vernier which is exactly opposite some mark on the limb. It is immaterial what the mark on the limb is—the mark on the vernier determines the *vernier* reading. The limb reading to which the vernier reading must be added is determined separately, as explained in the first part of the chapter.

230. Suggestions for Reading Verniers. (1) *Double verniers* like those shown on succeeding pages would be unnecessary if angles were always taken clockwise. *Always read that side of a double vernier which is ahead of the zero or indicator,* the direction in which the latter moved having been assumed in reading the limb. (2) When reading the limb estimate at a glance the approximate vernier reading. This not only saves time by indicating where to look on the vernier for the line of coincidence, but it often saves blunders such as reading the wrong side of a double vernier. (3) Form the habit of looking at one or two marks each side of the vernier mark that you think is in coincidence. If three marks coincide, take the middle one; if two marks coincide, take the average reading. In the latter case the reading will be closer than the least count of the vernier. For example, the reading will be to 30″ on a vernier which reads only to minutes. This means that *the error involved in reading any vernier is not greater than half the least count of that vernier.* (4) In setting the vernier at zero or at any angle, examine in the same way one or two marks each side of the indicator. (5) When there are two verniers 180° apart use the one under the eye end of the telescope. Avoid setting one at zero and reading the other. (6) Use a reading glass. (7) Practical suggestions for measuring angles are given in the next chapter.

231. Common Mistakes in Reading Verniers. (1) Mistakes due to reading the wrong half of a double vernier. (2) Arithmetical mistakes in adding the vernier reading to the limb reading, or, what is still more common, dropping the *minutes* from the original limb reading, as 68° 16′ instead of 68° 46′. (See Fig. 233.) Do not be so intent on reading the vernier as to lose track of the correct limb reading. Remember that the limb reading is by far the more important. (3) In verniers reading to seconds the mistake is sometimes made of calling seconds minutes, as 151° 40′ instead of 151° 0′ 40″. (See Fig. 236.) This is most likely to happen when the vernier reading is less than a minute.

232. Illustrations of Verniers. In studying the following illustrations of different verniers keep in mind: (1) The vernier and limb are enlarged to a little more than twice the usual size. (2) The lowest line of each illustration shows how the least count is obtained from the formula in Art. 228(*c*). (3) The whole process of reading the angle is indicated: first the limb reading, then the plus sign, then the vernier reading, and finally the total reading. The mistaken readings in parentheses are wholly mistakes in vernier readings or in dropping minutes from the original limb readings.

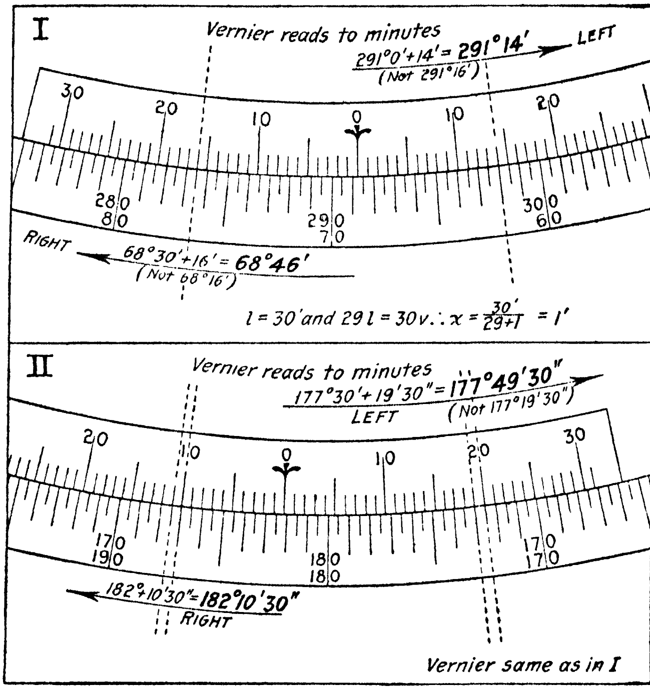


FIG. 233.

233. I and II. Verniers Reading to Minutes. (*a*). This is one of the forms of verniers most used in ordinary work. The mistake of 291° 16′ instead of 291° 14′ came from reading the wrong half of the double vernier. In the mistake of 68° 16′ instead of 68° 46′ the 30′ of the original limb reading was forgotten when it came to adding the readings. The mistake of 177° 19′ 30″ instead of 177° 49′ 30″ came in a similar manner from dropping the minutes from the original limb reading.

233(*b*). In II two marks on the vernier apparently coincide with two on the limb, and hence the mean of the two readings is correct. This illustrates how 30″ may appear in a reading though the smallest space on the vernier is *called* a minute. In a great deal of work it is not worth while to take account of seconds in a case like this—simply read to either whole minute.

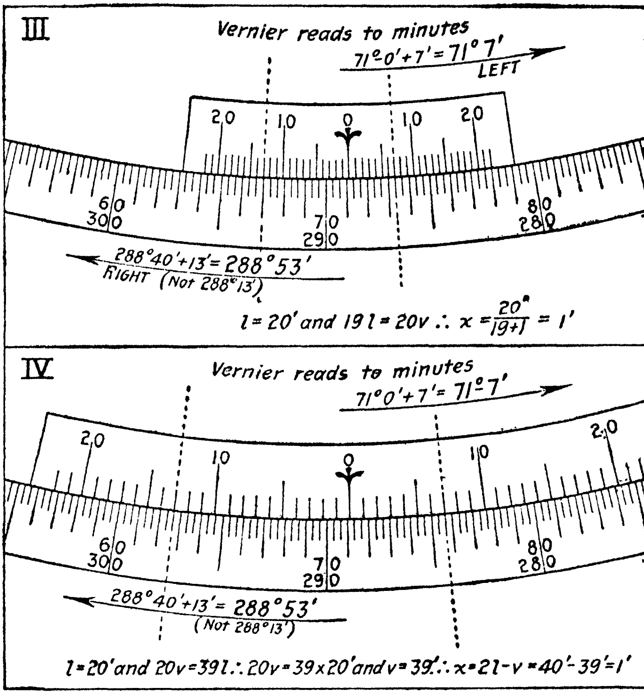


FIG. 234.

234. III and IV. Verniers Reading to Minutes. (*a*). The chief advantage of the form of vernier in III as compared with that in the preceding illustration is that it is shorter, and may therefore be used in a small space, as, for example, when set between the standards to go with the vertical circle.

234(*b*). In illustration IV the vernier is constructed like that in Fig. 235 except that every alternate mark on the vernier is omitted. Each division on the vernier is called *one* minute, but it is really 39′ long or 1′ short of two spaces on the limb. The advantage of this construction is that the division marks on the vernier itself are farther apart than those in I, II, and III, and hence are easier to read. Though this is an excellent form of vernier it is not so common a one as that in Fig. 233.

The mistaken readings in both III and IV are due to dropping the minutes from the original limb readings.

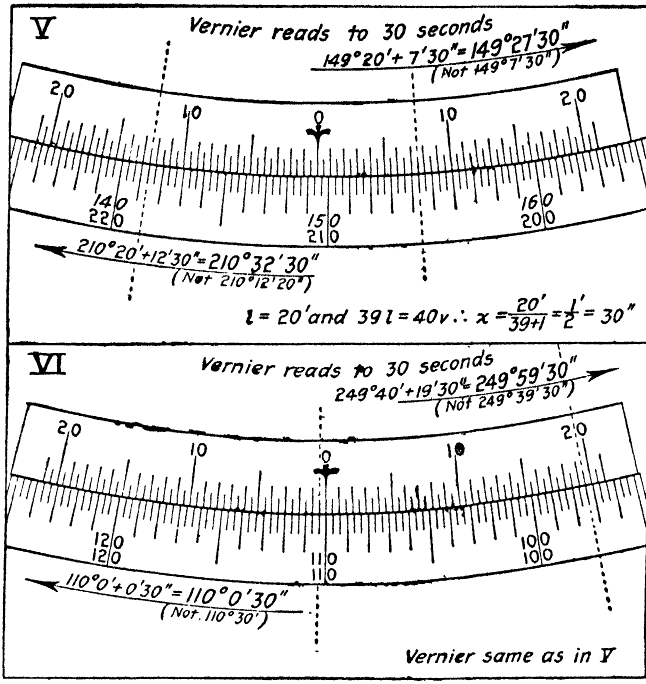


FIG. 235.

235. V and VI. Verniers Reading to 30 Seconds. The limb is graduated exactly like the limb in Fig. 234, but the vernier is more finely graduated, each of the small divisions being called 30″.

Both the mistakes in V are due to dropping 20′ from the original limb reading. Notice also in the reading 210° 12′ 20″ the mistake of calling the smallest division on the vernier 20″ instead of 30″. This is a natural mistake, since the smallest division on the limb is 20′. The mistake of 249° 39′ 30″ was also due to dropping 20′ because of the impression that the limb reading was 249° 20′ instead of 249° 40′.

The mistake of 110° 30′ instead of 110° 0′ 30″ came from adding the vernier reading as if it were 30′ instead of 30″.

The fact that 30″ and 20′, the smallest divisions on the vernier and the limb respectively, are unequal *numerically* leads to confusion as indicated above, and makes this form much less desirable than that in Fig. 236.

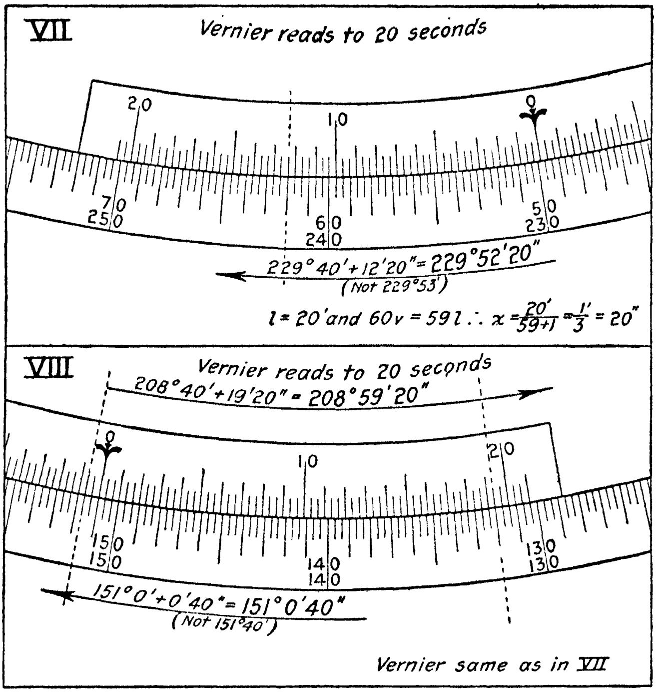


FIG. 236.

236. VII and VIII. Verniers Reading to 20 Seconds. This form of vernier, in which the smallest space is called 20″, is used for city work, simple triangulation, and other work of high accuracy.

The mistake of 229° 53′ instead of 229° 52′ 20″ came from observing the 12′ mark on the vernier and then counting the next space as if it were a whole minute instead of 20″, thus getting 13′ to add to the limb reading instead of 12′ 20″.

The mistake of 151° 40′ instead of 151° 0′ 40″ came from calling the vernier reading 40′ instead of 40″. This would probably not have occurred had the vernier reading been a full minute or more.

Question. Keeping the graduations on the limb the same, how could the vernier be changed so that its least count would be 10″?

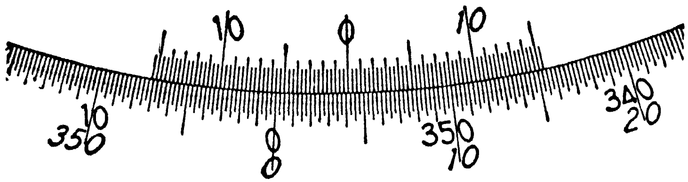


FIG. 236(*a*).

236(*a*). ANOTHER COMMON METHOD OF GRADUATING THE LIMB AND THE VERNIER TO READ TO 20″ is shown in Fig. 236(*a*).

The limb is divided to 15′ instead of to 20′ and the vernier is graduated to read to 20″. (Compare with Fig. 236.) Check by computation the least count of the vernier.

237. Verniers for Precision Instruments. Verniers reading to 10″ or less are common on instruments designed for the precise measurement of angles. On such a vernier several graduation marks may appear to coincide with marks on the limb; the reading should then be taken midway between the two extremes of the coinciding marks.

238. Eccentricity. If two opposite verniers of a transit do not read exactly alike regardless of their position on the graduated circle, it is due to eccentricity, the causes for which are explained in Art. 727. Errors due to eccentricity may be eliminated either (1) by always reading the same vernier, i.e., the one that was set at 0° or (2) by always taking the mean of the two vernier readings.

239. Special Forms of Verniers. Special forms of verniers may be devised at will, as, for example, verniers reading to a *decimal* of a degree (convenient for railway curves).

239(*a*). VERNIER ON A SEXTANT. A vernier sometimes used on the sextant has 60 spaces corresponding to 119 on the graduated arc. This makes each space on the vernier a little less than two spaces on the arc. It is used as a direct vernier. (If the smallest division on the graduated arc is 20′, prove that the vernier reads to 20″.)

239(*b*). RETROGRADE verniers are seldom used on transits.* A special form of retrograde vernier is used when it is desired to economize space as, for example, in the compass box for setting off the declination.† Thirty spaces on the vernier correspond to 31 on the limb, the vernier reading to minutes. There are two rows of numerals on the vernier arranged as follows:

	20	25	30	25	20	
15	10	5	0	5	10	15

The usual rules for reading do not hold good. Thus, if the indicator is to the *right* of the zero, read the lower figures on the *left*-hand vernier, unless the reading is over 15′, and then the upper figures on the right-hand vernier are used. If the indicator is to the left of the zero, read lower right-hand figures for less than 15′, upper left-hand for more than 15′. (See from the arrangement of the numerals that this must be so.) Note that this vernier is only half as long as the ordinary double vernier, and that the lower left-hand and the upper right-hand verniers form one vernier and vice versa.

* Verniers in which the smallest space is larger than the corresponding smallest space on the limb. (See Art. 703(*e*).)

† For setting declinations a similar "folded" vernier is often used which is direct instead of retrograde, 30 spaces on the vernier corresponding to 29 half degrees on the limb.

240. Reading Vertical Angles. No new principles or methods are involved in reading vertical angles from graduations on the vertical circle or the vertical arc of a transit, except that when measuring the angle that a line makes with a horizontal plane, the reading should be zero when the line of sight is level.

241. Special Method of Checking Angle Readings. There are several methods of checking angle readings to avoid mistakes. One of the most common is to repeat the measurement of the angle one or more times. Conditions may be such, however, that the time for measurement is too short to permit such repetition as, for example, when taking an observation on the sun apparently in motion. The following method of checking an angle reading, devised by Professor Claude H. Wall, is particularly useful when only one measurement of the angle is made. It will be explained by means of two illustrative examples.

241(*a*). TO CHECK THE ANGLE READ CLOCKWISE IN I, Fig. 233. (1) The reading is 68° 46′. (2) Using the left-hand 30′ mark of the vernier as the index (instead of the zero mark) and the left-hand portion of the vernier, a second reading of 83° 16′ is obtained. (3) The difference between the readings in (1) and (2) should be 14° 30′ (first check). (4) Using the right-hand 30′ mark of the vernier as an index and the right-hand portion of the vernier, a third reading of 65° 44′ is obtained (still using the outer row of numerals). (5) In the sum of the readings obtained in (2) and (4), the three right-hand numerals should be 9° 00′ (second check).

Second reading	= 83° 16′	Second reading	= 83° 16′
First reading	= 68° 46′	Third reading	= 65° 44′
Difference	= 14° 30′ (check)	Sum	= 9° 00′ (check)

This constant difference of 14° 30′ and constant sum of 9° 00′ will serve as a check on any angle reading taken either on the horizontal limb or on the vertical circle of a transit having 1′ verniers, provided the system of graduations is that shown in I.

241(*b*). TO CHECK THE ANGLE READ CLOCKWISE IN V, FIG. 235. The general procedure is the same as that just followed. Three readings are taken (1) with the zero mark as the index, (2) with the left-hand 20′ mark as the index, and (3) with the right-hand 20′ mark as the index.

Second reading	= 223° 32′ 30″	Second reading	= 223° 32′ 30″
First reading	= 210° 32′ 30″	Third reading	= 202° 27′ 30″
Difference	= 13° 00′ 00″	Sum	= 6° 00′ 00″

The constant difference of 13° and the constant sum of 6° will serve as a check on any reading taken on the limb of a transit having verniers reading to 30″, provided the system of graduations is that shown in V, Fig. 235.

The constant difference and the constant sum for any other system of graduations, such as that of any one of those shown in Figs. 233, 234, and 236, may be easily determined.

242. Accuracy in Reading Angles. The error in reading an angle is due partly to the error in the initial reading when the vernier is set at zero (or at some other initial reading) and partly to the error in the final reading after the angle has been turned. The error in *reading* an angle should always be understood, therefore, to be the total error that results from the combination of the errors

in both the initial and the final readings. The error is due solely to inaccuracy in reading. It does not include blunders and mistakes such as those described in Art. 231, nor does it include errors in field work such as errors in sighting and in manipulating the instrument.

Frequently in ordinary work it is neither necessary nor expedient to read angles with great accuracy. Readings may often be taken satisfactorily without the use of a reading glass. However, when in doubt it is best to use a glass. A series of experimental readings were taken on each of several transits having 1′ verniers. From these readings it was found that the average probable error of reading an angle without a reading glass was 22″, whereas the average probable error of reading with a glass was only 12″ or about one-half as great.* It should be added that these values do not represent the errors that would result in the *measurement* of angles since the investigation was carefully designed to eliminate all sources of error except those of setting the vernier and of reading the vernier after the angle had been turned.

In surveying with ordinary transits, the accuracy in vernier reading depends more upon the clarity or definition of the graduation marks than upon the spacing of marks or the least count of the vernier. However, in the precise measurement of angles, instruments are used which have fine, closely spaced graduations on both the limb and the vernier. It is not easy to read angles with the high degree of precision generally required in the use of such instruments. Several observers reading with care the same angle are likely to obtain different results. The error may be due partly to inherent, personal errors of observation that may be difficult to eliminate and partly to errors that can be greatly reduced by taking proper precautions in observing graduations.

242(*a*). Error due to parallax in reading verniers. A graduation mark on the vernier that is exactly opposite a graduation mark on the limb may not appear to be so if the eye of the observer is on either side of the vertical plane through the two marks. Moreover, if the eye is not in this plane, the observer may mistake some other mark on the vernier as one that is opposite to a mark on the limb. In order, therefore, to read an angle accurately, the eye of the observer should be brought into the proper plane. The error resulting from a failure to do this is called the *error due to parallax.* It is a common error and one that is not easy to avoid. For this reason, the better instruments are equipped with some de-

* From *Two Investigations of Transit Instruments* by Professor W. H. Raynor of the University of Illinois.

vice for bringing the eye into the proper position. On instruments used in the precise measurement of angles, this device consists of a reading microscope with an alignment index and a small field of vision.

Note. *Parallax* may be defined as the apparent displacement (or the difference in apparent direction) of an object as seen from two different points. Parallax is evident when the head is moved while observing two objects at unequal distances from the eye. There is also an effect of parallax when we alternately shut one eye and open the other.

When the eyepiece of a telescope is not properly focused, the image of an object may appear in an erroneous position with respect to the cross hairs. The error due to this apparent displacement (parallax) is often too large to be permissible even in surveying of ordinary accuracy.

Summary. For a summary of this chapter see Arts. 262–263.

CHAPTER 19

USE OF THE TRANSIT

THE only way of learning to use the transit is to practice with it in the field, but many suggestions and directions may be given which will help to make that practice intelligent and effective. After a preliminary reading, this chapter will best serve its purpose if the different articles are consulted *at the time that corresponding work with the transit is being done* in the field. Study, first, *methods* of work, and then consult the chapter frequently, thus gradually acquiring *habits* of work. It may be added that directions which often seem trivial to the beginner are nevertheless well worth following when learning to use such an important instrument as the transit.

Before beginning this chapter the student should study Art. 722 on the construction of the transit in order to become familiar with its various parts. It is particularly important to understand how the instrument may be revolved on either its inner or outer spindle and how the clamps and the tangent screws function.

243. Care of the Transit. The transit is a delicate as well as a costly instrument. It is particularly important that in learning to use it, one should learn first of all what precautions to take to reduce wear and tear, to avoid accidental injury, and to keep it in good working order.

243(*a*). SUGGESTIONS. It is often difficult if not impossible to restore an injured part of a transit to its original efficiency; yet only by taking *intelligent* precautions can injury be avoided. For this reason the student is advised to review the suggestions for the care of the level given in Art. 101; those suggestions, with slight modifications, apply equally well to the care of the transit. One important modification is as follows. Before taking the transit up to carry it, clamp both the upper and the lower plates together, tighten the lower clamp just enough to allow the two plates to revolve freely if accidentally hit; point the telescope straight up and clamp lightly so that it is free to turn on its supporting axis at the slightest knock; make sure that the compass needle is lifted off its point.

For more explicit instruction in the care of the transit see Chapter 55. For the adjustment of the transit see Chapter 54.

244. Definitions. *Transit Station.* A point over which the transit stands and is centered when in use. Some of the customary methods of marking such points are: a stake driven until its top is flush with the surface of the ground, with a tack or small nail in the top to mark the exact point; a small spike driven flush; a small but more or less permanent mark in a hard surface such as a pavement or natural rock; an iron pipe set vertically in the ground; a permanent stone or concrete monument with the point marked on a metal cap. *Transit Line.* An imaginary straight line between two transit stations. *Transit Angle.* Any angle measured with a

transit. It must, from the way in which the transit is constructed, be either a *horizontal* angle or a *vertical* angle. (See the description of the construction of the transit, Art. 722.) Unless otherwise specified it is usually understood to be a horizontal angle.

245. Plumb Bob. Do not begin the day's work by leaving the plumb bob behind. The string to the bob is looped over a hook under the center of the transit, and has a sliding knot by which to adjust its length. Figure 245 shows the sliding knot before it is drawn tight. But instead of putting the end of the string through the loop as shown in the figure, double it back upon itself so that the knot may be untied at any time by simply pulling the free end, as in the case of a bow knot. Avoid any other knot in the string, even one at the end. There are many devices that are used in place of the sliding knot. One that is easily made is that in which one end of the string is fastened to a piece of wood containing two or three holes, while the other end of the loop passes through the holes and down to the plumb bob. Friction and the crook in the string hold the device in any position as it slides up or down the plumb line. Bone or leather may be used in place of the wood. If the wind is blowing, shield the bob with something (a trouser leg will do) to keep it from swinging.

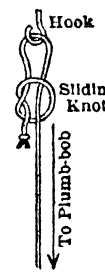


FIG. 245.

246. Setting Up the Transit. The transitman aims to do two things in setting up a transit, viz.: (1) to "center" it over some given point, such as the head of a small nail in the top of a stake; (2) to "level up," i.e., to make the axis of the transit's spindle truly vertical; this will bring the graduated circle into a horizontal plane, provided the manufacturer has made the plane of the circle and the axis of the spindle perpendicular to each other. The tripod, plumb bob, and shifting head are used to center the transit, the thumb screws and plate levels to level up. The general procedure may be outlined as follows.

1. Set the tripod, with shoes pressed firmly into the ground, in such a position that, when its head is *approximately level* (judged by the eye), the plumb bob will come to rest *nearly* over the point (tack). (A preliminary setup may be advisable. See suggestion 4, Art. 246(*a*).)

2. Loosen two *adjacent* thumb screws, and, using the shifting head, bring the point of the plumb bob over the required point (center of tack head). Use the sliding device on the plumb bob string, if necessary, so that the point of the bob just clears the stake.

3. Tighten the two thumb screws (previously loosened for shifting), and by means of all four screws level up.

4. See that the plumb bob still hangs over the center of the tack when the plates are level. If in leveling up it has been disturbed, bring it back to the center by using the shifting head and level up again.

Speed in setting up a transit is worth a great deal, for a slow transitman can delay the work of a whole party. Moreover, setting up over a railway track "between trains" often means quick work. It is true that the knack of setting up a transit comes from practice, but make it *intelligent* practice from the start. Consult frequently the practical suggestions which follow and practice until the methods suggested become second nature.

246(*a*). Practical suggestions for manipulating the tripod

1. Spread the tripod legs just far enough to bring the telescope at a convenient height; it saves stretching up or stooping down.

2. Manipulate the tripod so that its head is approximately level to begin with. The less the thumb screws are used the better; saves wear, saves time. On level ground set the shoes about equal distances from the tack. On a slope place *two* legs on the downhill side at about the same elevation, and make the tripod head approximately level by moving the third leg.

3. If the tripod head is too much out of level, the thumb screws in leveling up will bind; do not force them; start over again by moving the legs of the tripod so as to make the head level.

4. *Preliminary Setup.* Set up the transit at first approximately over the station without any attempt to bring the bob exactly over the tack, the main object being to get the plates *approximately* level, as judged by the eye, and the telescope at a convenient height, with the tripod shoes merely resting on the ground. This having been accomplished in two or three quick movements, lift all three shoes off the ground and move the transit bodily until the bob is over the tack, push the legs into the ground, and proceed to use the shifting head and leveling screws as already explained.

5. *Three Important Suggestions for Slightly Shifting the Tripod.* When the bob is within a few inches of the tack, but too far away to be brought over it with the shifting head, the legs of the tripod must be moved slightly. Keep in mind these three suggestions:

(*a*) If the plumb bob is off the tack, but the plates are nearly level, move a tripod leg on its own pivot towards or away from the tack. (Changes the position of the bob, but does not throw the plates greatly out of level. This is the condition most often met with.)

(*b*) If the bob is over the tack, but the plates are out of level, swing a tripod leg right or left without moving it on its own pivot. (Tends to level the plates without changing the position of the bob.)

(*c*) If the bob is only a little off the tack, first get it "on line" with one leg and the tack, and then bring it over the tack by moving that leg thus: If bob is at *a*, Fig. 246,5(*c*), bring it to *b*, or to *c* by moving legs *d* and *e*, and then bring it over the tack by moving leg *f*, or by moving both legs *d* and *e*.

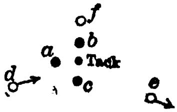


Fig. 246,5(*c*).

6. Until the work of setting up the transit is nearly completed, the plumb bob may be kept an inch or more above the point; but in the final centering with the shifting head, it should be lowered until it nearly touches the point. If the wind is blowing, shield the bob and string with a trouser leg, checking the swing at the top of the string. The shifting head is moved simultaneously with the stilling procedure.

7. The tripod must stand firmly. On soft ground push the legs well in, but not as if they were to stay for all time. Pushing against a leg crosswise is a waste of strength and may break it by bending; push lengthwise. On hard, smooth surfaces, such as floors and pavements, put the shoe points, if possible, in cracks or small holes, to prevent slipping.

8. Avoid setting a tripod leg on line where it might interfere with stretching the tape in chaining between stations.

Remark. The work of setting up a transit in a difficult position may be simplified, and time may be saved by the use of a tripod which has legs of adjustable length. With such a tripod some of the suggestions just given do not apply.

246(*b*). Suggestions for manipulating the leveling screws.

1. Two *diagonally opposite* screws are worked at the same time, first one pair, then the other, alternating until the plates are level. The thumbs always move toward or away from each other. The knowledge of which way to turn the screws should soon become instinctive. It may be helpful, however, to remember *that the bubble will move in the direction in which the left thumb is turned.*

2. If there is only one plate level, revolve the plate until the level bubble is parallel to the first pair of diagonally opposite thumb screws used, and bring the bubble nearly to the center; then revolve the plate until the bubble tube is parallel to the other pair of diagonally opposite screws and level with these screws. Repeat the process until the bubble remains in the center of the tube as the plate is turned through 90°. If there are two plate levels, each is placed parallel to a pair of diagonally opposite screws, and first one and then the other pair of screws are manipulated. When the plate levels are in adjustment, each bubble should remain in the center of its tube when the plate is turned through 360°.

3. Keep all four thumb screws bearing evenly and firmly, neither too tight nor too loose. If one of a diagonally opposite pair of screws is turned faster than the other, it will either bind or be raised off the plate. Stop turning this screw until the other catches up. If both screws of a pair bind, loosen one of them slightly. If all four are turning hard, the tripod may be too much out of level, or one pair may have been forced, or possibly the screw threads need cleaning. If this cannot be remedied without forcing a screw, consult an instructor. In no case should the screw be forced. The leveling screw should turn so easily that they may be *rolled* between the palms of the hands. This is quicker than turning them with the fingers.

4. Do not try to center one bubble exactly until the other has been approximately centered.

5. After leveling up, take hold of the graduated plate and see if it is firm. If it rocks, one or more of the thumb screws is not bearing properly.

6. In setting up to prolong a straight line, it is well to have one pair of diagonally opposite thumb screws in the line, the other pair at right angles to it.

7. If a plate level is not in adjustment, its bubble will not remain centered when the graduated plate is turned completely around, regardless of how the leveling screws are manipulated. If, however, the bubble comes to rest in the same position in its tube when the plate is turned, the vertical spindle of the transit is truly vertical. It is best, nevertheless, to keep the plate levels in good adjustment. (See also the method of bringing the spindle to a truly vertical position by observing the telescope level (Art. 247).)

Note. The suggestions just given, slightly modified, apply to the manipulation of the leveling screws where there are three instead of four leveling screws.

246(*c*). Accuracy required in setting up a transit

It is not always necessary to have the bob exactly over the tack or the plates exactly level. Like all surveying operations, setting up demands care in proportion to the accuracy required.

Accuracy in Centering. If angles are read to the nearest minute, and points sighted at are 50 to 100 ft. away, the point of the bob can be ¼ to ½ in. from the center of the tack without seriously affecting results. An experienced transitman, however, should be able to bring the bob to within less than ¼ in. of the point without special effort or waste of time.

Accuracy in Leveling Up. The following guiding principles should be remembered.

1. In measuring the horizontal angle between two points about on the same level, the plates need be only approximately level. (They can often be leveled by the eye alone.)

2. In measuring the horizontal angle between a high and a low point, more care should be taken in leveling up—especially should the bubble be in the center of that tube which is at right angles to the telescope.

3. In measuring the vertical angle between points in the same vertical plane, it is only that plate level parallel to the telescope which requires special attention.

The student will be well paid if he will experiment for himself along the lines suggested by this article.

Errors that are caused by not getting the center exactly over the tack or by not getting the plates exactly level are, as a rule, comparatively small. (See Art. 283.)

247. Special Method of Setting Up the Transit. In precise work, especially when the transit is to be used for leveling, or for measuring an angle when a line of sight is steeply inclined, the level attached to the telescope, being more sensitive than the plate levels, may be used in the final test. The procedure is as follows:

(1) Set up and level the transit as usual, guided by the plate levels. (2) Bring the telescope over a pair of diagonally opposite leveling screws. (3) Set the telescope bubble at the center of its tube by means of the clamp and the tangent screw which control the telescope (vertical motion). (4) Swing the telescope end for end. (5) Bring the bubble halfway toward the center by means of the pair of leveling screws. (6) Swing the telescope back to its original position and bring the bubble to center by means of the vertical motion. (7) Repeat until the bubble remains in the center in both positions. (8) Swing the telescope through 90° until it is over the other pair of leveling screws and by means of those screws bring the telescope bubble to the center of its tube. (9) Swing the telescope to several positions around the horizontal circle. The bubble should remain in the center throughout; if it does not, repeat steps (2) to (8).

The object of leveling the transit is to make its vertical spindle truly vertical. If the telescope level has not been adjusted to bring its axis perpendicular to the axis of the transit's spindle, the bubble will not remain in the center regardless of how the leveling screws are manipulated. If, however, the bubble comes to rest in the same position in its tube when the telescope is swung around the circle, the axis of the transit's spindle is truly vertical. It is best, nevertheless, to keep the telescope level in good adjustment (Art. 739(*i*).)

248. Use of the Telescope. (*a*). FOCUSING involves making the cross hairs clearly visible, focusing on an object by moving the objective slide in or out, and last but not least the *elimination of parallax*. These three operations are explained in Art. 103.

248(*b*). SIGHTING ON AN ESTABLISHED POINT. (1) If the point sighted at is near the transit, it may be possible to bring the intersection of the cross hairs directly on the point itself. For example, it may be possible to sight directly on the head of a nail in the top of a stake if the stake is near enough.

More often it is necessary to sight on a pencil, or on a sight pole held on the point, or on a plumb line held directly above it. In that case bring the line of sight down as near to the point as possible with the vertical hair bisecting the object sighted at. When only the upper portion of the sight pole can be seen, it is particularly important that the pole be held plumb. Do not use the lower part of the vertical cross hair for sighting when its intersection with the horizontal hair can be used just as well. The vertical hair may not be truly vertical. (2) To get the line of sight on an object quickly sight along the top of the telescope before sighting through it. (3) If the telescope has stadia hairs, do not mistake either of them for the regular horizontal hair. In some telescopes diagonal wires are added as an aid in avoiding this mistake. (4) Do not take hold of the telescope itself to move it sidewise; form the habit of turning it by taking hold of the plate on which the standards rest. (5) If the telescope is used in the dark, the cross hairs must be illumined by reflecting light down the barrel of the telescope. Some transits are furnished with an electrical illuminating device for this purpose. (See also suggestion 5 in Art. 516(*e*).)

248(*c*). PLUNGING, TRANSITING, OR REVERSING THE TELESCOPE. Synonymous terms for turning the telescope completely over on its supporting axis, thus bringing it upside down, and making it point in exactly the opposite direction. During the process, both the upper and the lower plates must be clamped so that the telescope cannot move sideways while it is being reversed.

249. Backsight—Foresight. A transit is at *B* (Fig. 249). It is desired to measure the angles *ABC* and *ABD*, and to produce the line *AB* to *E*. In measuring either angle or in producing the line, if the point *A* is the first point sighted at, then *BA* is called the backsight, while the sights on *C*, *E*, and *D* are called foresights. An indefinite number of sights to other points (all foresights) could be taken, the angles to which could all be measured from *BA* as a backsight.

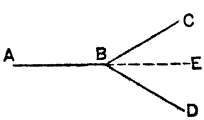


FIG. 249.

A backsight is a point of reference (or more strictly a line of reference from the transit to some other point); from it angles are measured to lines (foresights) to other points. If the angle is 180°, the line of backsight is simply produced. A backsight is usually the first sight taken after the transit is set up, and ordinarily only one backsight is used no matter how many foresights may be needed.

Note. The terms backsight and foresight are unfortunate in that they imply direction; they have nothing to do with direction. Thus, if only the angle *ABC* were to be measured, one might sight on *C* first, in which case *BC* would be the backsight.

In this book when three letters or numbers are used to denote an angle the MIDDLE *letter or number represents the* VERTEX *of the angle, i.e., where the transit stood when the angle was measured. Unless otherwise stated it may be assumed also that the* FIRST *letter or number represents the* BACKSIGHT *and the* THIRD *letter or number the* FORESIGHT.

Illustration. If $XYZ = 40°$, *Y* is the vertex, *X* the backsight, and *Z* the foresight.

249(*a*). PERMANENT BACKSIGHT. A backsight is usually taken on a pole or pencil held by one of the party. It is often desirable to establish a *permanent backsight* so that the transitman can sight to it as a check whenever he desires without keeping a man there

to give the sight. When the line of sight has been set on the backsight for the first time, before unclamping, move the telescope up or down. Usually the line of sight will strike some object definite and prominent enough to use as a backsight in place of the other, thus allowing the flagman to be called in. A crack in the fence, a mark on a tree, some definite corner of a house, or something similar, can be used, but if none is readily found, it often pays to have a flagman *make* a "permanent backsight" by sticking a flagpole or a chain pin into the ground, or by making some mark that can be easily distinguished through the telescope.

249(*b*). PERMANENT FORESIGHT. When, as is often true, a number of points are to be lined in with some point ahead, it is an excellent plan to establish a *permanent foresight* as far ahead as practicable, so that if the line of sight gets off the point the transitman will notice it and bring the line on again.

250. Use of the Upper and Lower Clamps. The *lower clamp* and corresponding slow-motion screw are used in taking a *backsight. These must never be disturbed while taking a foresight.*

The upper clamp and tangent screw are used in setting the indicator at zero and in taking a foresight.

250(*a*). NOTE. The importance of keeping the above in mind is readily seen. If, *after* taking a backsight, the transitman *will keep his hands off the lower part of the transit,* he will avoid the danger of disturbing his backsight. When in doubt, turn the upper screw —that can do little harm. Disturbing the backsight may mean beginning all over again. *Note that a tangent screw will not work until the corresponding clamp is set.* Errors often arise from *thinking* clamps are set when they are not, or from clamps slipping. Clamps should not be forced, but they must be set tight enough to hold. After setting on the backsight, it is well to try turning the upper plate *gently,* as a test before beginning to take angles. If the line of sight moves off the point and does not return, reset on the backsight and clamp again. The use of the upper and lower clamps is best understood by remembering that the spindle of the upper plate revolves inside the spindle of the lower plate. Study this construction in Art. 722.

251. Lining In, or Sighting in Line. It is frequently necessary in surveying to establish a point in a line of sight that has been brought to a fixed position, either by a sighting on a given point or by turning off a given angle. The general method may be illustrated as follows.

It is required to line in a stake and to set a nail on line in its top. Assume that the stake is far enough away to require the use of a

sight pole. (1) Make sure that the line of sight remains in its true position after both clamps have been set. (2) Look along the top of the telescope and motion the flagman approximately into line; then, looking through the telescope, help the flagman to move the sight pole into the line of sight. This is done by means of signals. Make the signal correspond to the distance the rod must be moved —a vigorous motion to the right, using the right hand, or to the left, using the left hand, means to move it quite a distance in that direction; a slight motion means a short distance. If possible, sight near the lower end of the rod; if this is not possible, by means of signals help the flagman to hold the pole plumb. When the rod is in line, signal to the rodman to drive the stake in that position. (3) When the stake has been driven, give a point for driving the nail by signaling until the sight pole, held on the stake, is exactly bisected by the vertical cross hair. (4) After the nail has been driven, sight again to make sure that it was driven on line.

251(*a*). SUGGESTIONS. (1) When the point to be lined in is near enough to the transit for a stake to be visible, sight directly on the stake instead of on a sight pole. Watch the stake through the telescope while it is being driven, and, by signals, help the flagman to keep the middle of the stake on line. A pencil can be used for sighting in setting the nail. Frequently it is not necessary to line in with such care as just indicated. For example, it would be a waste of time to split hairs in lining in chainmen. (2) When the point to be lined in is less than about 5 ft. from the transit, it is difficult if not impossible to focus the telescope on it. In such a case, using the transit, set a temporary point on line about 10 ft. away, stretch a string between this point and the point under the center of the transit, and locate the required point by means of this string. (3) If for any reason, the line of sight is not perpendicular to the horizontal axis of the telescope. it will not revolve in a vertical plane. This might result in a rather large error if, in setting the point, it becomes necessary to revolve the telescope through a large vertical angle. When, however, the vertical angle is small and the focus of the telescope remains unchanged, this error is usually negligible. In precise work the error may be eliminated by setting one point with the telescope normal and another point with it inverted. Halfway between these two points should be the true position of the point on line. (4) When it is desired to establish a number of points on the same line, it is well to establish a permanent foresight and to check on this foresight from time to time in order to make sure that the line of sight remains fixed.

252. Double Reverse or Double Centering. A technical term for the following process. A transit stands at *T*. Clamp the upper and lower plates together at the start and do not unclamp during the process. This means that *no angles are read,* and no use whatever is made of the vernier—it is not necessary even to set it at zero. (See Fig. 252.)



FIG. 252.

1. Backsight on *A*, using the lower clamp and tangent screw.
2. Plunge the telescope. If the transit is in adjustment, the line of sight will strike some point in the line *AT* produced as, for ex-

ample, the point *D;* if, however, it is out of adjustment, it will strike some other point as *B*.

3. Let the telescope remain bottom side up, unclamp the lower screw, turn the two plates together until the telescope (still bottom side up) again points at the backsight on *A*. Clamp the *lower* clamp and bring the cross hairs exactly on *A* with *lower* tangent screw.

4. Plunge the telescope. (The telescope now becomes right side up.) If the transit is in adjustment, the line of sight will again strike the point *D*, but if it is out of adjustment, the line of sight will strike a point *C* which is as far on one side of *TD* as *B* is on the other side. In practice, unless the instrument is badly out of adjustment, the points *B* and *C* are usually within an inch or so of each other, and the true point *D*, halfway between *B* and *C*, is easily established by measurement.

252(*a*). SUGGESTIONS. (1) No use is made of the graduations on the lower plate, thus eliminating error from incorrect graduation or from not setting the vernier at exactly 180°. Notice also that either the upper or the lower clamp can be used, provided both are tight when the telescope is plunged. It is better, however, *to set the upper clamp once for all before taking the first backsight and then leave it alone.* (2) If the transit has been set up in such a way that two diagonally opposite leveling screws are at right angles to the backsight *TA*, the error due to imperfect adjustment of a plate level may be eliminated by leveling for the bubble in that tube at right angles to the line just before each backsight is taken. (3) If in a length of sight of 1000 ft., points *B* and *C* are more than an inch or so apart, it will save time to adjust the transit. (4) Before plunging the telescope, but after the clamps have been set, always test the backsight to see if it is firmly fixed. (5) Take hold of the lower part of the transit and not the telescope in revolving it horizontally.

253. To Prolong a Straight Line. Let it be required to prolong a straight line *AB* by setting a point *C* on line beyond *B*.

253(*a*). FIRST METHOD. Set up at *A*, sight at *B*, and line in *C*. To prolong to a point *D*, too far from *A* to be visible, set up at *B*, sight at *C*, and line in *D*. Thus the process may be repeated for any number of setups.

If the transit is out of adjustment, it may be necessary in accurate work to take two sights at each setup, one with the telescope normal and the other with the telescope inverted as explained in Art. 252. If this is not done, and there are several setups, the prolonged line, instead of being straight, will be a series of straight lines curving in one general direction. When, however, the transit is in good adjustment, and the prolonged line is relatively short, it is usually sufficiently accurate to take only one sight in lining in a point.

253(*b*). SECOND METHOD. Set up at *B*, backsight on *A*, plunge the telescope, and line in *C*. To prolong the line still farther, set

up at *C*, backsight at *B*, and plunge the telescope to establish another point. Thus the process may be repeated for as many setups as may be necessary. This does not eliminate errors due to adjustment, and if continued for a number of setups, the prolonged line may be a curving series of straight lines in one general direction. When, however, the transit is in good adjustment, the method is sufficiently accurate for ordinary work.

253(*c*). THIRD METHOD. Set up at *B*, backsight at *A*, and establish *C* by the method of double reverse. If necessary to prolong a line still farther, set up at *C*, backsight on *B*, and establish another point by the method of double reverse. Thus the process may be continued for any number of setups. In precise work, this method of double reverse should be used as a precaution whether the transit is in adjustment or not. This third method is often called *double centering* or *double sighting*.

253(*d*). SUGGESTIONS. (1) In each of the three methods, the general procedure for lining in a point, once the line of sight has been fixed, is that explained in Art. 251. Note particularly the practical suggestions for setting a stake and a nail in the stake. (2) Permanent foresights may be useful in the first method, and permanent backsights and foresights in the other two, particularly when there are a number of points to be established from the same setup. (3) It is important in accurate work to have one bubble tube at right angles to the line *AB* and to bring the bubble to the center before backsighting as suggested in Art. 252(*a*), 2.

254. To Project a High Point to the Ground. Let *A* represent a point high above the horizon, such as the top of a church spire, and let it be required to find *any* point *B*, on the ground, which lies in a vertical plane through the line of sight from a transit to *A*. The most direct method is simply to backsight on *A*, depress the telescope, and establish a point *B*. This method, however, does not eliminate instrumental errors or errors in leveling up the instrument. Such errors may be practically eliminated by the following procedure which is similar to the method of double reverse. (1) Sight on *A*, with the telescope normal, and, depressing the telescope, establish a point *b*, on the ground. (2) Sight on *A* again, with the telescope inverted, and establish a point *b′* on the ground as close to *b* as possible. These two points should coincide, but if they do not, the required point *B* should be halfway between them.

254(*a*). TO FIND A POINT *C*, ON THE GROUND, WHICH LIES IN THE VERTICAL LINE THROUGH *A*. Set up at any convenient point *D* and establish, by the method just described, a point *B*, on the ground, which lies in a vertical plane through the line of sight to *A*. Set up at another point *E*, and establish on the ground a point *F* which lies in a vertical plane through the line of sight to *A*.

Determine the intersection of the lines *DB* and *EF* by the method explained in Art. 274. This point of intersection should be the required point *C*. The more nearly *DB* and *EF* are perpendicular to each other, the more definite will be the point of intersection. To check the position of the required point *C*, set up over a third point *G*, and establish on the ground a point *H* which lies in a vertical plane through the line of sight to *A*. The line *GH* should pass through the point of intersection of the lines *DB* and *EF*.

255. Setting the Vernier. A preliminary step in measuring an angle is to set the vernier. (1) Loosen the upper and lower clamps, and turn either the upper or lower plates or both until the zero mark of one of the verniers is so nearly opposite the zero mark of the graduated circle that a turn or two of the tangent or slow-motion screw will bring the two zero marks into coincidence. Clamp the two plates together with the upper clamp and do not touch this clamp again. (2) By means of the upper tangent screw, bring the two zero marks exactly opposite.

255(*a*). SUGGESTIONS. (1) In judging the exact coincidence of the two zero marks, it will be helpful to observe the nearest vernier mark on either side of the zero mark. The vernier mark on one side should be as far from the nearest mark on the graduated circle as the mark on the other side is from the corresponding mark on the circle, that is, the two vernier marks should be symmetrical with respect to the zero lines as judged by the marks on the circle. In finely graduated circles and verniers, such as those reading to 20″, it is well to examine for symmetry two or three vernier marks on either side of the zero marks. The eye should be directly over the zero marks in the final setting; otherwise they may appear to coincide when they really do not. (2) If the tangent screw does not bear firmly against the reacting spring, there may be slight play or lost motion which will be a source of error. It is best, therefore, that the last or final turning of the tangent screw should be in the direction that compresses the spring. This direction is to the right or clockwise. If turning the screw clockwise would move the zero of the vernier away from the zero of the limb instead of toward it, turn the tangent screw counterclockwise until the zero of the vernier moves to the other side of the zero of the limb; then when the tangent screw is again turned clockwise to bring the zero of the vernier to the zero of the circle, it will compress the spring. (3) Setting the vernier at zero is merely a matter of convenience. In some cases, it is better to set it at random, read it before and after measuring the angle, and take the difference of the two readings. For example, if when the vernier is set at random, it reads 30° 10′ and the final reading of the angle is 70° 21′, the true value of the angle is 70° 21′ − 30° 10′ = 40° 11′.

256. Reading Transit Angles. The general method of reading the graduated divisions on the limb, of reading the vernier, and of combining the two readings has been explained in the preceding chapter. Suggestions for reading angles and common mistakes in reading are given in Arts. 230 and 231.

256(*a*). USE OF THE READING GLASS. A small magnifying glass is useful in setting the vernier and in reading angles; one about an inch in diameter is a good size. Steady the glass by resting the hand on the plate near the vernier, and bring the glass slowly to-

ward the eye until the best results in clearness and magnification are obtained. In bright sunlight a good way is to use both hands, one resting on either side of the vernier, with the reading glass supported between them. This effectively shades the vernier. In the precise measurement of angles, a *reading* microscope with an alignment index is used.

257. Measuring Horizontal Angles. In measuring an angle keep in mind that the lower part of the instrument is a protractor, and the upper part merely a pointer with the zero mark of the vernier for indicator. The method of measuring an angle, as *ACB*, is outlined here (Fig. 257). (1) Set up at *C*. (2) With upper and lower clamps loose, revolve both plates until the zero on the vernier coincides as nearly as may be with the zero on the limb. Set the upper clamp and with the corresponding tangent screw make the zeros coincide. (This is called setting the vernier at zero.) Observe the marks each side of the zero. (3) Backsight at *A*. When the line of sight is nearly on *A* set the *lower* clamp, and, with the corresponding tangent screw (lower), bring the line of sight exactly on *A*. (4) Loosen the upper clamp, turn the upper plate until the line of sight is nearly on *B*, set the upper clamp and, with the *upper* tangent screw, bring the line of sight exactly on *B*. (5) Read the angle.

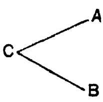


Fig. 257.

Using the wrong clamp or tangent screw is a frequent source of error. (See Art. 250.) Notice which is used for each operation, viz.:

Setting the vernier (2). Upper clamp and tangent screw.
Backsighting on *A* (3). Lower clamp and tangent screw.
Foresighting on *B* (4). Upper clamp and tangent screw.

It will help matters to remember that in measuring angles the only use made of the lower clamp and lower tangent screw is in *backsighting,* and that the upper clamp and the upper tangent screw are used for everything else, including *setting the vernier and foresighting. The upper clamp or tangent screw must never be disturbed when backsighting, and the lower clamp or tangent screw must never be disturbed when foresighting.*

For more accurate methods of measuring angles see Art. 265.

258. Angles Taken to the Right or to the Left. Any angle as *BCA* (Fig. 258) has two values, one measured clockwise, the other counterclockwise. If measured clockwise, it is said to be taken to the right; if measured counterclockwise, it is taken to the left. The difference lies wholly in *the direction in which* the angle is read; it makes no difference whatever in which direction the telescope is

turned when moved from the backsight to the foresight. This will be clear if the methods of reading right and left angles as explained in the preceding chapter are understood. In a good deal of ordinary surveying it is better on the whole to take all angles to the right, but in some lines of work it is desirable to take them right or left. Keep in mind that 360° minus an angle to the right equals the corresponding angle to the left and vice versa.

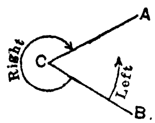


Fig. 258.

259. Measuring a Number of Angles from the Same Backsight. (See Fig. 259.)

1. Set the vernier at zero. (Upper clamp and tangent screw.)
2. Backsight on *A*. (Lower clamp and tangent screw.)
3. Foresight on *B*. (Upper clamp and tangent screw.) Read the angle.
4. Foresight on *C*. (Upper clamp and tangent screw.) Read the angle.
5. Foresight on D. (Upper clamp and tangent screw.) Read the angle.

Thus any number of angles may be taken from *A* as a backsight. Note again that the lower clamp and corresponding tangent screw are *not to be disturbed* after the backsight is taken. To guard against this, establish a permanent backsight and sight on it occasionally to see if the backsight reading is still zero. If it is not, the backsight setting has been disturbed, and it may be necessary to measure again the angles that have been taken.

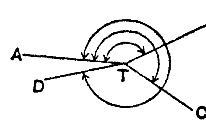


Fig. 259.

260. Suggestions for Measuring Horizontal Angles.

1. Remember that the telescope cannot be turned right or left without turning the upper plate with it.

2. If there are two verniers, it is sufficient in ordinary work to use only one, and that should be the one which is more nearly under the eyepiece of the telescope (saves walking around the transit). The other vernier across the circle should not be used at all except in special cases as, for example, when it is necessary to eliminate eccentricity. Avoid the mistake of using first one and then the other indiscriminately, or getting the two confused.

3. Form the habit of estimating by eye whether an angle is between 90° and 180°, or between 180° and 270°, or over 270°. This is one of the best suggestions that can be given, and if the beginner will check his readings in this way he will save himself absurd blunders of an even 90° or an even 180° due to reading the wrong vernier or the wrong number on the limb (Art. 225(*b*)).

4. After the backsight is taken see that all clamps are tight and that the vernier is set exactly at zero before loosening the upper clamp to take angles, but *don't force a clamp.* To make sure, test the lower clamp by trying *gently* to turn the plate. If the cross hairs move slightly off of the backsight but come back again when the hands are removed from the plate, the clamps are tight. Do not touch the lower clamp or tangent screw once the backsight has been taken.

5. Do not rest the hands on the tripod or upon any other part of the instrument.

6. In measuring a number of angles from the same backsight, a good transitman will find time, every few angles, to sight on the permanent backsight as explained in the preceding article. After all angles at a station have been measured, a final check on the backsight, before leaving the station, will reveal whether or not the lower part of the transit has been disturbed at any time, even during the measurement of the last angle.

7. In taking a number of angles from the same backsight, it is not necessary to set the upper clamp each time if the tangent screw is not used. Where angles have to be read rapidly and to the nearest minute or two, it may be better not to use the upper clamp at all.

8. If a number of angles are taken from the same backsight, and if, as is often the case, one is more important than the others, aim to take the important one first as soon as possible after backsighting, to avoid the error from slight disturbances of the transit.

9. In taking angles to the right, it is not necessary to turn the telescope to the right. Turn it through the shorter arc. However, in work of high precision it is well always to turn the upper plate in a clockwise direction.

10. Keep in mind that it is more important to have the transit's spindle truly vertical in measuring a horizontal angle between lines of sight to a high and to a low point than in measuring between lines of sight to two points about on the same level. (See special method of setting up the transit, Art. 247.)

11. Follow the suggestions for using the vernier and reading angles given in Art. 262.

261. Sources of Error in Measuring Horizontal Angles. If the transit is in good working order, well adjusted, and no parts loose, the most common sources of error are:

1. Careless setting up—not over the tack, or plates not level, or legs not firm. *Loose tripod shoes.*

2. The slipping of a clamp when it is supposed to be tight.

3. Rough or careless handling of the instrument.

4. Careless sighting—intersection of cross hairs not used, or sighting at the top instead of the bottom of the rod. Parallax not eliminated (Art. 103(*a*)).

5. Instrument disturbed or jarred out of position by walking around it, by passing vehicles or trains; vibration of telescope in strong wind.

6. Turning the tangent screw that moves the lower plate *after* a backsight has been taken, thus "throwing off" the backsight. (Checked by sighting on a permanent backsight to see if the two zero marks coincide.)

7. Mistakes in setting the vernier or in reading the angle.

262. Suggestions for Using the Vernier and Reading Angles. (A summary of Chapter 18.)

1. Use the upper clamp and tangent screw for setting the vernier. Loosen *both* clamps, upper and lower, until the setting is near enough to use the tangent screw, then tighten the upper clamp. After that remember that the upper tangent screw cannot be turned *without disturbing the setting of the vernier.*

2. If there are two verniers, 180° apart, use the one that is nearly under the eyepiece of the telescope. Do not use the other vernier at all except in special cases.

3. Ascertain from inspection whether the center (or inner) row of numbers on the limb is to be read for angles to the right. Use this row exclusively except in work where angles are taken to the left.

4. Do not make the mistake of setting the vernier at the wrong zero. If, for example, the outer row is graduated from 0° to 360°, and the inner row is graduated by the half-circle method or by the quadrant method (Arts. 226, 227), one of the zeros on the inner row will come opposite to the 180° mark on the outer row. A common mistake is to set the vernier at this zero and then read the angle as if it had been set at the 360° mark (or corresponding zero) of the outer row. This will make an error of 180° in the angle.

5. In setting the vernier as well as in reading angles, look at two or three vernier marks rather than at a single mark. *Avoid errors due to parallax* (Art. 242(*a*)).

6. Before reading the vernier make sure that you have the limb reading correct, especially the even number of degrees. Always keep in mind the value of the smallest division on the limb and do not neglect the fraction of a degree. Estimate roughly from inspection of the limb what the vernier reading should be before looking at the vernier. Remember that a mistake in reading the limb usually amounts to at least 20′ or 30′ and often to several degrees, whereas a mistake in reading the vernier is seldom over a minute or two.

7. In a double vernier read the vernier which is in advance of the indicator.

8. In reading the vernier keep in mind the value of the smallest division, and if it is in *seconds* do not call it *minutes*. Notice if the final reading agrees with that previously estimated from the limb.

9. Finally, the order in reading an angle is: (1) Note the limb reading *carefully*. (2) Estimate the vernier reading. (3) Read the vernier. (4) Add the vernier reading to the limb reading.

263. Common Mistakes in Reading Angles. (See Chapter 18.)

1. Setting the vernier at a zero of the inner circle, but not at the zero of the outer circle, and reading the angles thereafter from the outside circle, or vice versa.

2. Omitting the *fraction* of a degree in reading a limb.

3. Reading the inside circle of numbers on the limb instead of the outside circle, or vice versa.

4. Getting the number of degrees wrong when the limb is not numbered continuously from 0 to 360.

5. Reading the *B* vernier instead of the *A* vernier, or vice versa.

6. Reading the wrong side of a double vernier.

7. Calling the smallest division on a 30″, 20″, or 10″ vernier minutes instead of seconds.

8. Looking at a single vernier mark instead of at two or three.

9. Arithmetical mistakes in adding the vernier reading to the limb reading.

10. Reading an angle taken to the right as if it were taken to the left, or vice versa.

11. Finally, mistakes are more likely to occur in reading the limb than in reading the vernier, and mistakes in reading the limb are the more serious. Many such mistakes may be avoided by habitually estimating angles by eye as a rough check on the reading. (See suggestion 3, Art. 260.)

264. Measuring Horizontal Angles with a High Degree of Precision. A transit is never in perfect adjustment, nor can the graduations of the limb and of the vernier be perfect. Moreover, it is possible to sight on a point more closely than one can read the vernier, particularly with the ordinary transit reading to minutes, or, possibly, to half minutes. There are other sources of error in measuring angles which will be discussed in the next chapter and also in Chapter 27, "The Precise Measurement of Angles." In a single measurement of an angle, by the method explained in Art. 257, it is not possible to eliminate these errors. This is particularly true when there is a large difference in elevation between the two points sighted on, backsight and foresight. Consequently, the single measurement of an angle is not sufficiently accurate to meet

the requirements when a high degree of precision is required. The more accurate measurement of angles is based on multiplying the number of observations for each angle. Two methods commonly used are (1) *measuring angles by repetition* and (2) *measuring angles by series.* In each method, as explained later, it is possible to eliminate certain sources of error.

264(*a*). ADDITIONAL PRECAUTIONS IN MEASURING HORIZONTAL ANGLES. When measuring angles either by repetition or by series, one or more of the following precautions may be taken to eliminate errors due to lack of adjustment, eccentricity, faulty graduation, and other instrumental errors:

1. Set up the transit by the special method (Art. 247).
2. Read both (opposite) verniers each time.
3. Take angles to the right, then to the left.
4. Take angles with the telescope normal, then with it inverted.
5. When taking two or more independent measurements of the same angle, set the zero of the vernier at a different reading on the graduated circle at the beginning of each measurement so that a different portion of the graduated circle may be used each time that the angle is measured.

265. Measuring Angles by Repetition. To measure an angle by repetition means to measure it two or more times, allowing the vernier to remain clamped each time at the end of each measurement instead of setting it back at zero when sighting at the backsight. Thus an angle reading is mechanically multiplied two, three, four, or more times, depending upon the number of repetitions. In order to obtain the value of the angle, the final reading is divided by the number of times the telescope has been turned from the backsight to the foresight, i.e., by the number of repetitions.

Remark. It is customary to count the first time that the telescope is turned from the backsight to the foresight, i.e., the first measurement of the angle, as one of the repetitions though, strictly speaking, it is not.

265(*a*). DOUBLING AN ANGLE means to measure an angle twice by the method of repetition as follows. Measure the angle once in the usual manner, note the reading (merely as a check), and *allow the two plates to remain clamped at the reading.* Backsight again and measure the angle a second time in the same direction as before. The reading should now be double the real value of the angle.

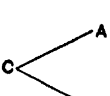


FIG. 265(*a*).

Illustration (Fig. 265(*a*)). Transit at *C*, backsight at *A*. Measure *ACB*; suppose vernier reads 30° 1′. Allow vernier to remain clamped at 30° 1′, backsight at *A*, and measure the angle again. The vernier will now read 60° 2′, which divided by 2 gives the value as before, 30° 1′. But suppose instead of 60° 2′

it reads 60° 3′. This would indicate that the angle is 30° 1′ 30″. But if the transit reads to minutes, this result could not have been obtained from a single observation without estimating the 30″. By letting the vernier remain at 60° 3′ and measuring for a third time, if the reading should be 90° 4′ the value would be 30° 1′ 20″. Thus on a transit reading to minutes results may be obtained to 30″ by doubling an angle, and to 20″ by tripling it.

It is good practice to double important angles as a check even when a high degree of precision is not required. By taking a reading at the end of the first measurement and then at the end of the second, any mistake in the first reading is almost sure to be revealed when the second reading is divided by 2.

265(*b*). More than two repetitions. In order to increase the degree of precision, an angle may be multiplied as many times as may be desired as, for example, four times or six times. More than six repetitions are seldom used.

265(*c*). Two important precautions to eliminate errors. In doubling, quadrupling, and sextupling an angle, certain sources of error may be eliminated by taking one-half the measurements with the telescope normal and the other half with it inverted. (See Art. 282(*a*), (*b*), and (*c*).) Eccentricity may be corrected by reading both verniers. (See Art. 282(*g*).)

265(*d*). Caution. Keep in mind the rule, Art. 250: Backsight—lower clamp and tangent screw; foresight—upper. In repeating an angle, turning the lower tangent screw when foresighting, or the upper tangent screw when backsighting, will make it necessary to start all over again.

265(*e*). Adding multiples of 360°. In repeating angles it is sometimes necessary to add 360° or 720° to a reading. *First illustration.* In doubling an angle of 200° the first reading is 200°, the second reading is 40°, or 40 + 360 = 400 and 400 ÷ 2 = 200°. *Second illustration.* An angle of 130° is sextupled, and the final reading is 60°, or 60 + 360 + 360 = 780, and 780 ÷ 6 = 130°. The indicator on the vernier has moved twice around the graduated circle and 60° over.

265(*f*). To check the final reading by short division. Divide the final reading by the number of repetitions and compare the result with the first reading, as illustrated by the following example.

Let the first angle = 126° 35′ 30″; sum of six angles = 39° 33′ 30″ (both corrected for initial). Ignoring multiples of 360° for the moment, divide 39° 33′ 30″ by 6 directly, thus 6 into 39° = 6°, and 3 to carry. Put the 3 directly into the tens part of the minutes of the answer. Then, 6 into 33′ = 5, and 3 to carry. The 5 goes directly into the minutes units of the answer, and the 3 into the tens of seconds of the answer. Then, 6 into 30″ = 5, which together with the carryover yields 35″ in the answer. The final quotient is 6° 35′ 35″. Now some multiple of 360° divided by 6 must be added to make the quotient equal the first angle. The multiple is always obvious. The complete answer, therefore, is 126° 35′ 35″, which is checked in the degrees and minutes, and closely enough in the seconds, by the value of the first angle.

266. "Sets" by the Method of Repetition. For the highest degree of precision in the measurement of a horizontal angle, several sets of repetitions are usually taken. There are two methods of taking a single set.

266(*a*). First method. The procedure for a single set is: (1) Measure the angle clockwise six times by repetition, keeping the

telescope normal throughout; divide the final reading by 6 in order to obtain the first value of the angle. (2) Invert the telescope and measure the angle counterclockwise six times by repetition to obtain a second value for the angle. Take the mean of the first and second values as the value of the angle for the first set. As many sets may be taken in this way as may be desirable. The mean of the values obtained from the different sets is taken as the final value of the angle.

266(*b*). SECOND METHOD. The procedure for a single set is: (1) Measure the angle clockwise by six repetitions, the first three with the telescope normal, the last three with it inverted, and thus obtain the first value for the angle. (2) Leave the vernier set as it was at the end of the first six repetitions and measure the explement of the angle (360° − the angle) *clockwise* six times by repetion, the first three times with the telescope inverted, and the last three with it normal. Theoretically, the vernier reading should come back to the zero mark or to the original setting whatever that may have been, and the sum of the two angles (angle + explement) should equal 360°. There will probably be a small error. One-half this error should be added to, or subtracted from, the value of the angle obtained in the first step, according to whether the sum of the angle and its explement is less or greater than 360°. The result is the corrected value of the angle for the first set. (See Art. 397(*a*) for a practical illustration.) The mean of the values found from several sets is taken as the final value of the angle. In order to eliminate possible errors of the graduation of the circle, the readings for different sets are distributed over different portions of the circle. This is done by increasing the initial reading of each set by adding a certain increment to the initial reading of the preceding set. This increment depends upon the number of verniers and the number of sets, and may be determined from the formulas given in Art. 399.

266(*c*). COMMENTS.

1. The second method is considered better than the first for measuring angles with a high degree of precision.
2. The number of sets in either method will depend upon the degree of precision desired. The number advisable may be limited by the degree of precision that can be attained in the measurement of a single angle with the particular instrument used, and by other limitations. In first-order work, five or six sets are usually required.
3. A modification of the second method is to keep the telescope normal throughout all six repetitions of the first step, and to keep it inverted throughout the six repetitions of the second step.
4. In using either the first or the second method, precautions to eliminate errors, such as those outlined in Art. 264(*a*), should be taken. A method of procedure in which these precautions are explained in detail, together with an illustrative example, is given in Arts. 397–398 and Art. 412. *Note that during an entire set of observations the transit should not be releveled.*

5. The method of measuring angles by repetition is an excellent one for measuring a single angle, particularly when an ordinary transit with rather coarsely graduated circle is used. It should be noted, however, that in repeating angles, operations such as sighting and clamping are multiplied, and hence opportunities for error are multiplied. This is particularly true of constant errors involved in the action of the clamps and tangent screws. Consequently this method, which seems so perfect in theory, is not so superior in practice as it at first appears to be, and the limit of precision in the measurement of an angle is ordinarily reached after the fifth or sixth repetition.

6. In the explanations of the first and second methods, six repetitions of each angle are specified; this corresponds to the general practice in first-order work. The number of repetitions may often be reduced, depending upon the degree of precision required, but the general procedures explained for the first and second methods may be followed regardless of the number of repetitions.

267. Measuring Angles by Series. This method, known also as the *direction method*, may be used to advantage where several angles have their vertices at a common point as, for example, at a station of a triangulation net. The method of procedure may be illustrated as follows.

The instrument stands at C in Fig. 267. After setting the vernier at 0° or at random, backsight on A. Without disturbing this backsight, take foresights successively on B, D, E, and A, thus closing the circle. Since the graduated circle remains in a fixed position throughout the entire process, the angle read for each foresight is the angle between that foresight and the backsight CA, measured clockwise from the backsight CA. On the final foresight to A, the reading of the vernier should be the same as it was to start with, i.e., the same as the original setting. It is probable, however, that there will be a small error, and this error is distributed equally between the angles irrespective of the size of the angle.

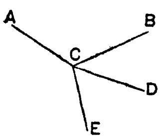


FIG. 267.

267(*a*). ILLUSTRATION. Transit at C (Fig. 267). Set vernier at random, say 20°. Backsight on A. Vernier reading 20°.

Fore-sight	*Vernier Reading*	*Angle*	*Subtraction*	*Value*
B	140° 21′	*ACB*	(140° 21′) − (20° 0′)	120° 21′
D	160° 32′	*BCD*	(160° 32′) − (140° 21′)	20° 11′
E	200° 40′	*DCE*	(200° 40′) − (160° 32′)	40° 8′
A	20° 1′	*ECA*	(380° 1′) − (200° 40′)	179° 21′
				360° 01′

Error = 20° 1′ − 20° = 1′ or 15″ to be subtracted from each angle, thus: 120° 20′ 45″ + 20° 10′ 45″ + 40° 7′ 45″ + 179° 20′ 45″ = 360° 0′ 0″.

268. Sets by the Direction Method. The method of procedure just outlined must be extended to meet the requirements in work of a high order. In such work several sets are taken. The procedure for each set is as follows. (1) Backsight on A, take the

reading, and then measure clockwise the angle between that backsight and each foresight, *CB, CD, CE,* and *CA*, exactly as already explained in connection with Fig. 267, but do not distribute the error. (2) Reverse the telescope, backsight on *A*, take the reading, and then foresight successively on *E, D, B,* and *A*, in order to measure in reverse order (counterclockwise) the angles between the backsight *CA* and the foresights *CE, CD, CB,* and *CA*, closing with a foresight on *A*. These two steps constitute a set of observations. From these observations two values may be obtained for each of the angles, *ACB, BCD, DCE, ECA*. The mean of the two values for each angle is taken as the unadjusted value for that angle. The sum of these unadjusted values should be 360°. There will probably be, however, a small error, and if only one set is taken, this error is distributed equally between the angles irrespective of size to obtain the final adjusted value for each angle.

When several sets are taken, the method of obtaining the final adjusted values is somewhat different. A general method of procedure for field work and for obtaining the final values from sixteen sets is explained in Art. 413.

269. Measuring Deflection Angles. A *deflection angle* is the angle which a line makes with the backsight produced through the point at which the angle is measured. For example, if in Fig. 269, a transit stands at *T* and a backsight is taken on *A*, the angle *CTB* is the deflection angle to the left which a foresight *TB* makes with *AT* produced. Similarly, *CTD* is a deflection angle to the right. Deflection angles, never greater than 180°, are usually less than 90°. It is necessary to indicate in the field notes whether a deflection angle is measured to the left or to the right. Three methods of measuring a deflection angle may be explained by applying each to the angle *CTB*.

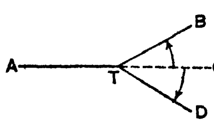


FIG. 269.

1. *By Plunging the Telescope.* Set the A vernier at 0°, invert the telescope, backsight on *A*, plunge the telescope (it now becomes normal), foresight on *B*, and read the angle *CTB* from the A vernier. The telescope is inverted when backsighting in order that it may become normal when foresighting, thus bringing the A vernier into a position where it can be read conveniently.

If the transit is not in adjustment, plunging the telescope results in a constant error in each angle of traverse. This error may be too small to be of importance in the measurement of a single angle, but, since it is cumulative from station to station, it may result in too great an error in the traverse as a whole. This cumulative

effect may be corrected to a large extent by taking one backsight with the telescope inverted, the next backsight with it normal, and so on, alternating in this way from station to station.

2. *By Direct Measurement.* Set the A vernier at 180°, backsight on *A* with the telescope normal, foresight on *B* with the telescope still normal, and read the angle *CTB* from the A vernier. In this method the telescope is kept normal throughout; hence the error due to plunging the telescope is eliminated.

3. *By Repetition.* Set the A vernier at 0°, backsight on *A*, invert the telescope, foresight on *B*, read the angle *CTB* as a check. Keeping the upper clamp tight and the telescope still inverted, loosen the lower clamp and backsight a second time on *A* as in the method of double reverse. Plunge the telescope (it now becomes right side up), foresight on *B*, thus measuring the deflection angle a second time; read the A vernier and divide this final reading by 2. In addition to the usual advantages of doubling an angle, this method eliminates errors that occur when the line of collimation does not revolve in a vertical plane. A deflection angle may, of course, be multiplied more than twice by the method of repetition if desired.

269(*a*). THE USE OF BOTH THE A AND B VERNIERS. In work of high precision, both verniers are read in order to eliminate errors due to eccentricity. When the A vernier is set at 0°, the B vernier is also read, and if it differs from 0°, the mean of the A and B vernier readings is taken as the initial reading. Likewise, after the second foresight on *B*, the mean of the A and B vernier readings is taken as the final reading. The difference between the initial reading and the final reading is then divided by 2 to obtain the value for the angle. A similar procedure may be followed when there are more than two repetitions.

270. To Establish a Line at a Given Angle with a Given Line. (*To lay off an angle.*) This is the reverse of measuring an angle. Set up the transit at the apex of the required angle. Set the vernier at zero and backsight along the given line. Loosen the upper clamp, bring the zero mark of the vernier nearly on the reading for the given angle, and, using the upper clamp and tangent screw, set the zero mark exactly. Line in the pole, stake, tack, or whatever is used for sighting on a point in the required line.

Deflection angles may be laid off by the same method either directly from the backsight, or, by inverting the telescope, from the backsight prolonged.

270(*a*). NOTE. Notice that the only use made of the lower clamp and tangent screw is in backsighting, just as in the measurement of an angle. A blunder sometimes made in setting the vernier for the foresight on the required line is to turn off the given angle on the wrong side of the given line. The angle most often laid off is 90°; it is necessary to do this more or less frequently in all kinds of surveying.

270(*b*). To LAY OFF AN ANGLE BY REPETITION. This method is used when it is desired to lay off an angle with a relatively high degree of precision.

With a transit set up at a point B and a backsight on a point A, it is desired to lay off an angle ABC by establishing a point C at a distance BC from the transit.

1. Lay off the required angle ABC by a single observation as just explained, but consider the point C' thus established to be a trial position of the required point C.

2. Measure the angle ABC' laid off in (1) by the method of repetition (Art. 266), using from three to six repetitions, depending on the degree of precision desired. The difference between the value of the angle ABC' thus obtained and the required angle ABC indicates the error in the angle ABC'. Let $C'BC$ represent this angular error, i.e., the angle at B from the line BC' to the true position of the required line BC.

3. Establish the position of C by laying off the angle $C'BC$. Since $C'BC$ is too small to lay off with the transit, the position of C is determined by measuring an offset from the temporary point C'. The value of this offset is practically the distance BC' multiplied by the sine, or the tangent, or the arc of $C'BC$. The value of $C'BC$ should be less than one minute; hence it is helpful to remember that the value of any one of the three functions is (very nearly) 0.0001 when $C'BC$ is 20″, 0.0002 when $C'BC$ is 40″, and 0.0003 when $C'BC$ is 60″.

4. Check the angle established in (3) by measuring it by the method of repetition. In precise work it is well to measure the explement of the angle as well.

Illustration. Required to lay off an angle $ABC = 36° 20' 40''$, correct to the nearest 10″, with a transit reading to the nearest minute. (1) Lay off an angle of 36° 21′ by a single observation and set a temporary point C'. (2) Measure this angle ABC' by six repetitions. Suppose the result is 36° 21′ 20″, correct, say, to the nearest 10″. The correction $C'BC$ is $36° 21' 20'' - 36° 20' 40'' = 40''$. (3) Suppose that the temporary point C' is 550 ft. from the transit; then the offset $CC' = 550 \times \sin 40'' = 550 \times 0.0002 = 0.11$ ft. Measure this offset from the temporary point C' in the direction that will reduce the angle $ABC' = 36° 21' 20''$ to the required angle $ABC = 36° 20' 40''$. (4) As a check, measure ABC by the method of repetition.

270(*c*). To LAY OFF FROM A TRANSIT STATION A SHORT OFFSET AT 90° TO A GIVEN LINE THROUGH THE STATION. It is assumed that the offset is too short to be "sighted in" through the telescope of the transit. (1) Backsight along the given line and turn off an angle of 90° to determine the direction of the offset. (2) Establish a point in line conveniently near the transit station but far enough away to be readily seen through the telescope. (3) Stretch a string

from the transit station to the point established in (2) and measure the length of the offset along this line.

Obviously, a short side of any angle can be laid off by a method similar to that just explained.

271. To Measure a Vertical Angle. (*a*). WITH THE HORIZONTAL. The vertical angle which a line of sight to any point makes with the horizontal, as measured with the transit, is the vertical angle between that line of sight and the horizontal plane of sight, i.e., the plane in which the horizontal line of sight revolves when the telescope, clamped in a level position, is turned through a horizontal circle. If the angle is above the horizontal plane of sight it is called an *angle of elevation* and is usually considered a *plus* angle; if it is below the plane, it is an *angle of depression* and a *minus* angle.

Just before measuring a vertical angle, level up the transit carefully, sight in the general direction of the point to which the angle is to be measured, and then level the telescope by centering the bubble in the telescope level tube. If the transit is in adjustment, the zero mark of the vernier should coincide with the zero mark of the vertical arc or circle. If it does not, the two zeros may be made to coincide when the line of sight is horizontal by adjusting the vertical circle vernier (Art. 739(*l*)). The vertical angle may then be measured as follows: Bring the horizontal cross hair nearly on the point to which the angle is to be measured, and clamp the horizontal supporting axis on which the telescope turns. With the corresponding tangent screw, bring the horizontal cross hair exactly on the point. Read the vertical angle by combining the reading on the vertical circle with the vernier reading just as in reading a horizontal angle. The vertical circle and vernier are not generally graduated to the same degree of fineness as is the horizontal limb and vernier.

271(*b*). INDEX CORRECTION. On some instruments, no provision is made for the adjustment of the vertical circle vernier, and even when there is such a provision, it is difficult to keep the vernier in perfect adjustment. In accurate work, therefore, instead of attempting to adjust the vernier exactly, it is better to correct all readings by the small amount that the vernier is out of adjustment. This *index error* or *index correction* may be determined, after carefully leveling first the transit and then the telescope, by reading from the vernier the amount that the two zeros are apart.

The amount of the index correction is to be subtracted from the vertical angle reading when it is on the same side of the vertical circle zero that the reading is; added, when it is on the opposite

side. In other words, if the vernier zero is on the side of the vertical circle zero that it would be for an angle of elevation, the index correction is subtracted from readings for angles of elevation and is added to readings for angles of depression. When, however, the vernier zero is on the same side of the vertical circle zero that it would be for angles of depression, the index correction is added to readings for angles of elevation and subtracted from readings for angles of depression.

Remark. The rules just given for applying the index correction are best understood by observing the relative positions of the zero on the vernier and the zero on the vertical circle when the telescope is level, and when it is turned from an angle of elevation to an angle of depression and vice versa. This is a good thing to do when in doubt as to whether an index error is to be added or subtracted.

Note. On some transits the vernier and a spirit level are mounted on a separate arm in such a way that the vernier can be moved until the bubble is centered in the level. After the telescope has been sighted in the general direction of the point to which the vertical angle is to be taken, the index correction can be made zero by centering the bubble in the vernier level. The angle can then be measured directly without correction.

271(*c*). ACCURATE MEASUREMENT OF A VERTICAL ANGLE WITH THE HORIZONTAL. When more than ordinary accuracy is required in the measurement of the angle which a line of sight makes with the horizontal, the following procedure is recommended. (1) Make sure that the transit is in good adjustment. It is essential that the axis of the telescope level and the line of sight should be parallel (Art. 739(*i*)), and it is highly desirable that both should be perpendicular to the axis of the spindle. (2) Set up the transit by the special method (Art. 247). (3) Measure the vertical angle by the usual method, but instead of correcting for an index error take an initial reading when the line of sight is horizontal and again when it is inclined. (4) Repeat the measurement two or more times and take the mean of the values thus obtained.

When there is a complete vertical circle on the transit, certain instrumental errors may be eliminated by measuring the vertical angle twice, once with the telescope normal and the second time with it inverted. The mean of the two readings is then taken as the value of the angle. Such a pair of measurements may be repeated as many times as may be desired to obtain an accurate value of the vertical angle.

271(*d*). TO MEASURE A VERTICAL ANGLE FROM ONE GIVEN POINT TO ANOTHER. Sight first at one point, note the reading on the vertical arc, then sight at the other and note that reading. Use the clamp and tangent screw attached to the supporting axis. Inspection of the vertical circle will make it obvious whether to add or to subtract the readings. If both sights are above or below the

horizontal, subtract; if one is above and the other below, add. When more than ordinary accuracy is required, the transit should be in good adjustment and should be set up by the special method (Art. 247). Several measurements of the angle should be taken, and if there is a full vertical circle, half the measurements should be made with the telescope inverted. (See the preceding paragraph.)

271(*e*). To LAY OFF A VERTICAL ANGLE. The procedure is the reverse of that for measuring a vertical angle. If the angle to be laid off is measured from the horizontal, the required angle should be turned off by moving the telescope up or down until it is brought into the correct position and clamped. A point in the line of sight may then be established. In laying off the angle, the reading should be corrected for the index error if there is any.

271(*f*). THE GRADIENTER. If the transit is equipped with a micrometer screw or gradienter, this may be used for either measuring or laying off a vertical angle. It is particularly useful for establishing grades. (See Arts. 181 and 188(*g*) for an explanation of its use.)

272. Special Field Operations. There are certain field operations in which the transit is used that are, perhaps, not as basic as those already explained, but with which, nevertheless, a transitman should be familiar. Some of the simplest as well as most useful of these operations are explained in the remainder of this chapter. Special problems in field operations which involve applications of geometric and trigonometric principles are given in Chapter 25.

273. Balancing In the Transit, i.e., to set up a transit on line between two given points. It is assumed that neither point can be seen from the other, as this is the most difficult case, but it is also assumed that both the given points can be seen from the required point at which the transit is to stand. As a preliminary step, set two sight poles on line by the method explained in Art. 91(*c*). The poles should be 30 to 100 ft. apart and so placed that the required transit point will be about halfway between them. The transit can now be set approximately on line by means of the two poles, and roughly leveled up. Sight on one of the given points, plunge the telescope, note how far to one side of the other point the line of sight strikes, and move the transit accordingly. Repeat until the transit is on line.

273(*a*). PRACTICAL HINTS. The preliminary use of the two poles (or short stakes if poles are not at hand) saves time. The nearer on line the transit is, the more carefully it should be leveled up, especially the level bulb at right angles to the line of sight. Usually after one or two preliminary setups the

transit can be brought into line by means of the shifting head. Do not split hairs in the preliminary setups. Do not backsight indiscriminately first on one of the given points and then on the other. The amount to move the transit can be estimated more easily if one of the given points is always used for a backsight, the foresight striking near the other. When the transit is apparently on line it may be well to test by the method of double reverse (Art. 252).

273(*b*). Alternative method (Fig. 273(*b*)). It is required to balance in a transit between two points *A* and *B* when neither point is visible from the other. Choose a point *C″* from which both *A* and *B* can be seen, preferably about halfway between those two points and as nearly on the line *AB* as can be estimated. Set up the transit at *C″*, backsight on one of the given points, say *A*, plunge the telescope, and establish a point *B″* opposite *B*. Estimate or measure roughly the distances *AC″*, *AB″*, and *B″B*, and from similar triangles, calculate approximately what the distance from *C″* to *C*, a point on line, should be. Move the transit this distance toward *C*. Since the measurements were made roughly, it is quite likely that the point thus established will not be exactly on line but in some such position as *C′*. Set up the transit over *C′*, backsight again on *A*, plunge the telescope, and establish the point *B′*. This should fall fairly near *B*. Estimate from the distance *B′B* how far to move the transit to bring it on line at some point *C*. Set up over this point, backsight again on *A*, plunge the telescope. The line of sight may now be so nearly on *B* that it can be brought exactly on by the use of the shifting head without disturbing the tripod. If this cannot be done, the process of moving the transit must be repeated until finally, by trial, it is found to be exactly on line.

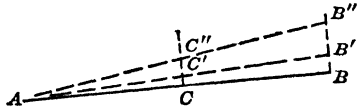


Fig. 273(*b*).

274. To Establish the Point at Which Two Straight Lines Intersect. Suppose that *AB* and *CD* are the two intersecting lines. Set up over *A* or *B* and line in two stakes on *AB* 6 to 10 ft. apart. One of these stakes should be on one side, the other on the other side of the required point of intersection as judged by sighting with the eye along *CD*. Mark the exact points on the stakes with nails driven *part way* in. In a similar manner, setting up over *C* or *D*, line in two stakes on the line *CD*, one on each side of the point of intersection. Find the required point by stretching two intersecting strings between the nails in the stakes.

Remark. It is evident that it is unnecessary to drive stakes on the line *CD;* it is often better simply to establish a single point, i.e., where the line of sight crosses the string (or the edge of a tape) stretched between the nails on *AB*.

274(a). TO ESTABLISH THE POINT OF INTERSECTION OF A GIVEN LINE AD WITH A PERPENDICULAR TO THE GIVEN LINE FROM A GIVEN POINT B (Fig. 274(a)). (1) Set the transit at A and sight at D. (2) Hold the zero mark of the tape at B, swing the tape about B as a center, and by trial find the point C on the tape which, when brought into the line of sight AD, makes the radius BC equal to the shortest distance from B to AD, i.e., C is at the least tape reading.

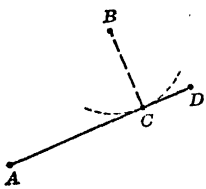


FIG. 274(a).

Another Method: Make the radius longer than BC so that when the tape is swung, the mark on the tape will be brought into the line of sight at *two* points. The required point C will be halfway between these two points (Fig. 87(f)).

274(b). SWING OFFSET. It is often required to establish a line AD (Fig. 274(a)) passing through a given point A and at a given distance BC from a given point B. (1) Set the transit at A. (2) With B as a center and a radius BC, swing the tape until the point C on the tape, as seen through the transit telescope, appears to be as far as possible from B, i.e., until the angle BAC becomes as large as possible. The line of sight is then in the required line. For measuring short swing offsets, a Philadelphia self-reading rod is often more convenient than a tape, particularly for offsets along lines, such as street lines, where many points are to be located. For a practical procedure in the application of this method of a swing *offset*, see Art. 277.

275. To Reference a Point. Any important point which has been established in surveying and which may be needed later on, should be referenced in such a way that if it is disturbed, it can be easily re-established. One of the simplest and best methods of doing this is to reverse the field operations explained in Art. 274. Establish four reference points in such positions that the line between two of them will intersect the line between the other two at the point to be referenced. The angle between the two lines should be great enough so that the point of intersection will be clearly defined. The point can then be re-established at any time by determining the intersection of the two lines.

275(a). COMMENTS. The method just explained is frequently used for referencing a transit station. A reference point is set on one side of the station, any convenient distance away, a sight is taken on this point, the telescope is plunged, and a second reference point is set any convenient distance away. The line through these two points is one of the two lines that intersect at the station. By a similar procedure, two reference points of the second line may be established.

In the method just explained no linear measurements were made. If any one of the four reference points should be disturbed, the point of the inter-

section of the two lines could not be determined. Unless, therefore, the reference points are of such a character that they cannot be disturbed, it may be desirable, as a precaution, to measure the distances from the point to be referenced to two or more of the four reference points.

The method of referencing a point by two intersecting lines is based on one of the seven methods for locating a point explained in Art. 10(*a*). Theoretically, any one of the other six methods may be used for referencing a point, but some are more practical than others. For methods of referencing a point solely by linear measurements, see Art. 92.

In some cases, the reference points must be placed quite a distance away from the point referenced in order to remain undisturbed during construction, whereas in other cases they may be only a few feet away. For example, it may be necessary in referencing a transit station to set stakes 50 ft. or more from the station. On the other hand, reference stakes for setting a monument may be driven a few feet from the spot where the hole is to be dug.

276. To Reference a Line. (1) Reference any two points in the line which are visible from each other and far enough apart to determine the line accurately. This may be done in two ways: (1) By offsets from buildings and other permanent structures, or by any of the other methods of referencing points. (2) By referencing one point in the line and measuring the angle which the line makes with some other line. This includes referencing a line by its azimuth or bearing.

Remark. The importance of carefully referencing at least one line of a survey is often overlooked. A map may be very accurate and complete and it may show one *point* carefully referenced; but unless some *line* is referenced also, the entire survey may be swung on the one known point as a pivot. This is the source of much of the trouble in rerunning old surveys, and gives rise to many boundary disputes. Objects used for reference should be as *definite* and as *permanent* as possible.

277. To Sight Parallel to a Given Line. A transit stands, for example, a few feet from the side of a building and it is desired to sight parallel to that side. *By swinging* the tape, find the *shortest* distance from the transit to the side of the building. Suppose this distance is 3.8 ft. At any convenient distance from the transit, the farther the better, hold one end of the tape against the building, and swing the tape with the thumb or pencil held 3.8 ft. from the end. As the thumb or pencil describes an arc of 3.8 ft. radius, it will seem to the transitman looking through the telescope to move *away* from the building and then *toward* it. Keeping the cross hairs on it by moving the telescope, the transitman should clamp just at the turn, i.e., when the thumb or pencil is farthest from the side of the building. After clamping, look through the telescope and observe the tape as it is swung again as a check. This method is based on the *swing offset* (Art. 274(*b*)). A Philadelphia self-reading rod is often more convenient than a tape for measuring swing offsets.

The same method may be used for sighting parallel to a fence or any other similar object. (Several methods of establishing a line parallel to a given line, based on geometric or trigonometric principles, are given in Art. 383(*b*).)

278. To Measure an Angle When the Transit Cannot be Set Up over the Vertex. Let it be required to measure the angle between two given lines when the transit cannot be set up over the vertex or over any point in either line. Establish a line parallel to each of the given lines and as close to it as may be convenient (Art. 277). Find the intersection of these two lines (Art. 274), and setting up over this intersection, measure the angle.

The method just explained is general; a special method requiring but one setup and applicable in many cases is the following. Set up at any point as close as practicable to the vertex, establish a line parallel to each of the given lines, and measure the angle.

279. To Read a Bearing. (Bearings are explained in Arts. 201 to 206.) (*a*). HOW TO OBSERVE THE NEEDLE. The compass box on a transit is attached to the upper plate, and when the telescope is turned sidewise the compass box must turn with it. When the telescope is right side up the letter N is directly beneath the front end of the telescope. A line between N and S is always directly beneath the axis of the telescope (or line of sight) *no matter in what direction the telescope may be pointing.* When the magnetic needle is lowered to its pivot it will come to rest pointing north, and if the telescope is also pointed to the magnetic north, the north end of the needle will point to the letter N (Fig. 279(*a*)). If, however, the telescope is turned in some other direction, the letters N, E, S, and W turn with it, but the needle will still point to the magnetic north, and each of its ends will usually rest *between* two of these letters. To read the bearing of a line, pay no attention to the telescope except to see whether it is normal or inverted, and do not try to think beforehand whether the bearing is going to be NE or SE, SW or NW. If the telescope is right side up, simply observe between which two letters the north end of the needle rests, and how many degrees it is from the letter N or the letter S as the case may be. The south end of the needle is observed instead of the north end when the telescope is inverted. *The north end of the needle is the end without the weight.*

On most transits the letters E and W are interchanged, the E being on the left and the W on the right. The reason for this will appear from the following illustrations.

279(*b*). ILLUSTRATION. In Fig. 279(*b*) suppose the telescope to be pointing N 30° 30′ E. In reading the bearing the *north* end of the needle will be

found between the letters N and E or 30° 30′ from N towards E. If W had been on the left in place of E, as one might naturally expect it to be, the north end of the needle would fall between N and W, which might lead to the mistake of calling the bearing NW instead of NE.

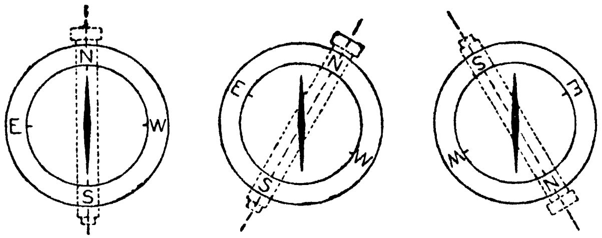


FIG. 279(*a*). FIG. 279(*b*). FIG. 279(*c*).

279(*c*). ILLUSTRATION. In Fig. 279(*c*) suppose the telescope to be pointing S 30° 15′ E. The *north* end of the needle is observed to be 30° 15′ from S *toward* E, as it should be. If E and W had not been interchanged, it would have fallen between S and W.

In transits where the E and W have not been interchanged one must be on guard against the mistake illustrated above of reading east for west or vice versa.

279(*d*). PRECAUTIONS.

1. Never read a bearing from E or from W. The first letter is always N or S, and the last letter is always E or W.

2. In recording a bearing do not omit either the first or the last letter. Of what value is a bearing recorded N 40°, or one recorded 35° E?

3. In some transits a vernier will be found within the compass circle, to be used, when desired, for setting off the magnetic declination, so that when the needle points to N, the telescope will point to the *true* north instead of the magnetic north. This makes it possible to read true bearings instead of magnetic bearings. ***When reading magnetic bearings be sure that this vernier is set at zero.***

CHAPTER 20

ERRORS IN TRANSIT WORK

In this chapter sources of error in transit work are explained, the relative importance of errors from different sources is discussed, and methods of correction or elimination are given. Limits of error, particularly in the measurement of angles, are suggested. Before beginning this chapter, the student should study the construction of the most important parts of a transit and understand how these parts enter into the construction of the instrument as a whole (Art. 722). Errors in the measurement of angles and methods of reducing such errors are treated in more detail in Chapter 27, "The Precise Measurement of Angles."

280. Sources of Error in Transit Work.

1. *Instrumental Errors.* (*a*) Errors of adjustment, (*b*) errors from defects in construction which cannot be eliminated by adjustment, such, for example, as eccentricity in the graduations on the limb, and (*c*) imperfections due to wear, such as loose-fitting parts, including loose tripod shoes.

2. *Personal Errors.* (*a*) Errors in setting up the transit, (*b*) errors in sighting, (*c*) errors in reading angles.

3. *Natural Sources of Error.* Such as those due to the effect of wind, atmospheric conditions, and temperature.

281. Fundamental Conceptions of Errors in Transit Work. In linear measurement the magnitudes of the errors from different sources are pretty well known, but in angular measurement some of the errors are so affected by varying conditions that it is more difficult to estimate their magnitudes in advance. Fortunately in ordinary surveying such errors can be practically eliminated, and then their magnitudes do not matter.

It is important for the surveyor to keep in mind the relation between the degree of precision in angular measurement and that in linear measurement, for, in any survey, the degree of precision for one should be consistent with that of the other. He should be able to think of the magnitude of an error in angular measurement in terms of units of linear measurement or vice versa. For example, if there is an error of 1′ in the measurement of an angle, it is important to know what, at any given distance from the vertex, the magnitude of this error is expressed in feet or decimals of a foot. There is no reason, generally speaking, for measuring angles with a greater degree of precision than that required for the corresponding measurements of distances. The ability to visualize

the magnitude of an angular error in terms of distance is helpful not only in keeping the accuracy of the transit work consistent with that of chaining, but also in determining to what extent any source of error in angular measurement should be eliminated.

The best transitman is not one who attempts at all times to eliminate completely all sources of error, but rather one who knows to what extent he must eliminate each source in order to attain a desired degree of precision, who knows what to slight and what not to.

282. Instrumental Errors. An instrumental error may be due to an imperfect adjustment of some part of the transit or to a defect in the construction of the instrument. The total instrumental error in an observation may be due solely to one or the other of these causes, or to a combination of both.

282(*a*). Imperfect adjustment of plate levels. If the upper and lower plates are not horizontal when the bubbles in the two plate levels are both centered, the vertical axis will not be truly vertical, and this will cause errors in measurements of both horizontal and vertical angles; it will also cause an error in prolonging a line by plunging the telescope. The error may be negligible in ordinary work over flat country, but may become serious in hilly country, where the line of sight for one side of an angle may be to a high point and the line of sight for the other side may be to a low point. Even in flat country, the error should be eliminated in work of high precision. This is best done by keeping the plate levels in good adjustment. (See Art. 739(*e*).) In addition, the error in the measurement of either a horizontal or a vertical angle may be reduced by taking the mean of two values, one obtained with the telescope normal and the other with it inverted, provided the plates are leveled immediately before each of the two backsights.

282(*b*). Imperfect adjustment of cross hairs. If the line of sight is not perpendicular to the supporting axis of the telescope, it will not revolve in a plane when the telescope is turned on that axis. Nevertheless, if the cross hairs are in reasonably good adjustment, the error in the measurement of a horizontal angle is usually negligible, unless, in sighting, first at one point and then at the other, it is necessary to turn the telescope through a large vertical arc, as, for example, when the two points sighted at differ greatly in elevation. The error is likely to be serious whenever the telescope is plunged in making an observation, as, for example, in prolonging a straight line. In any case, it should be eliminated in work of high precision. In the measurement of an angle, the error may be eliminated by taking the mean of two values, one ob-

tained with the telescope normal and the other with it inverted. In prolonging a straight line, the error may be eliminated by the method of double reverse (Art. 252).

282(*c*). Imperfect adjustment of the standards will prevent the line of sight from revolving in a vertical plane even when the transit, otherwise in good adjustment, has been carefully leveled up. This is because the supporting axis of the telescope will not be horizontal, i.e., parallel to the horizontal plates and perpendicular to the vertical axis of the transit. The resulting error in the measurement of a horizontal angle may be large if in sighting, first at one point and then at the other, it is necessary to turn the telescope through a large vertical arc. The error may be eliminated in the measurement of either a horizontal or a vertical angle by taking the mean of two values, one obtained with the telescope normal, the other with it inverted. In prolonging a straight line, the error may be eliminated by the method of double reverse.

282(*d*). Imperfect adjustment of the telescope level. If the axis of the telescope level is not parallel to the line of sight, it will cause an error in the measurement of any vertical angle which the line of sight makes with the horizontal; it will also be a source of error when the transit is used for leveling. Eliminated by keeping the telescope level in good adjustment (Art. 739(*i*)). Eliminated also by measuring the angle twice, once with the telescope normal and the second time with it inverted, to obtain a mean value (Art. 271(*c*)).

282(*e*). Imperfect adjustment of the vernier for the vertical circle. If the reading of the vernier is not zero when the line of sight is horizontal, the error in the measurement of an angle with the horizontal may be corrected by applying the index correction (Art. 271(*b*)).

282(*f*). Imperfect graduations. The graduations on a good transit are now so nearly correct that errors from imperfect graduations are negligible except in work of high precision. The error from defective graduations in the measurement of an angle may be eliminated by taking the mean of several measurements of the angle, provided the vernier is reset at a different reading for each backsight in such a way that the various readings for the value of the angle will be distributed over different portions of the graduated circle. (See Art. 399.)

282(*g*). Eccentricity. If the two opposite verniers do not read exactly alike in any position, the error from this defect may be eliminated by taking the mean of the readings of the two verniers. The error from this source is usually negligible except in work of high precision.

282(*h*). IMPERFECT MOVEMENT OF THE OBJECTIVE SLIDE. The error from this source is generally negligible in ordinary work, but may not be in work of high precision. It is eliminated when, in measuring an angle, the lengths of sights are such that the object-glass focus is the same, or nearly so, for both sights.

282(*i*). SUMMARY OF INSTRUMENTAL ERRORS. From the preceding discussion of instrumental errors, it is obvious that nearly all the important errors may be eliminated by taking a measurement or observation with the telescope normal and then repeating it with the telescope inverted. In eliminating the error due to imperfect adjustment of the plate levels, it is necessary, in addition, to level up just before each backsight. Errors due to imperfect graduations, to eccentricity, and to imperfect movement of the object-glass slide are generally negligible except in work of high precision. Errors due to imperfect adjustment of the telescope level and of the vernier for the vertical circle are involved only in the measurement of vertical angles, or in leveling.

283. Personal Errors. In addition to errors due to mistakes in manipulating the transit, such as those described in Art. 280, and mistakes in reading angles, such as those summarized in Art. 263, there are errors due to personal limitations, such as limitations of skill in manipulating the transit and limitations of the eye in sighting.

283(*a*). PLUMB BOB OFF CENTER. Although the plumb bob may be as much as ½ in. off a point or station, the resulting error from this cause alone will not exceed 1′ of arc if the lengths of sights are not less than 150 ft., whereas for sights 500 ft. in length, the error would be less than ½′. Consequently, the error from this source is not so great as it is generally thought to be. Moreover, since the plumb bob can be brought to within ⅛ in. of a point without special effort and still closer if desired, there is little excuse for not eliminating the error.

283(*b*). BUBBLES IN PLATE LEVELS OFF CENTER. If a transitman in leveling up fails to center exactly the bubbles in the plate levels, the resulting errors are akin to those due to imperfect adjustment of the plate levels, and what was stated in Art. 282(*a*) concerning those errors holds true in this case. If the limb is 6 in. in diameter, and one edge is 3/16 in. lower than the other, a maximum error of 1′ of arc might result. A difference of ½ in. in level might cause a measurement error of 6′ to 7′, but such a difference in level would be apparent to the eye. The error is the most serious in the measurement of a horizontal angle when it is necessary in sighting, first at one point and then at the other, to turn the telescope through a large vertical angle. The error may also be serious when the transit is plunged in prolonging a straight line, unless the method of double reverse is used. It is to be remembered also that in measuring an angle *the error is greater for short sights than for long sights, whereas in laying off an angle the reverse is true.*

283(*c*). ERRORS IN SIGHTING. Failure to sight exactly on a point may be due to parallax (eliminated by focusing the eyepiece, see

Art. 103); or to natural causes such as unfavorable atmospheric conditions (eliminated by taking important observations during a period of the day when conditions are more favorable); or to poor optical qualities of the telescope; or to inaccurate work either in manipulating the transit or in holding the sight pole. When all these sources of error have been eliminated, there still remain personal errors in sighting due to limitations of the eye and of skill in the use of the transit.

283(*d*). ANGULAR ERRORS IN TERMS OF LINEAR MEASUREMENTS. The practical question is: How much error is caused by not sighting exactly on a point? Suppose that in *measuring* an angle a line of sight which should strike a given point passes a short distance, say 1 in., to one side of that point. The nearer the given point is to the transit, the greater the angular error caused by this departure of 1 in. On the other hand, suppose that a small error, say of 1′, is made in *laying off* an angle. The line of sight will depart farther and farther from the true line as the distance from the transit becomes greater and greater. Some idea of the magnitude of errors from these two sources may be gained from the following *approximate* values.

ength of Sight in Feet	*Angular Error for 1 in. or 0.08 ft.*	*Deviation for Error of 1′*
100	3′	0.03 ft. or 3/8 in.
300	1′	0.09 ft. or 1 in.
500	1/2′ or 30″	0.15 ft. or 1 7/8 in.
1000	1/4′ or 15″	0.30 ft. or 3 5/8 in.
1500	1/6′ or 10″	0.45 ft. or 5 3/4 in.
5000	1/20′ or 3′	1.50 ft. or 18 in.

283(*e*). NOTE. It is well to fix two or three of these values in mind. For example, a point an inch out of line at 100 ft. from the transit causes an error of about 3′ of arc. This error varies very nearly *inversely* as the distance of the point from the transit, so that 1′ corresponds approximately to 1 in. 300 ft. from the transit. Again, an error of 1′ of arc brings a point 0.3 ft. out of line 1000 ft. from the transit. This error varies *directly* as the distance of the point from the transit, so that at 500 ft. the error is half as much, or 0.15 ft.

283(*f*). ILLUSTRATION. A rod is held so that the top is 1 in. out of plumb. It is 100 ft. from the transit. Sighting at the top results in an angular error of 3′. Hence the importance of sighting at the bottom of the rod or, when this is impossible, of holding the rod plumb. The nearer the rod is to the transit the more need there is for accurate sighting.

283(*g*). ILLUSTRATION. The following illustration shows how much more important it is to hold the rod plumb than it is to split it exactly with the vertical hair; incidentally it emphasizes also the importance of sighting at the bottom of a rod when it is possible to do so.

If a rod 1 in. in diameter is held on a point 300 ft. from the transit, the maximum error due to the vertical hair not splitting the rod will not exceed half a minute of arc, provided the hair covers the rod at all. On the other hand, in sighting at the top of a rod which is, say, 2 in. out of plumb, the error will be four times as great, or 2′. It is evident that it is easier to bring the vertical hair on the rod than it is to hold the latter plumb. Moreover, when all but the upper foot or two of a rod is invisible from the transit, the top may be considerably more than 2 in. out of plumb with a corresponding increase in error.

283(*h*). Errors in setting and reading the vernier may be due to mistakes in reading; or to indistinct graduations; or to the fact that the least count of the vernier may be too great. When all these sources of error have been eliminated, there still remain personal errors in reading due to limitations in observing the exact coincidence of a mark on the vernier with a mark on the limb, even though a magnifying glass is used. A large error may be avoided (1) by observing graduations on either side of the two that appear to coincide and (2) by keeping the eye directly in the vertical plane through the lines of coincidence to avoid parallax. When these precautions are taken, the error should not ordinarily exceed 30″, even though the least count of the vernier is only 1′ (Arts. 233 and 234). It should be less when the least count of the vernier is less. In any case, it should not exceed one-half the least count. In precise work, the error may be reduced by measuring an angle either by the method of repetition (Art. 265) or by series (Art. 267).

283(*i*). Summary of personal errors. (1) Errors due to imperfect centering over a station are not usually important unless sights are short, and in any case may be easily eliminated by bringing the plumb bob over the point. (2) Errors due to imperfect leveling up may be important when in measuring a horizontal angle it is necessary to turn the telescope through a large vertical arc; also when prolonging a straight line by plunging the telescope unless the method of double reverse is used. In measuring an angle the error from this source is greater for short sights than for long sights, whereas in laying off an angle, the reverse is true. (3) Errors in sighting are not likely to be great in ordinary work if the sight pole is held plumb. (4) Errors in setting and reading the vernier should not exceed one-half the least count of the vernier, and in precise work, they may be greatly reduced by measuring angles by the method of repetition or by series.

284. Natural Sources of Error. A strong wind, hazy atmosphere, lateral refraction, the sun shining on the instrument, and other natural sources of error are generally of little consequence in ordinary work unless the conditions are extreme. An exception is the unequal settlement of the tripod legs, which is most likely to occur on soft or thawing ground. An error from this source may be comparatively large. The precautions which may be taken to eliminate natural sources of error in work requiring great precision have already been discussed in connection with errors in leveling (Art. 167(*a*)). One of the most important is to carry on the work only during that portion of the day when conditions are most favorable, that is, directly after sunrise or from dark to midnight.

285. Elimination of Errors. *Constant Errors.* One of the reasons why the measurement of angles is so satisfactory is that constant errors may be largely eliminated by taking observations in pairs. Thus, for example, plunging the telescope between two observations eliminates certain instrumental errors, and reading both verniers eliminates errors due to eccentricity of graduation. Other

precautions are given in Art. 264(*a*). The extraordinary precautions which must be taken to eliminate errors in the precise (first-order) measurement of angles are explained in Chapter 27, "The Precise Measurement of Angles."

Accidental errors are eliminated by adjusting values of angles, by taking the mean of several observations, and by distributing the total error when several angles are related by some fixed condition (Art. 41(*c*), (*d*)).

286. The Precision of Angular Measurements may be judged in one of two ways: (1) by comparing the results of several independent measurements of the same angle, and (2) by observing how closely the values of different but *related* angles fulfill known conditions.

286(*a*). FIRST METHOD. In ordinary work if an angle is important enough to check at all, it is usually doubled. (Method of repetition, Art. 265(*a*).) In more important work an angle may be measured a number of times and the precision will then be indicated by the probable error (Arts. 42–47). The more completely the assumptions of least squares (Art. 44) are fulfilled the more nearly will the probable error represent the true error (Art. 44(*a*)).

286(*b*). SECOND METHOD. When the angles of a polygon are measured the *true* error of their *sum* is known (Art. 298(*a*)), but not the error of any one angle. The same is true of several angles having vertices at a common point, when, as in Fig. 267, their sum should equal 360°. When, however, the sum of two or more angles should equal a *measured* angle, the *discrepancy* between the sum and the measured angle is not an error in the true sense of the word, but simply the difference between two quantities both of which contain an error. (See Art. 37.) In all three cases errors in one angle may be compensated by errors in another angle, so that the error of the sum does not represent the actual precision of the work.

286(*c*). ILLUSTRATION. The three angles of a triangle often add up exactly to 180° when measured with a transit reading to minutes. This does not mean that no error has been made, but that the errors are exactly compensating. It is not likely that any one of the three angles can be expressed in even minutes. If, therefore, they are measured with a finely graduated transit reading to 10″, their sum is less likely to add up to 180° than if a transit reading to minutes is used, although the precision with which each angle is measured may be much greater. Thus, for example, three angles were measured with the following results:

	Transit Reading to 1′	*Transit Reading to 10″*
1	68° 10′	68° 9′ 40″
2	79° 40′	79° 40′ 25″
3	32° 10′	32° 9′ 40″
	180° 0′	179° 59′ 45″

The error of the sum in the second set is 15″, yet the precision for each angle is probably much greater than in the first set where the error of the sum is zero. In a similar manner the angles of a triangle as determined by reading a compass to the nearest 15′ will often add up 180°, while the same angles measured with a transit fail to do so, yet the precision of the work done with the transit is probably far greater.

287. Conditions Which Affect the Precision of Angular Measurements in addition to the sources of error already treated are these. (1) *Instrument Used.* How well made; how finely graduated; power of telescope, sensitiveness of levels, and rigidity of tripod. (2) *Method Used* in measuring the angle. (3) *Local Conditions.* All sorts of conditions which make it easy or difficult to attain a high degree of precision. (4) *Time Allowed.* If, because of cost or for any other reason, time is limited in which to make a survey, the degree of precision may also be correspondingly limited.

288. Customary Limits of Error. (*a*). RELATION OF ANGULAR ERROR TO LINEAR ERROR. Varying conditions render it difficult to state what constitutes good work with the transit. Moreover, as already pointed out in Art. 80, the limits of error are largely governed by the purpose of a survey, and even in the same survey angles are not all measured with the same precision. Suppose, however, that a limit of error has been fixed for chaining and that it is desired that the precision of angular measurement shall correspond. The question at once arises: What is the largest permissible error for each angle? In the following table the errors in feet caused by different angular errors are given for five lengths of sights, viz., 100 ft., 500 ft., 1000 ft., 1500 ft., and 5000 ft.:

Error	100	500	1000	1500	5000	Ratio
1′	0.0291	0.1454	0.2909	0.4363	1.4544	$\frac{1}{3440}$
30″	0.0145	0.0727	0.1454	0.2182	0.7272	$\frac{1}{6880}$
20″	0.0097	0.0485	0.0970	0.1454	0.4848	$\frac{1}{10,300}$
10″	0.0048	0.0242	0.0485	0.0727	0.2424	$\frac{1}{20,600}$
6″	0.0029	0.0145	0.0291	0.0436	0.1454	$\frac{1}{34,400}$
4″	0.0019	0.0097	0.0194	0.0291	0.0970	$\frac{1}{51,600}$
2″	0.0010	0.0048	0.0097	0.0145	0.0485	$\frac{1}{103,100}$

0.0104 ft. = ⅛ in. 0.0208 ft. = ¼ in. 0.0417 ft. = ½ in.
0.0833 ft. = 1 in. 0.2500 ft. = 3 in. 0.4167 ft. = 5 in.

This table has two uses. (1) It gives the permissible angular error for a given ratio of precision; (2) it gives an idea of the

error *in feet* at different distances from the transit caused by a given error in *minutes or seconds.*

288(*b*). PERMISSIBLE ANGULAR ERROR. The ordinary transit reads to minutes. If an error in an angle does not exceed 1′, the ratio of precision is about 1/3400. It is not difficult to keep within this limit in chaining, and in a large part of the surveyor's work greater precision is not required. A ratio of 1/5000 is classed as fair work for measurements with a steel tape. To come within this limit angles should be read to 30″ (see table), and this can easily be done with a transit reading to minutes. For a precision of 1/10,000 the angles should be read to 20″, therefore a transit graduated to 20″ is preferable. For the higher limits of 1/35,000, 1/50,000, and 1/100,000 it is seen that angles must be read to 6″, 4″, and 2″ respectively. This requires the use of accurate instruments, finely graduated, with corresponding precautions in sighting and manipulation. Moreover, a number of independent measurements of each angle must be taken (either by repetition or by series), from which the probable value of the angle may be deduced. This will be treated more fully in Chapter 27, "Precise Measurement of Angles," and in Chapter 28, "Triangulation." For permissible angular errors in first-order, second-order, third-order, and fourth-order traverses see the specifications for traverses in Art. 358(*a*). The permissible *error of closure* in a traverse, involving errors in both linear and angular measurements, is discussed in Arts. 357–358(*b*).

It is evident that in any given survey the limit of error for angles and the limit of error for linear measurements should be consistent. This is too frequently overlooked, one limit being set for chaining, another for transit work. An exception to be noted is in rough chaining when corresponding angular measurements can be made with greater precision than the linear measurements without special effort or loss of time.

288(*c*). THE PERMISSIBLE ERROR IN THE POSITION OF A POINT, if kept in mind, will often save unnecessary refinement in measuring angles in certain classes of work; in other work it will emphasize the necessity for accurate measurements. This is shown in the following illustrations.

288(*d*). ILLUSTRATION. In a topographic survey it is sufficient to locate contour points within, say, a foot or two. The longest sight is, say, 400 ft. From the table the error for 1′ for a 100-ft. sight is 0.029 ft.; hence for 400 ft. it is about 0.12 ft. To come within 1.5 ft. the angular error must not exceed 1.5 ÷ 0.12 = 12′. For sights less than 400 ft. the permissible error is still greater. As limb readings can easily be estimated to within 10′, this means that for 400-ft. sights, or less, it is not necessary to read the vernier—an obvious saving of time.

288(*e*). ILLUSTRATION. A certain point about 500 ft. from the transit must be located within ¼ in. From the table, ¼ in. (or 0.0208 ft.) for 500 ft. corresponds to an error of a little less than 10″. This is the maximum permissible angular error.

288(*f*). ERROR IN THE SUM OF ANGLES. Since it is comparatively easy practically to eliminate constant errors in the measurement of an angle, the remaining error in the measurement will be due largely to accidental errors. Consequently, when several angles are added together the total error of their sum may be expected to vary about as the square root of the number of angles. Limits of error may be indicated as in chaining (Art. 81); thus $E_s = a\sqrt{N}$, where E_s is the total error of the sum, a the probable error for one angle, and N the number of angles either in a polygon or about a point. From the following table the value of E_s in seconds can be obtained for different values of a and N.

PERMISSIBLE ERRORS IN THE SUMS OF ANGLES FOR DIFFERENT VALUES OF *a*

N	$a = 60''$	$a = 30''$	$a = 20''$	$a = 10''$	$a = 6''$	$a = 4''$	$a = 2''$
3	104″	52″	35″	17″	10″	7″	3.5″
4	120″	60″	40″	20″	12″	8″	4″
6	147″	73″	49″	24″	15″	10″	5″
9	180″	90″	60″	30″	18″	12″	6″
12	208″	104″	69″	35″	21″	14″	7″
16	240″	120″	80″	40″	24″	16″	8″

In ordinary work the value of a may be taken as 30″. (See Art. 288(*b*).) The error which may be expected in the sum of the angles of a triangle is 52″, or about 1′. For a four-sided polygon the table gives exactly a minute, while for one of sixteen sides it is 2′. Of course these values are not absolute—they are for guidance only. In selecting the value of a the correlation of angular and linear measurements already discussed should be kept in mind. The tendency is to measure angles with unnecessary accuracy, and to neglect important sources of error in linear measurements. (See also Art. 358(*a*) for maximum permissible error in the sum of interior angles for first-, second-, third-, and fourth-order traverses.)

CHAPTER 21

TRANSIT SURVEYING

METHODS OF RUNNING TRAVERSES

In this chapter four methods of running traverses are explained. The *theory* of each method is given without entering upon the practical details of the field work. (See Chapter 23 for *field work.*) In order that the different methods may be compared, the advantages and disadvantages of each method are stated.

289. Transit-Line Angle. The horizontal angle between any two transit lines that meet at a transit station. It may be either the larger or smaller angle, and each of these angles may be measured clockwise or counterclockwise. In practice, therefore, *it is not enough to note the value of an angle, but the backsight and direction in which the angle is measured must be noted also.*

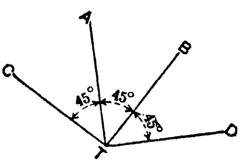


FIG. 289.

Illustration. In Fig. 289 let *A, T,* and *B* represent the *correct* location of three transit stations, the transit standing at *T*. Suppose the notes give the angle *ATB* as 45° from a backsight on *A,* but it is not stated whether the angle is measured to the right or to the left. If the draftsman assumes that it was taken to the left, plotting 45° he will get a line *TC* instead of the line *TB*.

Again, suppose that the notes say that the angle of 45° was taken to the right, but it is not stated whether *A* or *B* is the backsight. Should the draftsman assume that *B* was the backsight, plotting 45° to the right he would get a line *TD* instead of the line *TA*.

The notes, therefore, must not admit of ambiguity in respect to the backsight, or to the direction (right or left) in which an angle is taken.

290. Transit-Line Deflection Angle. The angle which a line makes with a transit line produced from the preceding station. Deflection angles are measured to the right or to the left; hence the *direction* should always be entered in the notes.

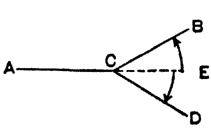


FIG. 290.

Illustration. A transit stands at *C,* Fig. 290; the backsight is at *A;* the telescope is plunged after backsighting, and then points at *E*. If turned to the right, any deflection angle, as *ECD,* may be measured to the right; if turned the other way, any deflection angle to the left may be measured, as *ECB*. When deflection angles are used at all, it is customary to measure each deflection angle to the right or to the left, according to which side of the line the point sighted at lies.

291. Running Transit Lines. A term for establishing transit stations and locating these stations with respect to each other. The work usually consists in chaining distances between successive transit stations and in measuring the angle which each transit line makes with some other transit line, or with some reference meridian.

Note. The field work of a survey may be divided into two parts: (1) running transit lines, and (2) locating details from those transit lines. This chapter has to do with the first part only.

292. Transit Lines as Lines of Reference. Transit stations are so chosen by the surveyor that the lines between them may be used as reference lines, by which and from which boundaries, buildings, streets, or whatever he may wish to survey are located. Methods of locating details of a survey such as these are given in the next chapter.

Transit lines, as a rule, will not be inked in on the finished map, and should be regarded merely as a means to an end.

293. Traverses. A *traverse* is a system of connected lines so chosen and laid out that they form a framework or skeleton of reference lines with respect to which all details of a survey are

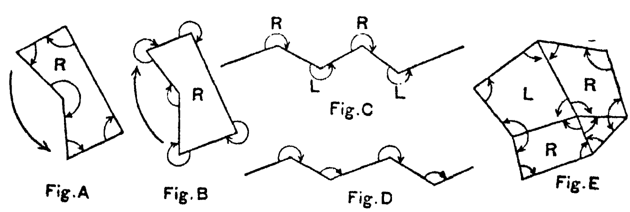


FIG. 293.

located. When the work of "running a traverse" has been completed, the direction and the length of each line are known.

The system may form a closed polygon (*closed traverse*) as in *A* and *B* of Fig. 293, or a series of lines which do not close (a *continuous* or *open traverse*), as in *C* and *D* of Fig. 293, or a network of lines as in *E* of Fig. 293.

In a *transit traverse* the directions of the lines are determined by means of the transit; in a *compass traverse,* by the compass; in a *plane-table traverse,* by the plane table. This chapter deals with transit traverses only. Lengths of lines are usually measured with a tape, or by the stadia method (*stadia traverse*).

It is desirable to have a traverse close if possible since the closure affords certain checks on the field work that are not possible in a traverse which does not close (Art. 299). A continuous

traverse is used in route surveying and in other surveys where a closed traverse is impracticable.

294. Triangulation. A network of connected triangles, in which the angles are measured, but not the sides. The latter are found by trigonometric calculation, starting from some measured base line. The principal distinction between traverse and triangulation is that the latter term implies the measurement of angles only.

295. Determining Directions of Traverse Lines. The directions of the transit lines in a traverse may be determined by measuring (1) the direct angles or (2) the deflection angles between transit lines at traverse stations; or by measuring (3) the azimuths or (4) the bearings of the transit lines.

296. The Transit-Line Angles—Right or Left. Angles may be measured all to the right from a backsight on the preceding station, as in *A, B,* and *D* (Fig. 293), or all to the left, as in two of the polygons in *E,* or right and left indiscriminately, as in *C.* (In Fig. 293, the direction in which the angles were measured is indicated by arrows; *R* in a polygon means that all angles in that polygon were measured to the right; *L* means that all angles were measured to the left.) For beginners, at least, it is better to measure all direct angles to the right. Deflection angles must, of course, be measured to the right or to the left, depending on which side of the backsight produced the angle lies.

297. Transit-Line Angles—Interior and Exterior. It is usually more convenient, but by no means necessary, to measure the *interior* angles of a polygon instead of the exterior. In order to measure interior angles to the right, it is necessary to proceed around the polygon *counterclockwise* (leave the polygon on the left), as in *A* of Fig. 293. In order to measure exterior angles to the right, it is necessary to proceed around the polygon *clockwise* as indicated in *B* of Fig. 293. If it is immaterial whether the interior angles of a polygon are measured to the right or to the left, obviously it is immaterial whether the procedure from station to station is counterclockwise or clockwise.

298. To Check the Sum of the Angles of a Polygon. (*a*). INTERIOR ANGLES. From the number, *N*, of the sides of the polygon, or of the angles, subtract 2 and multiply by 180°. The final result should be the sum, *S*, of the interior angles. $S = (N - 2) \times 180°$. Triangle: $(3 - 2) \times 180° = 180°$. Four-sided polygon: $(4 - 2) \times 180° = 360°$. Five-sided: $(5 - 2) \times 180° = 540°$. Six-sided: 720°. Any permissible error in the sum of interior angles (Art. 288(*f*)) is generally distributed equally among the angles.

298(*b*). Exterior deflection angles. If the deflection angles to the right are considered positive and those to the left negative, the *algebraic* sum of all the exterior deflection angles around any polygon should be ±360°.

299. The Error of Closure. When practicable it is an advantage to have a traverse close on the starting point or first station because of the check on the field work which such a closure affords. In any polygon of transit lines the location of each station with respect to any other station can be *calculated* from the angles and distances measured in the field. If the position of any station is assumed as a starting point, and then, following around the polygon, the position of this same station is found by calculation, the difference between the assumed position and the calculated position is the *error of closure*. This error is due to: (1) errors in measuring transit-line angles; (2) errors in measuring the *lengths* of transit lines. In practice a survey seldom closes exactly, because of errors from both sources. The method of determining the error of closure and its use in checking a closed traverse is explained in Art. 307. The permissible error of closure is treated in Arts. 356–358.

Note. The check on the field work afforded by the error of closure in a closed traverse is somewhat analogous to the check on the leveling afforded by the error of closure in a level circuit, that is, the error indicated by the difference between the known or assumed elevation of the starting point and the elevation of the same point as calculated from the circuit levels.

300. Types of Traverses. The methods of running transit lines in different types of traverses differ only in the methods of determining the *directions* of the transit lines; methods of linear measurements are the same for all types. Four common types of traverses corresponding to four methods of determining directions may be divided into two groups:

I	Traverse by Direct Angles	II	Traverse by Azimuths
	Traverse by Deflection Angles		Traverse by Bearings

In the first group the angle measured at each station is the direct angle or the deflection angle between the two transit lines that meet at that station. In the second group some reference meridian is adopted for the entire survey, and the direction of each transit line is determined by measuring (indirectly) the angle that it makes with that meridian.

301. Traverse by Direct Angles. This method consists simply in measuring each angle directly from a backsight on the preceding

station. If desired, the angle can be doubled, tripled, or repeated any number of times. The method may be used whether the transit lines form a closed polygon or not. On the whole it is better in this method to take all angles clockwise. When the transit lines form a closed polygon, and the interior angles are measured, the entire system is usually called an *interior-angle traverse*. The sum of the interior angles should equal $(N - 2) \times 180°$ (Art. 298).

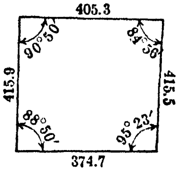


FIG. 301.

Illustration. In Fig. 301 each interior angle was measured to the right (clockwise) from a backsight on the preceding station. If more convenient, the exterior angles could have been measured (Art. 297).

Remark. A triangulation net is a special case of the angle method of running transit lines.

302. Traverse by Deflection Angles. Instead of measuring the angle at each station of the traverse directly from a backsight as explained in the preceding method, the *deflection* angle at each station is measured, to the right or to the left as the case may be (Fig. 302). There are three methods of running traverses by deflection angles corresponding to the three methods of measuring deflection angles, explained in Art. 269.

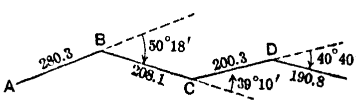


FIG. 302.

302(*a*). FIRST METHOD. Assume that the transit stands at *B*, invert the telescope, backsight on *A*, plunge the telescope, foresight on *C*, and read the angle 50° 18′ to the right.

The cumulative effect of errors due to plunging the telescope may be corrected to a great extent by taking one backsight with the telescope inverted, the next backsight with it normal, and so on, alternating in this way from station to station.

302(*b*). SECOND METHOD. Set the A vernier at 180°, backsight on *A* with the telescope normal, foresight on *C*, with the telescope still normal, and read the angle 50° 18′ from the A vernier.

302(*c*). THIRD METHOD. Each deflection angle is measured by the method of repetition described in Art. 269.

302(*d*). METHODS COMPARED. In the first method certain errors may be caused by plunging the telescope. In the second method these errors are eliminated. In either the first or the second method, the only check on the measurement of an angle is that obtained by comparing the calculated bearing or azimuth of the foresight with its magnetic bearing or azimuth as indicated by the needle. In the third method, not only is there a check on each

angle, but also by repeating the angle, a higher degree of precision may be obtained.

302(*e*). CHECK. For a closed traverse, the *algebraic* sum of the deflection angles should equal ±360. This check cannot be applied to an open traverse.

303. Traverse by Azimuths. Before beginning the traverse, it is necessary to adopt a reference meridian. The choice between a true meridian, a magnetic meridian, and an assumed meridian has been discussed in Art. 209(*e*). Whatever meridian is adopted, it must, of course, be the same for the entire survey.

303(*a*). ORIENTING THE TRANSIT. In running traverses by azimuths, the first step after setting up the transit at any station is to *orient* the transit. This is done by bringing the lower or graduated plate into such a fixed position that if the upper plate is revolved until the *vernier reading is 0°, the telescope will point in the same direction as that of the adopted meridian.*

303(*b*). TO ORIENT THE TRANSIT AT THE FIRST STATION OF A TRAVERSE. (1) For a *magnetic* meridian: With the upper plate clamped at 0°, point the telescope to the magnetic north, as indicated by the compass needle if azimuths are to be measured from the north, or to the magnetic south if azimuths are to be measured from the south, and set the lower clamp. (2) For a *true* meridian: If practicable, establish the first station at a point on a line, the true azimuth of which has been previously determined. With the upper plate clamped at the same reading as the azimuth of the line, sight along the line in the same direction as the azimuth, and set the lower clamp. When it is impracticable to start from a line of known azimuth the true azimuth of any line through the first station may be determined by one of the methods of observation explained in Chapter 34 and 35. With the upper plate clamped at this azimuth reading, the transit is then oriented by sighting along the line. (3) For an *assumed* meridian through the first station: With the upper plate clamped at 0°, orient the transit by sighting along this meridian.

Note. When the survey is to be tied into a system of permanent control points previously established, such as the systems described in Chapter 56, and the first station of the survey is one of the control points, the transit may be oriented by taking the first backsight along a line of known azimuth in the control system.

303(*c*). TO ORIENT THE TRANSIT AT ANY STATION EXCEPT THE FIRST. There are two methods. In the first, the vernier is set automatically for any station by leaving the upper plate clamped at the reading for the foresight that was taken while at the preceding station; the telescope is inverted, a backsight on the preceding sta-

tion is taken, and the telescope is then plunged to its normal position. In the second method, before backsighting, the vernier is reset at the reading for the *back* azimuth of the line from the preceding station to the station where the transit stands. (The back azimuth is the forward azimuth read at the preceding station plus 180°.) Both backsight and foresight are then taken with the telescope in its normal position.

303(*d*). THREE METHODS OF RUNNING TRAVERSES BY AZIMUTHS. One method is based on the first method of orienting the transit just described; another is based on the second method; a third is a modification of the first. For the purpose of explaining these methods more in detail, it will be assumed that azimuths are measured from the *north* end of a *magnetic* meridian. The general method holds good, however, no matter what meridian is adopted. The main things to be kept in mind are: (1) that the back azimuth of a transit line is equal to its forward azimuth plus 180°; (2) that at any station a backsight is taken *to orient* the transit, and a foresight is taken *to determine an azimuth;* (3) that if, after a transit has been oriented at *any* station, the upper plate is turned until the vernier reading is zero, the telescope will point to the magnetic north as indicated by the compass needle. The transit is *not* oriented, however, by means of the needle except at the first station. On the contrary, at any station except the first, the needle is consulted only as a rough check. Note that statements (1) and (2) hold good for any meridian, but that statement (3) is true only for a magnetic meridian.

303(*e*). FIRST METHOD BY AZIMUTHS. Let the first setup of the transit be at station *A* (Fig. 303(*e*)). With plates clamped at zero, point the telescope to the magnetic north as indicated by the needle. This is equivalent to *backsighting on the magnetic north, but this first station is the only one at which the needle is consulted except as a check.* Set the lower clamp as usual for a backsight. The transit is now oriented. Loosen the upper clamp and sight on *B*. Suppose that the vernier now reads 50°; then this angle, taken clockwise from the magnetic north, *must be the azimuth of AB. Allow the vernier to remain clamped at* 50°.

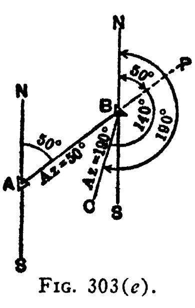


FIG. 303(*e*).

Setting up at *B, plunge* the telescope and backsight on *A*. Having backsighted, plunge the telescope again; it now becomes *normal* and points in the direction of *AB produced* along the line *BP*. If the azimuth of *AB* is 50°, the azimuth of *BP* must also be 50°; and

since the vernier has not been disturbed, it reads 50° just as it did when the transit stood at *A* and the telescope pointed at *B*. The transit is now oriented. Hence, if the telescope is pointed north, the vernier reading will be zero; and if it is pointed to any other station, the vernier reading will be the *azimuth* of the line from station *B* to that station.

Illustration (Fig. 303(*e*)). Set up the transit at *A*, and backsight on the magnetic north with the vernier at zero. Foresight on *B*, the vernier reading will be 50°, or the *azimuth* of *AB*. Without disturbing the vernier setup at *B*, backsight on *A* with *inverted* telescope, *plunge* the *telescope* (it now becomes normal again, the vernier still reading 50°), and foresight on *C*. The vernier will read 190°, or the *azimuth* of *BC*. Allow the vernier to remain clamped at 190, setup at *C*, *invert* the telescope, backsight on *B*, *plunge* the telescope, and sight on the next station. Thus the process can be repeated for any number of stations.

Note. Usually there are azimuths to *other points* beside transit stations which must be taken at each setup. If so, then the *last* sight before leaving a station must be on the next station, thus automatically setting the vernier for the next setup.

303(*f*). Second method by azimuths. In this method the telescope is *not* inverted for the backsight, but instead the vernier is set anew at each station so that its reading will be the *azimuth of the backsight*. The *azimuth* of the backsight is equal to the *back azimuth* of the foresight as taken at the preceding station, i.e., add 180° to the azimuth already recorded. For example, if the transit stands at *B* (Fig. 303(*f*)), before backsighting on *A*, the vernier is set so that the reading is the *back* azimuth of *AB*, or 230° (obtained by adding 180° to 50°). If, therefore, the vernier is set at 230° and a backsight taken to *A*, upon turning back to zero the telescope will point north; that is, the transit has been oriented. Hence, if a foresight is taken to any other point, as *C*, the reading of the vernier will be the azimuth of the line from the transit to that point, or the azimuth of *BC*.

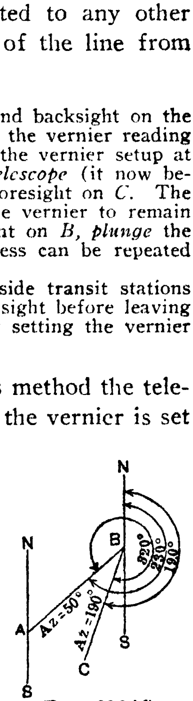


Fig. 303(*f*).

Illustration. In Fig. 303(*f*) the transit is set up at *A* and the azimuth of *AB* obtained exactly as in the first method. Set up at *B*, set the vernier at 230° (50° + 180°), backsight on *A*, foresight on *C*, the new reading will be 190°, or the azimuth of *BC*. Set up at *C*, set the vernier at 190° + 180° (i.e., 10°), backsight on *B*, and foresight on the next station. Thus the process is repeated at each succeeding station. The telescope is always normal, never inverted for backsight or foresight.

303(*g*). Third method by azimuths. The procedure at station *A* is exactly the same as in the first method. Note that when the transit is taken from station *A* to station *B* the plates are left clamped and that the A vernier reading is that for the forward azimuth of *AB*, namely, 50° (Fig. 303(*e*)).

Setting up at *B*, *invert* the telescope and backsight on *A* with the A vernier still clamped at 50°. Obviously the B vernier reading will be 180° + 50° or 230°, which is the back azimuth of *AB*. Consequently the transit is now oriented, provided the B vernier is used instead of the A vernier. *Keeping the telescope still inverted,* foresight on *C*. The B vernier reading will be the azimuth of *BC*, i.e., 190°. Note that the telescope has not been plunged between the backsight and foresight, as in the first method.

The procedure at the next station is exactly the same in principle. With the plates clamped at 190° for the B vernier and the telescope *normal*, a backsight is taken on *C* and the A vernier is used in obtaining the reading for the azimuth of the foresight. Thus at one station the B vernier is used, at the next the A vernier, and so on, alternating, from station to station. For convenience in reading, the telescope is kept inverted at a station where the B vernier is read, and normal at a station where the A vernier is used; but at no station is the telescope plunged during the measurement of the angle.

303(*h*). COMPARISON OF THE THREE AZIMUTH METHODS. The procedure at the first station is the same for all three methods. Moreover, the third method is a simple modification of the first. Consequently, only the procedures for the first and second methods, at some station other than the first, will be compared.

First Method	*Second Method*
1. Vernier is left clamped at the forward azimuth of the preceding line.	1. Vernier is reset at the forward azimuth of the preceding line *plus 180°*.
2. Telescope is inverted before backsighting.	2. Telescope remains normal when backsighting.
3. Telescope is plunged after backsighting.	3. Foresight is taken without plunging.

In both methods the azimuth can be read directly from the vernier, *whether in turning off the angle the telescope is moved to the right or to the left.* Transits should be graduated from 0° to 360°, preferably in one direction only, to correspond to angles taken to the *right*, or clockwise.

The first method has no disadvantages as compared with the second method except such as arise from plunging the telescope. If, for example, the line of collimation is badly out of adjustment, an error is introduced each time the telescope is reversed. The advantage of this method over every other method of running transit lines is that after the first setup it is not necessary (or de-

sirable) to set the vernier, since the *foresight* on each station automatically sets the vernier for that station. Thus the error which is so likely to occur in setting the vernier is eliminated. (For disadvantages see Art. 306, III.)

The second method is used when for any reason it is not best to plunge the telescope, or when, as on the theodolite, the telescope cannot be plunged. A disadvantage as compared with the first method is that the vernier must be set anew at each station, a possible source of error.

In the third method the principal advantages of the first and second methods are combined while the principal disadvantages of both are avoided. Thus the plates are kept clamped at the forward azimuth reading (as in the first method) to avoid the errors that arise from the continual resetting of the vernier (as in the second method), and, at the same time, the telescope is kept the same side up between a backsight and the corresponding foresight (as in the second method) to avoid errors caused by plunging (as in the first method). The chief disadvantage of the third method is that one must be continually on guard against sighting with the telescope normal when it should be inverted, or vice versa, and against reading the A vernier instead of the B vernier, or vice versa.

303(*i*). CHECKING AZIMUTHS BY THE MAGNETIC NEEDLE. Any azimuth may be easily reduced to a bearing. If, therefore, a *magnetic* meridian is taken as the reference line for a survey, the azimuth of *any* line may be roughly checked by glancing at the needle while the telescope is still pointing along the line and mentally noting whether or not the azimuth and bearing agree.

When a true meridian is taken as a reference line, azimuths are more readily checked by the needle, if the compass box is adjusted so that when the telescope is pointed to the true north, the needle will point to the *N* of the compass box, although the *N* will no longer be exactly underneath the axis of the telescope (Art. 279(*d*)).

When a meridian other than a magnetic or a true meridian is assumed, azimuths may still be checked by means of the needle, but not so readily, unless the assumed meridian is so nearly a north-south line that the compass box can be adjusted to it. In some transits no provision is made for this adjustment. When azimuths cannot be easily checked by the compass, one of the advantages of the azimuth method is lost.

303(*j*). WHEN AZIMUTHS AND MAGNETIC BEARINGS DISAGREE. When in any survey, azimuths are to be checked roughly by the magnetic needle, the back bearing of the *first* line should be read at

the second station. If it is consistent with the forward bearing as taken at the first station, it may be assumed that there is no local attraction at either place. If the two bearings are inconsistent, it is necessary, before proceeding with the survey, to establish some transit line the back bearing of which is consistent with its forward bearing. Having done this, a new start may be made from either end of the line, since in all probability the azimuth of the line taken at either end will be correct. Thereafter, should the azimuth of any transit line disagree with the magnetic bearing and no mistake be found in the azimuth, local attraction may be assumed as the cause. It often happens that such a disagreement will occur for several successive transit lines, but should bearings and azimuths continue to differ by about the same amount the original azimuth upon which all others depend is probably incorrect. If the magnetic needle is to be used as a check throughout the survey, it is obviously important that the meridian used for orienting the transit at the first station should be as nearly correct as possible. For this reason there is a decided advantage in starting from a true meridian, established from observation, or from a line whose true azimuth has been previously ascertained. The compass box may then be adjusted as previously explained.

303(*k*). A CHECK ON THE SUM OF THE INTERIOR ANGLES of a closed traverse run by azimuths may be illustrated as follows. Assume that the stations of a polygon of transit lines in order are *A, B, C, D,* and *E,* and that when the transit was at the first station *A* the back azimuth *AE* was measured as well as the forward azimuth *AB*. The azimuths of the remaining lines may be found without setting up at *E,* but this would be equivalent to not measuring the closing angle at *E*. By setting up at *E* the forward azimuth *EA* may be measured and compared with the back azimuth *AE*. Any discrepancy between the two would indicate the total error in the sum of the interior angles of the polygon.

303(*l*). THE ERROR OF CLOSURE may be determined as explained in Art. 307.

304. Traverse by Bearings. This method is essentially the same as the azimuth method except that *bearings* instead of azimuths are read directly from the transit. For this purpose the graduations on the limb should be numbered from 0° to 90° to correspond to quadrants of bearing in the compass.

The transit is oriented at the first station, whatever the meridian may be, just as explained for the method of azimuths (Art. 303(*b*)).

When the plates are left clamped at the *forward* bearing of a

transit line, the transit is oriented at the next station by backsighting with inverted telescope and then plunging; or it may be oriented by setting the vernier at the *back* bearing and then backsighting without inverting the telescope. (The two methods of orienting correspond to those used for azimuths.)

The reference meridian may be either *true* or *magnetic*. If the former is used, it is well to adjust the compass box as explained in Art. 303(*i*), so that bearings read from the limb may be checked by the needle without thinking of the magnetic declination.

Note. When the limb of a transit is graduated in accordance with the quadrant system, the two zeros on the limb (180° apart) may be called the *north* zero and the *south* zero. *The one with which the zero of the vernier coincides when the telescope points north (the transit having been oriented) is the* NORTH *zero.* (Why, in most cases, will this zero really be nearer the *south* end of a telescope pointed north?)

Illustration. In Fig. 303(*f*), if the transit is oriented at station *B* and is then pointed to station *C*, the vernier reading will be 10°, and since the vernier zero is to the *left* of the S zero on the limb, the bearing of *BC* is S 10° W. (Draw a sketch showing the relative position of the telescope, the N and S zeros on the limb, and the vernier zero. See (*b*) and (*c*), Fig. 279.)

Caution. In reading bearings, it is evident that an error of 180° will be introduced by mistaking the north zero for the south zero or vice versa. Thus in the illustration a common mistake would be to read N 10° E instead of S 10° W.

Note. Precautions should be taken to eliminate error due to local attraction in the bearing of the first line. (See Art. 303(*j*).)

305. Traverses by Angles and Calculated Bearings. This can hardly be called a separate method. Transit lines are run either by the direct-angle or by the deflection-angle method, but, in addition, the angle at each station is checked by comparing the *calculated* bearing of the foresight to the next station with the bearing of that line as indicated by the magnetic needle. The bearing of the *first* transit line is obtained from the needle, the angle which that line makes with the *second* transit line is measured, and the *calculated* bearing of the second line found by the method explained in Chapter 37, "Calculation of Bearings and Azimuths." Thereafter the bearing of each transit line is *computed* from the *calculated* (NOT the magnetic) bearing of the preceding transit line, and *is entirely independent* of the magnetic bearing. Since, however, the *first* bearing calculated, i.e., the calculated bearing of the *second* transit line, was obtained from the *magnetic* bearing of the first transit line, all other calculated bearings should agree *approximately* with the corresponding magnetic bearings as indicated by the needle, and hence any *large* error in a transit-line angle may be detected. The check thus afforded is a valuable one, especially in running the transit lines of a traverse which does not close, provided, of course, that the needle is not continually affected by local attraction.

Remark. If the cause of local attraction is not about the person of the observer, but remains stationary, an angle can still be checked by the needle provided two readings are taken at the station where the transit stands, one the back bearing of the preceding line, the other the forward bearing of the line ahead.

Note. Precautions to eliminate errors due to local attraction should be taken in assuming the bearing of the first line in order to use the needle throughout the survey as a rough check.

Illustration. The bearing of *AB*, Fig. 305, as read at *A* is N 20° E. The back bearing, read at *B*, is S 20° W; hence N 20° E is assumed as correct. The angles at *B*, *C*, and *D* are indicated on the figure. The calculated bearings of the different lines are also indicated.

Using the method explained in Chapter 37, the calculation is as follows:

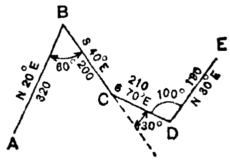


Fig. 305.

AB =	N 20° E	+	(SW)
ABC =	60°	−	
BC =	S 40° E	−	
BCD =	30°	−	
CD =	S 70° E	−	(NW)
CDE =	100	+	
DE =	N 30 E	+	

Notice that since the angle BCD = 30° is a *deflection* angle, the *forward* bearing of BC is used instead of the back bearing, i.e., the letters SE are not changed to NW before adding 30° algebraically.

For other examples of the calculation of bearings see Chapter 37.

Note. When a true meridian is used, the true bearing of the first transit line is calculated with respect to this meridian. All other calculated bearings will then be true bearings. In order to check roughly these calculated bearings by the magnetic needle, it is well to adjust the compass box (Art. 303(*i*)). The method of starting from a true meridian is illustrated in the sketch of transit lines on p. 643 by the angle which the first line *YV* makes with the true north.

305(*a*). Check for a closed polygon. The bearing of the first line assumed from the needle reading is sometimes called a calculated bearing. Hence it may be said that the calculated and magnetic bearings are *exactly* the same for the first line, although this is not likely to be true of other lines. If the transit lines form a closed polygon, a second calculated bearing may be obtained for the first line from the calculated bearing of the last side and the closing angle. The discrepancy between the assumed bearing of the first line and the final calculated bearing as thus obtained is equal to the total error in the sum of the interior angles, as found by the check of Art. 298(*a*).

Illustration. The assumed bearing of the first side of a triangle is N 30° E. The calculated bearing of the same side found by following around the triangle is N 30° 1′ E. This indicates that there is an error of 1′ in the sum of the interior angles. In a similar way the sum of the interior angles of any polygon may be checked no matter how many sides it may have.

306. Methods of Running Traverses Compared. It is important to know the advantages and disadvantages of the different methods of running traverses in order to use in any survey that method best suited to the work in hand.

I. Method by direct angles. Used in all kinds of work, particularly in surveys where there are but few transit lines and where the survey closes.

Advantages. (1) Angles at any station may be measured by repetition or by series, a decided advantage when a high degree of precision is required. (2) Stations may be occupied in any order. (3) The angle at one station is not affected by the angle measured at a preceding station. (4) Angles may all be measured in the same direction, i.e., clockwise.

Disadvantages. (1) In order to check angles by the needle, bearings of traverse lines must be calculated. (2) A distinction must be made in the field notes between right and left angles unless *all* angles are taken in one direction (clockwise). (3) It must be made plain in the field notes which station was used as a backsight for each angle. (4) The vernier must be set anew at each station.

II. Method by deflection angles. Used almost exclusively in continuous surveys in which each transit line makes a small angle (usually less than 90°) with the preceding transit line produced such, for example, as traverses in surveys for highways, railways, power lines, and pipe lines. Used in running in horizontal curves.

Advantages and disadvantages are practically the same as for the preceding method. It is necessary, however, to take angles both to the right and to the left, whereas by the direct-angle method, all angles may be measured, if desired, to the right. Deflection angles are usually small angles (less than 90°), and this is of some advantage in plotting. In the first method of running traverses by deflection angles (Art. 302(*a*)) errors may occur from plunging the telescope. Consequently for accurate work, one of the other two methods of measuring deflection angles is preferable.

III. Method by azimuths. An excellent method for a survey in which there are many details to be located from each station but in which the desired degree of precision is not so high as to require the measurement of angles by repetition. The method is quite generally used in stadia surveying and in topographic surveying.

Advantages. (1) When a magnetic or a true meridian is used, each azimuth is readily checked, with little or no calculation, by comparing it with the magnetic bearing as indicated by the compass needle. (2) The method is free from nearly all the disadvantages of the two preceding methods, i.e., since all angles are measured clockwise from the meridian as a backsight, it is not necessary to enter in the field notes the backsight for each station, or to state

whether an angle is measured to the right or to the left. Thus the mind is freed from these details, both in the field work and in the office work. Moreover, if either the first or the third method of running traverses by azimuths is used (Art. 303(*c*),(*g*)), errors due to resetting the vernier at each station are avoided. (3) The bearing of every line is known from its azimuth, yet the latter is recorded without letters that correspond to the compass points. (4) Field notes are simple, definite, and easily recorded. (5) By the use of azimuths, office calculations and plotting are somewhat simplified. (6) When a closed traverse is completed or when it is possible to close on any line whose azimuth has been previously determined, the total angular error of the traverse is evident from a comparison of the original and the final azimuths of the line.

Disadvantages. (1) The method does not lend itself readily to measuring angles by repetition or by series. (2) Each station must be occupied in order without skipping any. This may be a serious disadvantage if for any reason it is impossible or undesirable to set up over a transit station at any given time. (3) A mistake in the azimuth of one line affects the azimuths of all succeeding lines. This does not mean, however, that all the work is lost, for the azimuth of each succeeding line is affected by the *same* amount, and though all these succeeding lines are wrong with respect to the reference meridian of the survey, they may still be correctly located with respect to each other. If the mistake is discovered, the notes are easily changed by applying the correction to each succeeding line, though if there are many side shots this may cause considerable extra work. (4) If local attraction exists for any considerable number of stations, the advantages of checking azimuths by the needle is lost. (5) In the first method of running azimuths, plunging the telescope involves the same error in the adjustment of the line of sight that is involved in the deflection-angle method.

IV. METHOD BY BEARINGS. This method, probably a survival of the compass survey, is little used at present. It could, perhaps, be employed to advantage in special cases as, for example, in rerunning boundary lines or other lines, the directions of which were originally determined by magnetic bearings.

Disadvantages. It has no advantage over the preceding method except, possibly, a slight advantage in checking lines by the needle, while it has all its *disadvantages* and in addition: (1) a special numbering of the graduations on the limb (0° to 90°) is called for; (2) likelihood of mistaking the north zero for the south zero,

or vice versa; (3) letters corresponding to compass points must be used.

V. CALCULATED BEARINGS when used in connection with either the direct-angle method or the deflection-angle method serve as an excellent check on the transit-line angles, provided local attraction does not exist. Though mistakes of less than 15′ are not detected by the use of the needle, nevertheless in a survey which does not close there is no other check unless angles are measured by repetition.

VI. MEASURING ANGLES BY REPETITION. When running traverses either by the direct-angle method or by the deflection-angle method, it is good practice, as a general rule, to double all angles, if for no other reason than to guard against error. It requires more time in the field but often saves time in the end. In work of high precision, an angle may be measured in sets, each set consisting of as many as six repetitions (Art. 397).

307. Checking a Closed Traverse. In Fig. 307 let A represent the first station and D the last station in a closed traverse. DA will then be the closing line. It is desired to check the field work.

The first check is that for the sum of the interior angles, i.e., $S = (N - 2) \times 180°$, where N represents the number of sides, as explained in Art. 298. If the error in this sum exceeds the permissible error, the angles must be remeasured; otherwise, the angles are corrected by distributing the error equally among the angles.

The second check consists in calculating the *error of closure.* Let A be the origin of two rectangular coordinate axes X and Y. The rectangular coordinates of B, C, D, and A may be calculated in that order from the successive angles and distances around the traverse. (See Art. 563.) If there were no errors in angles or distances, each of the *calculated* ordinates of A would be zero. It is usually found, however, that the calculated ordinates are not zero but two values which bring the calculated position of A at a point, say, A'. The distance from A to A' is the *linear* error of closure. If the calculated ordinates of A, i.e., of A', are represented by x and y, the length of the error of closure $A'A$ is $\sqrt{x^2 + y^2}$. Although this error is measured in linear units, it is due to errors in the angles of the traverse as well as to errors in the lengths of the traverse lines.

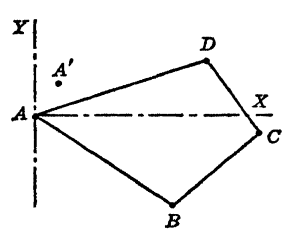


FIG. 307.

It is to be expected that the error of closure will be greater for a long traverse than for a short one, provided both traverses are run with the same degree of precision. In order that there may be a standard basis of comparison for determining the degree of precision of traverses of different lengths, it is necessary to reduce the total error to an error per unit of length, by dividing the error of closure by the length of the perimeter of the traverse. This *relative error of closure* is commonly expressed as a fraction, the numerator of which is unity. For example, if the error of closure is 1.5 ft., and the length is 3000 ft., 1.5 divided by 3000 is equivalent to $\frac{1}{2000}$. The greater the degree of precision, the greater will be the denominator of the fraction that indicates the relative error of closure.

Unless sources of constant error are eliminated, neither the total linear error of closure nor the relative error of closure is a true measure of the accuracy of the field work; indeed a small error of closure may be only an indication of *consistent* inaccuracy. For example, there might be a very large error in the length of the tape, say 10 ft., with a resulting consistent inaccuracy in the length of every traverse line, yet the closing error might be very small, say only a few inches. It is a common mistake to judge the degree of precision by the relative error of closure with little thought or inquiry as to whether or not sources of constant error were eliminated. For example, the relative error of closure for one traverse may be $\frac{1}{10,000}$ and for another $\frac{1}{5000}$, but if sources of constant error were not eliminated in the first traverse, the actual degree of precision of the second traverse might be greater than that of the first. *Since constant errors can never be eliminated entirely, no error of closure is an exact measure of the degree of precision attained.*

In the field work of running traverses practical questions pertaining to errors of closure arise, such as: What error of closure may be expected as the result of the combined errors from chaining and angular measurement? What are the customary limits for errors of closure? How are these limits usually specified? What precautions and methods of field work must be followed to attain a specified degree of precision? These questions are discussed in Arts. 356–359.

307(*a*). COMMENTS.

1. For purposes of explanation, the origin of the coordinate axes was taken at the first station *A*. A more general case is to assume it at some point not in any line of the traverse. The length of the error of closure will then be $\sqrt{d_x^2 - d_y^2}$, in which d_x represents the numerical difference between the original and the calculated *X* ordinates of the first station, and d_y, the numerical difference between the original and the calculated *Y* ordinates.

2. A meridian is often used as the Y axis, and the error of closure may then be calculated by *latitudes* and *departures*. (See Art. 563 for a numerical example.)

3. The sum of the interior angles should be checked in the field. The error of closure is usually calculated in the office.

4. The distribution of the error in the sum of the angles and the adjustments for the error of closure are treated in Arts. 566 and 570–572.

5. When there are many sides to a closed traverse it may be advisable to run a diagonal across the traverse in such a way as to divide it into two approximately equal parts and to treat each part as a separate traverse. Then, if there is any considerable error, it is possible to ascertain in which part of the traverse it occurred.

308. Checking an Open Traverse. An open traverse cannot be checked by any such method as that just explained for a closed traverse in which calculations are based *solely* on the measured traverse angles and distances. It is always well, therefore, to close a traverse when practicable, but in a great deal of surveying it is impracticable to do so. How then can an open traverse be checked?

It is sometimes possible in an open traverse in which there are abrupt curves to run a "cut-off" line between the beginning and the end of a curve in such a way as to make that portion a closed traverse. Thus the open traverse becomes a succession of closed traverses, but even in such a case, there are likely to be between these closed traverses straight stretches that cannot be checked.

Another method of checking an open traverse sometimes used is to select a distant land mark which can be seen from several traverse stations. At each station an angle or an azimuth to the land mark is measured. Several sets of the rectangular coordinates of the land mark can then be computed, and if the values of these coordinates agree closely, it is probable that no serious errors have been made in running that portion of the traverse.

Probably the most satisfactory method of running an ordinary traverse is to check measurements as the work progresses by methods already explained, such as the comparison of calculated bearings and magnetic bearings, in order to detect large angular errors; measuring angles by repetition; measuring the length of each traverse line at least twice and in opposite directions; and taking such precautions to eliminate constant errors as may be necessary in order to attain the required degree of precision.

In extended open traverses, in which a relatively high degree of precision is required, astronomical observations can be made at intervals along the line wherever a check may be desired. The bearing of a traverse line at any station, computed from the angles of the traverse, should agree with the azimuth of that line, determined by astronomical observation. If, however, the traverse runs for a

considerable distance east and west, adjustment must be made for the convergence of meridians.

The best method of checking an open traverse is by means of control points as explained in the next article.

309. Checking Traverses by Control Points. In many localities there are permanent points of reference or *control points*, the positions of which have been accurately determined by federal, state, or municipal surveys. When it is possible, the best method of checking a traverse is to begin at one of these control points and end at another, just as a line of levels may be checked by beginning at one permanent bench mark and ending at another. The line between the two permanent control points may be considered as the *closing line* of the traverse.

The position of each of the control points is fixed by two accepted coordinates determined in a previous survey. Starting with the coordinates of one of these points, the coordinates of the other point may be calculated from the traverse angles and distances. These calculated coordinates can then be compared with the accepted coordinates to determine the error of closure. The error of closure will be $E = \sqrt{d_x^2 + d_y^2}$, in which d_x represents the difference between the accepted and the calculated X ordinates of the control point on which the traverse closes, and d_y the difference between the accepted and calculated Y ordinates of the same point. Since the accepted coordinates of the two control points determine the direction and length of the closing line between the two points, the method just described for calculating the error of closure is similar to the general method already explained in Art. 307(*a*).

The total angular error in the traverse is equal to the difference between the *known* azimuth of the line between the two control points and the calculated azimuth of the same line. The latter is found by starting at the first control point with the known azimuth and by computing from successive traverse angles the azimuth of successive traverse lines until, at the second control point, the calculated azimuth of the line between the two control points is finally obtained.

309(*a*). Government control points. If a surveyor is to check a survey by means of control points, such points must be reasonably near the survey. Fortunately the U.S. Coast and Geodetic Survey, the U.S. Geological Survey, and state and municipal departments are extending surveys so rapidly that control points are being established in large numbers and relatively close together. In certain states, plane rectangular coordinate systems have been established for general use as a legal system of horizontal control. Not only can points of control in such systems be used for purposes of checking, but it is of great advantage to have a local survey tied in permanently with a state-wide system of both horizontal and vertical control which, in turn, is a part of a system covering the whole country (Chapter 56).

310. Traverses. Summary of the Chapter.

1. The transit lines of a traverse form a framework of reference lines with respect to which all details of a survey are located.

2. In a closed traverse, the transit lines form a closed polygon or a network of polygons; in a continuous or open traverse, the series of connected traverse lines do not close.

3. An error of closure in a closed traverse is usually due both to errors in the lengths of transit lines and errors in angles.

4. There are four common methods of running traverses, two of which depend upon measuring, at each station, the direct angle or the deflection angle between two transit lines; the other two depend upon measuring the angle which each transit line makes with a reference meridian, or with a line parallel to it. The four methods are: (1) direct angles; (2) deflection angles; (3) azimuths; (4) bearings.

5. In direct-angle traverses, the angle at each station is measured directly from a backsight on the preceding station. It is preferable to take all angles in the same direction either clockwise or counterclockwise. In a closed traverse, either the interior or the exterior angles of a polygon may be measured, but, as a general rule, there is an advantage in measuring interior angles. If it is desired to measure interior angles clockwise, it is necessary to proceed around the polygon from station to station counterclockwise. To check the sum of the interior angles, subtract 2 from the number of sides of a polygon and multiply the result by 180°, i.e., $(N - 2) \times 180°$. This result should equal the sum of the interior angles. A triangulation net is a special case of the direct-angle method.

6. In a deflection-angle traverse, the angle measured at each station is the deflection angle to the right or to the left. There are three methods of measuring the angles: (1) by backsighting with the telescope inverted and foresighting with it normal; (2) by setting the vernier at 180°, backsighting with the telescope normal, foresighting with it still normal, and reading the angle from the A vernier; (3) by using the method of repetition for deflection angles. The second method is preferable to the first, and the third method is, as a general rule, preferable to either. If deflection angles to the right are considered positive and those to the left negative, the algebraic sum of all deflection angles of a *closed* polygon should equal $\pm 360°$. This check cannot be applied to an open traverse.

7. In either the direct-angle or the deflection-angle method, angles may be checked roughly by comparing the calculated bearings of lines with corresponding magnetic bearings.

8. In an azimuth traverse, the direction of each transit line is determined by the angle it makes with a true meridian, or a magnetic meridian, or some assumed meridian.

9. In the azimuth method of running a traverse, the azimuth of the first line of a traverse may be determined in one of three ways. (*a*) By the magnetic needle; all azimuths will then be *magnetic* azimuths. (*b*) By establishing a true meridian by observation, and measuring the azimuth of the first line from that meridian. All azimuths will then be *true* azimuths. Often it may be possible to start from a point on a line, the truth azimuth of which is already known, thus avoiding the necessity for establishing a true meridian. If it is desired to check roughly true azimuths by means of the magnetic needle, it is well to adjust the compass box by turning off the declination arc. (*c*) If bearings are of no importance, the azimuth of the first line of the traverse may be assumed as the meridian, i.e., its azimuth will be 0°. The azimuths of all other traverse lines will then be measured with respect to this first-line meridian.

10. There are two methods of orienting the transit in running by azimuths. In one, the vernier is set automatically for any station by leaving the upper plate clamped at the reading for the foresight that was taken while at the preceding station; in the other, the vernier must be set anew at each station (at the back azimuth). In both cases, the transit is then oriented by backsighting on the preceding station.

11. There are three methods of running traverses by azimuths, differing only in methods of orienting the transit. The procedure at any station except the first is given for each method as follows.

First Azimuth Method. To orient the transit, allow the upper plate to remain clamped at the reading for the azimuth of the foresight taken at the preceding station, invert the telescope, backsight on the preceding station, and

plunge the telescope to its normal position. Foresight on the next station and read the azimuth from the A vernier.

Second Azimuth Method. To orient the transit, set the A vernier so that its reading will be the azimuth of the backsight, i.e., the azimuth of the foresight taken at the preceding station plus 180°, and backsight with the telescope normal. Foresight to the next station with the telescope still normal and read the azimuth from the A vernier.

Third Azimuth Method. This method is nearly the same as the first method, but the telescope is *not* plunged between the backsight and the foresight, and the forward azimuth for the foresight is read from the vernier that is 180° from the one used at the preceding station. Thus the A and B verniers are used alternately from station to station. For convenience in reading, the telescope is kept normal at a station where the A vernier is used and inverted at a station where the B vernier is used.

The third method avoids errors due to plunging the telescope in the first method, and those due to continually resetting the vernier in the second.

12. When a magnetic or a true meridian is used, the azimuth of any traverse line may be checked by comparing it with the bearing of the line as indicated by the magnetic needle. When any other meridian is assumed, the same check may be used but less readily, unless the assumed meridian is so nearly a north-south line that the compass box may be adjusted to it.

13. The sum of the interior angles of a closed traverse run by azimuths may be checked by comparing the original azimuth of a traverse line at the first station with the azimuth of the same line as found by completing the polygon.

14. Running transit lines by bearings is similar to the azimuth method, but calls for the quadrant system of numbering graduations on the limb. Errors are much more likely to occur and the method is but little used.

15. Any method of checking the directions of transit lines by the magnetic needle is useless in locations where local attraction exists.

16. Doubling angles, or measuring angles by repetition, is a more certain check than that afforded by the magnetic needle, and it is also a much more accurate one. By this method angles can also be measured with greater precision than by the ordinary method.

17. The first check to apply to a closed traverse is to compare the sum of the angles with the theoretical sum, $(N - 2) \times 180°$, for interior angles, and $\pm 360°$ for exterior deflection angles. The second check is to calculate the linear error of closure. The coordinates most often used in calculating this error are latitudes and departures.

18. The relative error of closure is equal to the linear error of closure divided by the length of the perimeter of the polygon. This is commonly expressed as a fraction, the numerator of which is unity.

19. Unless sources of constant error are eliminated, neither the linear error of closure nor the relative error of closure is a true measure of the accuracy of the field work.

20. Parts of an open traverse can sometimes be checked by "cut-off" lines; also by angles to a landmark measured at several successive stations. Usually the most satisfactory method is to check both angular and linear measurements as the work progresses. In extended open traverses astronomical observations can be made at intervals along the line wherever a check may be desired.

21. When control points are available, the best method of checking a traverse is to begin at one of these control points and end at another. The linear error of closure may be determined by comparing the calculated coordinates of the second control point with the accepted coordinates for that point.

CHAPTER 22

TRANSIT SURVEYING

METHODS OF LOCATING DETAILS

In this chapter are given the geometric methods in common use for locating boundary lines, street lines, buildings, fences, rivers, bridges, and other details of a transit survey. The different methods are outlined without entering upon the routine of the field work or discussing the practical questions which arise in connection with that work. (See the next chapter for field work.)

311. General Method of Procedure in a Transit Survey. Part of the work in a transit survey consists in running the traverse by one of the methods explained in the preceding chapter, and the rest of the work consists in locating details of the survey, using the traverse lines as lines of reference. The different geometric methods used for locating points and lines of details are exactly the same as those employed by draftsmen for locating points and lines on paper. Some involve only linear measurements, others are based entirely upon the measurement of angles, and still others combine linear and angular measurements.

311(*a*). Two methods of carrying on the work. Having chosen the method of running the traverse, there are two different methods of procedure. (1) The entire traverse may be completed before work is begun on the location of details. This means going over the ground twice, once for transit lines and once for details. (2) A more common method is to complete all the measurements at a station at one setup, carrying on the work of running the traverse lines and of locating the details simultaneously and establishing new stations as they are needed.

The first method is good for a survey of considerable extent, in which a network of transit lines must be established with care. The second method is better for a survey which does not close, or for one in which it makes little difference where the transit lines are run, or for small surveys, with comparatively few stations, where the work can be done in a short time.

As a compromise between the two methods it is often best before beginning a survey to decide upon the approximate positions for the different stations, and then, working by the second method, set the stations more exactly as they are needed.

312. Locating Details from Traverse Lines. Seven methods of locating a point have already been given in Art. 10(*a*). Locating details consists simply in locating two or more points of each object by one or more of these seven methods. Traverse lines form the skeleton upon which the survey is built. Objects are usually located from the nearest or most convenient traverse line. Enough points are taken to locate the object completely, with usually an additional point or two for a check. Thus, for example, it is enough to determine the position of two corners of a rectangular building directly from a traverse line, the other two corners being located indirectly by measuring the building itself. It is well, however, in accurate work to locate a third corner from a traverse line as a check.

Sometimes it is necessary to locate different parts of the same object from different stations or traverse lines. For example, part of a large building may be located from a station near one end, while other portions of the building are located from a station near the other end. Fence lines and street lines are nearly always located from at least two stations. For example, in a survey of a rectangular field each corner of the field may be located from a station near it, and each boundary line will thus be fixed by two points located from different stations, and possibly from different traverse lines also (Art. 326).

313. General Methods of Locating Points. Of the seven methods of locating a point in Art. 10(*a*), one depends upon the intersection of two straight lines, two depend upon linear measurements alone, and the other four involve the measurement of angles. Since the transit is an instrument for measuring angles, it is to be expected that these last four methods are used more or less freely in transit surveying, but not to the exclusion of the other three. Angles may be direct angles, deflection angles, angles of azimuth, or angles of bearing, according to the method used in running the traverse.

313(*a*). NOTE. The geometric methods of locating points hold good regardless of how distances and angles are measured. Thus, for example, in topographic surveying, distances may be measured with a tape or by the stadia method, or, in some cases, by a less accurate method such as pacing, and angles may be measured with a transit, or a plane table, or, in some cases, with a compass. In this chapter, however, it is understood that linear measurements are made with the tape and angular measurements with the transit.

314. Terms Used in Locating Details. *Offset distance.* The distance from a fixed point to a given line of reference measured on a perpendicular to that line. The term "offset" is quite generally used in place of "offset distance."

A *swing offset* is one in which the shortest distance from a point to a line of reference is found by holding the zero mark of the tape at the point and then swinging a portion of the tape in order to find by trial the radius of an arc that is tangent to the line of reference. When, as is often the case in transit surveying, the line of reference is a line of sight between two stations, the transitman, by observing through the telescope the arc as it is swung, signals to the chainman to lengthen or shorten the radius until the arc becomes tangent to the line of sight. The procedure is similar to that described in Arts. 274(*b*), 277.

Range Line. A line produced until it intersects some line of reference, usually a traverse line. A range line is often used for the purpose of locating two or more points through which it passes.

Tie Distance (or tie). The horizontal distance between a point of one line and a point of another line measured for the purpose of determining the relative positions of the two lines, i.e., as a *tie* between the two lines.

Measuring Up a Detail. Obtaining by measurement the principal dimensions that may be needed either for plotting a detail or for purposes of calculation. For example, the corners of a plot of land may be definitely located with respect to traverse lines, but in addition the lengths of the boundary lines themselves should be measured whenever possible. Similarly, all four corners of a rectangular house may be located with respect to traverse lines, but in addition the lengths of the sides of the house should be measured. *No matter how many points of a detail may be located, as a general rule, the detail itself should be completely "measured up."*

315. Location of One Detail from Another. It is frequently practicable to locate one detail by measurements from another whose location with respect to the line of reference has already been determined. For example, if two buildings are near together, and one has been located with respect to a line of reference, the second can often be located with reference to the first by a few simple linear measurements.

316. Indefinite and Inadequate Location of a Detail. A detail may seem to be completely and satisfactorily located during the field work, but it may be discovered later that its location is indefinite because of the methods used, or inadequate because certain measurements were not taken. To guard against this, it is well to test mentally the location of each point by imagining how the point could be plotted or how its position could be calculated.

317. Check Measurements. After a detail has been definitely located from some line of reference, one or more extra measure-

ments are often taken to check the work of location. For example, the location of a rectangular house with respect to a line of reference can be determined by locating merely two corners and by measuring up the house; but if there is an error in locating either corner, it might be difficult to determine the true position of the house. For this reason, check measurements may be made either to determine the location of the two corners by a different method or to locate a third corner. The dimensions obtained in measuring up a detail often serve as check measurements. For example, in the illustration just given, the location of the two corners of the house is to a certain extent checked by measuring the length of the side between the two corners.

318. Illustrations of Different Methods of Locating Details. In the succeeding pages are illustrations which are intended to give a general idea of the different methods used in locating details from traverse lines. Each method is based on one or more of the seven methods of locating points explained in Art. 10(*a*). The best way to study each illustration is to test mentally the location of each point by imagining how it could be plotted or how its position could be calculated. First those methods which combine linear and angular measurements will be given, and then those methods which involve linear measurements only. More complete illustrations in which different methods are combined will be found in the various forms of field notes given in Chapter 24.

319. Locating Details by Angles and Distances. (*a*). ILLUSTRATION. *Angles at and distances from one transit station.* Any number of points may be located from a station by taking an angle to each point and measuring the corresponding distance from the station to the point (method 3, Art. 10(*a*)). The angles are usually all taken from the *same* backsight (Art. 259). In Fig. 319(*a*) the transit is at station *B*, backsight on station *A*. Angles are taken to 1 and 2 (corners of the barn) and 3, 4, 5, and 6 (corners of the house), and the corresponding distances from station *B* to these corners are measured. Measuring up, i.e., measuring the lengths of the sides of the barn and the house, completes the field work of location. In many cases it would be sufficient to locate two corners of the house, as, for example, 3 and 4 or 5 and 6, as was done in locating the barn, but the

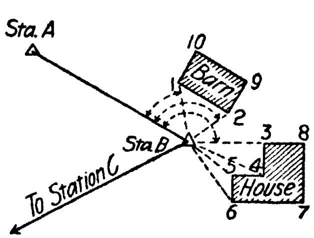


FIG. 319(*a*).

lengths 5 to 6 and 3 to 4 are so short that greater accuracy may be attained by locating all four points.

Check Measurements. The location of 3 and 4 is a check on the location of 5 and 6, and vice versa. The measured lengths between 1 and 2, 3 and 4, 4 and 5, and 5 and 6 are also checks. In addition, check measurements between 2 and 3 and between 9 and 8 might well be taken.

Comments. This is one of the best and most-used methods for locating points rapidly and exactly. When this method is used, the transit station should be reasonably near the detail to be located.

The points could have been located by azimuths instead of by direct angles. In some respects the former is the better method.

319(*b*). Illustration. *Locating curves by angles and distances.* The simplest way of locating a curve made up of *circular* arcs of small radii is to find by trial the radius and center of each arc, locate each center from the nearest transit station, and in addition, if necessary, locate the points where each curve ends or where it changes to a different curve. When, however, the radii are large or when the curve is irregular, i.e., not composed of *true* circular arcs, some other method must be used. One method, especially adapted to the location of small, sharp, irregular curves such as driveways, park walks, and other similar curves, is given in Fig. 319(*b*). Notice that for each angle measured two points are located. Thus, with transit at *E*, backsight at *D*, an angle to 7 is taken, 7′ is lined in, and the two distances to 7 and 7′ are measured, the object being to save work in plotting. The second point can usually be set in line with the first by eye. Chain pins or stakes can be stuck into the ground at intervals, close together where the curve changes rapidly, farther apart where the curve is less abrupt, and the angles and distances measured to these points. On the other hand, angles can be turned off, say every 15°, the corresponding points on the curve marked, and the distances to these points then measured.

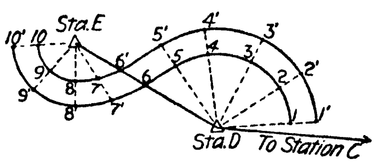


Fig. 319(*b*).

Check Measurements. Lengths of chords such as 10 to 5′, 10′ to 6, 5′ to 1′, and 5 to 1.

Comments. It often happens that, by making the same angle serve for the location of two or more points, work in plotting is saved not only in the location of curves, but also in the location of all kinds of objects. A method of locating curves by offsets is explained in Art. 322.

319(*c*). Illustration. *Locating a point by an angle and distance when the distance is measured from some fixed point other than the transit station* (method 4, Art. 10(*a*)).

In the two preceding illustrations the distance to any point located was measured from the station where the transit stood when

the angle to that point was taken (method 3, Art. 10(*a*)). Instead of measuring from the transit station it is sometimes more convenient to measure from some other point already located.

Assume that in Fig. 319(*c*) *a* and *b* are both visible from station *H*, but that an obstacle makes it difficult to measure the distance from station *H* to *a*. Corner *b* is located by an angle and a distance in the usual way. The angle to corner *a* is also taken from the backsight on station *G*, but the distance to *a* is measured from *b* instead of from station *H*. Note that the distance to *a* could have been measured from some other fixed point as, for example, from a suitable point on the transit line to station *G*.

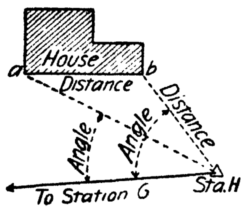


FIG. 319(*c*).

Check Measurements. What in your judgment would be the best check measurements?

Comments. The point *a* may be plotted by intersecting a line from station *H* to *a*, by an arc whose center is *b* and whose radius is the distance *ab*. Unless the intersection is sharp, *a* cannot be located accurately. The location of *a* would become more and more indefinite as the angle *baH* approaches 90°, because the arc swung from *b* as a center would become more nearly tangent to the line from station *H* to *a*. This means that when this method is used, the angle between the tie line to a point to be located and the line from the transit station to that point should be reasonably acute. Although the limitations of this method are obvious, it can sometimes be used to advantage.

319(*d*). ANOTHER ILLUSTRATION of the method just explained is given in Fig. 319(*d*), where it is used for determining the direction of a street which branches off from the main survey. Assume that only enough of the street will be shown on the map to indicate its general direction, and that, therefore, it will usually suffice to locate a single point, as, for example, *b* in Fig. 319(*d*). The corner *a* having been located by an angle and distance in the usual manner, an angle is taken to *b*, *but the distance from a to b is measured instead of* the distance from station *B* to *b*. This method might be used, for example, if because of fences it is much easier to measure from *a* to *b* than it is to measure from station *B* to *b*. If more convenient, produce *ba* to *e* and measure from station *B* to *e*, or if *b* has been located, as just explained, the distance from station *B* to *e* may be used as a check measurement. For another method of getting the direction of a street see Art. 323(*c*).

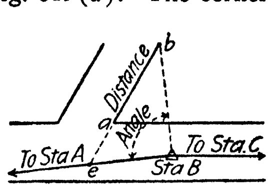


FIG. 319(*d*).

319(*e*). ILLUSTRATION. *Locating a detail by means of a range line.* In Fig. 319(*e*) a method is shown which is often used for locating a rectangular object. A point *c* is lined in with the transit

between stations *F* and *G*, and at the same time brought by eye into the range line through *a* and *b* by sighting along one side of the house. The point *c* having thus been established, the transit is set up at *c*, the angle *Fcb* is taken, and the distances *Fc*, *cb*, and *ba* are measured. One side of the house having been located in this way, the other corners are found by measuring up the house itself.

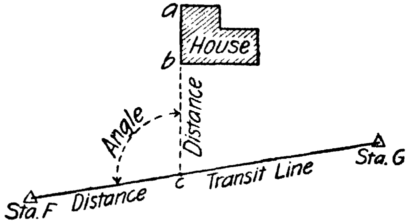


FIG. 319(*e*).

Check Measurements. The angle between the backsight and a line from *c* to the right-hand lower corner of the house, and possibly the distance to this corner as well.

Comments. This method is most used when the object is too far from a transit station to be located entirely from that station, and yet is comparatively near one of the transit lines. Alhough it is not the most accurate method, it will often answer the purpose. Note that the range line was taken through the longer side of the house because its direction could be more accurately determined than if it were through the shorter side. A method similar to this but one which does away with the setting up of a transit at *c* is given in Fig. 323(*a*).

320. Locating Points by Angles and Intersecting Lines. ILLUSTRATION. This method of locating points of details is a form of *triangulation*, and consists merely in repeated applications of the fifth method in Art. 10(*a*). Two transits are used, one at station *A*, the other at station *B* (Fig. 320). In locating any point, the transit line *BC* is used as a base line, and at each of the two stations the angle to the point is measured from a backsight along this base line. For example, in locating point 2, angles *CB2* and *BC2* are measured, practically at the same time. By thus locating points wherever there are marked changes in the direction of either bank, the position of the creek may be determined.



FIG. 320.

Check measurements different from the original measurements can sometimes be made to advantage, but the most common method of checking is to measure angles by the method of repetition. Frequently, the character of the work or the degree of precision required does not warrant even this check.

Comments. This method may be used for locating any point which can be sighted from two transit stations, provided the angle between the two lines of

sight is not less than 30°, i.e., is not so acute that it is difficult to determine the exact point of intersection. It may be the only practicable method of locating one or more points of an object that is remote from transit lines or is inaccessible. It is much used in plane-table surveying.

The angles can be measured with only one transit, but the work as a whole can be carried on more easily and rapidly when two are used.

321. Locating Points by the Three-Point Method (method 7, Art. 10(*a*)). Let *H, I,* and *K* in Fig. 321 be stations on shore the positions of which are known. Let 1, 2, and 3 be points on the water at which buoys are placed or soundings taken. At each of these points two angles between lines to the three stations on shore are taken, as indicated on the figure, thus locating each point. The peculiarity of this method is that *the angles are all taken at the point to be located.*

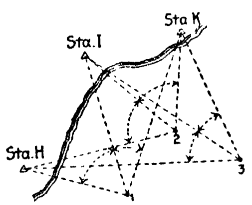


FIG. 321.

Comments. When, as in the illustration, the points to be located are on the water, angles are often measured with a sextant. On land they may be measured with a transit or laid off on a plane table. The points can, of course, be located from stations on shore by triangulation, as explained in the preceding article, and this method is often used in preference to the three-point method. Under certain conditions, however, the three-point method is better. In plane-table surveying it is often used for locating on the map the point over which the plane table stands. Likewise, it is sometimes used for determining the position of a transit station which cannot be conveniently located by one of the more common methods.

322. Locating Details by Offsets. (*a*). This method corresponds to the second method in Art. 10(*a*). Enough offsets are taken to locate the object completely, the distances *between* the offsets being measured *along the transit line.* In Fig. 322(*a*) only two offsets are needed to locate the building, provided the building is measured up as, of course, it should be. It is well, however, to make at least one check measurement. (What would be a good check?)

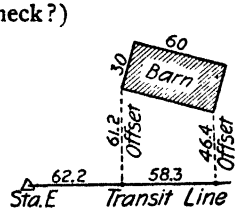


FIG. 322(*a*).

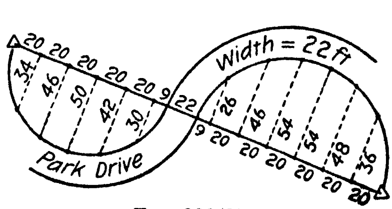


FIG. 322(*b*).

322(*b*). REGULAR OUTLINES. Anything which is ***regular in outline*** can be located by offsets taken at ***equal*** intervals along the transit line. Thus in Fig. 322(*b*) offsets were taken every 20 ft. Notice that distances (9 ft. and 22 ft.) were measured to the points

where the transit line crosses the curves. In keeping notes some engineers consider offsets on one side of the transit line *plus,* on the other side *minus.*

322(*c*). IRREGULAR OUTLINES are best located by offsets at *unequal* intervals, for in this way only can the points of abrupt change in the line be determined. The edge of a bluff, a river bank, or any other irregular line can be located by running a transit line near it and taking offsets as illustrated in Fig. 322(*c*).

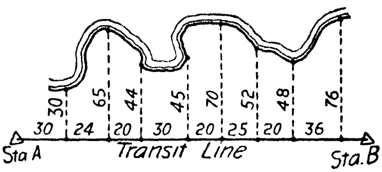


FIG. 322(*c*).

322(*d*). ESSENTIAL MEASUREMENTS. In locating objects by offsets remember three things:

1. The distance from one of the offsets to a transit station (or some other known point on the transit line) must be measured. If, for example, in Fig. 322(*c*) the distance 30 ft. from station *A* to the first offset were not measured, the rest of the work would be useless.

2. An offset should be taken at every point where there is an abrupt change in the outline to be located.

3. Offsets should be taken to all points of the outline which are *farthest from* or *nearest to* the transit line.

322(*e*). NOTE. It will facilitate plotting (and the field work, too) if, instead of measuring distances between offsets as shown in the figure, the total distance of each offset from the transit station is measured. Thus, starting from station *A*, the successive points would be numbered according to the system explained in Art. 337, +30, +54, +74, 1 + 04, 1 + 24, 1 + 49, 1 + 69, 2 + 05. This shows that the last offset is 205 ft. from station *A,* and, if the remaining distance to station *B* is now measured in the same way, the result should equal the total distance previously found by chaining between *A* and *B*. This is a valuable check on all the intermediate measurements along the transit line as well as upon the total measurement itself. In getting the plus stations *no arithmetical addition is necessary.* Simply stretch the tape its full length each time and read the plus for each point directly from the tape.

323. Locating Details by Tie Measurements from a Transit Line. In Fig. 323(*a*) and Fig. 323(*b*) are shown two different methods of locating a building, each of which, however, depends upon the same general principle (method 1, Art. 10(*a*)).

323(*a*). ILLUSTRATION. In Fig. 323(*a*) points *d* and *e* are two points lined in *on the transit line,* and the distance of one of these points from the nearest station is measured. The point *e* was taken on a range line through one side of the building, and point *d* is any convenient distance from *e* that will form one side of a well-proportioned triangle. Corner *f* is located by the two tie distances *df* and *ef*. A check distance may be measured from the correspond-

ing right-hand corner to some convenient point on the transit line a measured distance to the right of *e*.

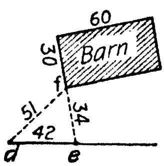


FIG. 323(*a*).

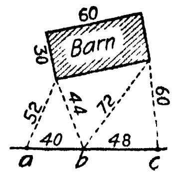


FIG. 323(*b*).

323(*b*). ILLUSTRATION. In Fig. 323(*b*), *a*, *b*, and *c* are lined in on the transit line in positions convenient for measurements to the two nearest corners. Each corner is then located by two tie measurements as indicated. Check distances may be measured from *a* to the right-hand corner and from *c* to the left-hand corner. The two corners could have been located in the first place by tie measurements from only two points, *a* and *c*, in which case check measurements to both corners could be made from some common point such as *b*.

If, in Fig. 323(*a*), the corner *f* had been a considerable distance from the transit line, instead of measuring a tie from *d* to the point *f*, one could have been measured from *d* to some other point on the range line a convenient distance from *e*. Such a tie would determine the direction of the range line; it is equivalent to measuring the angle *def* between the range line and the transit line. The result is usually not quite so accurate as if the angle were measured with a transit, as was done in Fig. 319(*e*); nevertheless the method can often be used to advantage. A similar use of a tie to determine direction is shown in the next illustration.

323(*c*). ILLUSTRATION. *Use of Tie.* In Fig. 323(*c*) the intersection *c* of two street lines is located from station *B* by an angle and distance in the usual way. No other point is located from station *B*, *the street line cef having been located by a second point taken from some other station not shown in the figure.* The point *d* determines the direction of a side street line. Measure from *c* to *d* and from *c* to any other point *e* on the line *cf*. The length of the tie *de* determines the angle *dce*. (As a check produce *dc* to *a*, and measure from *B* to *a*.) In a similar way if a point *n* has been lined in by eye with a division fence, a tie from

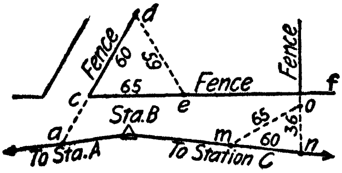


FIG. 323(*c*).

any point *m* to any point *o* will determine the direction of the fence and thus avoid a setup at *n*.

323(*d*). ILLUSTRATION. *Offsets on Two Range Lines.* In Fig. 323(*d*) the points *d* and *c* on the transit line were lined in by eye with two sides of a building, and the ties *ad* and *bc* were measured. The point *d* was located from station *A*, and the building was measured up. Notice that *dc* was *not* measured. This method is not geometrically correct, for, in plotting, the distance *ab* and *dc* are assumed equal.

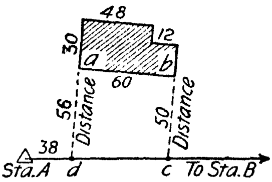


FIG. 323(*d*).

Comment. This method would not be used except in cases where the transit line is approximately parallel to the side of the house, and only then when the discrepancy between the lengths of *ab* and *bc* is of no importance. Thus, for example, when a building is located merely for the purpose of plotting it on a map and the scale is so small that the discrepancy will not be apparent, there is no point in measuring both *ab* and *dc*. The purpose of this illustration is to emphasize the fact that there are conditions which permit a departure from a strict geometric method and the omission of a measurement that at first sight may seem necessary.

324. Locating a Point by the Intersection of Two Straight Lines between Four Known Points (method 6, Art. 10(*a*)). It is sometimes possible and advantageous to locate one or more points of a detail by two straight lines that intersect at the point, each of which passes through two points that have been or can be located. This is one method of locating an inaccessible point, or a point to which it is difficult to measure distances. Note that in this method no angles or distances are measured to the point that is located.

325. Special Applications of General Methods. The following methods of locating a point of a detail are merely special applications of methods already given.

1. By a swing offset to the point from a transit line or some other known line of reference and a tie to the point from a known point on that line.

2. By two swing offsets to the point from two known lines of reference.

3. By two distances from a reference line to the point, measured along two range lines through the point. For example, two distances from a transit line to the corner of a building measured along two range lines that coincide respectively with the two sides of the building that form the corner. The advantage in this case is that the directions of the two sides of the building are determined by the range lines. Obviously this method could be used

only when the two sides of the building are at rather acute angles to the line of reference.

Note. The use of these and other special applications of the general methods of locating some point of a detail will often save work in locating other points of the same detail.

326. Locating Boundaries. In surveying boundary lines it is often not practicable to make the traverse lines coincide with the boundary or fence lines. They are therefore run around the field outside or inside as may be more convenient. It is not necessary to establish the traverse lines parallel to the boundary lines, although when it is desired to find the angle at the intersection of the boundaries, it may be well to do so. Even then, however, it is not really necessary, since the required angles can be calculated.

Illustration. In Fig. 326 three methods are shown for locating the boundary lines of the same plot of land. It is assumed that

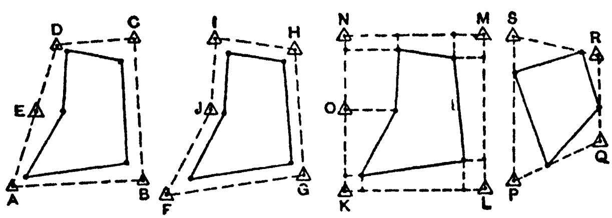


FIG. 326.

buildings, fences, or other obstacles prevent the traverse lines (represented in the figure by broken lines) from being established on the boundary lines.

1. Traverse lines *ABCDE* are run *at random,* and each corner of the plot is located from the nearest station by one of the usual methods—in most cases by angle and distance (Art. 319).

2. Traverse lines *FGHIJ* are parallel respectively to corresponding boundary lines, and the corners of the plot are located as in the first method; or each boundary line is located by offsets from the corresponding traverse line (Art. 322).

3. Any convenient rectangle of traverse lines, as *KLMNO,* is established, and each corner of the plot is located by perpendicular distances from the nearest traverse lines.

In the figure the traverse lines in each case are shown wholly outside the tract surveyed. If more convenient, they may be wholly inside or partly inside and partly outside.

The first method is the one most used because it involves the least field work. The second method calls for extra work in estab-

lishing the traverse lines parallel to the boundaries, but it gives the angle at each corner of the plot directly without computation. In the third method the difficulty lies in getting the points where the perpendiculars from the corners of the plot intersect the traverse lines, especially if this must be done with considerable precision; but when it is desired to save office work at the expense of field work, this method can be used. It is evident that much time is saved in plotting and in office computations.

In the second and third methods, buildings, fences, and other obstacles may render it difficult to establish the traverse lines, whereas in the first method, since the traverse lines are run at random, obstacles are more easily avoided.

This illustration shows how it is often necessary to locate different points of an object from different transit stations. Thus, for example, any one of the three methods could be employed for locating a large building. For staking out a small building, the third method is generally used (Art. 783).

4. The lines *PQRS* illustrate a fourth method, which is not a common one, but can be used sometimes to advantage where conditions are favorable. It consists simply in running a traverse line through each corner of the lot, and measuring along these lines to the respective corners. Not much work is saved in the field by this method, but the office work of plotting and of calculating the area is simplified. It has the further advantage of definitely locating each corner with the fewest possible measurements.

In the "Field Notes" of Chapter 24 will be found more complete illustrations of the use of traverse lines for locating boundaries.

327. Practical Application of Methods. The field work of locating details, involving such questions as which methods to use and how to apply them, is discussed in the next chapter. A careful study of the illustrations in the second part of Chapter 24, "Field Notes," especially the survey of a rural highway (Illustrations V and VI, pp. 368–373) and the survey of a city street (Illustration VII, p. 374), will suggest how several methods may be combined in practice. It is well worth while to plot the notes referred to in order to gain an idea of the actual field work of locating details.

328. Summary of Methods. Traverse lines form a skeleton upon which the survey is built. Details are located from the nearest or most convenient traverse line. There are seven methods of locating any point, four of which involve the measurement of one or more angles, two depend upon linear measurement only, and one upon the intersection of two straight lines. No one method should be used to the exclusion of all others, and the surveyor must be quick to see which is best adapted to his purpose.

1. The most common method of locating a point is that *by angle and distance*. It is quick and accurate and should ordinarily be used for points near a transit station, when measurements from the station to the points can be

made without difficulty. For a number of such points, angles should be taken from the same backsight. One angle will often serve to locate several points

2. A *second method* differs from the preceding in that the *distance* to the point located is measured from some other point instead of from the transit station. Used when it is more convenient to measure from a near-by point already located than to measure from the transit station itself.

3. A *third method* involves the use of a range line through two or more points to be located. The point in which the range line intersects the traverse line is first located. The angle between the range line and the traverse line is determined, and the distances from the traverse line to the points are measured along the range line. Used for locating a side of a building, or a boundary line, or a straight portion of some other detail, particularly when it is too far from a transit station to be located entirely from that station and yet is comparatively near one of the traverse lines.

4. A *fourth method* of locating a point is based upon the intersection at that point of two straight lines, one from one transit station, the other from another transit station. The line between the two stations serves as a base line, and the angles between this base line and the two intersecting lines are measured. For this triangulation, two transits are better than one, although the work can often be done with one. Used for locating inaccessible points or points too far from the traverse lines for linear measurements to be taken. Especially adapted to locating points on the water. If the angle between the two intersecting lines is too acute, the point of intersection is difficult to determine. Ordinarily the angle should be not less than 30°.

5. A *fifth method* involves the measuring of two angles *at the point to be located.* The angles measured are formed by three lines radiating from the unknown point to three known points. Can often be used to advantage in plane-table surveying, for hydrographic surveying, and in similar work. This method is the so-called three-point method.

6. A *sixth method,* locating points by offsets, involves linear measurements only. Used for locating objects which are near a transit line but not near a transit station. Especially adapted to locating curves, and irregular lines such as river banks, and edges of bluffs. Also used for locating buildings and other structures.

7. A *seventh method* is that in which each point is located by two distances or ties, one from each of two known points. The two points of reference should be so chosen that the two tie lines will intersect at an angle of at least 30°. This method is quite accurate and is often more convenient to use than one involving the measurement of angles. The use of a single tie to a range line, one form of this method, will often save setting up the transit, as it virtually determines the angle required.

8. An *eighth method* is that of locating a point by two intersecting straight lines from four known points. Used principally for referencing points, but it can also be used to some extent in locating points.

329. Summary of Practical Suggestions.

1. In choosing a method for locating a point, or a combination of methods for locating a detail, keep in mind the advantages and disadvantages in the office work of plotting and calculation as well as those in the field work of location.

2. Check measurements should be made in all important work. It is better to have a few such measurements well chosen than an unnecessarily large number.

3. Do not consider any point to have been completely located until, by a mental check, it is obvious that its position can be definitely plotted or calculated. Likewise, check mentally the location of a detail as a whole in order to make sure that there is no weak or inadequate location of any part of it.

4. It is important to measure up a detail no matter how many points of it have been located. This means that the length of every important line in a detail should be measured, when practicable, even though the ends of the line and possibly other points on it have been carefully located.

5. It is often easier to locate one detail from another than to locate it with respect to a transit line.

6. When a point is located by means of two straight lines that intersect at the point or by two tie distances to two known points, the angle of intersection should be at least 30° and as near 90° as practicable. On the other hand, when it is located by a tie distance from a known point to a line of

known direction, the angle of intersection should not be near 90° but acute enough to make the intersection definite.

7. It is generally better to locate a long side of a detail than a short side, particularly when a range line along the side is used.

8. In locating any object it may be well to use one method for locating some of the points, and another method for locating other points; thus a combination of two or more of the seven methods may often be used to good advantage.

9. If an object covers considerable ground, it may be necessary to locate different parts of it from different stations or different transit lines; in locating long lines, such as street lines and boundary lines, this is especially true.

10. Details are not generally located with so great a degree of precision as that required for running the traverse, and it may not be necessary to locate all details or all parts of the same detail with equal accuracy.

11. Practical suggestions for locating details, with special emphasis on methods of procedure in field work, are given in Art. 353.

CHAPTER 23

TRANSIT SURVEYING

FIELD WORK

It is comparatively easy to learn the theoretical methods of running traverse lines and of locating details given in the two preceding chapters. On the other hand, the judgment required for choosing methods best adapted to the work in hand, and the skill necessary for carrying out the field work with accuracy and despatch, can be acquired only from a long and varied experience in the field. It is not desirable, therefore, to lay down hard and fast rules for field work. The most that can be done is to outline the routine usually followed and to discuss in a general way practical questions which arise in the field.

330. Transit Party. Transitman, chainman, flagman, axeman, and others who work together as one party in making a transit survey. The size of the party depends upon the nature of the work. Two men may constitute a transit party, but usually three, a transitman, and two chainmen, can work to better advantage. In the usual party of three, the transitman not only runs the transit, but also acts as notekeeper and as *chief of party* to direct the entire field work. Under certain conditions, however, it will be easier and better for the chief of party to act as rear chainman instead of transitman. When it is desirable that he should be entirely free to direct the party and to keep notes, the number of the party may be increased to four. In country where it is necessary to clear brush and trees, one or two axemen may be required. Sometimes several transit parties are at work on the same survey. The duties of each member of the party are outlined at the end of this chapter.

330(*a*). Economic organization of a party. As a general rule, the greater the degree of precision required in a survey, the greater the number of men required in the party. For example, if linear measurements are to be made with a high degree of precision, it may require six or more men to do the chaining, including a levelman and a rodman to determine elevations of the two ends of the tape. Likewise, it may be necessary to have one or two men assist the transitman. A chief of party with, perhaps, one or two assistants, two recorders, one for the transit work and one for chaining, and men to carry on the manual work connected with the survey, are often included in large parties. The total number of men in the party sometimes may be fifteen to twenty or even more.

In an extensive survey, money can often be saved by having just the right number of men in a party. The economic number will depend upon the conditions under which the survey is made. It is often desirable, for example, to organize a party in such a way that the transit work and the chaining can be carried on at about the same rate of speed. If there is a large amount of transit work to be done and comparatively little chaining, it may be economical to use two transits with a corresponding increase in the number of men

required for the transit work. Likewise, if the amount of chaining to be done is large, it may be desirable to increase the number of men required for that work.

331. The Instruments and Equipment for a surveying party depend upon the work to be done. For ordinary surveys the different members of a transit party should be equipped as follows.

Transitman. Transit, plumb bob, reading glass.

Notekeeper. Notebook, pencil, eraser, straightedge, scribbling pad, and trigonometric tables.

Chainmen. Steel tape, chaining pins, two plumb bobs, two sight poles (also called range poles, line rods, or flags), stakes, axe, marking crayon, tacks or nails.

In addition to the above list there may be needed a hood for the transit in case of rain, and a cold chisel if points are to be marked on a hard surface. It is well for chainmen habitually to carry in the pocket a short steel tape, reading glass, sharp knife with screwdriver blade, pad of scribbling paper, string or spool of fine wire, pencil, and eraser.

Note. For the instruments and equipment required in work of a high degree of precision see Art. 360. For a more complete description of instruments and equipment consult Part III in this book.

332. Care of Equipment. All members of a party should cooperate in taking good care of the equipment to avoid unnecessary wear and accidental injury, particularly to the transit and to the steel tape. Some of the things to guard against are these.

Transit. Lifting it by the telescope; wing nuts of tripod so loose that the transit may fall; bumping the instrument when carrying it; transit set up with tripod legs so close together or so far apart that the transit may fall; leaving the instrument unguarded where it may be knocked over or where something may fall on it; forcing movable parts such as screws, objective slide, or eyepiece; rubbing scratches on the objective glass; allowing mud and moisture to accumulate on the tripod legs; failing to keep moving parts clean and slightly lubricated. Precautions that should be taken in the care of the transit are given more in detail in Art. 243.

Steel Tape. Pulling the tape when there is a loop or kink in it; stepping on it or allowing vehicles to run over it when it is on soft ground; bending it into or around sharp corners; failure to wipe it dry before putting it away.

Miscellaneous. Breaking sight poles; losing chain pins and plumb bobs.

Note. Members of the transit party should be thoroughly familiar with the directions for the care of instruments given in Chapter 53.

333. Linear Measurements. In transit surveying, linear measurements are usually made with a steel tape, though the stadia method is frequently used, particularly in topographic surveying. In this chapter, it will be assumed that all distances are measured with the tape.

No matter how accurately angles may be measured, a survey can be no more accurate than the chaining. Unfortunately, there is a tendency to regard chaining as drudgery calling for no special ability. As a matter of fact, it is generally more difficult to attain a fairly high degree of precision in chaining than it is to attain the same degree of precision in transit work. In order to realize the difficulty of chaining, one has only to appreciate the many precautions which must be taken to eliminate errors as explained in Chapter 6, and the extraordinary precautions in first-order linear measurement as outlined in Chapter 8.

Since the whole subject of chaining has been treated in Chapters 5–7 inclusive, no further explanation will be given in this chapter except wherever it is necessary to show how the chainmen work in cooperation with other members of the party. From a careful study of the chapters just mentioned, the chainmen should have a good working knowledge of the different methods of chaining, particularly on slopes; of how to read any kind of tape; of the common mistakes in chaining; of the different sources of error and the relative importance of errors; of the methods of procedure, including necessary precautions, which must be followed in order to attain any required degree of precision; and of the geometric and trigonometric methods used in the basic problems in chaining, such as constructing right angles, laying off an angle, and measuring between two points when an obstacle intervenes.

In addition to the measurement of linear distances, chainmen in a small party have other duties such as giving sights (holding a pencil or a sight pole for the transitman to sight on), establishing transit stations or other points, and clearing brush or other obstacles.

334. Signals in common use in transit surveying, and in many other types of field work as well, are as follows.

"Right" or *"Left."* When one member of the party desires another member to move right or left, he extends an arm in the direction desired, the right arm for a movement to the right, the left arm for a movement to the left. A slow motion of the hand over a distance of a foot or two means "move quite a distance," and a quick short motion means "move a short distance." The signal may be used by the transitman or by one chainman to another to keep the tape on line in measuring a distance. It may also be used by the transitman for indicating in which direction a sight pole, a stake, or a pencil should be moved to bring it into the line of sight.

"All Right." Both arms are extended horizontally, and the forearms are moved slowly up and then down. Another and more effective signal at a distance is to wave the arms in sweeping vertical arcs in such a way as alternately to bring the hands together above the head and then to separate them.

"Give a Sight." When the transitman or any other member of the party desires a sight pole or a pencil held on a point, he holds one arm vertically above his head.

"Plumb the Pole." One arm is held vertically above the head and moved slowly in the direction that the pole should be plumbed.

"Give Line." When a flagman desires to be lined in, he holds the sight pole horizontally in both hands above his head and then brings it down to a vertical position with the lower end on the ground. When a point is to be set so exactly that more than one sight may be needed, as, for example, in setting a tack in a hub, the flagman, keeping one end of the sight pole on the ground, waves it from side to side. (Another signal used for the same purpose consists in waving the upper end of the pole in a horizontal circle above the head before bringing the pole to a vertical position, or, if a pole is not being used, in waving the arm in a similar manner.) When the point has been completely set, the flagman signals "All right."

"Take a Sight." When a member of the party desires the transitman to take a sight on a point, he raises one arm vertically above his head and then holds a sight pole or pencil on the point.

"Move the Transit." When the chief of party or some other member of the party desires to have the transit moved to another point, he extends both arms outward and downward and then upward in such a way as to imitate the motions of picking up and shouldering the instrument.

Long-distance signaling is often done with a white handkerchief or with red or white signal flags attached to poles. In many cases two-way radio communication is a better method. For signaling at night see Art. 407(*d*).

Any system of signals may be used in place of those given. It is a matter of small consequence what the signals are, provided they are clearly understood by every member of the transit party. Avoid shouting; it is an evidence of "greenness."

335. Permanent Station Marks. It is frequently desirable to set a permanent mark for a transit station or for some other point such as a boundary mark or monument. The most common form is a metal tablet with such information stamped on it as may be necessary to describe the point.

The location of the station, depth of the soil, presence of rock ledges, and other local conditions will usually control the method of setting the tablet. In a rock outcrop, care should be taken that the rock in which a mark is set is a part of the main ledge, not a detached fragment. The tablet should be countersunk and well cemented in. When the tablet is set in a concrete post, the sides of the post should have a batter of, at least, 1 in. to the foot, and should extend to a depth of at least 30 to 36 in. or deeper, if necessary, to avoid frost action. Sometimes it is desirable for the post to have an enlarged base, 6 in. or more in thickness, and, at least, 4 in. larger in its horizontal dimensions than the post proper. If the concrete is cast in place, the enlarged base may be provided for by enlarging the bottom of the hole. Since monuments set in earth may be disturbed, an underground mark below the monument is often used for an important point. One method of construction is as follows. A hole is dug to a depth of 3½ ft. or more, concrete

is poured into the bottom of the hole to a depth of about 6 in., and a metal station mark or tablet is then set in the concrete. After covering this mark with a layer of 4 to 6 in. of sand, the concrete monument with its surface mark is either set or constructed directly above, care being taken not to disturb the lower mark. (For more details concerning permanent monuments see Arts. 790(*c*) and 822(*c*).)

Note. The adjective "permanent" when used in connection with a station or a monument may be misleading since movement may occur after a period of time despite precautions taken to prevent it. However, the displacement of a concrete monument, properly designed, set, and protected, is generally negligible except where the highest degree of precision is desired.

336. Establishing a Transit Station. Of all the different types of transit stations, a stake is the most common. Such a transit station or "hub" may be established in two ways: (1) Its position may be chosen, and the exact point marked before any sight is taken on it, or (2) it may be "sighted in" from the transit, as, for example, when a station is to be established "on line" with two other stations. When the transit station is to be lined in the flagman first holds the transit pole or, better still, the stake itself, and moves it into line in accordance with the signals from the transitman. (See Art. 334.) The required point having been determined, the stake is driven, and the pole (or a pencil if it can be seen) is again held to find the exact point on the stake for driving the tack. In rough work the tack is unnecessary, the stake itself answering every purpose. The procedure just described can be easily modified for use in establishing some other type of transit station.

337. Numbering or Lettering Stations. Stations are usually numbered or lettered in the order in which they are established, but this is not essential. Numbering is better than lettering where there are many stations. In route surveying and in some other kinds of work the first station is often numbered zero, and the other stations are numbered to indicate how many feet they are from zero *measured along the traverse line*—one number for every 100 ft. Thus: $15 = 1500$, $15 + 50 = 1550$, $15 + 83.5 = 1583.5$, $16 = 1600$ ft. from zero, and so on.

338. Guard Stake. When a station has been established, there should be some means of finding it at any time and of identifying it by letter or number. A common method is to drive a guard stake or witness stake partly into the ground, a foot or so from the station or hub. The station number should be marked on it, *from the top toward the bottom.* If the stake is driven on the left of the transit line and at a slant (with its top inclined away from the

station), the lettering can be easily read as one walks along the line. Another method is to drive the stake at a slant but in such a way that its top is nearly over the hub and with the lettering on the under side to protect it from the weather. If, for any reason, a guard stake near a station is likely to be knocked over, it may be driven a few feet away and marked accordingly as, for example, "Sta. 15 + 50 (12 ft. N)." Flat stakes are often used for guard stakes and square stakes for hubs. When it is impracticable to drive a guard stake, the station number and other lettering may be placed upon some other object near by.

Remark. The surveyor's crayon or "keel," used for marking stakes, differs very much in its waterproof qualities. Test by marking a stake and then trying to wash it off. Make all numerals bold and distinct, pressing the crayon into the surface of the wood.

339. Choosing a Place for a Transit Station. Do not choose a place for a station where the transit is easily jarred or disturbed. Avoid stations close to car tracks, on loose planking, soft or marshy ground, smooth pavements, or other places where the legs of the tripod are easily moved.

Do not choose a station where the line of sight will be continually obstructed by passing wagons, cars, or people, if it can be avoided. In a crowded city street the surveying may have to be done very early in the morning.

Illustration. Near the curb is a better place for a station than in the middle of the sidewalk, and, where there is much traffic, stations on the same side of a street are better than on opposite sides. Avoid establishing a station in a place where it will be destroyed when the work of construction begins.

Remark. It is frequently impossible to avoid all the unfavorable conditions suggested above. In fact, conditions may be so bad that the transitman must tax his ingenuity in manipulating the transit if he is to attain satisfactory results.

340. Referencing a Station. When a station is likely to be used more than once it should be referenced, so that if lost or destroyed it can be restored. This precaution is especially necessary when stations are where carting, excavating, grading, and other construction work is going on. The general method is to establish two or more points of reference by means of which the station may be relocated at any time. These reference points may be established by linear measurements only, or by angular measurements only, or by a combination of linear and angular measurements. Any one of the three methods of referencing a point by linear measurements only, explained in Art. 92, can, of course, be used for referencing a transit station. A method in which the transit is used to establish four points of reference in such a way that the straight line between two of them intersects the straight line between the

other two at the transit station, is described in detail in Art. 275. Other methods in which the transit or the transit and the tape may be used for referencing a station will be evident from a study of methods 3, 4, 5, and 6 in Art. 10(*a*).

Whatever method of referencing a station may be used, the points of reference should be established in such a way that they are more or less permanent, readily found, and easily used. A stake driven flush with a tack in its top, similar to a hub for a transit station, is often used for a point of reference. These reference hubs should, if possible, be set where they are not likely to be disturbed, and each should have a guard stake marked with the information that may be needed in the use of the reference hub. When a reference hub cannot be driven, a mark on a hard surface such as a pavement, or a spike, or any other device that would answer for a transit station under similar conditions, may be used for a reference point. Marks or nails on buildings, fences, and other objects are often used when only a point to be sighted on is needed for reference. In the case of a permanent station which is likely to be used for a long period of time, the reference points should be as permanent as the station itself.

341. Questions of Judgment are: (1) What *limits of error* to adopt, i.e., how close to measure angles and distances; (2) *where* to run traverse lines, i.e., the general plan of traverse lines; (3) *method* of running traverse lines—which one of the four to use; (4) *method* of locating details—which is the best for any given case; (5) method of keeping notes—which is the best adapted to the work in hand. These closely related questions are discussed in the succeeding articles.

342. Conditions That Govern Field Work. The purpose for which a survey is made is the first consideration governing the field work. It largely determines the methods used, the data required, the limits of error adopted, and the general plan of the field work. Among other considerations are: nature of the area surveyed—smooth or rough, open or wooded, city or country; time available for the survey; reasonable cost of the survey; correlation of field work and office work.

Note. In Art. 16 are given illustrations of how the purpose of a survey governs to a large extent the field work. In Art. 84 are given illustrations of how limits of error may be different for different parts of the work. In some of the chapters to follow are given the requirements for surveys for various purposes and for staking out work for different types of construction.

343. The Relation between Field Work and Office Work is so close that questions which arise in the field can often be settled by the bearing which they have upon work in the office. When,

for example, two or more methods of field work seem equally good, consider which will require the least office computation, or which is the best for plotting. To facilitate office work in this way, by forethought in planning field work, does not mean, however, that it always pays to do extra field work in order to save office work. Since the work of an entire transit party can often be plotted by one man an hour in the field for, say, three men may cost as much or more than three hours in the office for one man. Moreover, field work must often be done in a limited time and under favorable weather conditions, whereas the corresponding office work can be done more leisurely and regardless of the weather.

344. Obstacles Encountered in Field Work. In addition to difficulties in surveying that are due to the general nature of the area surveyed, minor obstacles are frequently encountered that make it necessary to use special methods for a portion of the field work, and to adopt certain precautions to obtain even an ordinary degree of precision. A line runs through a tree; a distance is wanted between two points, but a house corner intervenes; points to be located are inaccessible. Obstacles such as these give rise to special problems in the use of the tape or the transit or both. They may also influence the surveyor in deciding some of the questions suggested in the preceding article. Problems of this kind which occur in chaining are explained in Arts. 90 and 91. Similar problems which arise in transit work are explained in Arts. 381–382.

345. Where to Run Traverse Lines. Much depends upon the general plan of the traverse lines, for they form the skeleton of a survey. In general it is more important to plan carefully for a closed system of traverse lines, especially if it is a complicated network, than it is for an open survey. Nevertheless, even in an open survey, unnecessary work may be saved both in the field and in the office by judiciously choosing the location of transit stations.

When possible, transit stations should be so chosen that the work of running traverse lines, locating points of the survey, and plotting the map is reduced to a minimum. Before beginning a survey, go over the ground and note the most important features to be located. Place transit stations where they will be of the most use in locating those features. If the main purpose is to survey boundary lines, then a traverse line should run near each boundary (but not necessarily parallel to it). If buildings are to be located, plan to have stations near the buildings. Obstacles often prevent a surveyor from running his traverse lines just where he would like to run them, but he can usually find some good way out of the difficulty.

See if the traverse lines can be run readily. Changing a station a few feet one way or the other may save unnecessary field work or unnecessary work in plotting. Make a rough sketch of the traverse lines and see if the system as a whole is well balanced. Are there any unnecessary stations? Are distances between stations too long or too short for accurate sighting? Are there any unnecessary obstacles to sighting or chaining? Half an hour spent in choosing transit stations may save hours of field work.

345(*a*). ILLUSTRATION. *Boundary Survey.* Figure 345(*a*) illustrates how field work and office work are affected by the location of traverse lines. The lines 1–2–3–4–5–6–7–1 are boundary lines, and the question is *where* to run the traverse lines. They can be located in four different ways.

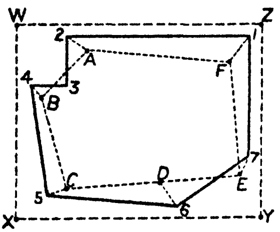


FIG. 345(*a*).

1. To *coincide* with the boundary lines.
2. To form a reference *rectangle*, as *WXYZ*.
3. To form a polygon each side of which is *parallel* to and a short distance outside or inside a corresponding boundary line. (This polygon is not shown in the figure.)
4. To form *any* convenient polygon, as *ABCDEFA*.

345(*b*). COMPARISON OF METHODS (FIELD WORK ONLY). (1) The first method, theoretically ideal, is often impracticable because of existing fences and other obstacles. (2) The second method facilitates office work, but is often unsatisfactory or impracticable in the field. (3) The third method involves considerable extra field work in establishing each traverse line parallel to the corresponding boundary line. (4) The fourth method is the one most used. Its chief disadvantage is that neither the length of boundary lines nor the angles at the corners are measured directly. They can be easily found, however, by computation. It is much better, however, to follow the general rule that a detail should be completely measured up, which, in this case, means measuring the lengths of the boundary lines.

Note. For general methods of locating boundary lines from traverse lines see Art. 326.

345(*c*). THE CHOICE OF STATIONS and its effect on field work and office work are further illustrated in the polygon *ABCDEFA*. Notice that station *B* was so chosen that *two* corners, 3 and 4, could be located from it, and that when the transit was at *C* station *D* and *E* were established "in line." In this way a five-sided polygon is made to answer in place of a seven-sided polygon, and a corresponding gain in both field work and office work is the result.

345(*d*). THE RELATION BETWEEN FIELD WORK AND OFFICE WORK discussed in Art. 343 is illustrated by the third and fourth methods. Suppose that the lengths of the boundary lines and the angles at the corners are required. If the fourth method is used, they must be computed. Is it worth while to save this extra office work by running parallel traverse lines according to the third method? The computation involves finding one unknown side in each of seven polygons. For example, if the lengths and directions of the lines *2A*, *AF*, and *F*1 are known, the length and direction of the missing side 1–2 can be computed. (See Chapter 39.) In the polygon 1–*F*–*E*–7 the missing side 1–7 can be found in like manner. The direction of 1–2 and 1–7 having been determined, the angle 2–1–7 is easily found. The extra office work involved is not so much as at first appears, especially if the latitude and departure method is used for plotting. It should not exceed in this particular case more than two hours *for one man.* Note that if the general rule of completely measuring up a detail is followed, the lengths of the boundaries will be measured in the field, even though all corners are located, and consequently the only unknowns left to be calculated are the angles at the corners. In city work it often pays to run traverse lines parallel to corresponding street lines, but with this exception the third method is not generally favored.

346. Methods of Running Traverses have already been compared in Chapter 21. No definite rule can be given for choosing the method best adapted to the work in hand, but if the advantages and disadvantages of each method as given in Chapter 21 are understood, it is not a difficult matter to decide which to use. In general it may be said that for work in the city where accuracy is required and where the magnetic needle is often useless, one of the first two methods (angles or deflection angles) is used. Angles can then be measured by repetition or in series. In country surveying where local attraction does not exist to so great an extent, and where less accuracy is required, the azimuth method is often used, the magnetic needle furnishing an approximate check on the azimuths of the different lines. This method is also especially adapted to stadia and topographic surveying. On the other hand, when the traverse angles of a survey must be measured by repetition or when the stations cannot be occupied in order, the azimuth method would not be used for running traverses, though possibly it might be employed later for locating details. The method of deflection angles is used in railway and highway work and is a favorite with many engineers for other kinds of work. The quadrant method has little to commend it. In extended surveys two methods are often combined. For example, in topographic surveys primary stations are often established by triangulation (a special case of the angle method), and from these controlling stations secondary traverse lines are run by the azimuth method. (See Chapter 31, "Topographic Surveying.")

347. Suggestions for Running Traverses. Whichever method of running traverses is adopted, by taking a few simple precautions the transitman can be reasonably sure of avoiding serious mistakes. Instead of depending entirely upon his own care and vigilance he should adopt some method of procedure by which a mistake will become self-evident as soon as it is made. This usually involves doing something as a check that otherwise would be unnecessary, but the extra time required is trifling compared with the assurance that his work is free from blunders. Indeed, with his mind thus relieved from unnecessary strain, the transitman will usually work more quickly and accurately than he otherwise would.

Precautions are taken for two reasons: (1) to insure accuracy and (2) to avoid actual mistakes or blunders. Common sources of error in manipulating the transit are pointed out in Art. 261, and corresponding precautions to avoid these errors can be taken. Common mistakes in reading angles are pointed out in Chapter 18 and summarized in Art. 263. At least four precautions are so im-

portant that they become automatic in the work of a good transit-man.

1. *To make sure that the lower clamp is tight.* Immediately after backsighting, try gently to turn the lower plates while watching the cross hairs to see if they remain on the backsight after the hands are removed from the plates.

2. *To avoid any movement or displacement of the transit between the backsight on one station and the foresight on the next.* Make the time interval between the two sights short. Leave the side shots until afterward.

3. *To check the measurement of an angle.* Double the angle if an accurate check is desired. For a rough check, observe the magnetic needle. A series of angles at any station may be checked by measuring and checking the large angle (often 360°), of which the angles to be checked are subdivisions. Any angle may be roughly checked by the magnetic needle regardless of local attraction (unless the latter exists in the transit or about the person of the transitman), provided the bearings of both lines are read at the same station, i.e., the back bearing of one line and the forward bearing of the other. Where local attraction does not exist, however, it is better to calculate bearings in continuous form. (See Art. 305.) When this is done make sure of some correct magnetic bearing at the beginning of the survey by reading the needle at each end of a line, and assume this bearing as the basis of all calculated bearings, Art. 305(*a*).

4. *To make sure that there has been no movement of the lower plate during the measurement of one or more angles.* Take a "check sight" as often as seems advisable *and always just before leaving a station,* or as soon as the transit work at that station has been completed.

A *check sight* is a sight taken on a fixed point (often the preceding station) to ascertain if the vernier reading is the same as it was when a previous sight was taken on the point. If it is the same, it can be assumed that no movement of the lower plate has occurred during the interval between the first and last sights. At each station, it is well to establish a permanent backsight, i.e., a range pole or some other object upon which to sight as a check; also, if several side shots are to be taken, to establish a permanent foresight on the next station.

After the angle between the traverse lines at any station has been measured, the vernier may be set at zero to see if the line of sight still strikes the permanent backsight. In this way if the clamp has slipped, or a wrong tangent screw has been turned, or the transit

has been disturbed, the movement of the lower plate becomes evident. A slight deviation from the original backsight due to inaccuracy in setting the vernier is, of course, to be expected. Instead of setting the vernier at zero, a check sight on the permanent backsight may be taken to see if the vernier still reads zero.

When side shots have been taken, before leaving the station the vernier may be set at the reading for the traverse angle to see if the line of sight strikes the permanent foresight on the next station. This is a double check: (1) Any movement of the lower plate while the side shots were being taken will become evident, and (2) any mistake in the original reading for the traverse angle is likely to be detected. When there is a large amount of transit work at any station, it is well to take a check sight from time to time as the work progresses.

Note. Sometimes instead of establishing a permanent backsight, it is more convenient to take for a permanent sight some other point that is prominent and easily sighted on, and measure the angle from the backsight to this point. A check sight can then be taken on this permanent sight at any time to see if the vernier reading for the angle remains the same. In order to make sure that there was no movement of the lower plate when the angle to the permanent sight was measured, it may be well, immediately after reading the angle, to take a check sight on the backsight.

It is to be noted that an important angle, such as a traverse angle, is often doubled by repetition. In that case, a check sight for the angle is unnecessary.

348. Methods of Procedure in Running Traverses. *Method of procedure* implies systematic precautions and checks, such as:

1. Precautions in the manipulation of the transit (Art. 260).

2. Precautions in reading angles (Arts. 262, 263).

3. Precautions to eliminate instrumental errors of adjustment and of faulty graduation or construction (Art. 264).

4. Precautions to prevent or forestall slight displacement of the transit.

5. Checks to detect displacement of the instrument or mistakes in manipulation.

6. Checks to detect mistakes in reading angles.

7. Check on the prolongation of a straight line (double reverse, Art. 252).

Any method of procedure may be used which accomplishes the desired result without waste of time, but it is wise to adopt some one method and stick to it, for then in many of the operations the mind will come to work automatically, which, in itself, is an efficient means of avoiding mistakes. It will be found necessary, however, to adopt different methods of procedure corresponding to the four methods of running traverses. Those suggested in articles

to follow are adapted to ordinary work where great precision is not required, and each method should be modified to suit conditions.

Remark. Surveyors differ as to minor details, but there is a general agreement as to the value of having a fixed method of procedure. Those outlined below are likely to seem somewhat intricate to the student, and hardly worth while following in all their details, but this is likely to be true where any operation in the field is described in print. As a matter of fact the methods are simple, and after a little practice the extra steps taken as precautions or checks become instinctive. This book contains no better pointer for inexperienced transitmen than this: *Adopt some method of procedure well adapted to your work and stick to it in all its details.* It will save mistakes, relieve the mind of unnecessary strain, and inspire confidence in results.

Methods of procedure corresponding to four different methods of running traverses are given in the next four articles. In each method the procedure indicates what the transitman actually does at any one station, from the time he has finished setting up the transit until he is ready to move to the next station. The extra steps taken as checks or precautions are indicated in italics to distinguish them from those that are absolutely necessary.

349. Method of Procedure for Direct-Angle Method. *Suggestions.* (1) Use a transit graduated clockwise from 0° to 360° and take all angles clockwise; saves confusion in reading and recording angles. (2) Invariably read the A vernier, consulting the B vernier, if at all, merely as a check; avoids mistakes of 180° in reading angles.

349(*a*). METHOD OF PROCEDURE WHEN ANGLES ARE NOT DOUBLED.

1. Set the A vernier at 360°, or, if at zero, *make sure that it is the zero corresponding to* 360°.

2. Backsight; *glance to see if A vernier is still at zero; see if the back bearing agrees with the forward bearing.*

3. *Test to see if clamps are tight;* foresight *immediately* on the next station before the transit is likely to be disturbed; *establish a permanent foresight.*

4. Read and record the angles.

5. *Read and record the forward bearing (from the needle).*

6. *Calculate and record the forward bearing and see if it agrees with the magnetic bearing.*

7. Take all side shots.

8. *Set the vernier at the original angle reading to the next station and see if the line of sight strikes the permanent foresight.* (Double check.)

Note. If there is any possibility that the lower plate has been moved while measuring the angle to the next station, a check sight on the backsight should be taken. This should be done before taking side shots.

Comments. Instead of relying upon the needle for a check, a traverse angle and any other important angle may be doubled; this is the surest check. If

the extra time required for doubling angles is not warranted and it is desired to use the needle as a check, the azimuth method is better adapted to the purpose, provided stations can be occupied in regular order.

349(*b*). METHOD OF PROCEDURE WHEN ANGLES ARE DOUBLED. The first four steps are the same as in the preceding method except that no attention is paid to the needle in backsighting, and a permanent *backsight* is established in (2) instead of a permanent *foresight* in (3).

5. *Double the angle; record the double value, and see if it agrees with the single reading.*

6. Set the A vernier at 360° again, backsight as before, and take all side shots.

7. *Set the vernier at 360° to see if the line of sight strikes the backsight. (Check sight.)*

350. Method of Procedure for Deflection-Angle Method. *Suggestion.* A transit graduated according to the quadrant method is most convenient unless deflection angles greater than 90° are to be measured (Art. 227). Make sure that the line of collimation is in adjustment.

350(*a*). METHOD OF PROCEDURE.

1. Set the A vernier at 0°, *glance at the B vernier to see that it is opposite* 180° (*and the corresponding* 0°).

2. Backsight; *glance to see if the A vernier is still at* 0°; *see if back bearing agrees with forward bearing.*

3. *Test to see if clamps are tight;* plunge; foresight *immediately* on the most important point, usually the next station; make a mental note whether the angle is "right" or "left"; *establish a permanent foresight.*

4. Read the angle from the B vernier, *verifying the direction (right or left) previously noted;* record the angle, *making sure it is in the proper column "left" or "right."*

5. *Read and record the magnetic bearing.*

6. Calculate and record the forward bearing, and see if it agrees with the magnetic bearing.

7. Take all side shots.

8. *Set the B vernier at the original deflection angle to the next station, and see if the line of sight strikes the permanent foresight.* (Double check.)

350(*b*). NOTE. If there is any possibility that the lower plate has been moved while measuring the deflection angle to the next station, a check sight on the backsight should be taken. This should be done before taking side shots.

350(*c*). NOTE. There are three methods of running traverses by deflection angles (Art. 302). The procedure just outlined is for the first method and may easily be modified to conform to either of the other two. The advantages and disadvantages of each of the three methods are discussed in Art. 302(*d*).

350(*d*). Additional suggestions.

1. It makes no difference whether the telescope is normal or inverted when backsighting if the limb is graduated in quadrants. When, however, the limb is graduated according to the half-circle method (Art. 226), make it a point to have the telescope *inverted* when backsighting and *normal* when foresighting, so as to bring the *A* vernier under the eyepiece when reading angles. In using such a transit a very valuable check is to observe and record both A and B vernier readings. Their sum should always equal 180°.

2. Instead of relying upon the needle for a check, it may be desirable to double the traverse angle (Art. 269(3)). This procedure also eliminates error in the line of collimation. In this case establish a permanent backsight in the second step instead of a permanent foresight in the third step.

3. When calculated bearings are kept the following check may be employed: *Check on the Calculated Bearings.* Add all angles to the left; add all angles to the right; the difference of these two sums applied to the bearing of the first line should give the calculated bearing of the last line.

Illustration. 1–2 = N 42° 18′ E. 2–3 = 16° 21′ (R.), 3–4 = 18° 20′ (L.), 4–5 = 28° 30′ (L.), 5–6 = 81° 19′ (R.), 6–7 = 8° 11′ (R.). Hence bearing of 6–7 = (16° 21′ + 81° 19′ + 8° 11′) − (18° 20′ + 28° 30′) = 59° 01′ to be applied to N 42° 18′ E. Since 59° 01′ is R. (clockwise) it is added to N 42° 18′ E, and the bearing of 6–7 is S 78° 41′ E. Verify this bearing by calculating it in the usual way.

351. Method of Procedure for the Azimuth Method. It is assumed that a magnetic meridian is used. The same method of procedure holds good, however, for a true meridian, but in that case, for convenience in reading bearings, the compass box should if possible be adjusted (Art. 303(*i*)).

Suggestions. Use a transit graduated clockwise from 0° to 360°, and invariably read the A vernier. As one advantage of this method is the ease with which azimuths are checked by the needle, make sure that the latter is in good working order and that proper precautions are taken in getting the correct azimuth of the first line, free from all local attraction (Art. 303(*j*)).

351(*a*). Method of procedure, first azimuth method (Art. 303(*e*)).

1. *Glance at the vernier to see if it is still set at the forward azimuth;* plunge and backsight on the preceding station.

2. *Test to see if the clamps are tight;* plunge; foresight on the next station; *establish a permanent foresight.*

Note. In these first two steps make it a point always to have the telescope *inverted* when backsighting and *normal* when foresighting.

3. Read and record the azimuth; *check the azimuth by the needle reading, and record this magnetic bearing.*

4. Take all side shots, *checking one now and then by glancing at the needle.*

5. Set the vernier at the forward azimuth and *see if the line of sight strikes the permanent foresight.* If, as is usually the case, the line of sight is *slightly* off, say the width of the pole, bring it on with the *upper* tangent screw, thus automatically setting the

vernier for the next setup. The vernier reading should still agree with the forward azimuth to the next station originally recorded. (Double check.)

351(*b*). Note. In the above method of procedure any movement of the lower plate occurring between the backsight and the first foresight to the next station would not be detected. The check for this would be to set the vernier at the azimuth of the preceding line, plunge, and see if the line of sight strikes the backsight. This should be done before taking side shots. When doubtful whether or not the transit has been oriented correctly, set the vernier at zero: the north end of the needle should point to N. This is only a rough check, but it will disclose any large error.

351(*c*). Note. There are three methods of running traverses by azimuths (Art. 303(*d*)). The procedure just outlined for the first method may easily be modified to conform to either of the other two. The advantages and disadvantages of each of the three methods are discussed in Art. 303(*h*). The choice of a meridian for any survey is discussed in Art. 203.

352. Method of Procedure for Traverses by Bearings. The method of running traverses by bearing (Art. 304) is so nearly like the azimuth method that the procedure outlined in the preceding article may, with only slight modifications, be followed step by step. The modifications will be obvious to one familiar with these two methods of running traverses.

353. Methods of Locating Details. The details of a survey may be located at the same time that the traverse is run or after the traverse has been wholly or partly completed. The question of when to follow one procedure and when the other has been discussed in Art. 311(*a*). Different methods of locating details have been given in the preceding chapter. The two most used are the method *by angle and distance* and the method of *offsets*, but one of the other methods can often be used to advantage. The accuracy required will frequently help to determine which method to employ. When, so far as accuracy is concerned, there seems to be no choice, it is usually a question of time. If, for example, the transitman is busy and the chainmen are not, it may save time to locate a point by some method which does not involve the measurement of angles. For the same reason, methods of locating details which save unnecessary setups of the transit are often used (Art. 323). Sometimes the surveyor is practically limited to one method as, for example, in locating inaccessible points by angles (Art. 320) or in locating irregular lines (such as river banks) by offsets (Art. 322). Some obstacle to measurement or to the line of sight often compels the use of a method different from the one which would ordinarily be used as, for example, in locating a point on the fence line (Art. 319(*c*). Frequently, it is best to locate some of the points of a detail by one method and other points by another; in fact, the complete location of even a relatively small detail may involve a

combination of several methods. A surveyor can get a good deal of satisfaction from habitually using his ingenuity in devising the most advantageous combinations. If the student will acquire a working knowledge of *all* the methods of locating details explained in the preceding chapter, he will soon learn by actual experience which method or combination of methods to choose in any given case.

In locating any detail, select points which will best serve the purpose. As a general rule locate points where the outline of a detail changes as, for example, corners of structures, points at which there are angles in a street line, and points at which straight lines change to curves. Frequently, only a portion of a detail is located with respect to transit lines, and the remainder of the detail is then located with respect to that portion as, for example, when one side of a rectangular building is located with respect to a transit line and the remainder is located with respect to that side. In such a case, select that portion of the detail which can be readily located from the transit lines and, at the same time, can be used to the best advantage in locating the remaining portion. For example, for a small building, select one of the sides that can be readily located from a transit line and, at the same time, is long enough to serve as a reference line in locating the remainder of the building.

Do not locate more points of a detail than are necessary. For example, do not locate points on both rails of a track when points on one rail will suffice; or an unnecessary number of points on a circular curve when the center can be located and the radius measured; or all points of a symmetrical structure when points on one-half will serve every purpose; or points unnecessarily near each other on the bank of a stream or on some other irregular line.

In filling in the details of a survey it is often a waste of time to take all measurements with equal accuracy. It is far better to keep in mind what each angle or distance will be used for and measure it with corresponding accuracy. Measurements required solely for plotting may be taken, as a rule, with less precision than those needed for permanent record, and measurements taken to fix the position of an object are usually less important than measurements of the object itself. This is especially true if the scale of the map is small. Thus, for example, suppose that several buildings stand on a piece of land. The map will be plotted to a scale of 1 in. = 50 ft. An error of ½ ft. in the relative positions of the buildings will hardly show on the map. Measurements made solely for plotting the buildings and the corners of the boundaries could, therefore, be made to the nearest *foot*. On the other hand, if the di-

mensions of the buildings and the lengths of the boundary lines are to be recorded on the map or are to be preserved elsewhere as a matter of record, they should be measured with greater accuracy. Rough measurements will, of course, suffice for *indefinite* details, such as banks of streams or stone walls, and for *temporary* objects, such as walks and driveways.

It is necessary constantly to guard against indefinite or inadequate locations. The location of a point by two intersecting lines may be indefinite if the angle between the two lines is much less than 30°; likewise, the location of a point wholly by two distances may be indefinite. The location of a point may be inadequate because of the omission of a single necessary measurement. The location of a detail as a whole may be indefinite because the location of some point is indefinite, or the location of a detail may be inadequate because too few points have been located. The best way of avoiding the weak location of any detail is to question how its position will be plotted or calculated from the measurements made in the field, and this mental attitude of questioning should be a habitual part of the field work.

353(*a*). PRACTICAL SUGGESTIONS. (1) When a definite object such as a building is located, remember that the work is not completed until the object is measured up, i.e., enough dimensions obtained to plot it. (2) If the two ends of a boundary line are located, it is a good rule to measure the actual length of the line as a check and as a matter of record. (3) Transitman, notekeeper, and chainmen must all cooperate to prevent the too common mistake of measuring to one point of an object and then through carelessness taking the angle to some other point or vice versa. (4) A common mistake in notekeeping is to interchange the angles or the distances to two points so that the angle to one point and the distance to another point are recorded as if they were taken to the same point. (5) In measuring from a transit station to a number of near-by points it is well to have the zero end of the tape at the station. If the tape is not graduated to decimals of a foot throughout, but only the end foot is subdivided, the chainman who is not at the station will simply read the *whole* number of feet; the chainman at the station, pulling the tape taut, notes the decimal of a foot, and after subtracting 1 ft. from the other chainman's reading, gives the final reading. (6) When at any station it is necessary to measure angles to a relatively large number of points of details, take a check sight occasionally to make sure that the lower plate of the transit has not been moved. (7) Keep in mind the relation between angular and linear measurement pointed out in Art. 288. Thus, for example, 1′ is equivalent to only 0.03 ft. at 100 ft. from the transit or 0.3 ft. at 1000 ft.; likewise 1 in. at a distance of 100 ft. from the transit may cause an angular error of 3′, while at 1000 ft. an inch is equivalent to only 15″ (Art. 283(*d*)).

Question. If a corner of a building is, say, 50 ft. from a transit and the building is to be plotted to a scale of 1″ = 100 ft., can the angle to the corner be read from the limb with sufficient accuracy for plotting, or is it necessary to read the vernier also?

354. To Establish Points and Lines for Purposes of Construction. *Staking out work,* i.e., setting stakes for purposes of construction, means to establish points which will be used in locating the principal lines of a structure. This is the reverse of

locating structures which already exist; hence much that applies to the location of details applies, in reverse, to staking out work. In addition to the lines used in locating the structure, it is generally necessary to establish points and lines which will be used in constructing different parts of the structure at the required elevations. Surveying for construction generally involves, therefore, both leveling and transit work. The greatest difficulty that the young surveyor encounters is in knowing what construction lines and what elevations will be needed during construction. This knowledge is best gained from long experience, from conversations with foremen, and from actual observation of what is actually needed in various types of construction. There are, however, general methods and procedures which are more or less common to all surveys for construction. These methods and procedures involve little that is new in the use of instruments or in the fundamentals of leveling and of transit work. They do, however, require a knowledge of certain *practical applications* of fundamentals which the young surveyor should acquire. This whole subject is so extensive and of such importance that it is treated in detail in a separate chapter (Chapter 57, "Surveys for Construction").

355. Field Notes. In the next chapter some of the forms of field notes used in transit surveying are given, and the different methods of keeping notes are discussed. It is not so important *which* method is used as it is essential that the notes be clear and self-explaining. There is little excuse for notes so obscure and confused that only the man who took them can understand them.

356. Errors and Limits of Error. To come within a given limit of error in transit surveying both linear and angular errors must be under control within fixed limits. The most important points brought out in the discussion of errors (Chapters 4, 6, and 20) may be summarized as follows.

1. The true error of any measurement is never known, but it can be estimated from the discrepancy between duplicate measurements or from *probable errors*. The more completely constant errors are eliminated the more nearly can the precision of the work be estimated.

2. Accidental or compensating errors, as a rule, are less important than constant or cumulative errors.

3. Not only should be the methods of eliminating or correcting errors from different sources be understood, but the relative importance of the errors should be known also (Arts. 40, 78, 283(*d*)).

4. Field work requirements corresponding to the adopted limit of error should be kept in mind (Art. 85).

5. Consistent accuracy implies the correlation of linear and angular errors (Art. 288(*a*)).

6. Customary limits of error should be adopted so far as the conditions governing a survey will permit (Arts. 83 and 288).

7. Not all the work of a survey is done with the same precision. For example, an error in the transit lines affects the whole survey, whereas an error in locating a detail may affect only a single point; hence the greater need of precision in running transit lines. *Make the accuracy of each step in the field work correspond to its importance.*

8. Limits of error should not be followed blindly. The best surveyor is not one who does all his work with great precision, but one who does it just accurately enough for the purpose without waste of time and money.

A surveyor soon becomes familiar with the common mistakes in the use of the transit and learns how to avoid them, but only from combined study and experience can he gain a thorough understanding of the sources of error in transit work. Yet a transitman must have this understanding if he is to know what precautions to take and what procedures to follow to attain the degree of precision that is required in any particular survey. Likewise chainmen soon learn how to avoid common mistakes in chaining, but they must also have a thorough understanding of the sources of error in linear measurement if they are to know what precautions to take in order to attain the same required degree of precision. Thus, in any survey, if the whole party works with consistent accuracy, both angular and linear errors can be kept under control within the limits of error adopted for the survey and without waste of time or money in unnecessary refinements.

356(*a*). General formula for error of closure. Let it be assumed that in running a closed traverse a coefficient of precision for chaining has been determined (Art. 82) and that a permissible error in the measurement of a single angle has been adopted (Art. 288(*b*)). What error of closure (Art. 307) may be expected as the result of the combined errors from chaining and from angular measurement? A general formula may be derived as follows.

Let N = number of sides to the polygon, also number of interior angles.
P = length of the perimeter, in feet.
c = coefficient of chaining adopted for one tape length, expressed in feet.
t = length of tape, in feet.
a = permissible error in one angle, expressed in minutes.
E_c = that part of the error of closure due to chaining, expressed in feet.
E_a = that part of the error of closure due to angular measurement, expressed in feet.
E = linear error of closure, in feet.

From Art. 81,

$$E_c = c\sqrt{N} = c\sqrt{\frac{P}{t}} \tag{1}$$

The error for any one side due to angular measurement will be proportional to the length of that side, and hence for the whole perimeter it will be the total angular error times the length of the perimeter. But from Art. 288(*f*) the total error in the sum of the interior angles may be expected to be $a\sqrt{N}$; hence $E_a = a\sqrt{N} \times P$. Since an angular error of 1′ = 0.0003 ft.,

$$E_a = 0.0003aP\sqrt{N} \qquad (2)$$

The error from the combined sources from the theory of errors may be the square root of the sum of the squares of the separate probable errors, Art. 45(4); hence

$$E = \sqrt{(0.0003aP)^2N + c^2\frac{P}{t}} \qquad (3)$$

In ordinary chaining the second term under the radical is often so small that it may be ignored and formula 3 then reduces back to formula 2, in which form it is usually used.

$$E = 0.0003aP\sqrt{N} \qquad (4)$$

Illustration. Suppose that the permissible error for one angle is 15″ or ¼′; that the total length of the perimeter of a ten-sided polygon is 4000 ft. The error of closure should not exceed

$$0.0003 \times \tfrac{1}{4} \times 4000 \times \sqrt{10} = 0.95 \text{ ft.}$$

In most surveys it is decided beforehand what relative error of closure is desired. The practical question then is what should be the values of *c* and *a* in order not to exceed the desired relative error of closure. Though the formulas just given are useful in deciding that question, the precautions to be taken and the methods of procedure to be followed in order to come within a specified relative error of closure have been pretty well determined from common experience in actual field work.

357. Limits of Error for Traverses. The degree of precision attained in running a closed traverse, or a traverse that begins at one accepted control point and ends at another, may be judged by the *relative* error of closure, as explained in Art. 307. Let E represent the linear error of closure, P the length of the perimeter of a closed traverse, and E_r the *relative* error of closure; then $E_r = \frac{E}{P}$. In order to compare the degrees of precision of traverses of different lengths, this ratio is reduced to a fraction the numerator of which is unity (Art. 307). Any desired permissible linear error can, therefore, be specified by stating that the relative error of closure E_r shall not exceed a certain fraction, the numerator of which is unity, such, for example, as 1/3000, 1/5000, or 1/10,000. The greater the degree of precision required the greater will be the denominator of the fraction. *The value of a relative error of closure as a measure of the accuracy attained depends upon the extent to which constant errors have been eliminated* (Art. 307). For this reason the field procedures and the instruments used should be such that constant errors will be reduced to a desired minimum. (See Arts. 358, 358(*c*).)

When a traverse begins at one accepted control point and closes on another, the relative error of closure E_r is usually determined by dividing the linear error of closure E (Art. 309) by the length

of the traverse between the two control points instead of dividing by the length P of the perimeter, which would include the length of the closing straight line between the two control points. This is equivalent to assuming that there has been no error in establishing the two control points, an assumption that is probably contrary to fact. The degree of precision actually attained may, therefore, be somewhat greater than it appears to be from the relative error of closure E_r determined by the usual method just explained.

The accuracy with which the angles of a closed traverse have been measured may be judged by comparing the sum of the measured angles with the theoretical sum. For interior angles this theoretical sum is $S = (N - 2) \times 180°$, in which N represents the number of sides or angles of the polygon (Art. 298(a)); for exterior deflection angles the *algebraic* sum should be $\pm 360°$ (Art. 298(b)). If constant errors are eliminated, the error in the sum of the angles may be expected to vary as the square root of the number of angles (Art. 288(f)).

Let a represent the probable error in the measurement of a single angle and E_s the permissible error in the sum of the measured angles. E_s for any number of angles may be expected to be $E_s = a\sqrt{N}$ (Art. 288(f)); the corresponding *average angular error* would be $\frac{a\sqrt{N}}{N}$. Any desired permissible error in the sum of the interior angles may, therefore, be specified by assigning a corresponding value to a. For example, assume that 30″ is adopted as the probable error in a single angle of a particular traverse; then $E_s = 30''\sqrt{N}$.

When a traverse begins on one accepted control point and ends on another, the error in the sum of the angles is determined as explained in Art. 309.

358. Specifications and Requirements for Field Work. Traverses are commonly classified as *first order, second order, third order,* and *fourth order.* For each order, the *maximum* allowable error of closure is generally specified. Frequently, either the maximum allowable error $a\sqrt{N}$ in the sum of the interior angles or the corresponding average angular error $\frac{a\sqrt{N}}{N}$ is also specified. Sometimes the maximum permissible error in a single angle is specified. (See Arts. 360(a), 360(b).) Thus a *minimum* degree of precision is established for each of the four orders. Any other degree of precision can, of course, be adopted to suit the requirements of a particular survey; in fact, in many surveys it is neither

necessary nor desirable to run traverses with as high a degree of precision as that specified for the lowest or fourth order.

358(*a*). THE MAXIMUM PERMISSIBLE ERROR IN THE SUM OF THE INTERIOR ANGLES of a closed traverse or the maximum permissible total angular error as indicated by the difference between the calculated and the accepted azimuth at the control point upon which a traverse closes is often but not always specified as follows: first order, $4''\sqrt{N}$; second order, $15''\sqrt{N}$; third order, $30''\sqrt{N}$; fourth order, $60''\sqrt{N}$. (N represents the total number of instrument stations or traverse angles. See table in Art. 288(*f*) for permissible errors corresponding to different values of N.) For city surveys the degrees of precision for second and third orders may be somewhat higher than those specified here (Arts. 360(*b*) and 360(*c*)).

358(*b*). THE ALLOWABLE RELATIVE ERROR OF CLOSURE for a traverse that closes, or for one that starts from one control point and closes on another, is often specified for each order as follows: first order, 1/25,000; second order, 1/10,000; third order, 1/5000; fourth order, 1/3000. It is understood that the significance of any one of these limits of error depends upon the extent to which constant errors have been eliminated. The error of closure is computed after the error in the sum of the interior angles has been distributed equally, as a general rule, among all the angles of the traverse. For city surveys the degrees of precision for first and second orders may be somewhat higher than those specified here (Arts. 360(*a*) and 360(*b*)).

358(*c*). ADDITIONAL SPECIFICATIONS. Unless systematic errors are eliminated or greatly reduced in a traverse survey, the computed relative error of closure may not be a true measure of the degree of precision actually attained. (See Art. 307.) There may be serious errors in the direction and length of every side of a traverse, and still the computed relative error of closure may be small. For this reason, it may not be sufficient to specify merely the maximum permissible relative error of closure and the permissible linear and angular errors. It may be necessary to specify also the field procedures and the instruments to be used in order to insure, as far as possible, the required degree of precision in the measurement of the direction and the length of *each* side of the traverse. (Note the procedures and instruments prescribed in the standard specifications for traverses, Art. 360.)

358(*d*). NOTE. A survey must often be made under conditions unfavorable for accurate field work, at a limited cost, and in a relatively short period of time. Under such conditions it is often difficult to eliminate or greatly to reduce systematic errors, particularly in linear measurement. Consequently the degree of precision is not likely to be so great as that attained in theoreti-

cal studies of purely instrumental accuracies under carefully controlled conditions. At best, the effects of combined errors in linear and angular measurements and the relations between these two types of errors are complex; hence there are many conditions affecting accuracy which no formula for the permissible error of closure can take into account. In considering the error of closure such questions as these may arise: Is the degree of precision likely to be greater when sides of a traverse have a mean length of, say, 1000 ft. (or more) than when the mean length is less than 1000 ft., other factors being equal? Is there a relationship of traverse accuracy to the total length of the traverse or only to the mean length of the sides of the traverse? If the transit work in a traverse is of a much higher degree of accuracy than that attained in chaining, as is often the case, will the degree of precision attained in the traverse be practically determined by and approximately equal to the degree of accuracy of the linear measurements. These and similar questions are discussed in a paper, "Relative Angular, Linear, and Traverse Accuracies in City Surveys," by S. W. Bauer, *Transactions, American Society of Civil Engineers,* Vol. 69, p. 577.

359. To Attain a Specified Degree of Precision. When a degree of precision has been adopted for any survey, it is then necessary to decide: What field procedure is to be followed? What equipment is to be used? What shall be the organization and personnel of the field party? It is impracticable to answer these questions in such a way as to attain exactly the specified minimum degree of precision. Consequently, in order to be reasonably certain that in no case will the maximum allowable error be exceeded, it is necessary so to plan the field work that the degree of precision on the average will be greater than that specified. This *margin of safety* between the specified *minimum* degree of precision and the greater *average* degree of precision actually attained may well be 30 to 50 per cent or even greater. The more difficult the conditions under which the field work is carried on, the greater the need for a reasonably large margin of safety.

The precautions to be taken and the methods of procedure to follow to attain a specified degree of precision in field work depend primarily upon the extent to which sources of error must be eliminated. The method of determining the extent to which errors in chaining should be eliminated to attain a required degree of precision is discussed in Art. 85. A similar discussion of the requirements that correspond to different limits of error in angular measurements will be found in Art. 288. This theoretical knowledge of errors in both linear and angular measurements is fundamental in planning to attain a specified degree of precision in running a traverse, but in the application of this knowledge practical difficulties and limitations make it impossible to specify with exactness to what extent each source of error should be eliminated. In the following general statements all numerical values are far from exact; they are intended merely as rough guides and should, therefore, be used with discretion. In every case, however, there is a reasonable

margin of safety, so that if the field work is kept within the limits indicated by the values given for a particular degree of precision, the average degree of precision actually attained should be considerably greater than the specified minimum.

359(*a*). ANGULAR MEASUREMENTS. For a degree of precision of 1/3000 or less, each traverse angle should be read to the nearest minute. (It may be desirable to double the angle as a check.) For a degree of precision greater than 1/3000, traverse angles are generally measured by the method of repetition (Art. 265). The *minimum* number of repetitions recommended are: for 1/5000, two repetitions, one with the telescope direct (*d*) or normal and the other with it reversed (*r*). For 1/10,000, two repetitions, one *d*, the other *r;* in addition, the explement of the angle is measured by two repetitions, one *r* and the other *d*. For 1/25,000, six repetitions, three *d* and three *r;* in addition, the explement of the angle is measured by six repetitions, three *r* and three *d*. (See Arts. 397, 398.)

Transits should be graduated to read directly to 1′ or less for 1/5000; to 10″ or 20″, preferably 10″, for 1/10,000; to 10″ or less for 1/25,000. Accuracy in reading angles depends as much upon the clarity of definition in graduations as upon the least count of the vernier (Art. 728(*d*)). In order to eliminate eccentricity of graduation, both verniers should be read. The precise measurement of angles (first and second orders) is treated in detail in Chapter 27.

359(*b*). LINEAR MEASUREMENTS. *Tapes* should be standardized and retested as often as the importance of the work, the specified degree of precision, and other conditions may require, at least once a week in work of high precision (Art. 95(*c*)). Invar types are recommended for use in first-order traverses.

359(*c*). For a degree of precision of 1/3000, slopes that do not exceed 2 ft. per 100 ft. may be disregarded; greater slopes should be measured with the tape held level, or along the slope and a correction made. For a degree of precision of 1/5000, it is preferable to measure along the slope, and the slope correction should be correspondingly accurate.

359(*d*). TEMPERATURE. For a degree of precision of 1/3000, no correction for temperature is necessary unless there is a variation of at least 15° F. from the standard temperature for the tape; for 1/5000, the variation should not exceed 10° F. For a greater degree of precision, the temperature of the tape should be accurately measured and the temperature correction made. (See Arts. 71 and 95(*g*).)

359(*e*). PULL. For a degree of precision of 1/3000, the pull can usually be kept sufficiently close to standard by estimating it care-

fully. For a degree of precision of 1/5000 or greater, the pull should be measured and kept at standard or a correction made (Art. 74).

359(*f*). Marking distances. For a degree of precision of 1/3000 or less, it is sufficiently accurate to set pins to within the width of the ordinary pin. For a greater degree of precision, distances should be marked more accurately.

359(*g*). Sag. For a greater degree of precision than 1/3000, correction should be made for sag whenever it occurs (Art. 74).

A more thorough analysis of the effect of various errors in chaining on the degree of precision and the precautions which are necessary to attain certain degrees of precision are given in Arts. 84 and 85. Precise linear measurement (first order and second order) is treated in detail in Chapter 8.

360. Standard Specifications and Requirements for Traverses. In the manual *Technical Procedure for City Surveys*, published by the American Society of Civil Engineers, specifications for each of the three highest orders of traverses are given. In the specifications for each order the equipment required, the organization and personnel of the field party recommended, and the field procedure to be followed are treated in considerable detail. The following abbreviated extracts, though only a small portion of this treatment of the subject, will illustrate standard specifications and requirements for traverses of the first, second, and third orders.

360(*a*). First-order traverse

A first-order traverse is logically a part of the triangulation system, its chief function being to furnish additional monumented points along the streets, where they will be more readily available for everyday use than the widely scattered and sometimes inconveniently located triangulation stations. Such a traverse is essentially a series of connected first-order triangulation base lines, intercepting angles measured by triangulation methods. The coordinates of traverse monuments are actual bases for map work and departmental and private surveys.

A first-order traverse may sometimes serve as the principal horizontal control, independent of triangulation. However, this is usually advisable only for small areas where triangulation is unduly expensive because of topographic conditions.

Specifications.

1. The location of first-order traverse lines and monumented stations shall be determined by a thorough field reconnaissance. In general, the distribution of permanent monuments shall be at the rate of at least one pair of monuments (a "pair" to be considered as two intervisible monuments, spaced 500 to 1000 ft. apart) to each linear mile of traverse. Intermediate stations, although considered as semi-permanent, shall be marked and referenced for future recovery.

2. All first-order traverse lines shall start from, and close upon, stations of a triangulation system surveyed in accordance with the specifications given herein, or shall be run in closed circuits.

3. For acceptable final results the following rules shall be observed.

a. The tapes used shall be tested frequently for absolute length, either at the National Bureau of Standards, or locally against a standard of acceptable accuracy. The coefficient of expansion of each tape shall also be determined by the Bureau of Standards.

b. All original field distances shall be corrected for absolute length, tension, sag, temperature, inclination, and the adopted datum plane.

c. The average angular error between accepted azimuths, or as computed from circuit closures, shall not exceed $\frac{4''.0\sqrt{N_i}}{N_i}$ per instrument station, and the maximum error shall not exceed $\frac{6''.0\sqrt{N_i}}{N_i}$ per instrument station, N_i being the number of instrument stations on the line.

d. The average position closure, after distribution of azimuth errors, shall not exceed 1 in 35,000, and the maximum shall not exceed 1 in 20,000.

e. In the case of a traverse controlled by triangulation, the final results shall be adjusted by the method of simultaneous weighted junction points. When the traverse net is without a triangulation control, the adjustment shall be made by the polygon method, or by the method of least squares.

Equipment. A theodolite of the repeating type reading directly to 10" or less; three tested tapes (preferably invar tapes), one to be kept in the office to check the lengths of the field tapes, or to check the length of the local standard tape; portable tape supports or "bucks"; spring balance; tape pullers or stretchers; several good tape thermometers, one or two of which should be tested at the National Bureau of Standards; small engineer's level and a leveling rod.

Field Procedure. Angle between the backsight and foresight turned clockwise from the backsight; one set of twelve repetitions. The procedure is identical with that described for triangulation observations (Art. 412); the circle reading after the twelfth repetition should preferably agree with the initial reading to within 30". The two mean angles of the two sets of six repetitions should agree with the other within 5", and the average value of the two is the final value of the angle. The field procedure in first-order chaining is similar to that for triangulation base-line measurement, except that generally each course is measured only once (Art. 100).

360(*b*). Second-order traverse

The principal purpose of the second-order traverse in a city survey is to furnish coordinates, bearings, and distances for the street monuments in the property survey. Its average accuracy of 1 in 15,000, or better (1 : 15,000 is sufficient for this purpose), makes it less expensive than a first-order traverse.

Specifications.

1. All lines of a second-order traverse, used to supplement triangulation or a first-order traverse, shall originate with and close upon stations of the triangulation system, or of the first-order traverse, executed according to the specifications given herein, or shall be run in closed circuits.

2. For acceptable final results, the following rules shall be observed.

a. The tapes used shall be tested frequently, for absolute length, by comparing them with a local standard.

b. All original field distances shall be corrected for absolute length, tension, sag, temperature, and inclination.

c. The average angular error shall not exceed $\frac{10''.0\sqrt{N_i}}{N_i}$ per instrument station, and the maximum shall not exceed $\frac{15''.0\sqrt{N_i}}{N_i}$ per instrument station on the line.

d. The average position closure, after distribution of azimuth errors, shall not exceed 1 in 15,000, and the maximum shall not exceed 1 in 10,000.

e. Final results shall be adjusted by the simultaneous junction-point method.

Equipment. A transit or theodolite, graduated to read directly to 20" or less, with two opposite verniers, and provided with a telescope level for the determination of the elevation of the tape ends. The equipment for chaining is practically the same as that for first-order traverse.

Field Procedure. With a theodolite graduated to read directly to 10" four repetitions are usually sufficient. The angle is turned clockwise from the backsight to the foresight. One repetition with the telescope direct and one with the telescope reversed, and both verniers are read. Next two repetitions

are turned clockwise from foresight to backsight, one with the telescope reversed and one with it direct. With an instrument graduated to read directly to 20", eight repetitions may be required.

The chaining procedure is similar to that described for first-order chaining, and the difference in the accuracy between the two procedures is a matter of refinement in calibration of equipment rather than in field operations. (See Art. 99, "Second-Order Chaining.")

360(*c*). Third-order traverse

A third-order traverse is used as a control for topographic details, especially in the outlying undeveloped areas where control by higher-grade triangulation and traverse is comparatively inadequate. To serve this purpose, its accuracy need not exceed that of plotting on the topographic map.

Specifications.

1. All lines of third-order traverses, used to supplement the triangulation and first-order or second-order traverses, shall originate with and close upon stations of these surveys, executed according to the specifications given herein, or they shall be run in closed circuits.

2. Criteria for acceptable final results shall be as follows.

a. The average angular error shall not exceed 15" per instrument station.

b. The maximum position closure, after distribution of azimuth errors, shall not exceed 1 : 5000.

c. The final adjustment of the results shall be made by the junction-point method or by the polygon method.

Equipment. An engineer's transit graduated to read directly to 1' or less; a 100-ft. steel tape, tested for absolute length on the local standard of length; a tension scale; a hand level, and a thermometer.

Field Procedure. The angles of a third-order traverse are measured by repetition just as in first- and second-order traverses, except that two repetitions will usually give the required accuracy. The angle is turned clockwise from backsight to foresight, once with the telescope direct and once with the telescope reversed. Both verniers are read, first at the initial pointing and again after the second angle is turned. For a check on the mean angle, the A vernier is read after the first pointing on the foresight rod.

With a transit equipped with stadia hairs the taped distance may be checked within 1 ft. or 2 ft. by reading the stadia distance. In this case, a stadia or level rod is a necessary part of the angle-measuring equipment.

The chaining is done in the ordinary manner, using a hand level to level the tape. It is sometimes advisable, however, to use a tension scale for securing proper tape tension, and to record approximate temperatures for corrections. In extreme cases it may also be desirable to note conditions of support and sag for each individual tape length, for subsequent computation of sag corrections.

When the traverse route runs over excessively steep grades, and horizontal taping becomes too expensive, excellent results may be secured by paralleling the tape to the slope of the ground, and measuring the vertical angle on a line approximately parallel to the slope of the tape. This method is much faster than leveling to determine the grade of the tape, and the application of the inclination correction is very simple. As the distance along the slope is usually 100 ft., the horizontal distance is the cosine of the vertical angle with the decimal point moved two places to the right.

361. The Duties of the Different Members of a Transit Party depend upon the number of men in the party. In class work, if the party is composed of five men the duties may be outlined as follows.

361(*a*). The chief of party plans the survey and directs the field work. Oftentimes in addition he acts as transitman or as notekeeper. He should understand thoroughly the advantages and disadvantages of the different methods of running traverse lines (Chapter 21; also Arts. 345–346) and the best field procedure for

the method selected (Arts. 348–351). He should be able to decide where to run traverse lines to the best advantage (Art. 345) and to choose the best method of locating any particular detail (Chapter 22; also Art. 353). He should have a thorough understanding of the sources and relative importance of errors in chaining and in transit work and should know how to eliminate errors from different sources (Chapters 4, 6, and 20; also Art. 356). He should be familiar with the customary limits of error for different orders of traverses and should know how to attain a specified degree of precision (Arts. 358-360(*c*)).

361(*b*). THE NOTEKEEPER often acts as chief of party, and in addition to directing the field work keeps the field notes in the party notebook. In class work, during the early part of the course *he should also read each angle as a check on the work of the transitman.* Read Chapter 3 for general directions and consult Chapter 24 for the different forms of transit notes. *Special Suggestions for Class Work* are given in Art. 33. The position of notekeeper is a responsible one, and upon his alertness and his ability to keep good notes will depend to a large extent the value of the work done by the whole party.

361(*c*). THE TRANSITMAN is responsible for all work done with the transit; also responsible for any injury done to the instrument either while in use or while being carried from one place to another. *The transitman must not leave the transit standing unguarded.* The use of the transit is explained in Chapter 19. Particular attention is called to the following articles.

1. *Care of the Transit,* particularly precautions to prevent injury (Art. 243).
2. *Setting up the Transit,* directions and suggestions (Art. 246).
3. *Use of the Telescope,* focusing eyepiece, getting line of sight quickly, etc. (Art. 248).
4. *Use of Clamps,* warning against using wrong clamps (Art. 250).
5. *Lining In,* sighting chainmen or flagman in line, *setting transit stations* (Art. 251).
6. *Method of Double Reverse,* to prolong a straight line accurately (Art. 252).
7. *Reading Angles,* use of vernier and common mistakes (Arts. 262–263).
8. *Measuring Horizontal Angles,* methods, suggestions, sources of error (Arts. 257–261).
9. *Measuring Angles by Repetition,* especially doubling an angle (Arts. 265–266).
10. *Measuring Angles by Series* or "sets" by the "direction method" (Art. 268).
11. *Measuring Deflection Angles,* three different methods (Art. 269).
12. *To Establish a Line at a Given Angle,* different methods of laying off an angle (Art. 270).
13. *To Measure a Vertical Angle,* general method; also by means of gradienter (Art. 271).
14. *The Sources and Relative Importance of Errors in Transit Work* (Chapter 20).
15. *To Read a Bearing* (Art. 279).

16. *Special Field Operations,* balancing in a transit; finding the intersection of two lines; referencing a point; sighting parallel to a given line (Arts. 272–278).
17. *Basic Problems in Transit Surveying* (Chapter 25).
18. *Methods and Suggestions for Running Traverses* (Arts. 346–347).
19. *Methods of Procedure in Running Traverses* (Arts. 348–352).
20. *Methods of Locating Details* (Chapter 22; also Art. 353).
21. *Methods of Checking Traverses* (Arts. 307–309).
22. *Limits of Error and Customary Degrees of Precision* (Arts. 356–360).
23. *Care and Adjustment of the Transit.* (Chapters 54 and 55).

In addition to the knowledge required in ordinary transit work, the transitman should acquire the knowledge which will enable him to act as chief of party. Without such knowledge, he cannot cooperate to the best advantage with the chief of party; with it, he can act as chief of party whenever it is desirable for him to do so.

361(*d*). THE CHAINMEN are responsible for the measurement of all distances. In addition they may be called upon to give sights, drive stakes, assist in setting up the transit, and to help in other ways to carry on the field work. They are also held responsible for any injury to the tape. Directions for chaining are given in Chapter 5. Special attention is called to the following articles.

1. *Care of the Steel Tape,* to prevent bends, breaks, and rust (Art. 52).
2. *To Do Up and Undo a Tape,* in a figure 8 (Arts. 53–54).
3. *General Methods of Chaining* (Arts. 56, 57, and 59).
4. *Common Mistakes in Reading Tapes* (Art. 58).
5. *Chaining on a Slope* (Art. 60).
6. *Sources of Error,* methods of correction and elimination (Chapter 6).
7. *Relative Importance of Sources of Error* (Arts. 78, 79).
8. *Customary Limits of Precision,* and the field-work requirements corresponding to different limits (Arts. 83–85).
9. *Summary*—of things which the chainmen should do and should not do (Art. 63).

361(*e*). THE AXEMAN OR FLAGMAN drives the stakes for transit stations, marks the guard stakes and sets them, holds the transit pole or pencil for backsights and foresights, and clears the way through underbrush, when necessary, for the line of sight.

The flagman should be thoroughly familiar with the system of signals (Art. 334), and should know how to cooperate with the transitman in such routine work as sighting a point in line (Art. 251), prolonging a straight line (Art. 253), and othei common field operations which involve giving sights.

Suggestions.
1. For the method of establishing a transit station, read Art. 336.
2. For the method of marking and setting a guard stake, read Art. 338.
3. In holding a sight pole be sure that it is *plumb.* Stand squarely behind it, and using both hands, suspend it so that it will swing freely until it becomes plumb, then keep it there. Oftentimes the vertical edge of a building or some other fixed vertical line will aid in keeping the pole plumb. Keep in mind how great an error in angular measurement can occur from sighting on a pole which is not plumb, or on one that is not held exactly on a point (Art. 283(*d*)).
4. There is a knack in driving a stake plumb; acquire it. Use a broad flat-headed hand axe, and strike squarely to prevent brooming. When a stake begins to get out of plumb do not bring it back by striking one side of it, but "draw" it over by striking it on the head with a peculiar blow, in which the

axe head moves towards the body as it hits the stake. If the stake is not too far down, pull it over with one hand while using the axe with the other.

Remark. Surveyors complain that axemen are either careless in driving stakes or that they do not know how—a constant source of annoyance.

5. If a backsight or foresight is wanted at a station where another transit is set up, hold a handkerchief behind the plumb-bob line, keep the bob from swinging, and see to it that members of the other party are not in the line of sight. Another way of giving a sight in such a case is to hold a pencil on top of the center of the transit telescope.

6. Do not "come in" from giving a sight until signaled to do so.

CHAPTER 24

TRANSIT SURVEYING

FIELD NOTES

In the first part of this chapter three general methods of keeping the field notes for a transit survey are discussed, and the essentials of good transit notes in any form are pointed out. In the second part of the chapter several methods of keeping notes are illustrated by reproductions of actual field notes. A study of these notes should also result in a clearer idea of how the methods of running traverses (Chapter 21) and of locating details (Chapter 22) are applied in actual field work. The comments accompanying the illustrations are intended to lead the student to a comparison of the different methods of field work and of keeping notes so that ultimately he will be better prepared to choose methods for himself, modify them, combine them, or devise new ones especially adapted to any work he may have in hand.

362. Different Methods of Keeping Field Notes. There are many forms of field notes in common use, some adapted to one kind of field work, others to other kinds. The best method of keeping notes for a given piece of work is a matter of opinion, and an experienced surveyor is likely to use his own method—one which is "the survival of the fittest." Any system may be used which results in a clear and accurate record of the field work without wasting time. A form of notes, however, which saves time in the field but wastes it when it comes to plotting is not, ordinarily, an economical form to use, although occasionally it may be.

It is well to use a system which can be modified to suit different methods of field work. Thus, for example, in the second part of this chapter there is a form of notes which not only can be adapted to any one of the four methods of running transit lines, but can also be easily modified to suit different kinds of surveys, such as a stadia survey, a compass survey, or a transit survey. Having once become familiar with some such method of keeping notes, it is an easy matter to adapt oneself to any other method.

363. Notes Pertaining to the Traverse Lines should be distinguished from all other notes because the traverse lines are usually plotted first and details given afterward. The field notes may be divided into two parts, viz.: (1) those pertaining to the location of transit stations, and (2) those pertaining to all other points. When traverse lines are run before details are located, the traverse line notes may be kept by themselves in compact form. When, however, details are taken at each transit station as soon as

the station is established the traverse line notes are not kept separate, but are scattered along mixed with other notes. In that case the notes should be kept in such a way that those pertaining to the traverse lines can be picked out without any trouble when it comes to plotting. One way is to *letter* all transit stations and *number* all other points. Another way is to enclose transit stations in a triangle or circle. Still another way is to keep traverse line notes in a separate column by themselves. (See Form 3, p. 354.)

364. Three General Forms of Notes. There are two methods of keeping notes which are radically different, viz.: (1) the method in which *all* measurements of angles and distances are *indicated on a sketch;* (2) the method in which no sketch is used, but measurements are recorded in some tabular form. The latter method can be used only where comparatively little description of the survey and its details is necessary. (3) A third method is a combination of the other two in which some of the data are recorded on a *sketch,* and the remaining data are *tabulated.* This combination method is, perhaps, the one most used.

365. The Tabulated Form of Notes without a Sketch can be used to advantage for traverse-line notes when kept by themselves, or for recording a number of measurements to points or objects, all of which can be described just as well without a sketch. In most cases, however, a sketch is the easiest way of making clear the details of a survey. A number of tabulated forms are illustrated in Art. 374.

366. The Sketch Method for Field Notes. When it can be done without crowding, there is probably no better way of keeping field notes than by making a freehand sketch with all angles and distances entered directly upon the sketch. The difficulty with this method is that a sketch made on the page of an ordinary field notebook will often be too small to provide for all the measurements in large, clear figures. It is then better to resort to the *combination* method.

It is not customary to draw the sketch to scale; indeed it is usually necessary to exaggerate certain portions of it to make room for recording measurements. In some forms of notes no attempt is made to sketch outlines as they appear on the ground. For example, a crooked highway may appear in the sketch as perfectly straight. (See Illustration VI, p. 373.)

367. The Combination Method of Keeping Field Notes, in the opinion of many surveyors, is the best all-round method. A sketch can be made on one page and such measurements as cannot conveniently be placed on the sketch itself can be recorded on the

opposite page in some one of the tabulated forms in common use. As far as possible it is well to put actual dimensions and offsets on the sketch, whereas angles and the corresponding distances used for locating points are put on the opposite page. For example, if a house is shown in the sketch it is usually well to put dimensions of the house on the sketch itself, whereas angles and distances taken to locate the corners of the house are given on the opposite page. (See Illustration III, p. 365.)

Points on the sketch are *numbered or lettered* when it is necessary to refer to them in the tabulated form on the opposite page. When it can be done readily it is well to number points *in order;* for example, the first one sighted at as the telescope is turned from the backsight is 1, the next 2, and so on (Fig. 319(*a*), p. 304).

368. When Should the Notes Read from the Bottom of the Page Up? In railway surveys or highway surveys or surveys of a similar kind it is often more convenient to record the notes *from the bottom of the page up.* This is especially true if the *sketch* method is used; otherwise the notebook must be held upside down to bring the points in the sketch in accord with the points which they represent on the ground. The general rule in making a sketch is to hold the book with its top toward the next transit station, and if a tabulated form on the left-hand page is used in connection with the sketch, this will read from the bottom *up* to correspond. (See Illustration V, p. 370.) On the other hand, in the notes of a closed survey, where the entire sketch is on one page, it may be more convenient to tabulate the corresponding notes on the opposite page, so that they will read from the top *down.* (See Illustration I, p. 358.)

369. What Should the Field Notes Include? In taking notes it is well to keep in mind (1) what will be needed for plotting, and (2) what it may be well to have in the notebook even though it may not be necessary for plotting. Measurements of angles and distances, sketches, and explanatory notes are of use in plotting, whereas the date of the survey, names of the men in the party, transit used, weather conditions, and notes indicating methods of work may be of value as a mere matter of record. Common sense should guide one in this.

In ordinary work, for example, the temperature is not recorded because there is no need of it, but in precise surveying it *is* recorded because corrections for temperature are to be made. A note which gives the methods used in a survey may indicate the limits of error, and if there is a chance that such information will be needed, it should be given in the notes, but usually it will not be

needed. On the other hand, the title of the survey, the date, and the name of the man in charge of the party should appear in the notes of almost every survey.

370. Allow Plenty of Room; Do Not Crowd the Notes. Make sketches larger, if anything, than may seem necessary. Usually there will be more dimensions to put on the sketch than will at first appear; make a generous allowance for this. It is particularly important that those portions of a sketch where most of the measurements are to be shown should be large and open so that small crowded figures may be avoided. Buildings, for example, should be sketched large enough to provide for dimensions. If a fence is located by offsets from a building, make the fence in the sketch far enough away so that the offsets may be recorded in clear figures, no matter how near to the building the fence really is. Do not try to put too much on one page; use several pages if necessary for different parts of a sketch. *Notebook paper is cheap compared with time wasted in deciphering crowded notes.*

In tabulated forms it is better to skip every other line when there is room, thus leaving space for corrections.

371. Repeating Notes on Different Pages. When tabulated notes extend over two or more pages it is customary to repeat on one page enough of the notes from the preceding page to make the whole continuous. Usually this means simply copying the preceding transit station (number or letter) and, perhaps, a traverse-line angle and distance.

372. General Sketch of Traverse Lines. When a survey is more or less extended and the traverse-line notes are scattered through several pages, it is a good plan to start a sketch of all the traverse lines on one of the first pages. This sketch, which should be completed as fast as the field work progresses, will show the relative positions of the different transit stations and will prove especially useful if there is a more or less intricate network of traverse lines. It is not necessary to enter any measurements on this sketch, at least until the notes are worked up, preliminary to plotting.

373. Five Things to Make Clear in Any Form of Transit Notes.

Linear Measurements. (1) Between which two points each measurement was taken.

Angles. (2) At which station the transit stood when an angle was measured.

3. Which station or point was used for a backsight.
4. In which direction (right or left) the angle was measured.
5. To which point as a foresight the angle was taken.

Illustration. A point 2 in a traverse survey was located from a traverse line *AB* by an angle of 45° and a distance of 30 ft. In order to plot the point 2, the notes must make clear whether the angle was taken at *A* or at *B*. If taken at *B*, then *A* must be entered as a *backsight*, the *direction* in which the angle was measured must be understood, and finally 2 must be recorded as a *foresight*. Should the notes fail to make clear any one of these points, a serious mistake might be made, as illustrated in Art. 289. It is especially irritating to find an angle or distance given in the notes and yet not be able to plot it.

If the sketch method is used and all distances and angles are indicated on the sketch exactly where they belong (see Illustration IV, p. 367), misunderstandings are not likely to occur in plotting—a strong argument in favor of the sketch method. In all tabulated forms the five points noted above must be made clear by the column headings, by the system of notation, and by explanatory notes and sketches, if necessary.

In an office where the methods of keeping notes and of taking angles are always the same, much in the notes may be taken for granted. For example, if it is customary to measure all angles clockwise, it is not necessary to record the direction in which angles are measured. It is well, however, to print once for all, near the beginning of a set of notes, some such note as: *"All angles taken to the right from a backsight on the preceding station."* (See Illustration I, p. 358.)

Suggestions. In order to make sure that everything in the notes is clear and unmistakable, try to put yourself in the position of a man in the office who is not at all familiar with the ground covered by the survey, and as you take the notes examine them critically from his point of view, keeping in mind what the notes will be used for and asking yourself continually the question: "Will this be clear to the man in the office?" Even though you expect to plot the notes yourself, remember that things which seem perfectly clear to you while you are recording the notes may become very obscure and indefinite by the time you begin plotting them. It is a poor notekeeper who can plot his notes only while they are "hot."

Reread, from time to time, the general suggestions for keeping notes given in Chapter 3.

Study the forms of field notes given in the next article, noting the advantages and disadvantages of each method. Study also the actual field notes reproduced on pp. 358–375, noting carefully all the little devices and methods for making the notes clear and unmistakable. The best way to study these notes is to plot them.

374. Different Forms of Tabulated Field Notes. When the usual form of field notebook is used, it is customary to keep the tabulated notes on the left-hand page, which is ruled in six columns for that purpose. Sketches, if there are any, are made on the right-hand page, which usually is cross lined and can be used as plotting paper if desired. Several of the forms for tabulating notes are given here. In most cases each form would be accompanied by an explanatory sketch on the opposite page, but for the present these sketches are omitted. The column headings and two or three lines of notes for each form will illustrate the different methods. The style of lettering used in these illustrations and in the reproductions of the field notes which follow is the free-hand, single-stroke system explained in Art. 29. Those forms which read from the top of the page down may easily be modified to read upward if desired.

FORM 1

Angle	Value	Line	Length		
BCD	89°29'	CD	280.1		
BC2	112°14'	C2	83.8		
		3-4	40.0		

Interpretation. Backsight at station *B*, transit at station *C*, foresight at station *D*, angle = 89° 29', and distance from *C* to *D* = 280'.1. Same backsight, same transit station, the angle to some point 2 (not a transit station) is 112° 14', distance from *C* to 2 = 83'.8. No angles taken to points 3 and 4, distance between them is 40'.0. In this instance *letters* are used for transit stations, numerals for all other points. Corresponding letters and numerals on a sketch (not shown here) on the opposite page of the notebook complete the notes. All angles are taken to the right. For a more complete illustration see Illustration I, p. 358.

The method of measuring an angle is not self-explanatory. A note should make clear once for all that the middle letter is the vertex, the first letter the backsight, and the last letter the foresight. This order is easily remembered since the backsight (first letter) is the *first* point sighted at and the foresight (last letter) is the *last* point sighted at in measuring an angle. It should also be noted that all angles were taken to the right.

Advantages. A simple and natural method of tabulating angles and distances. Easy to use in the field and not likely to be mis-

understood in plotting. When a system of traverse lines is established first and objects located afterward, this form is a very good one for the traverse-line notes, especially if there is a network of traverse lines.

Disadvantages. When stations are numbered instead of lettered the method may become cumbersome, especially if there are *plus* stations. It would be awkward, for example, to give an angle thus: (21 + 14) (24 + 30) (30 + 25).

FORM 2

Transit at	Sta. C,	backsight	on Sta. B		
Foresight	Angle	Line	Length		
D	89° 29′	CD	280.1		
2	112° 14′	C2	83.8		
		3-4	40.0		

Interpretation. This form differs from the preceding form in only one particular: the station at which the transit stood and the backsight are noted once for all at the head of each set of angles. The third column can be and often is omitted.

When magnetic and calculated bearings are kept, the last two columns may be used for that purpose, as shown in Form 4. As in the preceding method it should be noted that all angles are taken clockwise from the backsight. Should it be desired to take angles right and left, two columns headed "Right" and "Left" respectively may replace the single column shown (see Form 5), or else R or L may be written after each angle.

Advantages. When a number of angles are taken at each station this form has, in addition to the advantages of the preceding form, the merit of being self-explanatory: no doubt can arise as to which station is the backsight, which the vertex, and which the foresight. Moreover, the form can be used for plus stations. This form, or some modification of it, is a very common one. It is especially useful in topographic surveying, where a large number of side shots are taken at each station, and where vertical angles are measured. By omitting the third column three columns are available for recording vertical angles and corrected elevations or distances.

Disadvantages. When only one or two angles are taken at each station, it becomes necessary to reprint the words *"Transit at Sta. —, backsight on Sta. —,"* on almost every other line. In that case Form 1 is more compact and convenient.

FORM 3

Angle	Distance	Transit Line	Angle	Distance	
		△			
		Sta. D			
		280.1			
4=310°18'	44.6	89° 29'	1=40°30'	28.1	
		△			
3=211°11'	40.3	Sta. C	2=112°14'	83.8	
		320.1			
		108°12'			
		△			
		Sta. B			
		510.6			

Interpretation. The notes are to be read from the bottom up. All notes pertaining to the traverse or transit line are put in a column by themselves, each *distance* inserted *between the two stations* where it was measured, each *angle* directly *over the station* where the transit stood when the angle was taken, the backsight being the preceding station *below*, the foresight the next station *above*, thus: backsight at station *B*, transit at station *C*, foresight on station *D*, angle = 89° 29′, distance from *C* to *D* = 280′.1.

When at any station side shots are taken to points not on the traverse line, the corresponding angles and distances are entered near the point in the notes that represents that station. Thus points 1, 2, 3, and 4 were located by angles taken at *C* from the same backsight on *B*, and by distances measured from *C*. An explanatory sketch on the opposite page is implied in this case.

Remark. If it is assumed that all angles were measured clockwise, the angle 40° 30′ indicates that point 1 is on the *left* of the line *CD*. It would be an improvement, therefore, to enter 40° 30′ and 28.1 ft. in the *left-hand* columns; then, to be consistent, the measurements corresponding to the points 2, 3, and 4 should be entered in the *right-hand* columns. This would be the relative positions of the four points if the measurements were placed directly on a sketch. (See Illustration VI, p. 373.)

Advantages. Traverse-line notes are *separate* from other notes. In a survey progressing in one general direction as, for example, the survey of a highway, this form is useful, especially if combined with a sketch (as in Illustration VI, p. 373).

Disadvantages. Notes are not self-explanatory, more or less complicated, and not adapted to surveys where only one or two points are located from each station.

FORM 4

Angle	Value	Line	Length	Bearing Mag.	Cal.
	2)198°44′				
CDE	99°22′	DE	410.8	N26°30′E	26°24
	2)148°53′				
DEF	74°26′30″	EF	328.5	N79°15′W	79°9′30″

Interpretation. This form is a modification of Form 1. Two additional columns provide for magnetic and calculated bearings. Thus, N 26° 30′ E and N 26° 24′ E are the bearings of the line *DE*. The notes indicate that the angles were doubled by repetition, every other line being left blank to provide space for the arithmetical work of division. Notes read downward.

Frequently the first reading or single value of an angle is entered as well as the double value. The third column may be omitted if it is understood that the corresponding letters in the first column represent the line from the station to the foresight. Frequently the only letters entered in the first column are those shown in the third, it being understood that the backsight was on the preceding station. For example, if in the first column *DE* is entered instead of *CDE* and *EF* instead of *DEF*, *D* and *E* still represent transit stations, and the backsight for an angle taken at *E* would be understood to be on the preceding transit station *D*.

The *advantages* and *disadvantages* are the same as for Form 1.

FORM 5. DEFLECTION ANGLES

Sta.	Deflection Left	Deflection Right	Line to	Length	
E	56°24		F	379.4	
E		41°18′	1	38.3	
D	28°10′		E	220.6	

Interpretation. One of the forms used when *deflection* angles are taken to the left or right. Notes read from the bottom up. For example, backsight at *D*, transit at *E*, deflection angle to the right of 41° 18′ to point 1, distance from *E* to 1 = 38′.3.

The fourth column may be combined with the first to read *EF*, *EL*, and *DE*, it being understood that the first letter is the transit station, that the second letter is the foresight, and that the back-

sight is on the preceding station as explained above in connection with Form 4. This modification is very common in practice.

Advantages. Right and left deflection angles are in separate columns.

Disadvantages. Two columns are used for angles instead of one. The extra column is often needed for another purpose. The mistake most likely to occur in using this form of notes is that of entering an angle in the "Left" column when it should be in the "Right" column, or vice versa.

FORM 6. DEFLECTION ANGLES

Station Occupied	Point Sighted at	Deflection Angle	Distance	Bearing Mag.	Cal.
+20.6	a	41°18'L	38.3		
+20.6	36	56°24'R		N36°30'E	36° 25'
32.	32+20.6	28°10'L		N20°0'W	19°59'
30	32	12°11'R		N8°15'E	8°11'

Interpretation. This form is similar to the preceding form, except that deflection angles are kept in one column instead of in separate columns, the letters L and R denoting whether they are taken to the left or to the right. Notes read upward.

When no side shots are taken, i.e., when the notes pertain only to traverse lines, the fourth column may be omitted and often the second.

Advantage. A column is saved for some other purpose. Thus, for example, two columns are left in which magnetic and calculated bearings may be entered separately if desired.

Disadvantage. If the letter R or L is omitted in any case it may cause trouble in plotting. This is more likely to occur than the mistake of entering the angle in the wrong column, in the preceding form.

FORM 7. AZIMUTHS

Line	Azimuth	Length	Mag.		
CD	190°10'	281.6	S10°15'W		
DE	280°48'	310.8	N79°15'W		
D2	291°18'	28.9			

Interpretation. This form is a modification of Form 1, azimuths being recorded in place of angles. Magnetic bearings are also recorded as a rough check on the azimuths. The notes are read downward.

Calculated bearings are frequently entered in the fifth column, and sometimes an additional column is used for points sighted at.

When many azimuths are to be recorded at each station, a modification of Form 2 may be used, the words "Transit at Sta. —" being placed above each set of azimuths. Backsights, of course, are not recorded in the notes.

Advantages. This form of notes is so simple and self-explanatory that it (or some slight modification of it) is in general use for the azimuth method. It is especially useful for stadia notes when a large number of points are to be located. (See Art. 434.)

Form 8. Bearings

The form used for notes when traverse lines are run by bearings (Art. 304) may be a modification of Form 7, simply substituting a column of bearings as read from the vernier for the column of azimuths, and keeping the other three columns as they are. (Art. 215(*a*)).

375. Modified Forms of Field Notes. Although the forms of notes for transit surveying thus far given are more or less standard, they can be and are modified in many ways to suit the personal preference of the surveyor or to meet a special need. For example, many engineers prefer a different order of columns, particularly one in which, reading from left to right, lengths come first, usually in the first or second column. A modified form of notes is necessary when both the A and B verniers are read, as illustrated on p. 405; or when vertical as well as horizontal angles are measured, as illustrated in Art. 434. Special forms are required when each angle is measured in "sets" either by repetition or in series. (See Arts. 398, 412(*a*), and 413(*a*).) These examples merely suggest the great variety of notes for transit surveying that are in common use, but practically all forms are based on those already given.

376. Illustrations of Field Notes. In the following pages several sets of field notes are given; they illustrate not only different methods of keeping field notes, but also different methods of field work. The size of a page in the ordinary field notebook is somewhat wider and longer than the pages shown in the illustrations. Hence the notes are shown correspondingly reduced, but in other respects they are like the original field notes. They should be studied from two points of view: (1) to see how the methods of Chapters 21 and 22 for running traverses and locating details are applied in actual field work; (2) to gain a clearer idea of how field notes are kept.

The student is urged to study the notes and comments in the order in which they are given and with unusual care. They have been chosen and arranged with the hope that this may prove one of the most useful portions of the book. It must be borne in mind, however, that for every method of running a traverse line or locating a point given in the notes, there is probably another method equally good which could have been used instead. Moreover, the methods shown of keeping notes can be and are modified and changed to suit the work in hand, and the remarks in Art. 362 should be kept in mind.

It is suggested that a most effective way of studying the different illustrations is actually to plot the notes. If this can be done before going into the field, it will be of great assistance to the student not only in keeping notes but also in choosing methods for locating details.

ILLUSTRATION I

18 Angle	Value	Line	Length	Mag.	
ABC	101°05'	BC	376.9	N 9°15'E	
AB2	152°51'	B2	11.4		
BCD	73°47'	CD	338.3		
BC3	235°42'	C3	27.1 ~~127.1~~		101°05'
CDA	96°47'	DA	339.8		73°47' 96°47'
CD4	207°29'	D4	43.3		88°21'
DAB	88°21'	AB	274.2	Check→	360°00'
DA1	312°21'	A1	8.1		

All angles to the right from a backsight on preceding station.

ILLUSTRATION I

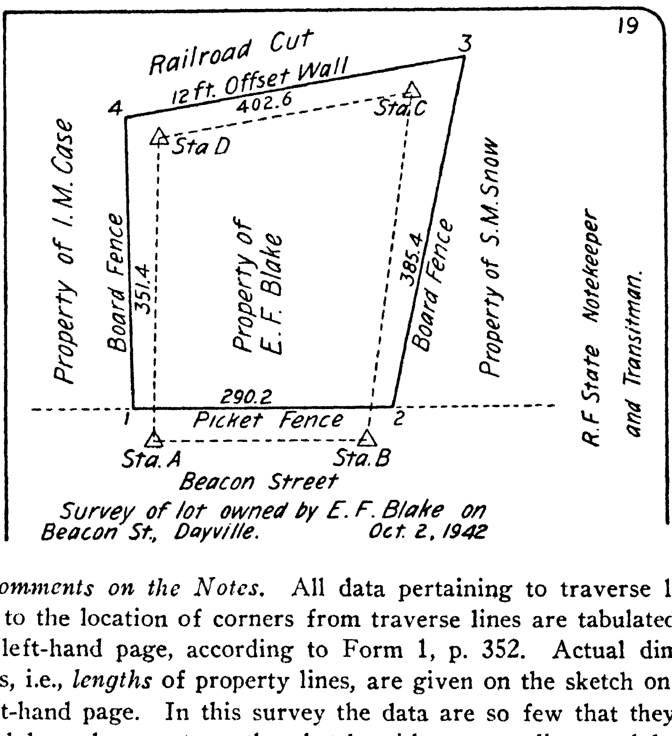
Comments on the Survey. This is an illustration of a survey to determine the boundary lines of a building lot. Traverse lines are run at random within a few feet of the fence lines, but not parallel to them. There are four transit stations, and each corner of the lot is located from the nearest station by angle and distance. Notice that in addition to traverse-line measurements the length of each boundary line was also measured, although a map of the lot could have been plotted without these four measurements. It is always best, when practicable, to measure boundary lines directly. The four angles at the corners of the lot were not measured—they can be easily calculated. See Art. 345(*d*).

Since angles were not doubled there was apparently no check on their values until the last or closing angle was measured. The fact that the sum of the interior angles is exactly 360° does not

mean that there is no error in angular measurement, but that probably there is no serious error, and that the small errors compensate each other.

Though the survey is complete in itself, no one line seems to be referenced with respect to permanent objects. Consequently, if any three of the four corners should become obliterated, it would be difficult to restore them.

Illustration I



Comments on the Notes. All data pertaining to traverse lines and to the location of corners from traverse lines are tabulated on the left-hand page, according to Form 1, p. 352. Actual dimensions, i.e., *lengths* of property lines, are given on the sketch on the right-hand page. In this survey the data are so few that they all could have been put on the sketch without crowding, and hence the sketch method (see the next illustration) would have been the natural one to use. Notice that the mistaken entry, 127.1 ft., was not corrected by erasing that value but by crossing it out for the reason explained in Art. 25(5).

Frequently explanatory notes, in addition to those shown, would be necessary, such, for example, as the degree of accuracy with which the survey was made, the magnetic bearing of at least one line, existing monuments or markers for corners if any, and other information, depending upon the purpose for which the survey was made.

ILLUSTRATION II

Comments on the Survey. The lot surveyed is the same as that in the preceding illustration. This time, however, the transit stations were chosen so as to make each traverse line parallel to the corresponding boundary line, and any convenient distance away from it. The angle at each corner of the lot is therefore equal to the angle between traverse lines at the corresponding station. No angles are taken to the corners of the lot, but the boundary lines are located by offsets from the traverse lines, and their lengths measured as usual.

The *advantages* of this method are that the boundaries are more easily plotted by offsets than by angles, and the angle at each corner is known without computation. The *disadvantages* are in the field work itself. It takes time to get the traverse lines established exactly parallel to the fence lines, even if there are no obstacles to measurements or to lines of sight. When there are obstacles, as there usually are, the extra field work involved is out of all proportion to results gained. The time of the draftsman in computing "fence angles" is usually less expensive than the time consumed by a whole transit party in measuring them.

Comments on the Notes. This is a simple illustration of the sketch method of keeping notes. It has decided advantages over the method used in the preceding illustration when so few measurements are taken. Extra care, however, should be taken to show very definitely where each measurement belongs. This is accomplished oftentimes by the use of arrowheads. Notice that the distance between each traverse line and the corresponding fence line is entered in two places; this shows without question that the two are parallel. Notice also that the distance between each traverse line and the corresponding fence line is greatly exaggerated in the sketch to allow space for recording the offset distances in large clear figures.

The note telling what instruments were used and the degree of accuracy with which measurements were taken is useful as a matter of record, and might have weight in relocating boundaries at some future time, or in settling a boundary dispute should one arise.

ILLUSTRATION II

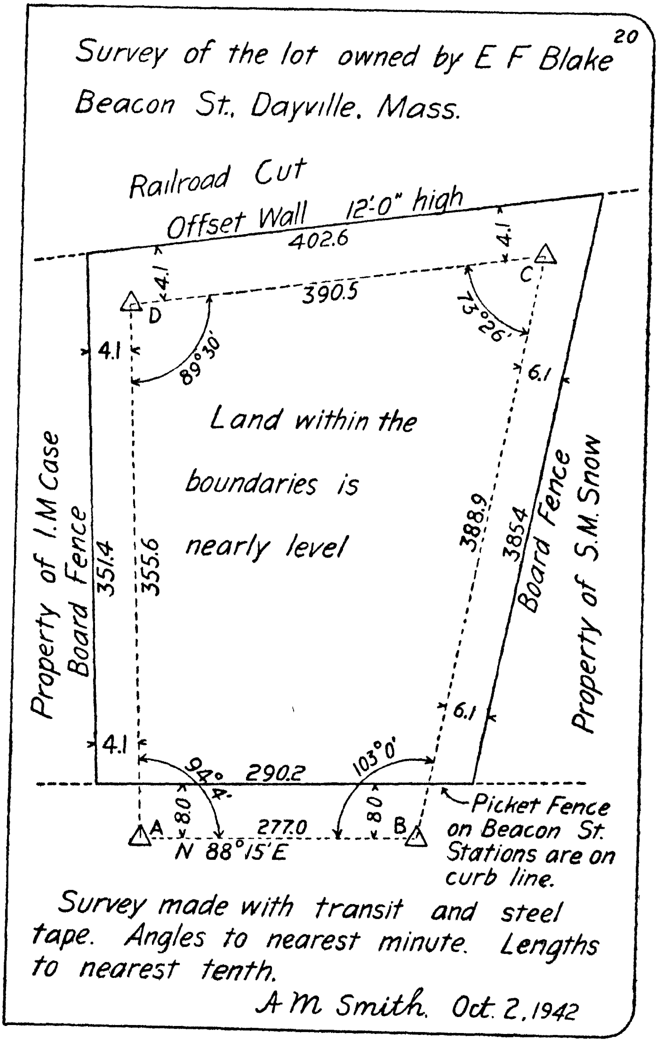


Illustration III

Comments on the Survey (p. 365). The object of this survey was to locate the house, the barn, and the property lines. The first step was to establish transit stations where they would be of the most use. (See Art. 345.) Station *A* was chosen in such a position that the four nearest corners of the house (1, 2, 3, and 4), two corners of the barn (5 and 6), and a corner of the lot (7) could all be located from that one station. Stations *B* and *C* were chosen so that the traverse line between them could be used for locating the river bank by offsets. Corner *D* was chosen in any convenient position near the southwest corner (11) of the lot, where both *C* and *A* would be visible from *D*. A corner of the house (10) was located from *D* as a check. All points were located by angles and distances except the point *c* on *CD* and a point on *AB*, 54.9 ft. from *A*. These two points, one on a line with the front of the barn, the other on a line with the back of the house, were located by measurement, as indicated on the sketch, merely as a check. It will be noticed that the traverse-line angles were doubled. The calculated bearing was also kept, and the magnetic bearing observed for each traverse line. This, perhaps, was an unnecessary precaution since the angles were doubled. Notice also that each building was measured up, and the lengths of three boundary lines were obtained by direct measurement.

The interior angles of the traverse lines happen to add up to 360°. This does not mean that no error was made in angular measurement, but that the errors were small and compensated each other.

The traverse lines might, perhaps, have been run to better advantage close to and outside the fence lines, provided there were no obstacles. In that case some other method might be used for locating the buildings in place of angles and distances.

No one line of the survey seems to be referenced with respect to permanent objects. Consequently, if three of the four corners should become obliterated, it would be difficult to restore them.

ILLUSTRATION III (*Continued*)

Comments on the Notes (pp. 364–365). The form used corresponds to Form 4, p. 355. All angles with corresponding distances are tabulated on the left-hand page. All dimensions and offsets are given directly on the sketch. *DA* is the backsight for so many angles that, instead of repeating the letters *DA* in the first column, a vertical line is drawn. A vertical line is also drawn in the third column instead of repeating the letter *A;* this column is often omitted.

This is a good illustration of combining the sketch and the tabulated form to advantage. To put all the angles and distances directly on the sketch itself would have required considerable care to avoid crowding. That it can be done, however, is shown in the next illustration.

The explanatory notes, while not so complete as would be required in many cases, are sufficient for this particular survey. The note giving the accuracy with which measurements were taken is one often omitted. Nevertheless, it might be of considerable value, should any dispute arise concerning the boundary lines.

The mistake of entering 162° 12′ for 162° 42′ was probably due to an error (afterwards discovered) in reading the angle. (Omitting the half degree, see Art. 231.) The error in recording 40.5 for 60.5 was also probably due to careless field work and not to careless notekeeping. The chainmen evidently read the tape from the wrong end and afterward discovered their mistake.

Notice the method of recording magnetic and calculated bearings. In this case they could just as well have been recorded in separate columns.

In giving offsets it would, perhaps, have been better to use the system of "*plus*" distances from *B* instead of giving distances between offsets. (See Art. 322(*c*).) There are so few offsets in this case, however, that it makes little difference.

Reference numbers are sometimes enclosed thus (6) to avoid mistaking them for dimensions.

In this survey it makes little difference whether the notes are kept from the bottom of the page up or from the top down. Since, however, it is sometimes advantageous to keep the notes from the bottom up, some surveyors prefer to keep *all* notes in that way for the sake of uniformity.

ILLUSTRATION III

8

Angle	Value	Line	Length	Bearings	
2	274°10'			N37°30'E = M	All angles to the right from a backsight on the preceding station.
DAB	137°05'	AB	214.8	N37°35'E = C	
1	7°09'	A1	74.5		
2	25°27'	2	75.4		
3	28°12'	3	60.5 ~~40.5~~		
4	41°51'	4	66.3		
5	162°42' ~~162°12'~~	5	87.9		
6	175°24'	6	142.4		
7	221°12'	7	153.6		
2	109°12'			N87°45'W	
ABC	54°36'	BC	339.1	N87°49'W	
AB8	178°27'	B8	38.5		
2	169°16'			S3°15'E	
BCD	84°38'	CD	214.7	S3°11'E	
BC9	234°03'	C9	25.3		
2	167°22'			N80°30'E	
CDA	83°41'	DA	198.4	N80°30'E	
CD10	73°01'	D10	91.7		
CD11	262°02'	D11	78.2		

Illustration III

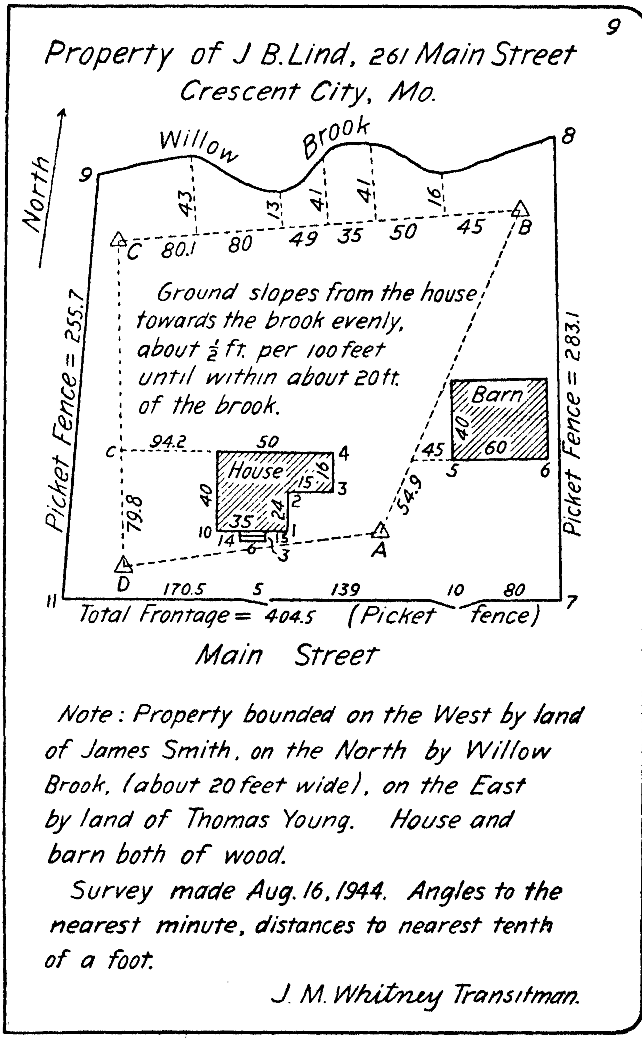


Illustration IV

Comments on the Survey. The property surveyed is the same as that in the preceding illustration. The same transit stations were used, but the buildings were located by a slightly different method. The two rear corners of the house were located by finding two points, one on the traverse line *AB*, another on *CD*, both in line with the rear of the house. The distances from each of these two points to the nearest corner of the house and to the nearest transit station (viz., 78.7, 36.8 and 94.2, 79.8), together with the dimensions of the house, are all the measurements needed to plot the house. The angles and distances to the two front corners of the house were taken as a check. Notice the different methods of locating the two front corners of the barn. Other ways of using one or more of the methods of Chapter 22 for locating fences and buildings in place of those that were used will suggest themselves to the student. It will prove good practice to study out as many different methods as possible that might have been used in this particular case.

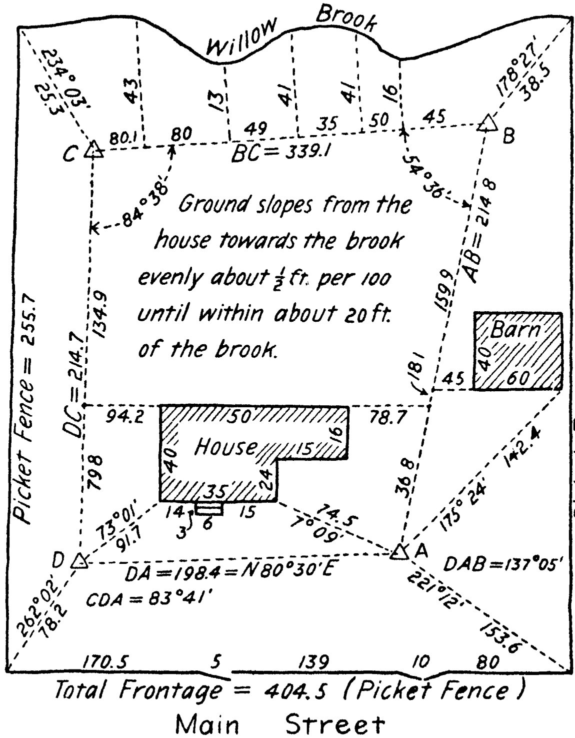
Comments on the Notes. In this case the sketch method is used; the sketch is made freehand without the use of even a straightedge. All measurements, including angles, are given on the sketch itself. Notice how the sizes of the buildings and the distances of the traverse lines from the fence lines are exaggerated to make room for recording measurements. In the same way the other distances are exaggerated, no attempt being made to draw to scale, it being far more important to have the notes open and clear. When a sketch is as clear as this illustration the notes are more easily plotted than if kept in tabulated form, but it is rather difficult to keep such a sketch from becoming crowded when there are many dimensions.

The direction, right or left, in which all angles were taken is not noted. It should be, unless previously noted elsewhere once for all.

Notice how the distance 18.1 ft. near the barn is recorded by drawing an arrow and writing the figures to one side. It is frequently necessary to do this (just as in making working drawings) when the space where the number is to go is too small for large clear figures.

ILLUSTRATION IV

Property of J. B. Lind, 261 Main St., Crescent City, Mo. 12



Property bounded on the west by land of James Smith, on the north by Willow Brook (about 20 ft. wide), on the east by land of Thomas Young. House and barn both of wood. Survey made Aug. 16, 1944. Angles to nearest minute, distances to nearest tenth of a foot

J. M. Whitney, Transitman.

ILLUSTRATION V

Comments on the Survey (pp. 370–371). This is a portion of a preliminary survey for the improvement of a rural highway. Boundaries of the present road, property lines, bridges, streams, and other topographic features for a considerable distance on each side of the highway must be located. The sketch, though not drawn to scale, represents approximately a portion of the map as it will appear when plotted. Great accuracy is not required. Houses, for example, may be located to the nearest foot. Property lines should be located with more care.

In this illustration a number of different methods were employed for locating objects, a greater variety, perhaps, than would ordinarily be used. For this reason the illustration should be studied with more than ordinary care, since it shows the practical applications of so many of the methods of Chapter 22. The points to be noticed are:

1. Stations are numbered according to the system explained in Art. 337.

2. All angles measured are deflection angles.

3. The boundaries of the present road were located by offsets perpendicular to the transit lines taken at irregular intervals.

4. The fence at *A* is located by finding where if produced it would strike the traverse line, and by locating a point 1 (on a line with the front of the house) by angle (50° 58′) and distance (122.6).

5. The Bond house was located by a range line along one side of the house, intersecting the traverse line at point *a* (22 + 7.3). The angle at *a* was determined by measuring the tie line (127.8). This was probably more convenient than to set up the transit at *a* (22 + 7.3) and measure the angle. Point 1, already located on a line with the front of the house, serves as a check.

6. The fence at *C* was located by finding where, if produced, it would strike the traverse line, by producing the front line of the house until it intersects the fence, and by measuring the distance (95.1).

7. The Sykes house was located from station 23 + 89 by angles and distances to two front corners (2) and (3).

8. The fence at *B* was located by two offsets (48.5 and 46.5) measured on lines produced from the front and rear sides of the Sykes house.

9. The fence at *D* was located by the tie-line method (method of Art. 323(*c*)).

10. The four corners of the bridge (14, 15, 16, and 17) were located from station 27 + 9 by angles and distances.

11. Each of the points 4 to 13, inclusive, were located by two angles taken at stations 23 + 89 and 27 + 9, respectively, with the traverse line between these two stations as a base line (method of Art. 320).

12. The fence at *E* was located by the tie-line method, the point *b*, however, being taken on a line with the front of the Morris house.

13. The fence at *F* was located by producing to a point on the traverse line, and locating another point 19 from station 30 + 32.5 by angle and distance. Afterwards, as a check, the distance to *c*, a point on line with the front of the house, was measured (69.1).

14. The Morris house was located from *b* and *c* (already established on line with its front) by the distance from *c*, 84.4. The angle (but not the distance) to one of its corners (18) was taken at Station 30 + 32.5, merely as a check. A better check would have been to measure from 18 to *b*.

In several cases the location of a detail was not checked, as, for example, in the location of the Sykes house, of the bridge, and of some of the division fences. The small number of check measurements required would have taken little time and would have been worth while.

Comments on the Notes. The form used corresponds to Form 5, p. 355. The notes read from the bottom up. Station 30 + 32.5 at the top of the left-hand page would be repeated at the bottom of the next left-hand page to make the tabulated form continuous; just as station 20 + 50 was repeated from the preceding left-hand page. In thus repeating a station, only the deflection angle of the traverse line is given, no other notes are repeated.

Only angles and corresponding distances are given in the tabulated form. All other linear measurements are entered directly on the sketch. Bearings are taken for traverse lines only, and each is entered between the two stations where it belongs. Thus, for example, S 81° 15′ E is entered between 23 + 89 and 27 + 9, and is the magnetic bearing of the line between those two stations. All bearings are understood to be *forward bearings.*

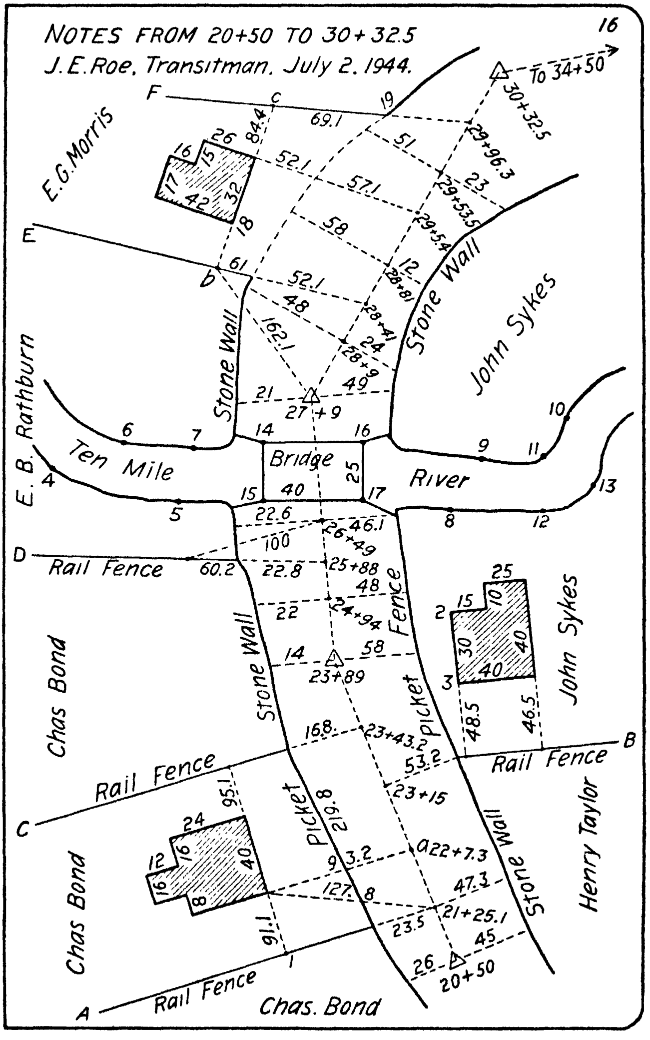
Notice that each piece of property is marked with the owner's name. In front of the Morris house a broken line indicates that there is no fence.

In this illustration the sketch covers too much ground; it would have been better to put the sketch as far as the bridge on one page and the remainder on another. Little is gained by sketching the

ILLUSTRATION V

15 Station	Deflection Left	Angle Right	Line to	Length	Bearings
	144°57'		18		
	134°27'		19	53.5	
30 + 32.5		40°09'	34 + 50		M = S 57°45' E C = S 57°50' E
		146°33'	17	57.2	
		125°33'	16	39.6	
		124°54'	8		
		117°39'	12		
		111°42'	9		
		107°48'	13		
		105° 0'	11		
		90° 0'	10		
	169°54'		15	47.1	
	159°45'		14	22.5	
	109°27'		5 and 7		
	96°15'		4 and 6		
27 + 9		23°21'	30 + 32.5		M = S 81°15' E C = S 81°11' E
		43°27'	13		
		40°24'	12		
		39°15'	11		
		37°33'	10		
		34°12'	9		
		26°09'	8		
	1°24'		7		
	10°39'		5 and 6		
	16°51'		4		
		106°45'	3	94.9	
		88°15'	2	93.8	
23 + 89		10°58'	27 + 9		M = N 87°45' E C = N 87°51' E
	50°58'		1	122.6	
20 + 50		3°34'	23 + 89		

ILLUSTRATION V



roadway as it really is instead of making two straight lines represent the boundaries. The next illustration (VI) is an improvement in both these respects.

In both this and Illustration VI notice that the distances to points on the traverse line at which offsets were taken are entered as *"plus"* stations instead of measuring the actual distances between successive points. This saves time in the field and in plotting since several distances can be measured or plotted at one stretching of the tape or at one setting of the scale as the case may be. When this method is used the distance should be measured from the last offset point preceding each transit station to the station itself, as this is a check on the total distance previously measured between the two stations. (See note, Art. 322(*c*).)

Notice that at the top of the page there is a running title, as, for example, "Notes from 20 + 50 to 30 + 32.5." This is invaluable as an aid in finding notes quickly, especially when they are indexed properly.

Illustration VI

In Illustration VI a portion of the notes of Illustration V have been reproduced according to the sketch method. All notes pertaining to the traverse line are entered in the middle space by themselves according to Form 3, p. 354. This middle space therefore represents the traverse line and may be considered *as having no width,* i.e., points on the edges opposite to each other actually coincide on the ground. Thus, for example, offsets marked 23.5 and 47.3 were really measured from the same point on the traverse line. Notice how the sides of the roadway, although in reality curved, are represented by straight lines. These lines were ruled, but all other lines were drawn freehand. Compare this page of notes with those of Illustration V on pp. 370, 371. Although it is true that one set of notes seems as clear as the other, it should be kept in mind that much more of the survey is included in Illustration V than in Illustration VI. The notes of Illustration VI, on the other hand, are perhaps easier to plot than those of Illustration V. The two illustrations afford an excellent opportunity of comparing the sketch method with the combination method of keeping notes.

ILLUSTRATION VI

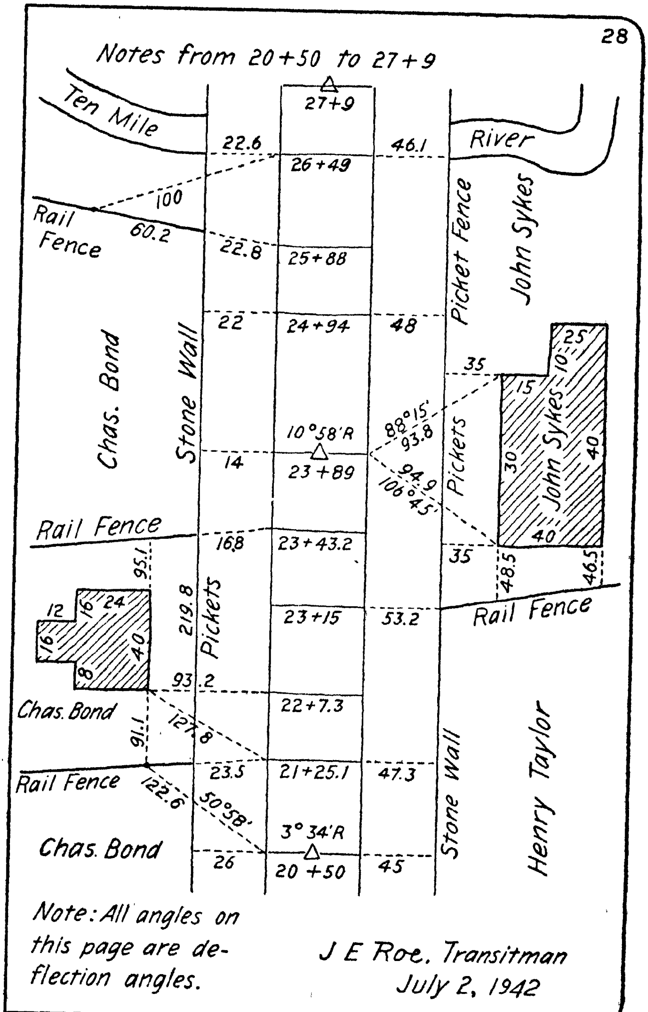


Illustration VII

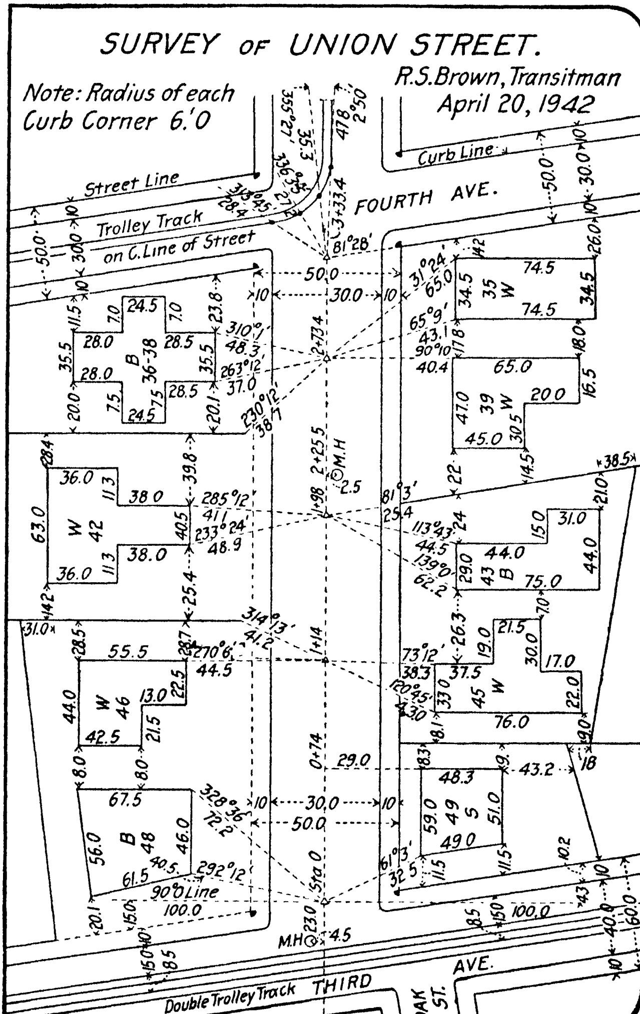
Comments on the Survey. The method illustrated in the notes on the opposite page is, perhaps, the most general method for making a survey of a city street. Station 0 was found by stretching the tape between merestones and taking the middle point. Station 3 + 33.4 was found in a similar manner, thus fixing the center line of the street. The substation 1 + 98 was set at the intersection of this center line with a fence line. Each of the other substations was established at the intersection of the center line with the side of a house produced. The method of angle and distance was used for locating most of the house corners, the backsight in every case being from a permanent sight on the center line *produced* beyond station 3 + 33.4. Distances between houses, front and rear, were also measured. Fence lines were located by offsets from houses, and fence corners by distances along the fence lines. Manholes were located by offsets from the traverse line, and street-car tracks by offsets from the curb to the gauge line of the nearer rail. The curved track was located by angles and distances to points about the same distance apart on the nearer rail. Widths of sidewalks and widths from curb to curb were measured and checked by the total measurement from street line to street line. The angle of the street line on Fourth Avenue (81° 28′) was measured with the transit at station 3 + 33.4, sighting on a monument a block away, but on Third Avenue the angle of the street line was found by laying off 90°, measuring 100 ft. on this line, and then from this point by measuring the offsets (4.3 ft.) to the curb and (4.3 + 10.2) to the street line. Sometimes one of these methods and sometimes the other can be used to advantage.

Other Methods. In the above method the distances from the street line to the fronts of houses are not measured, and can be found only by scaling the map or by calculation. A survey is often made to locate veranda lines and front lines of houses, and in such a case the best method is probably that of offsets measured from a line run on the sidewalk *parallel* to the street line. Sometimes offsets are taken from the curb line or, if there are no fences in the way, from the street line itself.

Suggestion. Where buildings are near the street line offset rods may be used to advantage in place of the steel tape. A stadia rod will answer the purpose. One end of the rod is held against the building, and the measurement is read at the point where the line of sight strikes the rod.

In a survey of an entire city block or of several adjacent blocks it is often necessary to run traverse lines on the sidewalks or in the

ILLUSTRATION VII



street or wherever they can be established to the best advantage. Even in the survey of one street for a single block, such as that shown in the illustration, it may be impracticable to run the traverse line in the center of the street or even parallel to the center line.

Remark. In a series of letters to an engineering periodical, the following methods for locating houses were advocated by different surveyors: (1) plane table; (2) angle and distance; (3) offset; (4) tie-line; (5) each front corner by two intersecting distances; (6) by intersecting angles. This shows the diversity of opinion regarding methods of location, and indicates that each method has its advantages.

Comments on the Notes. This is an example of the sketch method. Each angle is entered above a line drawn from the station to the point sighted at, and the corresponding distance is entered below the same line. Portions of the houses are exaggerated to accommodate dimensions. Each house is marked with its street number, and with W, S, or B to denote whether it is wood, stone, or brick.

When it is desired to distinguish between a fence along a street and the street line itself, the former may be indicated by a full line and the latter by a broken line. Thus in the sketch there is a fence along the right-hand side of the street, but only a street line along the left-hand side.

The notes present a somewhat crowded appearance, but in this respect they are typical of city notes kept by the sketch method. By using the combination method all angles and corresponding distances would be transferred to the left-hand page, thus relieving the crowded appearance, but making it a little more difficult to plot the notes.

CHAPTER 25

TRANSIT SURVEYING

BASIC PROBLEMS

THERE are certain geometric and trigonometric problems which occasionally arise in transit surveying, but which occur too infrequently to be considered a part of the routine field work. These problems have been grouped for reference in this chapter.

377. Field Operations. The field work involved in the basic problems explained in this chapter consists mainly in laying out geometric figures and in the measurement of the angles and distances in these figures. This may require a combination of two or more of the following field operations.

Measuring Angles. In ordinary work it will be sufficiently accurate to measure angles with a transit reading to minutes. It is well to double each angle as a check even when a single reading would otherwise suffice. When the required degree of precision is high, a more finely graduated transit may be required and a greater number of repetitions for each angle may be necessary.

Laying Off Angles. In accurate work, angles should be laid off by the method of repetition as explained in Art. 270(*b*).

Note. If two lines starting from the same point make an angle of 1′ with each other, they will be approximately 0.03 ft. apart at 100 ft. from their intersections and 0.3 ft. apart at 1000 ft., i.e., *the error for 1′ of arc is 0.03 ft. per 100 ft.* (See Art. 283(*d*).) These quantities, if kept in mind, will often suggest the degree of precision with which angles should be measured or laid off in order to meet given requirements.

Base-Line Measurements. Some of the problems involve the measurement of a base line. Methods for measuring base lines with a high degree of precision are explained in Chapter 8.

Prolonging a Straight Line. Three methods are explained in Art. 253. In accurate work the third method (double reverse) should be used.

Sighting Parallel to a Given Line. The method of swinging offsets can often be used to advantage (Art. 277).

To Establish the Point at Which Two Straight Lines Intersect. The method is explained in Art. 274.

378. Triangulation. Some of the basic problems in this chapter involve simple triangulation. Though it is not generally neces-

sary to resort to methods of measuring distances and angles as accurate as those explained in Chapters 8 and 27, certain fundamentals emphasized in those chapters should be kept in mind. It is particularly important to remember that the degree of precision attainable depends to a large extent on the proportions of the triangles, and that very oblique intersections are undesirable; consequently, angles less than 30° or more than 150° should be avoided.

379. Choice of Methods. In choosing among the methods given for each problem two things are to be considered: *accuracy* and *time*. When one method seems as accurate as another and it is simply a question of which is the quicker, it should be noted that constructions which require the least time when worked out on paper sometimes require the most time when worked out in the field. Thus, for example, the intersection of two straight lines is found quickly on paper, but not so quickly in the field. Perpendiculars and parallels are drawn quickly on paper, but corresponding operations in the field may involve considerable work.

380. Obstacles in Transit Surveying. *Obstacles to alignment* and *obstacles to linear measurement* give rise to a special set of problems in transit surveying. Many of the methods already given for obstacle problems in chaining (Arts. 90 and 91) can be used in transit surveying, and with greater accuracy. The problems treated in this chapter are those most likely to arise in practice. Two or three constructions are explained for each problem—enough, it is thought, to suggest other methods not only for problems given here, but also for those problems which occur so seldom in practice that they have been omitted.

381. Obstacles to Alignment. (*a*). TO PROLONG A STRAIGHT LINE THROUGH AN OBSTACLE. Four methods are shown in Fig. 381(*a*). In all except the second method, each point at which the transit was set up is indicated by a small triangle, and the angles actually laid off are noted on the figure. The lines to be prolonged are respectively *AB*, *GH*, and *LM*.

First Method. Lay out a rectangle *BCDE*, the fourth side of which, *BE*, is in the required line.

Second Method. This method is a modification of the first. Parallel to the given line *AB*, establish a line *A'C*, and on this line prolonged, set points *D* and *F'* in convenient positions. Parallel to *DF'*, establish the required line *EF* by offsets equal to *BC* and *AA'*.

Third Method. Lay out an equilateral triangle, *HIJ*, the third side of which, *HJ*, is in the required line.

Fourth Method. Lay out a square *MNPO*, the diagonal of which, *MP*, is in the required line.

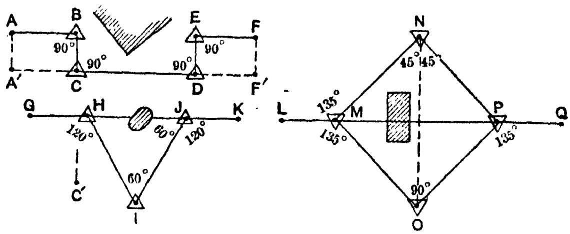


FIG. 381(*a*).

Comments and Suggestions. In the *first method* the backsights, *CB* and *ED*, used in laying off 90° at *C* and *E* respectively are likely to be too short for accurate work. This disadvantage can often be overcome by setting extra points to be used as backsights. For example, when the transit is at *B*, an extra point such as *C′* can be established to be used as a backsight in turning off the angle of 90° at *C*. Similarly, an extra point may be set in *DE* prolonged, to be used as a backsight in turning off 90° at *E*.

Second Method. When the distance *BE* is required, the transit should generally be used in establishing the offsets *BC* and *DE*, if *CD* is to be an accurate measure of *BE*. On the other hand, when an offset, such as *AA′*, does not affect the measurement of the length of the main line, it can usually be laid off by the swinging method (Art. 277). The length of the offset and the accuracy required will determine whether or not the right angle for an offset should be laid off with a transit. When an obstacle is small, such, for example, as the trunk of a tree, offsets are usually short and a small error in laying off the right angle is relatively unimportant. When, however, it is necessary to use a transit in establishing an offset less than about 5 ft. in length, an offset point close to the transit may be located by means of a temporary point and string as explained in Art. 270(*c*). Of the four methods just explained, the second is, in most cases, the simplest and most accurate, particularly when the distances *AB*, *A′C*, and *DF′* can be long enough to make possible the accurate determination of the direction of *EF*. Three setups of the transit, namely, at *B*, at *C*, and at *D*, may be desirable, but the work can be done accurately with two, one at *B* and the other at *D*. (How?)

The *third method* or the *fourth method* can sometimes be used when the open space around the obstacle is such that neither the first nor the second method can be applied easily or accurately.

Convenient distances arbitrarily chosen may be used for the sides of the figures in all four methods, but, if possible, they should be great enough to provide adequate lengths of sights.

There are other methods than the four given which, in some cases, may be equally good. Any system of triangulation may, of course, be used, but it is well to avoid the calculation involved in the use of oblique triangles.

Checking the Field Work. In the four methods just given and in geometric constructions to follow, the field operations may usually be checked with very little additional work by methods easily devised. For example, in the second method, an additional offset to the line *AB* may be laid off, and the point thus obtained should be in line with *A′* and *C*; likewise an additional offset to *DF′* should give a point in line with *E* and *F*. In the fourth method, the length of the diagonal *NO* = *MP* may be calculated, and checked by measurement; *QP* produced should bisect *NO*. How could the third method be checked?

381(*b*). TO RUN A LINE BETWEEN TWO GIVEN POINTS, WHEN AN OBSTACLE INTERVENES.

First Method. (Fig. 381(*b*).) *C* is any convenient point which is visible from the two given points *A* and *B*. Make *CD* = *AC* and

$CE = CB$. Measure ACB and CDE. Then lay off the angle $CAB = CDE$, and the angle $CBA = 180° - (ACB + CDE)$. (Or the value of CBA can be found by measuring CED. This affords a check.) Transit points and angles actually measured or laid off are indicated on the figure by arrows.

Another Method. Measure AC and BC, also the angle ACB, and solve for the two angles CAB and CBA by trigonometry.

Second Method. Choose H, any convenient point visible from the two given points F and G. Produce FH to some point J and measure GHJ. Turn off at G the angle $HGI = 90° - GHJ$. This makes

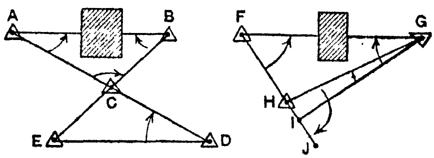


FIG. 381(*b*).

GIF a right angle. Hence tan $HFG = GI \div FI$. The angle thus calculated is laid off at F, and its complement at G, the angle at G being laid off first to save a setup.

Practical Hints. The first method requires four setups, the second three, but trigonometric computation is involved in the second. The second method can be used sometimes where there is not room on the ground to lay out the first.

Offset Method. This method often is much more practicable than either the first or second method, particularly in municipal surveying where this type of obstacle problem is constantly encountered. Only two transit setups are required.

Let A and B, the two given points, represent, for example, city monuments at the ends of a block. (Draw a figure to correspond to the following directions.) Lay off two equal offsets AA' and BB' as nearly perpendicular to the line AB as can be judged. The offset distance may be 5, 10, or 20 ft., or as great as may be necessary for the line $A'B'$ to clear the obstacles between A and B. Although the directions of the offsets are judged solely by the eye, the lengths are so short that the line $A'B'$ will be parallel or very nearly parallel to AB. With the transit at A', backsight on B' and turn 90°. The line of sight should now pass through A. If it does not, the point A' is corrected to a position A'' on the true offset line through A, and the linear measurement is then started from A'' toward B'. With the transit at B', backsight on A'', turn 90°, and set B'' by a procedure similar to that followed in establishing A''.

The linear measurement is completed to B''. If the line $A'B'$ is parallel or very nearly parallel to AB (it usually is), the transit at B' is used in determining intermediate points along the block. By this simple method with only two setups, both line and distance are obtained as accurately as desired.

Random-Line Method. When woods, underbrush, or some similar obstacle intervenes between the two given points, a method similar in principle to that in Art. 91(*b*) may be used. Let it be required to run a line through underbrush from K to L. (1) With the transit at K, run a random line KM as nearly in the direction of KL as can be estimated. (2) Determine the point N at which the perpendicular from L intersects KM. This is done by swinging the tape from L as a center (Art. 274(*a*)). (3) Calculate the angle MKL from tan $MKL = LN \div KN$. (4) From M as a backsight, lay off the angle MKL. If the line does not pass through L, correct the error by linear measurements (similar triangles KOP and KLN). The offset correction at any convenient point O is to KO as the offset error at L is to KL (Art. 91(*b*)). Frequently the angle MKL is so small that after the length of the swing offset LN and the point N have been determined, it is better to follow the method explained in Art. 91(*b*) than to lay off the angle MKL. When O has been established by that method, the line KO may be produced with the transit.

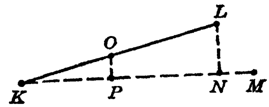


FIG. 381(*b*),4.

382. Obstacles to Measurement. (*a*). TO MEASURE BETWEEN TWO POINTS WHEN ONE IS INACCESSIBLE. There are several methods, three of which are given here.

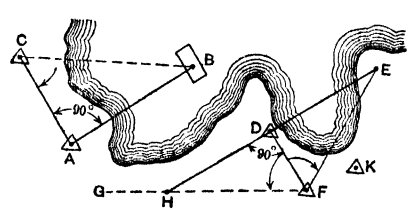


FIG. 382(*a*).

First Method. Let AB be the required distance. Lay off a base line AC perpendicular to the required line AB, and of *any* convenient length. Then $AC \times \tan BCA = AB$.

Second Method. Let DE be the required distance. Lay off DF perpendicular to the line DE, and of *any* convenient length. Measure DFE and lay off $DFG = DFE$. Then $DH = DE$.

Third Method. When both ends of the lines are accessible a third point K may be chosen. In the triangle DKE measure enough parts to obtain DE as, for example, two sides and included angle, or one side and two adjacent angles.

Practical Hints. The second method appears on paper to be the easier, as no calculation is involved. Considerable work is involved, however, in laying off accurately the angle DFG and in finding the precise point of intersection at H, i.e., the intersection of FG with ED produced. Moreover, many times there is not room to lay off DH.

To check either the first or the second method repeat the process, using a perpendicular or base line of different length. To check the third method, choose a new point in place of K, and repeat the process. Sometimes it is better to check one method by using an entirely different one.

382(*b*). SPECIAL CASE. TO MEASURE THE DISTANCE BETWEEN TWO GIVEN POINTS WHEN ONE IS INVISIBLE FROM THE OTHER AND IS ALSO INACCESSIBLE.

One Solution. Choose two auxiliary points visible from both of the given points and from each other. The inaccessible point and the two auxiliary points form a triangle, and by measuring the angle at each of the auxiliary points and using the distance between them as a base, the sides of the triangle can be calculated. An auxiliary point, the given point, and the inaccessible point form a triangle one side of which has just been found by calculation, another side (from the given point to the auxiliary point) can be measured, and the angle between these two sides can also be measured. The third side (the required distance) can then be found from these two sides and the included angle.

Remark. This is an example of numerous special cases which now and then arise in practice, the solutions of which depend on various forms of triangulation. The student should draw a figure illustrating the method here given.

382(*c*). TO MEASURE BETWEEN TWO POINTS WHEN BOTH ARE INACCESSIBLE. Let AB, Fig. 382(*c*), be the required distance. Lay off any convenient base line, as CD, and measure its length. At C measure the angles ACD and BCD. At D measure ADC and BDC. In each of the triangles ACD and BDC one side and two adjacent angles are now known, and hence the distances AC, AD, BC, and BD can be calculated. AB can then be found from either of the triangles ACB or ADB, one computation serving as a check on the other. (Two sides and included angle. See formulas on page 1101.)

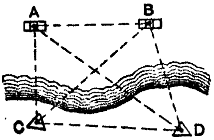


FIG. 382(*c*).

382(*d*). TO MEASURE THE HEIGHT OF AN INACCESSIBLE POINT. General method: In Fig. 382(*d*), B is the point the height of which

is required. The transit is set up at any convenient point *A*, and the vertical angle *BAC* is measured, *AC* being a horizontal line.

Case I. When *AC* can be measured, $BC = AC \times \tan BAC$.

Case II. When *C* is inaccessible, find the length of *AC* by one of the methods of Art. 382(*a*), and proceed as in Case I.

Note. The method for Case I is exactly the same as that for trigonometric leveling, the field work for which is explained in detail in Art. 180. For the degree of precision which may be expected and for practical suggestions see Art. 180(*a*).

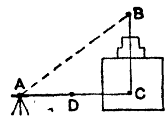


FIG. 382(*d*).

383. Perpendiculars and Parallels. (*a*). TO ESTABLISH A PERPENDICULAR TO A GIVEN LINE FROM ANY GIVEN POINT.

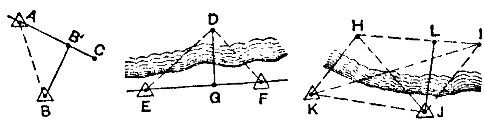


FIG 383(*a*).

Case I. When the given line AC and the given point B are both accessible. At *any* point *A* on *AC* measure *CAB*, and at *B* lay off $ABB' = 90° - CAB$.

Case II. When the given line EF is accessible but the given point D is inaccessible. At *any* two points *E* and *F* measure *DEF* and *EFD*. Then $EDF = 180° - (DEF + EFD)$ and $EG = (EF \times \sin EFD \times \cos DEF) \div \sin EDF$.

Case III. When the given point J is accessible but the given line HI is inaccessible. Choose some point *K*, and with *KJ* as a base line find the angle *JHI* by a procedure similar to that of Art. 382(*c*). Turn off $HJL = 90° - JHI$.

383(*b*). TO ESTABLISH A LINE THROUGH A GIVEN POINT PARALLEL TO A GIVEN LINE.

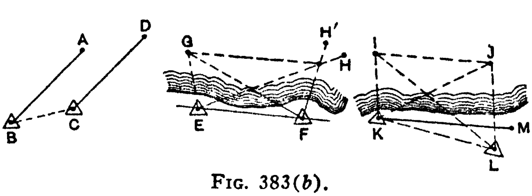


FIG. 383(*b*).

Case I. When the given line AB and the given point C are both accessible. At *any* point *B* measure *ABC*, and at *C* lay off $BCD = 180° - ABC$. This method may be used when, for any reason, the usual method based on offsets cannot advantageously be applied.

Case II. When the given line EF is accessible but the given point G is inaccessible. At *any* two points E and F measure GEF and EFG. At F lay off $EFH' = GEF$, and at E lay off $FEH = EFG$. The intersection of EH and FH' is a point in the required line through G.

Case III. When the given point K is accessible but the given line IJ is inaccessible. Choose some point L, and with KL as a base line find the angle IJK by a procedure similar to that of Art. 382(*c*). Lay off $JKM = IJK$.

CHAPTER 26

TYPICAL TRANSIT SURVEYS

THE examples of comparatively simple transit surveys given in this chapter have been selected as typical of much of the work that is done in the general practice of plane surveying. Surveys for construction are not included since they are treated in Chapter 57.

384. Surveys for Maps. Introduction. In making surveys for maps, two general questions continually arise, namely: (1) *where to run the transit lines* in order that they may be used to the best advantage as reference lines; (2) *what details should be located and what information should be secured and noted.* The answer to the first question depends largely on the answer to the second. The answer to the second depends primarily on the purpose for which the map or description is to be used, but a mere knowledge of the purpose is not enough. If, for example, an engineer is making a survey for a deed, he must have a knowledge of what, in general, should be given in the description of the land or shown on the plat that is recorded as a part of the deed. Thus, in addition to the technique of chaining, leveling, and transit surveying, the engineer must have judgment based on a fund of practical knowledge. This knowledge, for the most part, can be gained only from extensive experience. However, certain methods and procedures, such, for example, as the methods of locating boundaries, are used in nearly all map surveys. Similarly, there are certain common requirements to be met. For example, in a survey of a building site, certain requirements must be fulfilled to enable the architect properly to locate and to design the building, but many of these requirements are the same as those which must be met in numerous other types of surveys. In seeking a knowledge of common methods and requirements, it is usually more helpful to the inexperienced engineer to study a few comprehensive surveys thoroughly explained in detail than to study a large variety of surveys superficially explained. The relatively few types of surveys in this chapter have been chosen and explained with this in mind.

Note. In many simple surveys, leveling as well as transit surveying is required. Surveys in which leveling predominates have been treated in Chapter 14. In the surveys explained in this chapter, there is either no leveling or leveling is subordinate to transit surveying. It is frequently necessary to obtain data for plotting topographic maps even in simple surveys, a subject treated in detail in Chapter 31, "Topographic Surveying."

384(*a*). GENERAL SUGGESTIONS. (1) No new principles are involved in the special surveys explained in this chapter. (2) In many surveys it is necessary to set monuments; different types are described in Arts. 335, 790(*b*), (*c*). (3) Important control points of a survey should be referenced so carefully that if they are disturbed or lost they can easily be re-established. For methods of referencing points and lines see Art. 340. (4) When making a survey for a special purpose, it is frequently desirable or necessary for the engineer to secure from his client a clear statement of the requirements to be fulfilled. For example, in surveying a building site, the engineer, in the beginning, may obtain from the owner, or from the architect, or from both a statement of general requirements, and then as the work progresses, he may seek additional information concerning specific requirements. When warranted by the extent and character of the work, there should be a formal agreement based on specifications for the survey.

385. Boundaries. The boundary lines or property lines of a parcel of real or landed property may be straight, or curved, or irregular. A boundary corner is an established point at the intersection of two boundary lines. A corner monument is an object, more or less permanent, used to mark a boundary corner.

Descriptions of the boundaries of real property are given in deeds. Boundaries are frequently shown on official maps and further described in the notes of the survey. Deeds are recorded, maps are filed, and other official records are generally kept in the office of the city or town clerk, or in the county registry of deeds.

385(*a*). NOTE. In a great variety of surveys, it is necessary for the engineer to locate existing boundaries, to establish new boundaries, or to re-establish old boundaries. When there is uncertainty with respect to the location of a boundary line, the engineer has no judicial function, though it may be possible for him to develop evidence which may result in a legal location of the line. Even when boundary lines are clearly marked by monuments, it may be necessary to check the positions of the monuments in order to be sure that they are in the positions fixed by the deed. Since an engineer may be held liable for incompetent services, he must exercise reasonable care and skill in surveying boundaries. Obviously he must have some knowledge of the laws pertaining to boundaries; he must know how to look up recorded deeds and how to interpret them; he must know also the different methods of marking the location of boundary lines, and what to look for as evidence of the original location of a boundary line when the objects which marked the line seem to have disappeared. This whole subject is treated in Chapter 58, "Surveys of Land Boundaries."

In the special surveys in this chapter which involve the survey of boundary lines, it is assumed that there is no uncertainty with regard to the correct location of the lines.

386. Survey of Existing Boundaries. The general method of surveying the boundaries of a plot of land is to run a closed traverse around the plot and to locate the boundary lines with respect to this traverse. It will be assumed for the present that all boundary lines are straight and that all boundary corners are well marked on the ground. In any given case, it is necessary to decide where it is best to run the traverse lines.

386(*a*). TRAVERSE LINES MADE TO COINCIDE WITH BOUNDARY LINES. When it is practicable to set up the transit over each boundary corner, to sight from one corner to the next, and to measure the length of each boundary line, the

traverse lines can be made to coincide with the boundary lines, provided the latter are straight. The error of closure of the traverse is a check on the measured angles at the boundary corners and the measured lengths of the boundary lines. Obviously, this is by far the simplest and best method of locating boundaries with respect to a traverse. Even when, because of some obstacle such as the brow of a hill, it is impossible to sight from one end of a boundary line to another, the method can still be used if an intermediate point on the line can be located by some field procedure such as balancing in the transit. When it is impracticable to set up the transit over a boundary corner, the angle at that corner can be obtained by measuring the angle between two lines that are established parallel respectively to the two boundary lines that meet at the corner (Art. 278).

When it is impracticable to run a traverse so that it will coincide with property lines, one of several other general methods of locating traverse lines with respect to boundary lines may be used. Four of these general methods are explained and compared in Art. 326. Of these four, the two most often used will be outlined more in detail here.

386(*b*). TRAVERSE LINES RUN PARALLEL TO BOUNDARY LINES. (1) Establish each traverse line parallel to and conveniently near the corresponding boundary line (second method, Art. 326). (2) At each station measure the angle to the boundary corner and the angle between traverse lines. (3) Measure the distance from each station to the corresponding boundary corner and the length of each traverse line.

It is assumed, as the most general case, that the lengths of the boundary lines cannot be measured directly. The length of each boundary line can easily be computed, however, from the measurements made in steps 2 and 3, and the angle between the two boundary lines at each corner is the same as the corresponding traverse angle. The error of closure of the polygon formed by the boundary lines may be computed from the adjusted values of the interior angles and the *computed* lengths of the boundary lines and the error distributed as explained in Arts. 570–571. The area may be calculated as explained in Chapter 40. Note that all the traverse lines may be outside the boundary lines, or all may be inside, or some of them outside and some inside. In any case, each traverse line and the corresponding boundary line are parallel sides of a trapezoid; the other two sides are the lines from the transit stations at the ends of the traverse line to the corresponding boundary corners.

Exercise. Explain how a tape survey of boundaries could be made by a method similar to that just explained.

386(*c*). TRAVERSE LINES RUN AS NEARLY PARALLEL TO BOUNDARY LINES AS MAY BE PRACTICABLE. The field work in this method does not differ materially from that in the preceding method except that the traverse lines, instead of being made parallel to the boundary lines, which is often impracticable, are run wherever they can be established to the best advantage. (See the first method, Art. 326.) This traverse may be wholly outside or wholly inside the boundary lines or partly outside and partly inside. The field measurements are similar to those in steps 2 and 3 of the preceding method.

When it is necessary to run only one traverse line for each boundary line, the boundary line is one side of a quadrilateral whose opposite side is the traverse line. When the length of the boundary line cannot be measured and its direction is unknown, both can be computed, since the lengths and directions of the other three sides of the quadrilateral have been determined by the field measurements. (Case I of Chapter 39, "Omitted Measurements.") In locating a boundary line it is often impracticable to run a single straight traverse line from the station near one end of the boundary line to the station near the other end. It then becomes necessary to run a traverse from one of these stations to the other with a number of intermediate stations and traverse lines. The boundary line in such a case is one side of a polygon with more than four sides; but its direction and length can still be computed, provided the direction and length of each of the other sides have been determined in the field. The error of closure of any polygon formed by boundary lines may be determined from the computed directions (adjusted) and the computed lengths of the boundary lines and the error distributed as explained in Arts. 570–571. The area may be calculated as explained in Chapter 40.

386(*d*). COMBINATION OF METHODS. Instead of using only one method of establishing traverse lines in a boundary survey, it is often best to combine methods. For example, when it is practicable to set up the transit over some of the boundary corners but not over others, it is often advantageous to use

one method of establishing traverse lines for one part of the survey, another method for another part, and still another method for another part. Ingenuity in combining methods may save time both in the field and in the office.

386(*e*). SPECIAL METHOD. Occasionally it is possible and desirable to complete a boundary survey from a single setup of the transit. For example, in Fig. 94, p. 78, by setting up a transit at *S*, the angles between the lines to all the boundary corners and the lengths of these lines may be measured. Each boundary line will then be a side of a triangle, two sides and the included angle of which are known. If it is not practicable to measure the length of the boundary line, it can be computed. The angle at each boundary corner may also be calculated. The closing error of the polygon formed by the boundary lines and the area within these lines may be computed and distributed by the usual methods.

386(*f*). WHEN A BOUNDARY LINE IS CURVED OR IRREGULAR, it may be located by offsets from a reference line, usually one of the traverse lines (Art. 322) or by one of the other common methods of locating such a line.

386(*g*). SUGGESTIONS. (1) The length of a boundary line should always be measured whenever it is practicable to do so. When there is an obstacle to the direct measurement of a boundary line, the length of the line can often be measured indirectly by one of the methods explained in Arts. 90 and 91, or in Art. 382. (2) In choosing a method of locating traverse lines from which to locate one or more boundary lines, it is well to consider the amount of office work as well as the amount of field work involved. (3) When a transit station is near but not over a boundary corner, it should be far enough away to permit focusing on the corner. (4) When the area within the boundaries is large, it may be desirable or necessary to run traverses in such a way as to divide the area into a network of polygons each of which is a closed traverse. (5) When an area, such as a city block, is well covered with buildings, it is sometimes difficult to survey the interior boundary lines. It may be necessary to run special traverse or transit lines from the outside to the inside through alleys, hallways, windows, or wherever it may be practicable to run them. In some cases it may be best to establish a system of coordinate reference lines similar to the system described in the article on surveying for the construction of buildings (Art. 788(*c*). (6) In beginning a boundary survey, it is often highly desirable to tie the survey to an existing system of permanent horizontal control as explained in Chapter 56. When this is not practicable, it may be desirable to determine the *true* bearing of one or more of the boundary lines (Chapters 34 and 35). (7) Unless boundary corners are well marked or well referenced, it is often difficult to determine just where a boundary line is, even though its position seems to be fixed by a fence, or by a wall, or by some other means. For example, how much of the fence or of the wall is on one side of the line and how much on the other? Such questions are discussed in Chapter 58, "Land Boundaries." (8) The required bearings of property lines are those calculated from *adjusted* angles and not those obtained from observed angles unless no adjustment of the latter is necessary. (9) For examples of simple surveys of established boundaries see Illustrations I, II, III, and IV, pp. 358–367.

387. Survey to Establish Boundaries. In order to establish new boundaries it is necessary to know: (1) the position of the two ends of one of the new boundary lines with respect to some other line, such as a street line or an old boundary line, which is clearly marked on the ground and can be used as the starting line of the survey; (2) the positions of the new boundary lines with respect to each other.

The field work is, to a large extent, the reverse of that in a survey of existing boundaries. Instead of measuring angles and distances to points, such as boundary corners, that are already located on the ground, measurements are laid off to establish new points. The simplest procedure is: (1) Establish the two ends of

one of the new boundary lines by laying off measurements from an existing line. (2) Run a closed traverse, the first line of which is the boundary line established in (1) and the other lines of which are made to coincide with the other new boundary lines by laying off the proper angles and distances. Thus each station at a corner of the traverse marks a boundary corner. (This is the reverse of the field procedure explained in Art. 386(*a*).)

The simple field procedure just outlined is often impracticable. The method next in simplicity is to locate the new boundary lines from traverse lines that have been run parallel to those boundaries. It will be necessary to start the traverse from an existing line, as in the first method, and then to lay off angles and distances such that each traverse line will be parallel to the corresponding boundary line and at a known distance from it. (The reverse of the field procedure explained in Art. 386(*b*).) It may be practicable to establish some of the boundary lines by this method and some by the simpler method first explained.

When it is impracticable to establish a boundary line by either of the methods just described, a procedure may be followed which is the reverse of that in Art. 386(*c*). Beginning at the end of the boundary line to be located, run a traverse from this end until a station has been reached conveniently near the other end. The direction and length of the traverse line from this station to the required end of the boundary line are unknown, but both these quantities can be computed since this last line is the closing line of a traverse in which the directions and lengths of all the other lines are known, including the direction and length of the boundary line itself. (Case I of "Omitted Measurements," Chapter 39.) The position of the end of the boundary line is then determined by laying off the computed direction and the computed length of the closing line. Although it may be difficult to check the position of a boundary corner located in this way, the method is often the best available when obstacles to sighting and to chaining make it impracticable to use more direct methods. Fortunately it is not often necessary to resort to this method except for some portion of the survey.

387(*a*). To RE-ESTABLISH BOUNDARY LINES which are not well marked or not marked at all, the engineer must seek evidence that will enable him to determine the true location of each line. As stated in Art. 385(*a*), this often requires a knowledge of laws, of deeds, of methods of marking boundaries, and of other matters pertaining to land surveying. (See Chapter 58.) Once the true positions of boundary lines have been determined, the field work

in re-establishing them does not differ materially from that just explained for establishing new boundaries.

387(*b*). SUGGESTIONS. (1) It is often practicable and highly desirable to tie in the boundaries with a permanent system of horizontal control (Chapter 56), or with a true meridian (Chapters 34 and 35). (2) It may be desirable, particularly when the area within the boundaries is relatively large, to establish some of the boundaries by running part of the traverse in one direction around the area and to establish the remaining boundaries by running the remainder of the traverse in the opposite direction. (3) As a general rule, the engineer is responsible for the accurate setting and referencing of monuments at boundary corners. (For different types of monuments see Arts. 335, 790(*b*), (*c*).) (4) Some of the suggestions in Art. 386(*g*) hold good also for establishing boundaries.

388. Rural Surveys. The degree of precision required in rural surveys depends to a large extent on: (1) the purpose for which the survey is made; (2) the time available in which to make it; and (3) the permissible cost. Frequently the required degree of precision is such that a stadia survey meets the requirements better than one made with a transit and tape.

A survey of a farm may be taken as an example of a rural survey. The following general method is recommended, regardless of whether line measurements are made with the tape or by the stadia method. Run traverse lines conveniently near the boundaries by the azimuth method, locating stations to the best advantage. Take side shots to each corner, and if boundaries are straight, chain the lengths of boundaries as well as of traverse lines. If boundaries are not straight, locate them with respect to the traverse lines by offsets or by some other method. The azimuths and lengths of the traverse lines and side shots to corners as found above are the only data required for the office work of calculating the azimuth and length of each straight boundary line. (The measured length of a boundary line serves as a check.) From the azimuths and lengths of the boundary lines thus calculated the area may be computed, as explained in Chapter 40. When boundary lines are irregular, it may be well to determine the area by means of a planimeter (Art. 589).

When buildings, driveways, interior fences, or other details that should be located are not conveniently near the main traverse, an additional traverse may be run. It is well to make this a closed traverse which begins at a station of the main traverse and ends at another such station. When the area is large, it may be best to run several interior traverses which form with the main traverse a network of traverse lines.

Frequently the position of a boundary line is uncertain. It then becomes necessary to re-establish the boundary. (See Art. 387(*a*).)

389. Urban Surveys. Because of the higher degree of precision generally required in urban surveys, sources of error in measurement must be eliminated to a greater extent than in rural surveys. Procedures designed to eliminate sources of error are explained: for linear measurements in Chapters 6 and 8; for leveling in Chapters 12 and 15; for angular measurements in Chapters 20 and 27, and in Arts. 359 and 360.* It is economical to use surveying instruments of the best quality and to keep them in good adjustment. Tapes should be standardized. When cities maintain a standard of length for this purpose, it is well to compare tapes with this standard even though they may have been tested elsewhere. Urban surveying must often be carried on under such difficulties as obstacles to measurement, varying surface temperatures, interruptions due to street traffic, and vibration of surveying instruments.

The engineer is more likely to find an established control survey (municipal, state, or federal) in an urban community than in a rural district. Most cities and many towns have monuments, usually at street intersections, for horizontal control, and bench marks for vertical control. It is not only highly desirable but also in many cases mandatory that an urban survey be tied in with such systems of control, particularly if it is a boundary survey. Maps and descriptions of primary and secondary systems of control, property maps, topographic maps, wall maps, maps which show sewers, mains, conduits, and tunnels, and various other maps and records kept in the office of a department of municipal engineering furnish many of the data needed in making urban surveys.

Requirements and specifications for an urban survey differ in different localities even for surveys of the same general type, such, for example, as property surveys. Moreover, the whole field of municipal surveying is one in which practice is so highly specialized that no attempt will be made to treat it in detail here. Instead, on the pages immediately following, two examples will be given of typical urban surveys such as any surveyor in general practice may be called on to make, namely, (1) a survey for a deed, and (2) a survey for an architect.

390. Survey for a Deed Description. A description of a tract of land by *metes and bounds* is one which includes: (1) a description of the exact position of some boundary corner as a starting point; (2) a statement of the length and direction (bearing) of each boundary line given in consecutive order around the tract; (3) a description of the kind of object which marks each corner.

* See also *Technical Procedure for City Surveys*, a manual published by the American Society of Civil Engineers.

A description by metes and bounds is the oldest type of deed description. Then came the practice of filing in the same place where the deed is recorded a map or plat of the tract of land which the deed describes. In some localities it is now the procedure to require a description that is in the form of an engineering drawing *without* a metes-and-bounds description. In any case, the deed description is generally based on a boundary survey made with transit and tape by one of the methods previously explained.

As stated in Art. 385, in surveying boundaries the engineer should have a knowledge of the laws pertaining to boundaries and of land surveying in general. This whole subject, including deed description, is treated more in detail in Chapter 58, "Surveys of Land Boundaries."

A deed description has two functions, namely, (1) proper identification and (2) precise location. Hence to be adequate a description should satisfy these two tests:

1. Does the description completely identify the property so that the title can be surely traced?

2. Does it include all the data necessary to mark the boundaries on the ground?

390(*a*). SIX GENERAL RULES FOR LAND DESCRIPTIONS.*

"*Rule 1.* Each description in a recorded deed should bear the name of the person responsible for the description.

"*Rule 2.* All recorded descriptions should include the date when the person whose name appears upon it was assured of its accuracy. (To eliminate the practice of recopying ancient descriptions through transfer after transfer, without a field checkup.)

"*Rule 3.* The source of the survey data upon which the description is based should be definitely stated. For example, in an extreme case, the wording might be: 'From computations by Richard Roe, Land Surveyor, License 1501, April 1937, based on survey by John Smith, Land Surveyor, License 1502, made January 1936, description by Mary Jones, Real Estate Broker, License 1503, February 1939.'

"*Rule 4.* The identity of the property must be clearly stated. For example, 'Being the same tract (or part of the same tract) conveyed to *A* by *B* by warranty deed dated June 1934, and recorded in Book 100, page 100, July 1934.'

"*Rule 5.* The description should include survey ties to at least two durable monuments. If the street is monumented, the street monuments can be used as long as distances and bearings are given to both. If such do not exist, then at least two monuments must be set marking the longest property line. Obviously, it is better to base the description on more than two monuments. All monuments should have sufficient ties to permit reestablishing one that is lost. In other words, the best method is to base the description on a system

* These six rules are suggested by Professor Philip Kissam in an article entitled "Proposed Standards for Land Descriptions," which is printed in *Civil Engineering*, Vol. 10, No. 11, November 1940, published by the American Society of Civil Engineers. In Professor Kissam's opinion, a description in the form of an engineering drawing is the ideal form, clear in its meaning and understood as easily by the layman as by the land surveyor. He believes that a tracing on cloth should be substituted for the metes-and-bounds description. Reasons for the six rules are given in the article. For additional recommendations and procedures pertaining to deed descriptions see Art. 798, "Deed Descriptions."

of monuments interconnected by surveys. The state system of plane coordinates offers the most perfect system of monuments; in many localities such monuments have been established along the streets and highways, ready for immediate use. When all land surveys are made part of such a system, many of the difficulties will disappear (Chapter 56).

"*Rule 6.* All dimensions of property lines and all bearings shall be stated, and no 'more or less' distances shall be used. All dimensions shall be based on a closed survey which has been checked by latitudes and departures."

390(*b*). SPECIFIC REQUIREMENTS. (See also Art. 798.) In addition to the general requirements stated in the six rules just given, there are specific details which should be included in a deed description. Among the most important details that should be shown on a plat are: (1) The direction of the *meridian* with reference to which all bearings or azimuths were calculated should be stated, i.e., whether it was the true meridian, or the magnetic meridian, or the central meridian of the state coordinate system. When practicable, the true meridian or the central meridian of the state coordinate system should be used. When the magnetic meridian is used, the directions of both the true and the magnetic meridians should be shown with the magnetic declination at the time of the survey clearly indicated. (2) The exact location and a description of each corner monument and of each reference point used in referencing the monument should be given. (3) The length and bearing or azimuth of each boundary line should be clearly indicated along the line. If the line is a part of a circular arc, the radius of the arc and the bearing and length of the chord can be given. (4) Names of the owners of all adjacent property should be stated. (5) Scale of drawing should be given.

Among the details that may or may not be required on a particular plat are: city, town, or county lines, railroads, highways, water courses, street lines, building lines, fences, walls and other objects that mark the boundaries, passageways, and statements of public or private restrictions on the use of property. It is sometimes required to show the buildings that are on the property and in some cases such parts of buildings or other physical features on adjoining land as are within a specified distance, say 10 ft., of a boundary line. Lines on the plat which represent physical features should be so marked as to indicate what they are. When fences, walls, etc., are represented by conventional signs, such as those on p. 802, a key or legend near the title should explain the signs, or better still in many cases, the nature of the fence, wall, or whatever it may be, should be marked along the line.

The degree of precision with which a survey must be made is often specified in a set of rules or laws in which the degree of precision is adjusted to be consistent with the value of the property, from that in cities to that in timberland and other isolated properties. In some cases, therefore, it is well to state on the plat the degree of precision adopted for the survey.

390(*c*). SPECIFICATIONS FOR A PLAT.* Before beginning a survey for a deed description, the engineer should obtain information which will help him to decide what details should be included in the description. In some localities he will be governed, to a large extent, by laws which are more or less in the form of specifications. The following rules for governing property surveys and maps are from a by-law adopted by a relatively small town. They are given, not as a model set of rules, but because they specify in considerable detail what should be shown on a plat, and because they constitute a fair example of specifications under which the surveyor must often work.

"A carefully drawn plat shall be submitted to the Board of Survey in order that it may be approved before it is filed for record with the County Registrar.

"The plat shall be drawn accurately to scale. The minimum scale shall be 100 feet to the inch.

"The title of the plat shall include its name, an accurate description of its location, scale, date, name and address of alloter, its certification and the name and address of the subdividing engineer or surveyor.

"The following data shall be shown on the plat:

"Section 1. Direction of north (preferably the true north).

"Section 2. All plat boundary lines with lengths of courses to hundredths of a foot and calculated bearings to half minutes. The error of closure shall not exceed 1 in 10,000. The names, exact location and widths along the

* Specifications for a plat issued by the land court of a state are given in Art. 798(*d*).

property line of all existing or recorded streets intersecting or paralleling the plat boundaries.

"Section 3. Any town limits shall be accurately located on the plat, and their names shall be printed on them.

"Section 4. All streets as shown on the Official Town Plan, wherever they lie within the plat.

"Section 5. The exact layout, including: (*a*) Streets, and alley lines; their names, bearings or widths. (*b*) The lengths of all arcs, radii, and bearings of tangents. (*c*) The grades of all streets or parts of streets over five per cent. (*d*) All easements, or rights of way where provided for or owned by public services; the limits of the easement rights should be definitely stated on the plat. (*e*) All lot lines with dimensions in feet and tenths. (*f*) The accurate location and material of all monuments or bounds. (*g*) All public property on the tract, accurately outlined and described with existing or proposed use designated. (*h*) The location and use of all property reserved by covenant in the deed for the common use of all property owners. (*i*) Proposed drainage and disposal of surface water.

"Section 6. Both By-Law and Private Restrictions including: (*a*) Boundaries of each type of use restrictions. (*b*) Set back building lines.

"Section 7. A copy of private restrictions for each differently restricted section of the plat."

391. Survey for an Architect. A map of a building site and its surroundings from which an architect is to obtain many of the data for locating and planning a building should show a number of details not found on an ordinary map. Moreover, much of the information is given by dimensions and explanatory notes as on working drawings, so that the architect will not have to depend upon scaling the map. Such a map is often called an *original plat* to distinguish it from an *architect's plat,* which shows the location of the proposed building on the site.

Many architects prefer that the map be drawn to a scale that corresponds to that used in drawing plans for the building, such as: 1 in. = 32 ft., 1 in. = 16 ft., 1 in. = 8 ft., and occasionally 1 in. = 4 ft. Other scales in more or less general use are 1 in. = 10 ft., 1 in. = 20 ft., and 1 in. = 40 ft. (See choice of scales, Art. 653.)

391(*a*). Location of details. When the building site is comparatively open to sighting, details may be located with respect to traverse lines by the usual methods. When, however, it is difficult to run a satisfactory traverse, as it frequently is in the interior of a city block, it may be better to establish a system of coordinate lines. For example, an offset line, parallel to the street line, may be established on the sidewalk, and then reference lines perpendicular to this offset line may be run through alleys, hallways, or windows into the interior of the block. Details can then be located with respect to these reference lines. The use of coordinate reference lines in locating points is explained in detail in Art. 788(*c*), (*d*).

The following lists of details, dimensions, and elevations are intended to remind the engineer of what may be desired by the architect.

391(*b*). Details located by horizontal measurements. (1) Street lines on opposite sides of the street. (2) Curbs, sidewalks, and driveways. (3) Boundary lines of the site. (4) Monuments at street corners. (5) Legal building line (or lines if the site is on a street corner). (6) Center lines of storm and sanitary sewers, gas and water mains, conduits and tunnels. (7) Manholes near the site. (8) Existing structures on the site and near the boundaries of the site. (9) Trees, poles, hydrants, street lights, and similar objects near the site.

391(*c*). DIMENSIONS. Among the dimensions which should be clearly indicated by dimension lines or by explanatory notes are: (1) widths of streets, sidewalks, and driveways; (2) lengths of boundary lines and angles at boundary corners; (3) lengths of property frontages; (4) distances of building lines from street lines; (5) angles between street lines at street corners; (6) sizes of sewers, mains, conduits, and tunnels; (7) such dimensions of existing structures as may be of use to the architect; (8) diameter of tree trunks and the approximate height and spread of tree tops.

391(*d*). ELEVATIONS may be required of the following: (1) the reference bench mark; (2) surface points at intervals of say 25 or 50 ft. along curbs, sidewalks, street lines, and boundary lines, including boundary corners; (3) flow line of sewer taken at a manhole; (4) mains, conduits, and tunnels; (5) points at corners of squares laid out on the ground surface of the site to be used in locating contours. The size of the squares will depend upon the character and size of the surface. When the surface is quite irregular, contour lines at 1-ft. or 2-ft. intervals will best show the surface elevations. (6) Elevations of water tables and other parts of adjacent buildings that may affect the architect's design; (7) depth of foundations of buildings on or near the site.

391(*e*). GENERAL INFORMATION that may be useful to the architect may be given by notes on the map or in a supplementary statement accompanying the map. Such information may include: (1) a description of every monument or marker, including those of boundary corners, and of every bench mark; (2) character of the soil, its bearing power, and other information useful in estimating costs of excavation and in designing foundations; (3) type of pavement, sidewalk, and driveway; (4) kind of pipes used for sewers and mains; (5) gradients of sewers and the direction in which the grade ascends or descends (indicated by a plus or a minus sign and an arrow parallel to the center line); (6) description of trees—kind and condition; (7) names of streets; (8) the legal description of the tract, such as lot numbers, name of tract, city or town, county, and state; (9) descriptive title, the name of engineer, the scale, and the date; (10) direction of the *true* north (usually shown by an arrow); the *true* bearing of a boundary line is even better, for it helps to reference that line; (11) the datum used for elevations.

When the building is planned by an out-of-town architect, the engineer may be requested to furnish such additional information as: local water pressure; zoning restrictions that apply to the site; a copy of the city building code or ordinances; information concerning adjacent objects or buildings that should be considered in properly locating the new building on its site, in planning some of its features such as entrances and windows, and in otherwise designing the building.

391(*f*). CAUTION. (1) The liability of the engineer for damages resulting from an inaccurate or erroneous survey, discussed in Chapter 58, particularly in Arts. 793, 794, should be kept in mind in making a survey of a building site, particularly under certain conditions that exist in a city. (2) The engineer must be on the alert to detect any condition that may interfere with the normal design and construction of a building. For example, if an adjacent building has a wall along a boundary line and the corresponding wall of the new building is to abut on it, the engineer should ascertain whether or not the wall of the old building is plumb and whether the wall or any part of the building encroaches on the site of the new building. When a new building is to fit exactly between two existing buildings, measurements of distances between the two existing buildings must be made with great care not merely at the ground level but elsewhere up and down the buildings, including foundations.

392. Survey to Locate Soundings. The general method is to run an open traverse along the shore, and to locate each point at which a sounding is taken by determining its position with reference to one or more of the traverse lines.

392(*a*). BY THE USE OF TWO TRANSITS. (1) Choose a base line with respect to which some, if not all, of the soundings can be located. This base line may be between two traverse stations, or

between two points which later are tied to the traverse. (2) Measure the length of the base line if it is not already known from traverse measurements. (3) Set up a transit at each end of the base line, and locate each sounding by measuring the angles between the base line and the two lines which intersect at the sounding, as explained in Art. 320. (4) When as many soundings as practicable have been located, choose another base line and repeat the procedure.

Suggestions. (1) The base line between the transits should be so chosen that the angles of intersection at the soundings are not too oblique or too acute; otherwise the point of intersection may be indefinite. In some cases, this may make it necessary to establish a supplementary base line tied to the traverse; in other cases, instead of using a traverse line as a base, the transits may be set up over two traverse stations which are not successive. The length of the base line between the two stations must then be either measured or calculated. (2) In most cases, as in taking soundings from a boat, the work must be done rapidly. Each transitman follows the boat with his telescope, keeping the line of sight on the sounding line or pole, and at a given signal from the man in the boat reads and records the angle. *It will save time not to clamp the upper plate at all.* In recording angles some system must be used so that when the notes are plotted, the angle to a given point recorded in one transitman's book can be used with the angle to the same point recorded in the other transitman's book. It is just here that mistakes are likely to occur. One transitman, for example, misses a signal and fails to take the angle. This puts him one angle behind the other transitman, and this causes trouble in plotting. One method of avoiding this is for each transitman to record the exact time at which each angle is taken in a column reserved for that purpose. If angles are taken very rapidly, record the time, say, of every fifth angle. Of course the watches of the two transitmen should be set to keep the same time. Some method of recording should also be followed such that the depth of a sounding for a given point can be identified as belonging with the pair of angles to that point. As an additional precaution, flags of different colors may be used by the boatman as, for example, a white flag each time a new set of ten soundings is begun, a red flag each time a new set of one hundred, and a green flag for all other soundings. (3) At frequent intervals, each transitman should sight on a permanent backsight to make sure that the instrument remains oriented. (4) The chief advantages in the use of two transits are that the soundings may be taken at random, wherever it may seem desirable to locate them, and that each sounding can be located so quickly that it is not necessary to keep the boat in position for any great length of time. Frequently, however, it is better to use a simpler method.

392(*b*). BY THE USE OF ONE TRANSIT. (1) Run a traverse and choose a base line for each set of soundings as explained in the preceding method. (2) At measured intervals along the base line, lay off range lines perpendicular to that line and set two range poles or flags on each range line as signals. (3) Set up a transit at one end of the base line. (4) Take as many soundings along one of the range lines as may be desired, keeping the boat on line with the two signals on shore that mark that particular range line. As each sounding is made, the transitman reads the angle between the base line and the line of sight to the sounding. This line of sight and the portion of the base line between the transit and the range line are, respectively, the hypotenuse and the base of a right triangle: the third side is the offset distance from the base line to

the sounding. This offset distance can be plotted or calculated from the measured angle of the triangle and the length of the base of the triangle. (5) Repeat the procedure until soundings on each range line have been located.

Suggestions. (1) Intervals between range lines may be equal or unequal. (2) Signals on a range line should be far enough apart to permit bringing the boat into line with the desired accuracy. (3) The position of each range line should be recorded in the notebook, each designated in accordance with some systematic notation. (4) The line of sight from the transit to the boat should intersect the range line at an angle which will make the point of intersection definite. (5) In recording angles, some system should be used such that an angle to a given point and the depth of the sounding for that point can be identified in the notes as belonging to the same point. (See suggestions for the preceding method.) (6) Range lines may be laid off oblique to the base line, but computations are simplified by making them perpendicular.

392(*c*). Other methods. Among the other methods of locating soundings are these. (1) The three-point method explained in Art. 321. (2) The stadia method, provided conditions are such that the stadia rod can be kept sufficiently still to permit readings. (3) The method in which soundings are taken at known intervals along a wire stretched between opposite shores. The intervals may be marked by metal tags or other device. This method is used chiefly in obtaining data for cross sections of streams. (4) Lay out a "grid" on the ice, bore holes, and take soundings through the ice—a convenient method for use in winter in cold climates.

CHAPTER 27

PRECISE MEASUREMENT OF ANGLES

IN order to measure angles with precision, errors in angular measurement must be reduced to a far greater degree than in ordinary transit work. This chapter is devoted largely to a discussion of how this reduction of errors is accomplished.

393. Introduction. The measurement of an angle with a high degree of precision is not a simple procedure. It involves the selection of the type of instrument, the mounting of the instrument, the protection of the instrument from disturbing elements, the selection of conditions under which to measure the angle, the adoption of suitable methods of manipulating the instrument, and a program of procedures in observation designed to reduce errors. The observer should have a clear understanding of the errors from different sources and of their relative importance (Chapter 20). He should also weigh the value of maximum accuracy against the cost of obtaining it. In triangulation, for example, there are generally weak portions of the triangulation network where conditions make the cost of uniform accuracy out of all proportion to the value to be derived from it, particularly if these weak portions do not detract from the strength of the network as a whole.

394. The Instrument. Horizontal angles can be measured with a relatively high degree of precision by means of an ordinary transit; a high degree of precision can be attained with a precise transit; for the highest degree of precision, the theodolite is better adapted than any other type of instrument. The *repeating type of theodolite* has double spindles, the inner spindle turning in a socket in the outer spindle. The direction theodolite has only one spindle. Theoretically, the repeating type of theodolite should minimize the effect of errors of adjustment, of graduation, of pointing, and of reading. Actually the slipping of the clamps and the eccentricity in centers offset, to a large extent, these advantages. The *direction theodolite* has only one movement and one vertical axis from which practically all eccentricity can be removed by painstaking workmanship; fitted with micrometer microscopes, directions can be read on such an instrument to a high degree of fineness.

394(*a*). SELECTION OF THE INSTRUMENT. In the selection of a transit or theodolite, it is important to consider carefully the optical characteristics of the telescope as explained in detail in Chapter 51, Arts. 711, 728(*a*). The diameter of the horizontal limb and the least count of the vernier are discussed in Art. 728(*b*), (*c*). It is to be noted that the accuracy of vernier reading depends more upon the clarity or definition of the graduations than upon the diameter of the limb, the spacing of the graduations, or the least count of the vernier. For all except measurements of the highest precision, a least count of the vernier of 10″ is sufficiently accurate; many instruments reading to 20″ are used, since such a vernier can be read with greater assurance. Verniers should be provided with high-power, focusing reading glasses

of small field and with a centering arrow to prevent parallax errors. The selection of a transit is discussed quite fully in Chapter 53, Art. 728; to a large extent, this discussion applies also to the selection of a theodolite.

Ordinary transits are generally used for traverses and for triangulation of third- or fourth-order accuracy, and precise transits or repeating theodolites for traverses and for triangulation of second-order accuracy. A precise transit, a repeating theodolite, or a direction theodolite may be used for traverses or for triangulation of the first order; but when there are several angles to be measured at a station, as often is the case in a triangulation net, the direction theodolite is preferable.

395. Errors in Angular Measurement due to different causes are treated in Chapter 20. The supplementary treatment of errors which follows may repeat to some extent portions of that chapter but from a different point of view. The emphasis in this chapter is on the somewhat extraordinary precautions that must be taken in order to reduce errors in precise angular measurement, some of which, in ordinary surveying, are often unimportant or negligible. These errors may be classified as follows: (1) errors from external causes; (2) errors due to incomplete adjustment; (3) errors in the instrument itself; (4) errors due to improper manipulation. The following discussion of these classes of errors is taken from a paper entitled "The Measurement of Angles," by Lester C. Higbee, W. and L. E. Gurley, Troy, N. Y.

395(*a*). ERRORS FROM EXTERNAL CAUSES are due to phase and eccentricity of target, to lateral refraction, to wind vibration, to heat waves, or to changes in the constant error of the observer due to changes in the other external causes. These errors are usually corrected by night observing, by properly shielding the instrument, and by providing suitable lights for targets.

Under this heading we may also include errors in the instrument mounting. With the standardized steel towers of the Coast and Geodetic Survey, this error is negligible. However, in the frames and tripods constructed by inexperienced field parties, there are frequently opportunities for error. The twisting and untwisting of members under the sun's rays; vibration from blasting operations in the vicinity, from strong winds in exposed locations, or from the observer moving around the instrument; settling of leg members if built on soft ground; these are conditions which will introduce errors the source of which is generally unsuspected.

395(*b*). ERRORS THROUGH INCOMPLETE ADJUSTMENTS are minimized by careful adjusting and by the proper program of observing. The mean of two sets of readings, one with the telescope direct and one with it reversed, is free from errors arising from imperfect collimation adjustment and from inequalities in the heights of the standards and in the diameters of the telescope axles. Parallax errors in the telescope, reticules that are not vertically aligned, and plate levels that are not accurately centered are errors due to incomplete adjustments which cannot be washed out by any observing program.

Parallax errors in the telescope are due to failure of the observer to adjust the eyepiece and objective so that the common focusing plane coincides with the plane of the cross hairs. Many observers are careless about this adjustment, being content simply to focus the eyepiece on the cross hairs, without checking to see that cross hairs do not appear to move when the head is shaken slowly and the objective is focused carefully on a distant point.

An error in the alignment of the vertical hair or hairs would introduce no error in results if the target were always sighted on the same point of the hair. This is not always possible so that this adjustment should be carefully checked.

Since there is a position of the bubble in which it will remain stationary when rotated about a plumb spindle, there is no need to have it exactly centered. This position of the bubble should be accurately determined, and per-

haps recorded, so as to facilitate leveling up. With some observers it is easier to determine when the bubble moves off if it is adjusted to the center position, and in such cases this should be done carefully.

395(*c*). Errors in the instrument itself. It is interesting to note that size is no longer a measure of accuracy in a theodolite. The Parkhurst theodolite of today, which has a 9-in. circle, is much more efficient than the Roy theodolite of 1787, which had a 36-in. circle, although both read to the same fineness. The difference lies in improved tools and equipment, in better materials, and in workmanship.

Instrument errors lie in the spacing of the graduations of the limb, of the vernier if of the repeating type, and of the micrometers of the direction type. These errors may be analyzed by plotting curves of the differences in readings between the A verniers and microscopes and the B verniers and microscopes.

A loose objective slide, either of the internal or of the external type, will cause either constant or systematic errors. Such may be due to too full fitting or to a change of lubricant. The error becomes more apparent when some stations are out of focus, and there will be a different reading with the telescope reversed from what it was direct. If there is no looseness in the slide, the error due to change of focus is eliminated by the reversal of the telescope during the observations.

Telescope axles of unequal diameter cause no error in the angle measurement if they are not reversed in their standard bearings during the observations. This condition can be detected by noting the position of a striding level when placed on the axle first in the normal position and then with the axle reversed.

The accuracy and fit of the centers are probably the cause of most inconsistencies in the angle reading. The eccentricity of the centers is the least important since by a proper program of observation the angle can be freed from the error caused.

The real source of trouble lies in the uneven manipulation by the observer and the variations in drag on the clamps which are caused by variations in friction in the bearings themselves.

Another variable in the bearings is the lubricant used. To begin with, the lubricant should be thoroughly tested to determine whether or not it will hold its body under the weight on the bearing, its viscosity throughout the temperature range of field service, and that it will not readily oxidize and become gummy.

The bearings should be fitted for the lubricant to be used and any change in lubricant may necessitate a refitting of the bearings. On instruments of the highest precision, an adjustable screw and nut are sometimes provided at the lower end of the spindle which lessen the weight of the plate, standards, and telescope upon the conical bearing surfaces. This adjustment, you can appreciate, is a very delicate one as too little lift will cause drag in the centers, and too much lift will cause play in the bearing surfaces, either of which will result in a loss of accuracy.

Clamps and tangents require just as careful centering as the limb and its vertical axis and just as precise fitting as the bearings themselves, if maximum accuracy is to be achieved. The fitting of the tangent screws, their freedom from backlash and point eccentricity, their perpendicularity with the tangent arm, and their alignment with the line of pressure of the counter spring are niceties of precise theodolite design that only the skilled workman and the skilled observer appreciate.

395(*d*). Manipulation errors are those due to the actions of the observer, which may or may not overemphasize errors in the instrument itself. Such errors will be discussed as they are listed.

1. *Pointing errors.* These may be due to poor targets, poorly lighted targets, obliquely lighted targets; to unstable atmospheric conditions; to poor optics in the telescope; to inequalities, irregularities, and to walk in the telescope axle and bearings; to collimation errors both in adjustment and in looseness of the objective slide. However, as far as the observer is concerned, there are two sources of error which can be overcome by careful

manipulation: first, the error of parallax due to careless focusing on the cross hairs, and second, rapid and jerky movements in rotating the telescope and in operating the clamps and tangents.

2. *Errors due to spindle being out of plumb.* Angles so measured are not in a horizontal plane, and an error is introduced which cannot be eliminated by any method of observing. Tending the plate levels is especially important during daytime observing since the heat of the sun on one end of a vial may cause the bubble to run out of adjustment. As previously discussed, a plate bubble need not be in adjustment, but its position with the spindle vertical must be noted carefully.

3. *Parallax in reading verniers* on a repeating instrument is a frequent cause of error, particularly with observers who are breaking into triangulation work from the regular engineering surveys. Many such observers are impatient with the small field of the reading microscope with its alignment index, and frequently resort to the hand glass with fatal results.

4. *The strain of the four-screw base* is frequently given as a source of instrument error. As far as present-day instruments go, this excuse is greatly overworked as few repeating theodolites can be bound in the centers by excessive tightening of the leveling screws. The reputation no doubt gained headway from the careless handling of inexperienced observers who felt that they had to demonstrate their full strength when leveling up. More careful handling would have saved the reputation of this sturdy and excellent leveling device, especially for lightweight instruments.

5. *Looseness of the three-screw base* can be as productive of errors as overstrain on the four-screw base. This is particularly true of theodolites which depend upon their own weight for their stability. Hence observers must not forget to tighten the leveling screw clamp screws when making a setup.

6. *Unequal temperature changes* are much overrated as a source of error as the most successful triangulation is done at night. When daytime observation is mandatory, the instrument *must be protected from the sun.* Uniform changes of temperature of a wide range do not affect adjustments, provided the parts are symmetrically made, but the sun shining on one side of an axis will throw it out of adjustment very quickly, no matter what the material or quality of workmanship.

7. *Constant error of the observer.* This in itself introduces no error in observations as long as the observer maintains the same place in the schedule of observations. The hazard is the variation which takes place in an observer's constant error due to variations in observing conditions, targets, etc.

8. *The drag on clamps and tangents* is perhaps the least understood error of all. Few observers understand the need for slow and uniform motion when rotating a theodolite, for approaching a target always from the same direction without overrunning it, and for manipulating tangent screws and micrometer screws against the counter springs. These precautions are particularly necessary in the manipulation of repeating theodolites since errors caused by drag are in each turning of an angle and are not reduced by any program of multiple reading.

9. *The lack of a program of observation* is an error of planning which might be a part of manipulation. Each triangulation survey has its own problems and no standardized set of instructions will meet every condition, although standardized sets are better than none at all. Such a program should have constantly in mind the kind of errors that are to be overcome, the climatic conditions, the length of time available, the men and tools to do with, and the degree of accuracy desired.

Many errors are caused by forgetfulness or by a general discarding of their causes as having little effect on the final results. In precise angle measurement, no source of error is too small to be overlooked, and only by experience does the observer learn how properly to weigh errors and to provide for a program of observation which will overcome them.

396. Prerequisites for Precise Measurements. (*a*). A SUITABLE INSTRUMENT. The selection of an instrument has already been discussed.

396(*b*). Adjustment of the instrument. To keep the instrument in adjustment it may be necessary to test it frequently, in some cases every day, before using it. This can be done quickly if permanent points are established for use in the tests for adjustments. (See "Permanent Standards for Testing Adjustments," Chapter 54, Art. 740.)

396(*c*). Stability of the instrument. When an instrument, mounted on a tripod, is set up on the ground, every precaution should be taken to keep it from settling or moving in any direction. Sometimes it may be necessary to provide a rigid support for the tripod such as rigid posts for the tripod shoes, set flush with the ground, or a rigid plank platform. When an instrument is to be used continuously at a station, a more permanent support may be advisable. For example, in surveying for the construction of a bridge, a rigid pedestal upon which to mount the instrument may be constructed at each end of each base line. When it is necessary to mount the instrument above ground, the platform or tower should be rigidly constructed and firmly anchored. It is desirable to have two independent unconnected platforms or towers, one within the other. (See "Towers," Art. 406.)

396(*d*). Suitable targets or signals. The essential feature of a target or signal is a vertical line upon which the vertical cross hair may be set when a sight is taken. This line must be fixed rigidly, and must be clear and distinct. Its length and width should be such that the cross hair may be set on it repeatedly with negligible variation in the pointing of the telescope. For night observations, a rod illuminated as described in Art. 720 or a target with an illuminated vertical slit may be used. The illumination of the slit is obtained by placing a light behind the target with a reflecting surface behind it. The light should be nearly enclosed so that its rays will not appear except through the slit. In order to increase accuracy in pointing, extra cross hairs in the form of an X are often used. For sighting on an illuminated slit at night, two parallel vertical cross hairs, closely spaced, are better than a single vertical hair. The target should preferably "fit the cross hairs," in whatever form they may be. (See "Signals," Art. 407.)

396(*e*). A suitable procedure in measuring an angle or a direction. The method of repetition and the method of direction have both been explained in Chapter 19, Arts. 265-268. The method of direction is the better method when it is desired to determine with a high degree of precision the direction of several lines which meet at the same station. It is used mainly in triangulation, particularly when the network is extensive and more or less com-

plex. The method of repetition is in general used for measuring the angles of a traverse or of a simple triangulation network. In each method a *set* of observations is taken from which one value of the angle is determined. Several such values may be obtained by taking several sets, and from these values the probable value of the angle can be computed. The number of sets taken in a given case will depend mainly upon the degree of precision desired (Art. 401). The field procedure in the precise measurement of directions is explained quite fully in Chapter 28, "Triangulation"; therefore it will not be given here. The remainder of the chapter is devoted to an explanation of the use of the method of repetition in the precise measurement of horizontal angles.

397. Measurement of Angles by Repetition. Two procedures for this method have been outlined in Art. 266. The second procedure is recommended. This procedure is designed to reduce the errors in the measurement of an angle (1) by repeated measurements of the angle and (2) by making one half of the measurements with the telescope normal and the other half with it inverted. The number of repetitions in a set as well as the number of sets taken depends mainly on the degree of precision desired. For the purpose of explanation, it is assumed that two sets of *twelve* repetitions each are taken, *six* repetitions of the angle and *six* of its explement. In turning the telescope to sight upon a point, *the direction of revolution should always be clockwise,* never counterclockwise, regardless of whether the sight is to be a foresight or a backsight. This precaution is taken to reduce errors. (See Art. 395(*d*), paragraph 8.)

The general procedure for one set, in brief, is to repeat the angle six times to the right, inverting the telescope at the end of the third repetition. Then, *without resetting the vernier,* measure the explement (360° − angle) also to the right, with the telescope still inverted for the first three repetitions, but normal for the last three. Theoretically the vernier reading should come back to the original setting, and the sum of the two angles (angle + explement) should equal 360°. The discrepancy, equal to the difference between the final reading and the original setting, represents the total cumulative error due to both accidental and constant errors, and this error has been found constant for a given instrument and given conditions within the probable error of pointing. It seems probable, therefore, that when the angle and its explement are corrected each by half the amount of the cumulative error, the errors from all sources have been as far as possible eliminated. This is confirmed by the fact that when angles measured by this method are so cor-

rected they are in close accord, even though they are taken under conditions which give closing errors of wide range.

It is to be noted that all the angles were turned in the same direction, clockwise, and that although there are in all twelve repetitions, six for each angle, the verniers are set only once. Both verniers should be read to eliminate errors of eccentricity.

397(*a*). METHOD OF PROCEDURE. (Illustrated by notes on p. 405.) *First Step.* (1) Set the A vernier at 0°, or at random, and record the readings of both the A and the B vernier. (2) Take the angle to the right, record the single reading as an approximate check, and repeat two more times, three repetitions in all. (3) *Invert* the telescope and continue the repetitions for three times, or until six repetitions in all have been made. (4) Record both vernier readings.

Second Step. Without resetting the vernier, measure the explement of the angle clockwise in exactly the same way, with the telescope inverted for the first three repetitions and normal for the last three.

These two steps constitute a set, and from two to four sets for each angle are usually sufficient in ordinary work.

397(*b*). PRACTICAL SUGGESTIONS. (1) Great delicacy of manipulation is required, as a slightly awkward touch will often spoil a set. Keep the hands off the tripod, turn the plates gently, taking hold of the plate, *not* the telescope, and do not force the clamps. (2) Notice that in the first step the backsight is always on the left-hand station, whereas in the second step it is always on the right-hand station. Remember that the lower clamp and tangent screw are used for backsighting, and that turning the wrong tangent screw will spoil a set. (3) Half the observations are taken with inverted telescope to eliminate, as far as possible, errors in the horizontality of the supporting axis of the telescope as well as errors of collimation. Hence do not touch the leveling screws during the progress of a set. (4) Be sure in making all pointings to turn the tangent screws *against* the opposing springs. (5) With the exception of the single check reading no readings are taken until the six repetitions have been completed. Read one row of graduations exclusively. (6) This method does not involve the precaution of measuring an angle first to the right and then to the left, but the error is allowed to accumulate and then is corrected. There are several other methods of procedure in use which do involve this precaution, but the method given here is used in the U.S. Coast and Geodetic Survey, and is considered one of the best of the methods of repetition. (7) The suggestions just made relate to only some of the more important sources of error. In triangulation it is particularly important to understand all the various sources of error in angular measurement and how to eliminate them (Chapter 20).

397(*c*). ILLUSTRATION. The following is an outline of the work that was done in the measurement of an angle *ACB* when the notes on p. 405 were taken: (1) Set the A vernier at 0° 0′ 0″; found the reading of the B vernier to be 180° 0′ 0″; entered these two readings on the fourth line from the bottom of the page, opposite *A*. (2) Measured the angle *ACB* once and recorded the single reading (42° 5′) on the second line from the bottom, opposite *B*1; repeated the angle five times in addition, the last three with telescope inverted; recorded both vernier readings (252° 31′ 0″ and 72° 31′ 0″) opposite *B*6. These two readings happen to agree in minutes and seconds. (3) Allowed the verniers to remain at these two readings (which are again recorded opposite *B*, three lines up the page), backsighted on *B*, and made six repetitions of the explementary angle *BCA;* recorded the two vernier readings (0° 1′ 0″ and 180° 1′ 0″). Evidently the total error which has accumulated

Point Sighted	Readings Vernier A	Readings Vernier B	Mean Reading	Corrected Value	Mean of Two Sets
Transit on Station C. Measured Angles ACB and its Explement,—Both Clockwise					
A 6	25°21′0″	205°20′30″	2185°20′45″		
B	277°51′30″	97°51′30″	277°51′30″		
6×BCA			6)1907°29′15″		
A 1			317°54′52.5″	317°54′50″	
B 6	277°51′30″	97°51′30″	277°51′30″		
A	25°20′30″	205°20′0″	25°20′15″		
6×ACB			6)252°31′15″		
B 1	67°26′ 25°20′30″ 42°5′30″		42°5′12.5″	42°5′10″ 360°0′0″	10 5 15
A 6	0°1′0″	180°1′0″	2160°1′0″		42°5′7.5″
B	252°31′0″	72°31′0″	252°31′0″		
6×BCA			6)1907°30′0″		
A 1			317°55′0″	317°54′55″	
B 6	252°31′0″	72°31′0″	252°31′0″		
A	0°0′0″	180°0′0″	0°0′0″		
6×ACB			6)252°31′0″		
B 1	42°5′		42°5′10″	42°5′5″ 360°0′0″	

from all sources is 0° 1′ 0″. This completes the field work of the first set. (4) Worked out the values in the fourth column as indicated. Since the readings from both verniers agree exactly throughout the work in this first set, the values in the fourth column are exactly the same as the corresponding values in the second column except 2160° 1′ 0″, which was obtained by adding 6 × 360° to 0° 1′ 0″, since the indicator has moved six times around the graduated circle. (See Art. 265(*d*).) (5) Worked out the values in the fifth column. Since the sum of 42° 5′ 10″ and 317° 55′ exceeds 360° by 10″, and since, presumably, the angle and its explement were measured with equal precision, each value was reduced 5″, the corrected values 42° 5′ 5″ and 317° 54′ 55″ were recorded in the fifth column, and the two values were finally added to make sure that their sum is 360°.

The second set was carried out in exactly the same way except that in the beginning the verniers were set at random (25° 20′ 30″ and 205° 20′ 0″) to avoid bias from the readings in the first set, and also to use a different portion of the limb. In this set the readings from the two verniers disagree in two instances and hence the corresponding values in the fourth column are ***mean*** values. The value 2185° 20′ 45″ was obtained by adding 6 × 360° to 25° 20′ 45″.

The values of the required angle as found in the first and second set were 42° 5′ 5″ and 42° 5′ 10″, and since no other sets were taken the final value is the mean of these two or 42° 5′ 7.5″, as indicated in the last column.

Note. Had the procedure outlined in Art. 399 been followed, at the beginning of the second set the vernier would have been set at the initial reading of 90° instead of 25° 20′ 30″, i.e., the increment would have been $i = 360° \div nm = 360° \div 4$. If it had been desired in addition to eliminate errors in the graduations of the vernier, there would have been added to 90° the quantity $v = d \div m$. Assuming d to be 20′, $v = 20' \div 2 = 10'$, and the initial reading would have been 90° 10′.

Checking readings. A useful method of checking a final reading after a given number of repetitions of an angle is explained in Art. 265(*f*).

398. Field Notes—Method of Repetition. Many forms are in common use. The advantages of the form on p. 405 are: only six columns are required, all arithmetical work is shown, and the notes are practically self-explanatory. Notice the following points: (1) The first three columns are completed in the field; the last three may be worked up in the office. (2) The vernier readings occupy only two lines for any one angle, but leave three lines at the bottom of the page and three between angles for the arithmetical work. The notes are entered from the bottom of the page up as a matter of convenience in subtracting vernier readings. (3) The fourth column is the mean of ***minutes*** and ***seconds*** read from verniers A and B. The degrees are taken from A. Ignore the degrees in the B vernier reading: sometimes they are not recorded at all. When they are recorded, however, they serve as a check on blunders in reading vernier A. The two readings would be 180° apart, provided there were no eccentricity. (4) The single reading in the second column is entered simply as an approximate check.

398(*a*). SUGGESTIONS. (1) Always enter at the head of the notes the station occupied and the angles measured. (2) Sketches of the triangulation, explanatory notes, and other data are put on the opposite page. Data pertaining to the weather, to atmospheric conditions, to the kind of instrument used, etc., are more important in triangulation than in other kinds of work. (3) When a set of notes is manifestly incorrect, cross them out neatly and write across them "Abandoned" or "Incorrect."

The form of field notes just described can be used to advantage when, as in a traverse, there are only one or two angles to be measured at a station and when there are only one or two sets for each angle. It becomes cumbersome when, as in triangulation, there are many angles and many sets. A more condensed form is shown in the following example of field notes which is similar to the field notes for triangulation shown on p. 425.

398(*b*). EXAMPLE OF FIELD NOTES. At a station called "Mellon," three angles were measured. Several sets were taken of each angle, but the notes given here are for the third set only. Each set consisted of six repetitions of the angle and six repetitions of its explement; in each case, three repetitions were with the telescope normal and three with it reversed.

In the notes, D_0 is the initial vernier reading for the angle; D_6 is the reading after six repetitions of the angle and also the initial reading for the measurement of its explement; R_6 is the "return" reading after six repetitions of the explement; R_1 is the reading for the angle after the first foresight (first repetition). This last reading is not used in comput-

ing the value of the final angle, but it should be recorded because it furnishes a check on the degrees and minutes of the value of that angle and also because it facilitates the work, especially when the angle exceeds 360°.

Station Sighted	*Tel. Rep.*	*Circle Reading*	*Verniers* A	*Verniers* B	*Verniers* *Mean*	I_m	$D_6 - I_m$	*Mean Angle* $((D_6 - I_m) \div 6)$
Lynn	D_0	50° 00′	$\overline{55''}$	00″	$\overline{57''.5}$			
Homewood	D_1	128° 21′	20″					
Set 3	D_6	160° 08′	40″	30″	35″.0	$\overline{56.2}$	110° 08′ 38″.8	78° 21′ 26″.5
	R_6	50° 00′	$\overline{55''}$	$\overline{55''}$	$\overline{55''.0}$			
Homeo	D_0	50° 00′	00″	00″	00″.0			
E. End	D_1	101° 14′	10″					
Set 3	D_6	357° 25′	35″	25″	30″.0	08.8	307° 25′ 21″.2	51° 14′ 13″ 5
	R_6	50° 00′	15″	20″	17″.5			
Bat	D_0	00° 00′	10″	10″	10″.0			
Kow	D_1	27° 34′	10″					
Set 3	D_6	165° 26′	20″	30″	25″.0	02.5	165° 26′ 22″.5	27° 34′ 23″.8
	R_6	00° 00′	$\overline{50''}$	00″	$\overline{55''.0}$			

A line *over* a numeral in seconds indicates that the recorded value of the corresponding angle is 1′ greater than its true value, and that consequently 1′ must always be subtracted from the recorded value before it can be used. For example, the A vernier reading $\overline{55''}$ for D_0 means that 1′ should be subtracted from 50° 00″; hence the actual reading is 49° 59′ 55″. The mean between this reading and the reading of 50° 00′ 00″ of the B vernier is 49° 59′ 57″.5, but it is recorded as 50° 00′ and $\overline{57''.5}$.

At the end of a *set* of twelve repetitions, D_0 and R_6 should be identical; if they are not, the shorter arc between D_0 and R_6 is the error of horizon closure. This error may be distributed equally between the measured value of the angle and the measured value of its explement. The correction may be applied to the angle by changing the initial reading D_0 to a point on the arc halfway between D_0 and R_6. This new initial reading I_m (instead of D_0) is then subtracted from D_6 before D_6 is divided by 6 in computing the final mean angle.

To find the mean angle at Mellon between sights taken on Lynn and Homewood:

$D_0 = 49° 59' 57''.5$ $R_6 = 49° 59' 55''.0$; hence

$$I_m = 49° 59' + \tfrac{1}{2}(57''.5 + 55''.0) = 49° 59' 56''.2$$

Only the seconds are recorded in notes, i.e., $\overline{56\ 2}$. Note that this value may be obtained merely by inspection of the vernier-means column. For the second angle, $I_m = 50° 00' + \tfrac{1}{2}(00''.0 + 17''.5) = 50° 00' 8''.8$. For the third angle, $R_6 = 359° 59' 55''$; hence the arc between R_6 and D_0 is $5'' + 10''$, and $I_m = D_0 - \tfrac{1}{2}(5'' + 10'') = (00° 00' 10'') - 7''.5 = 00° 00' 02''.5$.

The final mean value of the angle is obtained by subtracting from D_6 the corresponding I_m and then dividing the result by 6 as indicated in the notes. For example, the mean value of the first angle is:

$160° 08' 35''.0 - 49° 59' 56''.2 = 110° 08' 38''.8$ and

$$(360° + 110° 08' 38''.8) \div 6 = 78° 21' 26''.5$$

(See also the method of short division, Art. 265(*f*).) The mean value of each of the other two angles may be found in a similar manner.

The first two angles in the notes are the same as the first and fifth angle in the notes on p. 425. The third angle is added to illustrate the case in which D_0 and R_6 are on opposite sides of the zero graduation of the limb.

398(*c*). *Note.* In the notes on p. 425, two "arcs passed over" have been obtained in each case, each of these arcs has been divided separately by 6, the two results have been added, and one-half the small difference between this sum and 360° has been applied, with the proper sign, to one-sixth of the arc passed over by direct measurement. As compared with that method of computation, the one given here has the following advantages. (1) It is simpler and less laborious. The calculation of the value of I_m and the subtraction $D_6 - I_m$ are so simple that the work may be done mentally by inspection. (2) This simpler method is less susceptible to error for the reasons that there are fewer computations and that it does not involve the application of a small correction at the end with its proper algebraic sign. The final values of the mean angles are

correct to the nearest 0.1″, whereas in the longer method they are not, because of the dropping of decimals in the larger number of computations.

399. Initial Vernier Readings for Different Sets. In order to eliminate possible errors of the graduation of the circle, the readings for different sets are distributed over different portions of the circle. This is done by increasing the initial reading of each set by adding a certain increment to the initial reading of the preceding set. This increment (i) depends upon the number of verniers (n), and the number of sets (m), and may be determined from the formula, $i = 360° \div nm$. When, however, a theodolite has three verniers or micrometers, the formula $i = 360° \div 2nm$ may be used. In order to eliminate errors in the graduation of the vernier, the index of the vernier is set ahead at the beginning of each set by an amount (v) equal to the smallest division of the circle (d) divided by the number of sets, or $v = d \div m$. The total increment (I) for settings is $I = i + v$. Assume, for example, that a two-vernier instrument has a circle the smallest division of which is 10′ and that six sets are taken with this instrument. The increment for the circle in degrees is $i = 360° \div 2 \times 6 = 30°$, and the increment for the vernier is $v = 10' \div 6 = 1'\,40''$; then $I = 30°\,01'\,40''$. The initial reading for the first set would be 00° 00′ 00″; for the second, 30° 01′ 40″; for the third, 60° 03′ 20″, and so on. If the initial reading had been some other value than 00° 00′ 00″, the same increments would have been added.

400. The Accuracy Attainable in the measurement of angles depends upon so many variable conditions, such as those summarized in the article "Prerequisites for Precise Measurement" (Art. 396), that it is practicable to give only a general idea of the accuracy that may be attained under specified conditions.

The term *set* is used sometimes (1) to denote a set of repetitions of a *single* angle, sometimes (2) to denote a set consisting of a given number of repetitions of an angle and the same number of repetitions of its explement. To avoid ambiguity here, the first will be called a *single set* and the second a *double set*.

400(*a*). ACCURACY ATTAINABLE WITH A GOOD "MINUTE" TRANSIT.

1. The error in doubling an angle should not exceed 30″.

2. A single set, three repetitions with telescope normal and three with it reversed: The error should not exceed 10″ to 12″. When additional sets are taken, the error should not exceed $12 \div \sqrt{n}$, in which n is the number of sets. (See formula 2, Art. 45.) For two single sets, this value is 8″.5; for six single sets, it is 5″.0.

3. A double set, six repetitions of the angle and six of its explement: The error in the horizon closure should not exceed 20″, and hence the error in the angle should not exceed 10″. When additional double sets are taken, the error in the angle should not exceed $10 \div \sqrt{n}$. For two double sets this value is 7″.0; for three double sets, 5″.8; for four double sets, 5″.0; for five double sets, 4″.5.

400(*b*). ACCURACY ATTAINABLE WITH A GOOD 30″ TRANSIT.

1. A single set of six repetitions: The error should not exceed 5″ to 6″. Additional sets 5″ or 6″ $\div \sqrt{n}$. For four single sets: 2″.5 to 3″.0; for six single sets: 2″.0 to 2″.5; for ten single sets: 1″.6 to 2″.0.

2. A double set, six repetitions of the angle and six of its explement: The error in the angle should not exceed 4″ to 5″. For additional sets it should not exceed 4″ or 5″ $\div \sqrt{n}$.

Note. The values just given for maximum permissible errors should not, of course, be taken as absolute; they are intended merely to serve as a guide under average conditions. For additional values which indicate the permissible angular errors in traverses of the first, second, and third orders, see the *Specifications* in Arts. 360(*a*), 360(*b*), and 360(*c*). See also Art. 415.

400(*c*). THE ACCURACY THAT CAN, IN GENERAL, BE ATTAINED IN TRIANGULATION with precise instruments, as well as with ordinary transits, is indicated by the permissible average angular error of closure in the triangles of a triangulation net and the permissible maximum angular error of closure in any triangle, as specified in seconds of arc by the U.S. Coast and Geodetic Survey.

	First Order	*Second Order*	*Third Order*
Average triangle closure	1″	3″	6″
Maximum triangle closure	3″	8″	12″

401. The Number of Sets required, when each set consists of six repetitions of the angle and six of its explement (sometimes called a double set), is generally specified as follows: for first order, 5 to 6; for second order, 2 to 3; for third order, 1 to 2. When precise instruments, reading, say, to 10″, are not used, the number of sets may be increased, particularly for the higher orders. For fourth-order accuracy and a 30″ transit one or two sets are generally sufficient.

CHAPTER 28

TRIANGULATION

THE aim in this chapter is to give methods for simple triangulation, such, for example, as that which forms the basis of an ordinary topographic survey. Many of the suggestions and directions apply equally well to the special problems in triangulation given in Chapter 25. No attempt has been made to treat the subject as comprehensively as would be required in a work on geodetic surveying.

402. Triangulation. If, in any triangle, one side and the two adjacent angles are measured, the lengths of the other two sides may be computed by the methods of trigonometry. This is the basis of all triangulation. The simplest form is a single triangle, but, generally speaking, this is used only in finding distances to inaccessible points, distances between inaccessible points, and similar problems such as those given in Chapter 25. A more general form of triangulation is that in which there is a network of triangles.

Spherical Excess. When three points on the mean surface of the earth are vertices of a triangle, the surface within the triangle is very nearly a spherical surface and the sides of this spherical triangle are arcs of great circles. The sum of the interior angles of a spherical triangle is always greater than 180° by an amount directly proportional to the area of the triangle. This increase above 180° is called the *spherical excess.* Except for triangles of very large areas, it may be taken as 1″ of arc for every 76 sq. miles. One third of the correction is subtracted from each of the angles.

Spherical excess may ordinarily be disregarded entirely, even in work of high precision, when the area within the triangle is less than 10 sq. miles. The triangle may then be regarded as a plane triangle, and the methods used in triangulation are those of plane surveying. When the area within a triangle is sufficiently large to be treated as a curved surface, the methods used in triangulation are those of geodetic surveying.

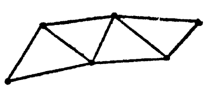
402(*a*). IN A NETWORK OR SYSTEM OF TRIANGLES the side of some one triangle is measured, and from this *base line* and the two adjacent angles the remaining two sides of the triangle are *computed.* The *calculated* lengths of these two sides may be used as new base lines for two other triangles, and the *calculated* lengths of the sides of these new triangles used as base lines for still other triangles. Thus it is possible to extend this system indefinitely solely by the measurement of angles. Theoretically, it is necessary to make only one linear measurement, that of the original base line of the first triangle, but in practice the length of at least one other line as, for example, one side of the last triangle, is generally measured as a check.

The lengths of the sides of the triangles in a network may be only a few hundred feet or many miles, depending primarily upon

the purpose of the survey. For a description of the U.S. Coast and Geodetic Survey triangulation system which covers the United States see Chapter 56, Art. 764.

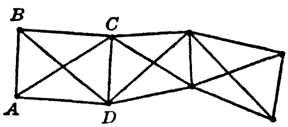
403. Three Different Forms of Triangulation Systems are illustrated in the accompanying figures.

403(*a*). THE FIRST FORM is that in which there is a simple chain of triangles. In this form there is only one possible set of triangles by means of which the calculations of the lengths of the lines between stations can be carried throughout the chain.



Chain of Triangles

FIG. 403(*a*).

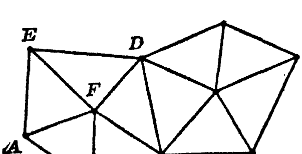


Chain of Quadrilaterals

FIG. 403(*b*).

403(*b*). THE SECOND FORM is that in which there is a chain of quadrilaterals, each of which is divided into triangles by its two diagonals. The intersection of the diagonals is not, however, generally used as a triangulation station. Consequently the four smaller triangles, with apices at the intersection of the diagonals, are not involved in the computations. In each quadrilateral, the length of one side can be calculated from the length of the opposite side by the use of any one of four sets of triangles, each set being composed of two triangles. For example, assuming that the length of *AB* is known, the calculations for determining *CD* may be carried through the two triangles, *ABC* and *BCD;* or *ABD* and *BDC;* or *BAD* and *ADC;* or *BAC* and *ACD.* This may be regarded as *carrying computations through different routes.*

403(*c*). THE THIRD FORM of triangulation network is composed of polygons, each of which is divided into triangles by lines from a station within the polygon to the stations in its perimeter. The polygons may have three, four, or five sides. A network of three-sided or four-sided, central-point polygons has certain advantages, though one with five-sided polygons, like that illustrated in the figure, is often used. The lengths of lines between stations can be calculated by the use of different sets of triangles. For example, if the length of *AB* is known, the length of *CD* can be calculated by the use of the tri-



Central-Point Polygons

FIG. 403(*c*).

angles *AFB, BFC,* and *CFD;* also, through the triangles *BFA, AFE, EFD,* and *DFC.*

There are many other systems of triangulation in common use, but most of them are based on or are combinations of the three that have been illustrated.

403(*d*). SYSTEMS COMPARED. The chain of simple triangles is particularly adapted for covering a comparatively long, narrow area, but the lengths of lines between stations can be calculated through only one route or set of stations, and the number of rigid geometric conditions is relatively small. Consequently neither the field work nor the calculations can be so well checked as in either of the other two systems. Nevertheless, the chain of simple triangles can often be used to advantage when a high degree of precision is not required. On the whole, the chain of quadrilaterals with both diagonals of each quadrilateral observed is the strongest and best system, but if the area to be covered is wide in proportion to its length the system of central-point polygons often makes possible a better-planned network.

403(*e*). RIGID GEOMETRIC CONDITIONS must be fulfilled in any network. The most common conditions are: (1) The sum of the interior angles of a figure having three, four, or five sides should be respectively 180°, 360°, and 540°. (2) If an angle is subdivided into two or more smaller angles, the sum of the smaller angles should equal the value of the large angle. (3) When *all* the angles at a station are measured, their sum should equal 360°. (4) When the length of a line has been calculated through two or more routes, the results should agree. Owing to errors, not all the rigid conditions are likely to be exactly fulfilled until field measurements have been adjusted. (For the method of adjustment see Art. 418.)

403(*f*). STRENGTH OF FIGURE. The strength of any triangulation network depends on such factors as the number of angles or directions observed, the number of geometric conditions in the network, and the proportions of each triangle.

No matter what system of triangulation is adopted, a single, poorly proportioned triangle weakens it. A triangle is poorly proportioned when any angle in it is too small or too large. Angles less than 30° or more than 150° should be avoided, not because it makes any difference in the field work—one angle is as easily measured as another—but because a given error in a small angle affects the computed lengths in a triangle more than does the same error in an angle which approaches 90°. This is because the tabular differences in a table of sines approach zero as the angle approaches 90°, whereas they are very large for angles near 0° or 180°. The ideal triangle is of course an equilateral triangle. It is

often difficult to select stations so that the triangles will be properly proportioned; weak triangles are a very important source of error and one sometimes overlooked.

It should be observed that angles smaller than 30° or larger than 150° do not weaken a network provided they are not used in the computations of the lengths of sides. Moreover, an angle which would be too small if used by itself may be permitted as a subdivision of a larger angle used in computation. For example, two adjacent angles may be respectively 15° and 40°; if the angle used in computation is 55° (the sum of the two), the angle 15° is, of course, permissible.

403(*g*). Criterion for strength of figure. In planning a triangulation system during reconnaissance, it is often desirable to compare the relative strength of two or more possible systems of figures. Likewise, it may be desirable to compare the relative accuracy with which a length of a side in a network may be computed through different routes, starting from the same line of known length. The strength of figure depends on (1) the number of directions observed, (2) the number of geometric conditions which must be satisfied in the least squares adjustment together with the number of stations at which angles (or directions) were measured, and (3) the sizes of *distance angles* used in the computations. (The distance angles A and B in each triangle are the angle opposite the side of known length and the angle opposite the side whose length is to be computed.) The criterion based on these three factors is expressed in the following formula: *

$$R = \frac{D - C}{D} \times \Sigma(\delta_A^2 + \delta_A\delta_B + \delta_B^2)$$

The smaller the computed value of R, the greater is the strength of figure.

D = number of directions observed, not including the known (fixed) side of the figure.

C = number of geometric conditions.

δ_A = units in the sixth place of a table of logarithmic sines corresponding to a change of 0° 0′ 1″ in the distance angle A.

δ_B = the corresponding difference in logarithmic sines for distance angle B.

The value of C may be found from $C = (n' - s' + 1) + (n - 2s + 3)$, in which n' = number of lines observed in both directions, including the fixed line; s' = number of occupied stations; n = total number of lines, including the fixed line; s = total number of stations.

Some of the values of the quantity $(D - C)/D$ for the more common triangulation figures are given in the following table:

Triangulation Figure	Number of Directions, D, Observed	Number of Conditions, C, in Least Squares Adjustment	$\frac{D - C}{D}$
Single triangle	4	1	0.75
Completed quadrilateral	10	4	0.60
Three-sided central point figure	10	4	0.60
Four-sided central point figure	14	5	0.64
Five-sided central point figure	18	6	0.67
Six-sided central point figure	22	7	0.68
Four-sided central point figure, with one diagonal observed	16	7	0.56

* See A.S.C.E. Manual, *Technical Procedure for City Surveys*, p. 14.

The value of the quantity $(\delta_A^2 + \delta_A\delta_B + \delta_B^2)$ may be taken from the table on p. 414. This table is entered by finding the value at the top of the table which corresponds most nearly to the smaller of the two distance angles A and B and by finding the value in the first column which corresponds to the larger of the two angles.

Illustrative Examples. Given the quadrilateral in Fig. 403(*g*). Required to compute the strengths of figures, first for carrying lengths from the side AD to the opposite side BC and then for carrying lengths from the side AB to the opposite side CD.

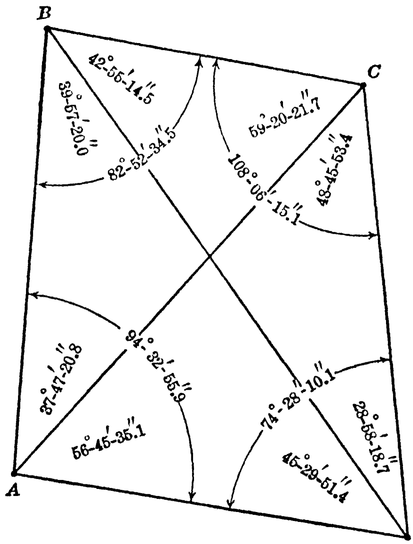


FIG. 403(*g*).

COMPUTATION FOR STRENGTH OF FIGURE

Triangle	Route	Distance Angles*		$\left(\frac{D-C}{D}\right) \times (\delta_A^2 + \delta_A\delta_B + \delta_B^2)$	R
		A	B		
ACD	*AD* to *AC*	49°	74°	0.6 × 4.6	2.8
ABC	*AC* to *BC*	83°	38°	0.6 × 8.2	4.9
				For the best route *AD* to *BC* R_1 =	7.7
ABD	*AD* to *BD*	40°	95°	0.6 × 6.0	3.6
BCD	*BD* to *BC*	108°	29°	0.6 × 12.4	7.4
				For the second-best route *AD* to *BC*, R_2 =	11.0
ABC	*AB* to *AC*	59°	83°	0.6 × 2.0	1.2
ADC	*AC* to *CD*	74°	57°	0.6 × 2.8	1.7
				For the best route *AB* to *CD*, R_1 =	2.9
ABD	*AB* to *BD*	45°	95°	0.6 × 4.0	2.4
BCD	*BD* to *CD*	108°	43°	0.6 × 4.0	2.4
				For the second-best route *AB* to *CD*, R_2 =	4.8

* Distance angles A and B are not necessarily angles at A and B in Fig. 403(*g*).

The R of the best-shaped chain of triangles is generally represented by R_1 and that of the second best by R_2. For first-order triangulation, R_1 for any figure in the direct chain between bases should rarely exceed 5. For any line of a triangulation system which is located outside the direct chain of figures between bases, the summation of R_1 from the nearest line in that chain should rarely exceed 15. The strength R_1 should be computed for all triangulation figures during the reconnaissance so that the unusually weak figures may be discarded and the strongest alternative figures may be adopted.

RELATIVE STRENGTH OF FIGURES IN TRIANGULATION *

°	10°	12°	14°	16°	18°	20°	22°	24°	26°	28°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
10	428	359																					
12	359	295	253																				
14	315	253	214	187																			
16	284	225	187	162	143																		
18	262	204	168	143	126	113																	
20	245	189	153	130	113	100	91																
22	232	177	142	119	103	91	81	74															
24	221	167	134	111	95	83	74	67	61														
26	213	160	126	104	89	77	68	61	56	51													
28	206	153	120	99	83	72	63	57	51	47	43												
30	199	148	115	94	79	68	59	53	48	43	40	33											
35	188	137	106	85	71	60	52	46	41	37	33	27	23										
40	179	129	99	79	65	54	47	41	36	32	29	23	19	16									
45	172	124	93	74	60	50	43	37	32	28	25	20	16	13	11								
50	167	119	89	70	57	47	39	34	29	26	23	18	14	11	9	8							
55	162	115	86	67	54	44	37	32	27	24	21	16	12	10	8	7	5						
60	159	112	83	64	51	42	35	30	25	22	19	14	11	9	7	5	4	4					
65	155	109	80	62	49	40	33	28	24	21	18	13	10	7	6	5	4	3	2				
70	152	106	78	60	48	38	32	27	23	19	17	12	9	7	5	4	3	2	2	1			
75	150	104	76	58	46	37	30	25	21	18	16	11	8	6	4	3	2	2	1	1	1		
80	147	102	74	57	45	36	29	24	20	17	15	10	7	5	4	3	2	1	1	1	0	0	
85	145	100	73	55	43	34	28	23	19	16	14	10	7	5	3	2	2	1	1	0	0	0	0
90	143	98	71	54	42	33	27	22	19	16	13	9	6	4	3	2	1	1	1	0	0	0	0
95	140	96	70	53	41	32	26	22	18	15	13	9	6	4	3	2	1	1	0	0	0	0	
100	138	95	68	51	40	31	25	21	17	14	12	8	6	4	3	2	1	1	0	0	0		
105	136	93	67	50	39	30	25	20	17	14	12	8	5	4	2	2	1	1	0	0			
110	134	91	65	49	38	30	24	19	16	13	11	7	5	3	2	2	1	1	1				
115	132	89	64	48	37	29	23	19	15	13	11	7	5	3	2	2	1	1					
120	129	88	62	46	36	28	22	18	15	12	10	7	5	3	2	2	1						
125	127	86	61	45	35	27	22	18	14	12	10	7	5	4	3	2							
130	125	84	59	44	34	26	21	17	14	12	10	7	5	4	3								
135	122	82	58	43	33	26	21	17	14	12	10	7	5	4									
140	119	80	56	42	32	25	20	17	14	12	10	8	6										
145	116	77	55	41	32	25	21	17	15	13	11	9											
150	112	75	54	40	32	26	21	18	16	15	13												
152	111	75	53	40	32	26	22	19	17	16													
154	110	74	53	41	33	27	23	21	19														
156	108	74	54	42	34	28	25	22															
158	107	74	54	43	35	30	27																
160	107	74	56	45	38	33																	
162	107	76	59	48	42																		
164	109	79	63	54																			
166	113	86	71																				
168	122	98																					
170	143																						

* From *Manual of Triangulation, Computation, and Adjustment*, by W. F. Reynolds, Special Publication 138, U.S. Coast and Geodetic Survey, p. 235; or from *Manual of First-Order Triangulation*, by C. V. Hodgson, Special Publication 120, U.S. Coast and Geodetic Survey, p. 6.

403(*h*). Checking triangulation by measuring base lines. Any line in a system of triangulation whose length is measured may be regarded as a base line. In a system covering a small area, one side of the last triangle may be used as a second base line. The discrepancy between the measured length of this base line and its computed length is one indication of the degree of precision attained in the triangulation network between the original base line and this second base line. In a more extensive system, base lines are established at intervals and that portion of the system between two adjacent base lines may be checked by computing the length of one of the base lines, beginning with the known or accepted length of the other, and then comparing this computed length with the measured length.

403(*i*). The classification of triangulation systems corresponds to the classification of traverses (Arts. 358, 360). The requirements for the different orders are based primarily on: (1) the permissible discrepancy between the computed and measured lengths of a base line; (2) the permissible average angular error of closure in the triangles; (3) the permissible maximum angular error in any triangle. The permissible angular errors are usually expressed in seconds of arc. The corresponding requirements for the first three classes as specified by the U.S. Coast and Geodetic Survey are:

	First Order	*Second Order*	*Third Order*
Base line discrepancy	$\frac{1}{25,000}$	$\frac{1}{10,000}$	$\frac{1}{5000}$
Average triangle closure	1″	3″	6″
Maximum triangle closure	3″	8″	12″

Fourth-order triangulation is used as a system of horizontal control in stadia surveys, plane-table surveys, and transit surveys, in which the required degree of precision is not high. The accuracy required will depend on the purpose of the survey. If, for example, the survey is merely for the purpose of mapping an area to a small scale, the degree of precision may be relatively low. In general, the permissible errors in triangulation of the fourth order may be assumed to be about twice as great as the corresponding permissible errors in triangulation of the third order.

404. The Purpose of a Triangulation Network is to establish a system of horizontal control in which the stations are control points. By the use of one or more of these control points, any survey, or different surveys, or different parts of the same survey can be tied in to the triangulation network. For example, a traverse may begin at one control point and end at another.

The U.S. Coast and Geodetic Survey is gradually covering the whole country with a triangulation network. (See Chapter 56.) Any survey that is tied in to this network in one part of the country is, therefore, coordinated with any survey tied in to the network in any other part of the country. In many cities and in some states, plane rectangular coordinate systems have been established for general use as systems of horizontal control, and these coordinate systems are in turn tied in to the network of the U.S. Coast and Geodetic Survey. (See Chapter 56.) Many cities have triangulation systems from which points of control have been established throughout the city areas, to be used not merely by city engineering departments but by private engineers as well. Similarly, triangulation networks are used as bases of control in surveys for highways, railroads, tunnels, waterways, and in other extensive surveys. In a topographic survey covering a comparatively large area, a triangulation network is often absolutely necessary to prevent the accumulation of errors as the survey progresses.

When the sides of the triangles are so long and the precision required is so great that the curvature of the earth must be taken into account, the work is no longer plane surveying—it is geodetic surveying. The question arises as to how large an area it is permissible to cover in plane surveying. This depends, of course, upon the required limits of error. The errors which result from neglecting curvature are indicated by the examples given in Art. 5(*a*). (See also *spherical excess,* Art. 402.)

405. Station Marks. In triangulation of a temporary sort, stations may be marked by any of the methods used for ordinary transit stations, such as a wooden hub, a pipe driven in the ground, or a point chiseled on a rock or other hard surface. For more permanent stations, bronze or copper markers cemented into solid rock or concrete monuments with metal caps or tablets are generally used. Since monuments set in earth may be disturbed, an underground mark below the monument is often used. This is equivalent to setting one monument on top of another. Each station should be carefully referenced, usually by at least two reference points within a distance of 100 ft., more or less, from the station. The markers for these points should be practically as permanent as that used for the station except that usually no underground marks are set. The azimuth and distance to each reference mark should be measured and recorded.

When starting a traverse survey from a triangulation station as a point of control, it is not always convenient or practicable to backsight on another triangulation station in order to orient the

survey. For this reason, a permanent azimuth reference mark, similar to an ordinary reference mark, is often established, and the azimuth to it carefully measured. It is not necessary that this mark should be set on one of the triangulation lines; it may be in any convenient place where it will be clearly visible from the ground at the triangulation station. It should be at least 100 yd. from the station and often a much greater distance in order to furnish a length of backsight adequate for accurate orientation. The description of this azimuth mark and its location, the value of the azimuth of the line from the station to the mark, and other information that is kept on file should be so clear and complete that it will enable any competent surveyor to set up his instrument at the triangulation station, backsight on the azimuth mark, and start his survey with this line of known azimuth as readily as if he had determined the azimuth of the line himself. Sometimes two or more permanent azimuth reference marks are established at a station.

It is frequently desirable in a triangulation system to establish a station or a *substation* at which no angles will be measured, but which can be located from other stations as a definite point in the system. It is often practicable to use for such a station some permanent, well-defined object such as a church steeple or a high chimney. Such an object will also often serve as an azimuth reference mark.

Note. In the U.S. Coast and Geodetic Survey's Special Publication 158, specifications for marks for triangulation stations and directions for their construction are given in detail. (See also Chapter 56 for descriptions of triangulation stations and azimuth marks.)

406. Towers. When it becomes necessary to elevate an instrument above a triangulation station in order to obtain a clear sight above intervening obstacles, a platform or a tower may be erected over the station. For accurate work, it is desirable to have two independent, unconnected structures, one within the other. An instrument set up on the inner one is then free from any disturbance that might be caused by the movements of the instrument man on the outer one. Platforms on simple supports can be used when the required height above the ground is not too great; otherwise some form of a tower, rigidly constructed and firmly anchored, is necessary. In order to minimize the effects of refraction, it is desirable to have the instrument elevated to such a height that the line of sight will be well above surfaces over which it passes.

Note. In extensive systems of triangulation, towers which can repeatedly be erected, taken down, and moved from station to station are often used. The well-known Bilby Steel tower, described in the U.S. Coast and Geodetic Survey's Special Publication 158, is a tower of this type.

407. Signals for Stations. At each triangulation station there should be a signal, more or less permanent, which can be sighted from near-by stations. At a station where no angles will be measured, the signal may be a pole, with one end in the ground, held rigidly in a vertical position by guys, or by a braced framework, or by some other device. At a station where angles will be measured, provision must be made for setting up the instrument over the station. One of the simplest devices is a pipe set vertically in the ground or

in rock to mark the station and a pole set in the pipe as a signal. The pole can be removed temporarily when an instrument is to be used at the station.

In the accompanying figure is shown another simple form of signal which may be used in ordinary work. The requirements are: (1) The mast should be plumb and over the station. (2) The three legs should be set firmly in the ground and be cross-braced to make them rigid. (3) The cross bracing and the bottom of the mast should be high enough from the ground to allow the instrument to be placed underneath and the work to be carried on without disturbing the signal.

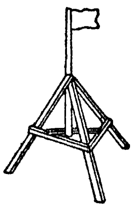


FIG. 407.

407(*a*). PRACTICAL SUGGESTIONS. (1) To anchor the legs dig holes 2 ft. or so into the ground, and when the legs are planted drive a stake into the ground beside each leg and nail the leg to the stake. If a signal is on a ledge, use anchor bolts driven into drill holes in the rock. (2) Plumb the mast by observing it from four different positions, holding a plumb line before the eye. If it is desired to plumb the mast more accurately, use the transit. (3) Signals may often be constructed from poles, 4 in. or 5 in. in diameter, cut in the woods. (4) It is sometimes difficult to set a *quadripod* signal so that its legs will not interfere with lines of sight to other stations; for this reason a *tripod* signal is often preferred. (5) Put a small hook on the bottom of the mast from which to suspend the plumb bob.

407(*b*). PRACTICAL QUESTIONS.

1. *What should be the diameter of the mast?* Make it as small as possible and yet have it distinctly visible, so that it may be accurately bisected by the cross hair. The latter will ordinarily cover a strip 0.3 in. wide at a mile, or about 3 in. for a 10-mile sight. Use a pole 1 in. in diameter for sights less than a mile, and poles up to 3 in. in diameter for 5-mile sights. A pole 4 in. in diameter can be seen 20 miles with a good telescope and clear atmosphere.

2. *What error is allowable in centering a mast or an instrument over a station?* From the table in Art. 283(*d*) it is seen that in order to make an error of 1″ in an angle the instrument or masts must be out of position the following amounts:

$\frac{1}{32}$ in.	for sights	500 ft. long.
$\frac{1}{16}$ "	" "	1000 " "
$\frac{5}{16}$ "	" "	a mile long.
3 "	" "	10 miles long.

If the lengths of sights are known approximately, and limits of error for angular measurement have been adopted, the above table will serve as a guide in centering the mast, setting up the instrument, and in pointing.

407(*c*) NOTE. In place of a simple signal mast, some form of target is often used, particularly when the lengths of sights are great. In triangulation systems of the first order, observations are usually taken at night, and special illuminated signals must, therefore, be used. Descriptions of various forms of signals, used in work of high precision, may be found in books on geodetic surveying and in publications of the U.S. Coast and Geodetic Survey.

407(*d*). COMMUNICATION. When triangulation stations are too distant from each other to permit the use of ordinary methods of signaling, some special means of communication must be established. A two-way radio system is excellent for this purpose. For communication at night, the Morse code is in common use; the dots and dashes are transmitted by means of signal lights. (See *Manual of First-Order Triangulation*, U.S. Coast and Geodetic Survey, Special Publication 120.) The following explanation of the Morse code and its use is taken from Manual 20, *Horizontal Control Surveys*, published by the American Society of Civil Engineers.

A ·—	J ·———	S ···	1 ·————
B —···	K —·—	T —	2 ··———
C —·—·	L ·—··	U ··—	3 ···——
D —··	M ——	V ···—	4 ····—
E ·	N —·	W ·——	5 ·····
F ··—·	O ———	X —··—	6 —····
G ——·	P ·——·	Y —·——	7 ——···
H ····	Q ——·—	Z ——··	8 ———··
I ··	R ·—·		9 ————·
			0 —————

Instructions in the Use of the Morse Code. *BT* (—·· ·—) should be used between angles instead of the word "stop." A question (?) is signaled as ··— —··. All angles should be transmitted by numerals, but before sending numerals, send *NF* to signify that "numerals follow." When the transmitting of numerals has been completed, send *LF* to signify that "letters follow."

A series of one-half-second dots means, "I have made a mistake, will begin again." A series of one-second dots means, "I understand your message." *R* means, "Repeat your message, I could not get it."

408. The General Order of Field Work in triangulation is generally as follows:

1. A reconnaissance of the terrain to be covered by the triangulation in order to gain the knowledge necessary to plan the best system and to locate tentatively, at least, the points where stations will be established.

2. The marking of stations, the erection of signals at stations, and, if necessary, the construction of elevated platforms and towers. It may be necessary to complete some of this work from time to time as the survey progresses.

3. The preparation of the base line for measurement by clearing and, in some cases, by driving stakes; the measurement of this base line with a tape. When it is practicable, it is well to use, as a base, some line whose length and location have been accurately determined by a previous survey.

4. The measurement of each angle that is to be used in calculation. Theoretically, if three angles of a triangle are to be used in calculation, only two of them need be measured, but in practice it is customary to measure all three as a check and for purposes of adjustment. When a station is at the vertex of each of two or more triangles, several combinations of measurements of angles are possible; the angle of each triangle may be measured, then the sum angle of any two angles or any three angles, and so on. Although this multiplication of measurements furnishes more conditions for the final adjustment of all the angles at the station, it is not advisable because the extra time required is seldom warranted by the increase in the accuracy of the results. It is better to use the time available in the careful measurement of only those angles which will be used in the computations.

5. The determination by astronomical observations of the true azimuth of at least one side in the triangulation system. This may be done at the beginning of the survey or later. When the triangulation system is tied in with a previous survey, the azimuth of a line in that survey may be used instead of one determined by observations. In triangulation where true azimuths are of no consequence, astronomical observations, of course, are not taken. In geodetic surveying, astronomical observations are made to deter-

mine the geographical coordinates, namely, the latitude and the longitude, of points in the triangulation system.

The adjustment of observations and the calculation of the lengths of the sides in the triangulation system are a part of the office work.

409. Reconnaissance. A thorough reconnaissance of the ground to be covered is essential in planning any system of triangulation. Since the immediate purpose of such reconnaissance is to determine the location of stations, and since the number and positions of stations will determine the accuracy, cost, and future usefulness of the triangulation, it pays to give careful consideration to all feasible schemes. Under favorable conditions, reconnaissance may be simple, but under unfavorable conditions, reconnaissance may be complex and difficult even when the area covered is relatively small. Among the requirements to be considered are these. (1) The system of triangulation best suited to the purpose of the survey (Art. 403(*d*)) and the strength of figure of that system (403(*f*)). Aim at the simplest, best-balanced system of triangles practicable. (2) The number of stations to be occupied and the number of angles to be measured. (3) Stations in any one triangle must be clearly intervisible, and all stations as accessible as possible. (4) Avoid, on the one hand, high, costly towers and, on the other hand, excessive clearing. When one or the other is necessary, determine which is preferable. (5) Suitable locations for base lines. (6) Stations located as far as possible where they will be most useful either in the location of prominent features, or as starting and closing points for traversing, or for establishing secondary systems of triangles. (7) Lengths of sight suited to the telescope used. (8) The greatest possible area in view from each station if there are a large number of substations to be controlled. (9) Firm ground or stable support for the instrument, free from vibration, sheltered from wind. (10) Freedom from atmospheric disturbances. For example, if a line of sight passes over a hilltop close to the ground or too close to the surface of a body of water, it will be affected by the "boiling" of the air, or if it passes near a chimney continually emitting smoke, it may be almost useless.

It is very seldom that all the above requirements can be realized. Usually the surveyor has to take stations where he can get them. The most favorable conditions for triangulation are generally found in hilly country, with summits comparatively clear and unobstructed upon which stations may be established. The least favorable conditions are in flat wooded country, where it is either necessary to do a great deal of clearing or else to build high towers

upon which to set up the instrument. When at any station, either clearing or the construction of a tower is necessary, it is well to consider which would be less costly and more practicable. Contrary to a general impression, clearing is often preferable. In cities, stations may be established on the tops of buildings, and church spires or other prominent objects may be taken for substations.

During the reconnaissance, it is often helpful to make sketches from points of highest elevations, noting particularly all high points which might be favorable positions for triangulation stations. By taking magnetic bearings and estimating distances to such points, a rough sketch map may be made which will be helpful in planning the triangulation. Existing maps and aerial photographs, even though they may not be very accurate, are also helpful. Information which may be of future use to parties in the field should be noted.

409(*a*). THE SIZE OF TRIANGLES in a triangulation network is largely determined by the purpose for which the network is used. For example, in a topographic survey, stations should be near enough together to afford adequate control for the traverses. In a geodetic survey, on the other hand, stations may be miles apart.

The size of triangles may also be determined to some extent by the varying character of the terrain. For this reason it may be advisable or necessary to use triangles of different sizes in different portions of the same network. In portions of the terrain where conditions make triangulation difficult or expensive as, for example, in the flat wooded areas, it may be best to plan comparatively large triangles, whereas in portions which are open and where conditions make triangulation easy and inexpensive, it may be best to plan comparatively small triangles.

In geodetic surveying the curvature of the earth limits the length of a line of sight; hence the size of triangles may be limited by the maximum height to which it is desirable or practicable to erect towers.

409(*b*). TRIANGULATION SYSTEMS AND TRAVERSES COMPARED. By the use of modern methods and equipment, traverses may be run under favorable conditions for chaining with a degree of precision comparable to that attained in triangulation. A system of traverses is often less expensive and better suited to the purpose of the survey than a triangulation net. For example, in a certain survey of the terrain along a river, approximately parallel traverses were run, one on each side. These traverses were connected at intervals by lines run across bridges. Not only was this system of traverses less expensive than triangulation would have been, but also, by its use, it was possible to locate control points more advantageously. The belief that triangulation is the best and only accurate method of establishing control points has sometimes led to its use where a traverse would have been better. The question where to use a system of triangulation and where to run traverses should be given careful consideration in reconnaissance.

410. The Base Line. (*a*). LOCATION. Try to select a place for the base line: (1) where the ground is level and well suited for accurate measurement; (2) where the country is open, so that sights may be taken in any desired direction from each end of the line; (3) where a system of well-proportioned triangles can be built up which meet the requirements of the preceding article. It is particularly desirable that a base line should be so located that the computation to the next base line can be carried through two strong routes (Art. 403(*g*)).

Note. Often the surveyor will be unable to find a suitable place for a base line and must overcome natural obstacles as best he can. Sometimes it is necessary to lay out a broken base line and find the direct distance between the two ends by indirect measurements. The required distance corresponds to an omitted measurement, and is found by computations. (See Chapter 39.)

Occasionally, it may be necessary to locate the base line in a place where it is impossible to establish signal stations at the ends as, for example, on a railroad track. The signals may be established elsewhere, and the ends of the base transferred by some accurate system of check measurements.

When the lengths of sides of a triangulation network are relatively short, one side of one triangle may be used as the first base line. When the lengths of sides are long, as in geodetic surveying, the measurement of one side as a base would usually be difficult and expensive. A relatively short base line is, therefore, established in such a position that it may be connected to one or more sides of the triangulation network by means of a smaller network, known as the *base net*. This base net may be simple or quite complex, depending upon conditions. An example of a simple base net is shown in Fig. 410(*a*). From the measured base line *AB*, the length of *CD*, one side of a triangle *CDE* in the main triangulation network is calculated. By locating the base line so that it extends across the side of the main triangle, as shown in the figure, the strength of this form of base net is made greater than if the base were wholly on one side.

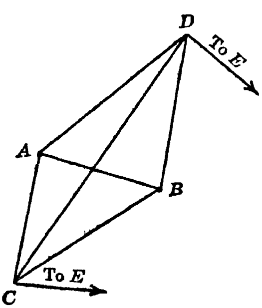


FIG. 410(*a*).

Since, in an extensive network, the original base line can be measured with accuracy greater than can be sustained throughout the triangulation, it is customary to introduce a new base line whenever the accumulated errors are likely to exceed the permissible limits of error. The number of base lines or the intervals between them will depend upon the strength of the triangulation system. In first-order triangulation, this strength may be deter-

mined (Art. 403(*f*)). Usually the work can be carried through at least ten triangles and often through twenty or thirty before introducing another base line. In ordinary triangulation covering a relatively small area, the original base line and a check base line at the end of the chain of triangles are usually sufficient.

410(*b*). THE LENGTH OF BASE LINE. When the base line is one side of a main triangle, its length will depend, of course, upon how large the triangles are. This, in turn, depends upon the character of ground to be covered and the points where stations are needed. The base line must be of such a length that properly proportioned triangles can be built up. To adopt a base line several hundred feet long and make each of the two other sides of the first triangle several miles long is an extreme illustration of what should not be done.

When in an extended system a base net is used, the length of the base line will generally be one-sixth to one-quarter the length of the side of the main triangle with which it is connected.

410(*c*). THE MEASUREMENT OF THE BASE LINE. Since any error in the length of the base line affects every side of every triangle in the network between it and the next base line, precautions must be taken to measure it with a degree of precision corresponding to the order of the triangulation. This whole subject has been treated in Chapter 8.

410(*d*). USE OF A PREVIOUSLY DETERMINED LINE AS A BASE. When it is practicable, it is well to use for a base line the line between two control points already established and accepted. In such a case, if the distance between control points is much longer than the sides of the new triangulation system, the process of expanding a base by means of a base net may be reversed. For example, assume that in Fig. 410(*a*), the points *C* and *D* are accepted control points and that the known distance *CD* is to be used as a base for a triangulation system in which the sides are much shorter than *CD*. By a system of triangulation, a shorter line such as *AB* may be determined, and this line may serve as one side of a triangle in the new triangulation system.

411. Measurement of Angles. The instrument and the method used in measuring the angles of a triangulation network depend mainly upon the degree of precision required. The instrument used may be an ordinary transit for ordinary precision; a precise repeating instrument for intermediate precision, either a precise transit or a repeating theodolite; a repeating instrument or a direction theodolite for high precision; a direction theodolite for the highest precision.

Two general methods for the precise measurement of angles, namely, the *method of repetition* and the *direction method,* have been explained in Arts. 264–268. For all except triangulation of the highest precision, the method of repetition is sufficiently accurate. The method of direction is generally used for triangulation of the highest precision, and some surveyors prefer it whenever the degree of precision required is relatively high. Whichever method is used, errors in angular measurement must be reduced to the extent necessary in order to attain the required degree of precision. This reduction of errors in angular measurement is treated in Chapter 27, "Precise Measurement of Angles."

412. The Method of Repetition. The general field procedure in the method of repetition, the field notes, the number of repetitions in a set, the number of sets required, and the degree of precision that can be attained have all been treated in Chapter 27. Consequently there will be given here only an example of the field notes for the measurement of the angles at a single triangulation station. The accompanying explanation of these notes is also, incidentally, an explanation of the field procedure in the method of repetition. The notes are given in one of the standard forms; a simpler form for the same notes is given in Art. 398(*b*). A careful comparison of these two forms is recommended.

412(*a*). Example of field notes.* At a station called Mellon, five angles were measured with a 7-in. repeating theodolite. Each set consisted of six repetitions of the angle and six repetitions of its explement. The notes shown include only those of the third set for each angle.

Note. For the number of sets required for each angle see Art. 401. For the method of increasing the initial reading of each set see Art. 399.

In the measurement of the first angle, the initial reading for the backsight on station Lynn was approximately 50° 00′ (column 4), and after six repetitions, the foresight reading on station Homewood was approximately 160° 08′ (column 4). This last reading was also the approximate initial reading for the backsight on station Homewood in measuring the explement of the angle, and after six repetitions, the corresponding foresight reading on station Lynn was approximately 50° 00′ (column 4, third quantity). In columns 5 and 6 are given the vernier readings that are to be combined with the circle readings.

In the initial reading for the angle, the A vernier reading is $\overline{55}''$; the line over the 55 indicates that the final reading is 5″ less than the circle reading of 50′ 00″, i.e., 49° 59′ 55″. Note that the circle reading becomes 49° 59′ or 01′ less than the recorded circle reading.

In column 8, the heading "Arc passed over" means the arc, measured clockwise, between the initial reading for an angle and the corresponding final reading *after six repetitions. This arc is not the value of the angle.*

The angle is computed as follows. To the arc passed over add the number of complete circles (360°) passed over by the indicator of either vernier during the six repetitions, each circle beginning and ending at the initial setting of the vernier. Divide the sum thus obtained by the number of repetitions. (See also the method of short division, Art. 265(*f*).)

* Taken from *Technical Procedure for City Surveys,* published by the American Society of Civil Engineers.

Form of Field Notes, Repetition Method, for Two-Vernier Theodolite

Station occupied, Mellon

Station Sighted (1)	Tele-scope Direct—Re-versed (2)	Number of Repe-titions (3)	Circle Reading (4)	Ver-nier A (5)	Ver-nier B (6)	Mean (7)	Arc Passed Over (8)	Angle (9)	Mean (10)
Lynn	D	0	50° 00′	$\overline{55}''$	00″	$\overline{57}''.5$			
Homewood	D-R	6	160° 08′	40″	30″	35″	110° 08′ 37″.5	78° 21′ 26″.2	26″.4
Set 3	R-D	6	50° 00′	$\overline{55}''$	$\overline{55}''$	$\overline{55}''$	249° 51′ 20″.0	281° 38′ 33″.3	
								−0″.5	
Homewood	D	0	50° 00′	$\overline{55}''$	$\overline{55}''$	$\overline{55}''$			
Synagogue	D-R	6	221° 43′	45″	50″	47″.5	171° 43′ 52″.5	28° 37′ 18″.7	19″.5
Set 3	R-D	6	50° 00′	$\overline{45}''$	$\overline{45}''$	$\overline{45}''$	188° 15′ 57″.5	331° 22′ 39″.6	
								−1″.7	
Synagogue	D	0	50° 00′	00″	00″	00″			
Negley	D-R	6	20° 51′	40″	35″	37″.5	330° 51′ 37″.5	55° 08′ 36″.2	37″.2
Set 3	R-D	6	50° 00′	$\overline{50}''$	$\overline{45}''$	$\overline{47}''.5$	29° 08′ 10″.0	304° 51′ 21″.7	
								−2″.1	
Negley	D	0	50° 00′	$\overline{50}''$	$\overline{45}''$	$\overline{47}''.5$			
Homeo	D-R	6	288° 35′	55″	55″	55″	238° 36′ 07″.5	39° 46′ 01″.2	00″.2
Set 3	R-D	6	50° 00′	00″	00″	00″	121° 24′ 05″.0	320° 14′ 00″.8	
								+2″.0	
Homeo	D	0	50° 00′	00″	00″	00″			
E End	D-R	6	357° 25′	35″	25″	30″	307° 25′ 30″	51° 14′ 15″.0	13″.6
Set 3	R-D	6	50° 00′	15″	20″	17″.5	52° 34′ 47″.5	308° 45′ 47″.9	
								+2″.9	

The arithmetical work for the first angle and its explement is as follows.

Measurement of the Angle

The average initial reading for the backsight on Lynn is:

(50° 00′) + (−0.5″ + 00″) ÷ 2 = 49° 59′ 57″.5

After six repetitions, the average reading for the foresight on Homewood is:

160° 08′ + (40″ + 30″) ÷ 2 = 160° 08′ 35″

The arc between the initial and final readings (clockwise) is:

160° 08′ 35″ − 49° 59′ 57″.5 = 110° 08′ 37″.5 (column 8)

The angle is:

(360° + 110° 08′ 37″.5) ÷ 6 = 78° 21′ 26″.2 (column 4)

(Check 78° 21′ 26″.2 by the method of short division, Art. 265(*f*).)

Measurement of the Explement

The vernier setting remains the same as it was at the end of the measurement of the angle, i.e., 160° 08′ 35″.

After six repetitions, the average reading for the foresight on Lynn is: 49° 59′ 55″.

The arc between the initial and final readings (clockwise) is:

(49° 59′ 55″ + 360° 00′ 00″) − 160° 08′ 35″ = 249° 51′ 20″ (column 8)

The angle is:

(1440° + 249° 51′ 20″) ÷ 6 = 281° 38′ 33″.3 (column 9)

The error is:

360° − (78° 21′ 26″.2 + 281° 38′ 33″.3) = 0″.5

The adjusted value of the angle is:

78° 21′ 26″.2 + (0″.5 ÷ 2) = 78° 21′ 26″.45 (column 10)

413. The Direction Method for one set of observations has been explained in detail in Arts. 267, 268. The number of sets or *positions* generally specified for the measurement of angles of direction at any station are: for triangulation of the *first order,* sixteen with a 1″ direction theodolite, and twenty to twenty-four with a 2″ instrument; for *second order,* four with a 1″ and four to eight with a 2″ instrument; for *third order,* two with a 1″ and two to four with a 2″ instrument. At the beginning of each set, the initial reading is changed in order to distribute the readings for different sets in different positions around the circle. The term *position* is, therefore, commonly used in place of *set.* Initial readings may be determined as explained in Art. 399. Approximate initial settings when sixteen positions are required may be taken from the following table.

APPROXIMATE INITIAL SETTINGS OF MICROMETER *A* FOR THE REQUIRED SIXTEEN POSITIONS OF THE DIRECTION OBSERVING PROGRAM

Position	Setting	Position	Setting	Position	Setting	Position	Setting
1	0° 00′ 40″	5	45° 00′ 40″	9	90° 00′ 40″	13	135° 00′ 40″
2	11° 01′ 50″	6	56° 01′ 50″	10	101° 01′ 50″	14	146° 01′ 50″
3	22° 03′ 10″	7	67° 03′ 10″	11	112° 03′ 10″	15	157° 03′ 10″
4	33° 04′ 20″	8	78° 04′ 20″	12	123° 04′ 20″	16	168° 04′ 20″

The entire procedure of measuring the angles of direction at any station will be better understood from a study of the field notes that follow, but it should be kept in mind that the notes are for triangulation of a higher order than is necessary in many cases.

413(*a*). EXAMPLE OF FIELD NOTES.* At a station called Pine Rue, angles of direction to four other stations were measued with a two-micrometer theodolite. The notes shown are those for the first set or position. The initial pointing to station Swartz is equivalent to a backsight, and the angle of direction to each of the other stations is measured from this backsight. The *direction* of a station is to be understood, therefore, as the angle measured from the initial pointing or setting.

The mean readings for the initial pointing on station Swartz are 00° 00′ 13″.0 from micrometer *A*, and 00° 00′ 10″.0 from micrometer *B*. The mean of these two readings is 00° 00′ 11″.5. After measuring directions clockwise to Glasgow, Island, and Park stations, and closing on Swartz, the mean readings, given at the bottom of the table, are from micrometer *A*, 00° 00′ 14″.5 and from micrometer *B*, 00° 00′ 15″.0. The mean of these two readings is 00° 00′ 14″.8. This differs from the mean of the initial readings on Swartz by 14″.8 − 11″.5 or 3″.3.

The telescope is reversed and a new initial pointing is taken on Swartz for which the mean readings are: from micrometer *A*, 180° 00′ 09″.0 and from micrometer *B*, 180° 00′ 07″.0. The mean of these two readings is 180° 00′ 08″.0. After measuring directions counterclockwise to Park, Island, and Glasgow, and closing on Swartz, the mean readings are: from micrometer *A*, 180° 00′ 06″.0, and from micrometer *B*, 180° 00′ 05″.0. The mean of these two readings is 180° 00′ 05″.5. This differs from the mean of the initial reading on Swartz by 08″.0 − 05″.5 or 2″.5.

There are now two mean readings or directions for each station differing only in seconds as shown in column 8. One was obtained when the telescope was normal, and the angle was turned clockwise; the other, when the telescope was reversed and the angle was turned counterclockwise. The mean of each pair of readings in column 8 is given in

* Taken from *Technical Procedure for City Surveys*, published by the American Society of Civil Engineers.

column 9. An adjustment of 1″.4 is made in column 9 in order that each direction in column 10 may correspond with an initial setting of 00° 00′ 00″.

The final values for directions given in column 10 were found by subtracting from each direction the initial setting, using the values for seconds given in column 9. For example, the direction for Glasgow is 45° 55′ 01″.2 − 00° 00′ 09″.9 = 45° 54′ $\overline{51}$″.3. The line over 51.3 is a method of indicating that 1′ should be subtracted from the value for minutes given in column 3. The direction for Island is 104° 13′ 40″ − 00° 00′ 09″.9 = 104° 13′ 30″.1.

Form of Field Notes, Direction Method, for Two-Micrometer Theodolite

Station occupied, Pine Rue — Position No. 1

Station Sighted	Telescope, Direct or Reversed	Circle Reading	Micrometer A: Forward Backward	Micrometer A: Mean	Micrometer B: Forward Backward	Micrometer B: Mean	Mean, A and B	Mean, D and R	Final Angle
			15″		09″				
Swartz	D	00° 00′	11″	13″.0	11″	10″.0	11″.5	08″.5	00″.0
			06″		04″			+1″.4	
	R	180° 00′	06″	06″.0	06″	05″.0	05″.5	09″.9	
			06″		08″				
Glasgow	D	45° 55′	05″	05″.5	06″	07″.0	06″.2	01″.2	$\overline{51}$″.3
			54″		58″				
	R	225° 54′	57″	55″.5	56″	57″.0	56″.2		
			42″		44″				
Island	D	104° 13′	39″	40″.5	44″	44″.0	42″.2	40″.0	30″.1
			38″		38″				
	R	284° 13′	36″	37″.0	39″	38″.5	37″.8		
			27″		26″				
Park	D	167° 24′	24″	25″.5	25″	25″.5	25″.5	22″.2	12″.3
			20″		20″				
	R	347° 24′	16″	18″.0	20″	20″.0	19″.0		
			14″		14″				
Swartz	D	00° 00′	15″	14″.5	16″	15″.0	14″.8	11″.4	00″.0
			08″		08″			−1″.4	
	R	180° 00′	10″	09″.0	06″	07″.0	08″.0	09″.9	

From this first set, or position 1, one value for the direction of each of the four stations sighted has been finally determined. From sixteen positions, sixteen such directions for each station are obtained. The final direction for any station is taken as the mean of all the values for the sixteen directions for that station. In first-order triangulation, any value of an individual direction which differs more than 05″ from the mean should, in general, be rejected and a new observation made. The value of the final triangulation angle between the lines from Pine Rue to two stations is equal to the numerical difference between the final directions of these two stations.

414. Reduction to Center. It is sometimes desirable to use as one of the stations of a triangulation network some prominent object, such as a tall chimney or a church spire. Since, in such a case, it is not possible to occupy the station, it is necessary to determine the angles at the station indirectly. A temporary station is assumed as near to the main station as practicable, and at this *eccentric station,* angles to the adjacent stations of the main network are measured. From these angles and

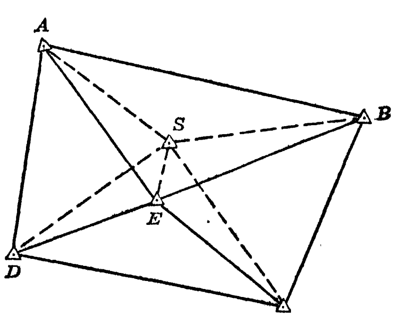


Fig. 414.

from values of the network already known, the angles at the main station can be calculated. This procedure, known as *reduction to center*, may be illustrated as follows. Suppose that in the triangulation network, shown by the full lines in Fig. 414, the angles at stations A, B, C, and D have been measured, but that it is not possible to set up an instrument at E. It is desired to determine the angles AEB, BEC, CED, and DEA. From angles at A, B, C, and D, and from the lengths of the sides AB, BC, and CD, the lengths of the sides EA, EB, EC, and ED may be calculated tentatively without the use of the angles at E.

414(*a*). THE FIELD WORK for reduction to center may be outlined as follows:

1. Assume a temporary station S, say 8 or 10 ft. from E, from which all five stations of the network are visible. (The distance ES is greatly exaggerated in the figure.)
2. Measure the distance SE.
3. Assume a meridian through SE. With the instrument at S and the initial pointing toward E, determine the direction or azimuth of each of the lines SA, SB, SC, and SD, using the method of direction explained in the preceding article. This work should be done with the same degree of precision required for the main triangulation. Any convenient meridian through S can, of course, be assumed in place of one through SE.

Note. If it is desired to use the method of repetition instead of the method of direction, measure the angles ESA, ESB, ESC, and ESD, and from these angles, determine the directions or azimuths of EA, EB, EC, and ED.

414(*b*). COMPUTATION OF THE ANGLES AEB, BEC, CED, and DEA. From the directions or azimuths of SA, SB, SC, and SD obtained in (3), determine the angles ESA, ESB, ESC, and ESD. Determine by computation the angles EAS, EBS, ECS, and EDS as follows. Let d represent the distance from the main station E to the eccentric station S. This distance is a common base of the triangles EAS, EBS, ECS, and EDS. In each of these triangles the angle at S is known and the lengths of two sides, namely, the base SE and the length (previously computed) of a side of the main triangulation network. Consequently the angle opposite to the base in each triangle may be calculated. For example, in the triangle EAS:

$$\sin EAS = \frac{d \times \sin ASE}{AE}$$

Since the distance d is very small compared with the lengths of the sides of the triangulation, the angle EAS is very small and the sine of EAS is practically equal to the arc. The value of EAS in seconds may, therefore, be obtained by dividing both sides of the equation by the sine of 1″ or

$$EAS = \frac{d}{\sin 1''} \times \frac{\sin ASE}{AE}$$

The value $\frac{d}{\sin 1''}$ may be determined once for all and used also in the calculation of the angles ESB, ECS, and EDS. The sine of 1″ is 0.000004848 and the colog sin 1″ is 5.314425. The log of the distance AE is taken directly from the computation of the sides of the main triangulation network.

The directions or azimuths of SA, SB, SC, and SD with respect to a meridian through ES having been measured in the field and the angles EAS, EBS, ECS, and EDS having been calculated, the directions or azimuths of EA, EB, EC, and ED, with respect to the same meridian are easily found, and from these directions, the angles AEB, BEC, CED, and DEA may be determined.

415. Limits of Error in Triangulation. In order to come within the limits of error already specified for first- and second-order triangulation (Art. 403(*i*)), it is necessary to use special equipment and methods of procedure which involve extreme precautions. It is interesting to inquire what degree of precision may be attained in triangulation of a lower order with an ordinary transit. In a test of a 6-in. transit reading only to 30″, sights were taken from the same station to five distant points. Angles were measured in succession, each angle being given twelve repetitions, six times with the telescope normal and six with the telescope in-

verted. Upon completing the circle the sum of the angles exceeded 360° by 5″, or an average error of 1″ per angle.

In an example taken from actual practice eight angles of a triangulation system were measured with a 6-in. transit reading to 20″. Twelve repetitions were given each angle, following the method of procedure outlined in Art. 397(*a*). Angles were adjusted by the method of logarithmic residuals, and the maximum difference between the measured value of an angle and its adjusted value was 3.98″, the minimum difference was 0.57″, and the mean difference for the eight angles was 1.86″. In a second system of eight angles the differences were: maximum 0.83″, minimum 0.07″, mean 0.44″. In the survey from which these results are taken work was continued throughout the day, regardless of atmospheric "boiling" or of wind, save in extreme cases, and the number of sets measured per day varied from ten to twenty-five, the time occupied in going from one station to another being an important factor. (See also Art. 400.)

The above results show what can be done with an ordinary transit. They represent a higher degree of precision than is ordinarily required in simple triangulation covering small areas. In any given case the allowable angular error is determined by the allowable error in the position of a triangulation point, and this depends not only upon the angular error, but also upon the size and proportions of the triangle and upon the precision of the base line measurement.

416. Triangulation and Trigonometric Leveling. Let A and B be the ends of a base line and C the third station of the triangle. If the distance AC has been found by triangulation, and the vertical angle was taken to C when the instrument was at A, the difference in elevation between A and C may be computed (Art. 180). In like manner the difference in elevation between B and C may be found, and since the base line is either level or the difference in elevation of its two ends is known, this affords a check on the elevation of C. When the instrument is moved to C the vertical angles and triangulation distances to two other stations will give the elevations of these two stations, and so on indefinitely. As the horizontal angles must be measured anyhow, the only additional field work consists in measuring vertical angles; hence it is an economical method. As the elevation of each station may be determined by independent observations from at least two stations, there is a good check on the work. The results in many cases compare favorably with results obtained by spirit leveling.

417. The Computations for a Triangulation Net consist mainly in applying to each triangle the trigonometric method of solving an oblique triangle when the base and two adjacent angles are known. This is generally done by the use of logarithms or by means of a calculating machine and tables of natural trigonometric functions. The degree of precision in the calculations should be consistent with the degree of precision in the field work. Some good form for computation should be adopted, and this form should be completed as far as possible before calculations are begun. One form is given in Art. 573, together with more specific directions for computations.

418. The Adjustment of a Triangulation Net. The object of adjustment is to eliminate the discrepancies in the field measurements, particularly in the measurement of angles, and to make each part of the triangulation system satisfy certain geometric conditions so that the system as a whole can be geometrically consistent. In work of a high degree of precision, this adjustment is based on the method of least squares. The complete treatment of this method may be found in books on least squares or geodesy, or in the publications of the federal government.* When the required degree of precision is not so high, a triangulation net may be adjusted by an approximate method. Some of these approximate methods give results which are very nearly the same as those obtained by the method of least squares. Two approximate methods are explained in the two following articles.

418(*a*). ADJUSTMENT OF THE OBSERVED ANGLES IN A CHAIN OF TRIANGLES. The geometric conditions to be satisfied in a single chain of triangles are: The sum of the angles around each station should equal 360°; the sum of the interior angles of each triangle should equal 180°.

In the following approximate adjustments, it is assumed that all angles were measured in the same way and with the same degree of precision.

Station Adjustment. Add together all the angles around a station; subtract the sum from 360°; divide the difference by the number of angles; apply the correction thus found to the observed value of each angle.

Triangle Adjustment. The values of all angles to be adjusted are the values obtained by the station adjustment. In each triangle add together the three interior angles; subtract the sum from 180°; divide the difference by 3; apply the correction thus found to each angle.

Note. The triangle adjustment affects the interior angles at each station but not the exterior angle. Consequently after the completion of the triangle adjustments, the sum of the angles at a station will not be 360° unless the sum of the interior angles at that station happens to remain the same as it was at the completion of the station adjustment. Despite this discrepancy in the sum of all the angles at a station, the values of the interior angles at the end of the triangle adjustment are sufficiently accurate for use in computing the lengths in most single chains of triangles.

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* Special Publications, such as those numbered 28 and 138, U.S. Coast and Geodetic Survey. See also *Technical Procedure for City Surveys,* Manual 10, published by the American Society of Civil Engineers, and *The Adjustment of Observations* by T. W. Wright and J. F. Hayford, D. Van Nostrand Company, Inc., New York.

418(b). ADJUSTMENT OF THE OBSERVED ANGLES IN A QUADRILATERAL. For many surveys, the best triangulation system is one composed of quadrilaterals, such, for example, as the system shown in Fig. 403(b), p. 410. The small errors which inevitably occur in the measurement of angles may be corrected and the corrections distributed throughout the system by adjusting the observed angles in one quadrilateral, then in an adjacent quadrilateral, and so on throughout the system. Whatever method is used in the adjustment of the first quadrilateral is used also in the adjustment of each of the others.

The angles of a quadrilateral may be adjusted by the rigorous method of least squares or by approximate methods. No rigorous method should be undertaken unless a high degree of precision has been attained in the field work. In the theory of least squares it is assumed that *systematic* errors have been eliminated (Art. 42). The method accomplishes nothing beyond the computation of the *most probable* values of the measured angles. It affords no guaranty of accuracy —it merely distributes the *accidental* errors better than any other method.

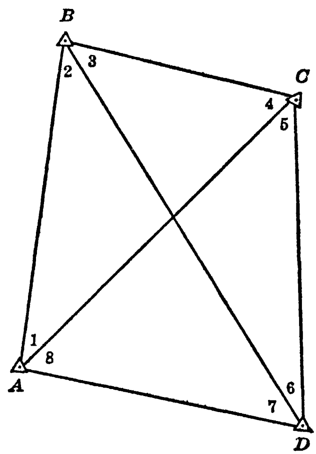


FIG. 418(b).

For many triangulation surveys, an approximate method of adjustment, less complex than the method of least squares, can be used to advantage. The approximate method explained here is simple and easily applied. The results obtained by this method are very nearly the same as those obtained by the method of least squares, and are sufficiently accurate for the adjustment of any triangulation system of ordinary precision and extent.

The angles at the intersection of the diagonals of a quadrilateral are seldom measured. The angles to be adjusted, therefore, are the angles measured at the four corners A, B, C, and D of the quadrilateral (Fig. 418(b)). At each of these stations, the external angle (explement) is measured as well as the two internal angles. Assume that AD is the only line whose length is known.

There are three steps in the method of adjustment:

I. *Station Adjustment.* The adjustment of the three angles (including the explement angle) at each of the four stations A, B, C, and D so that the sum will equal 360°.

II. *Triangle Adjustment.* The adjustment of the four angles of each of the four triangles ABC, BCD, CDA, and DAB so that the sum will equal 180°.

III. *Final Adjustment of Angles.* The adjustment of all eight interior angles of the quadrilateral so that, starting from the known length of the side AD, the computed length of the side BC will be the same whichever combination of triangles (route) is used in the computation.

Step I. Station Adjustment. Add the three angles (including the explement angle) and distribute the difference between the sum and 360° equally among the three angles. (It is assumed that the three angles are of equal weight.)

Assume that *after* the station adjustments have been made the values of the angles in Fig. 418(b) are as follows:

1 = 37° 47′ 20″.8	5 = 48° 45′ 53″.4
2 = 39° 57′ 20″.0	6 = 28° 58′ 18″.7
3 = 42° 55′ 14″.5	7 = 45° 29′ 51″.4
4 = 59° 20′ 21″.7	8 = 56° 45′ 35″.1

The sum of these eight interior angles of a quadrilateral should be 360°; it actually is 359° 59′ 55″.6. This error of 4″.4 will be distributed in the two remaining steps of the adjustment.

Note. For the computation of strength of figure when the values of the eight angles are those just given, see Art. 403(g).

Step II. Triangle Adjustment. Let the corrections still to be made to the adjusted values of angles obtained in step I be represented as follows:

v_1 for 1; v_2 for 2; v_3 for 3; v_4 for 4; v_5 for 5; v_6 for 6; v_7 for 7; v_8 for 8

Let the difference between 180° and the sum of the four angles in a triangle be represented by C; this is the total correction for a triangle. Then

$$\begin{aligned} \text{Correction for triangle } ABC &= v_1 + v_2 + v_3 + v_4 = C_1 \\ BCD &= v_3 + v_4 + v_5 + v_6 = C_2 \\ CDA &= v_5 + v_6 + v_7 + v_8 = C_3 \\ DAB &= v_7 + v_8 + v_1 + v_2 = C_4 \end{aligned}$$

If any three of these equations are satisfied, the remaining equation must necessarily be satisfied also.

Note. If C_1, C_2, C_3, or C_4 is so large that it indicates a degree of precision which is too low for the purpose of the survey, the only way in which a higher degree of precision can be obtained is to remeasure those angles which are involved in the error.

From the equations just given the following expressions may be derived: *

$$v_1 = v_2 = \tfrac{1}{4}C_1 - K \qquad v_5 = v_6 = \tfrac{1}{4}C_3 + K$$
$$v_3 = v_4 = \tfrac{1}{4}C_1 + K \qquad v_7 = v_8 = \tfrac{1}{4}C_3 - K$$

in which the constant $K = \tfrac{1}{4}(C_2 - \tfrac{1}{2}C_3 - \tfrac{1}{2}C_1)$.

From the values of the eight angles found in step I, the values of C_1, C_2, and C_3 are computed. (The algebraic signs of the corrections C_1, C_2, and C_3 should be observed with care.)

Triangle	*Angles*	*Sum*	*Total Correction*
ABC	1 + 2 + 3 + 4 =	180° 00′ 17″.0	$C_1 = -17''.0$
BCD	3 + 4 + 5 + 6 =	179° 59′ 48″.3	$C_2 = +11''.7$
CDA	5 + 6 + 7 + 8 =	179° 59′ 38″.6	$C_3 = +21''.4$

$$K = \tfrac{1}{4}(C_2 - \tfrac{1}{2}C_3 - \tfrac{1}{2}C_1) = \tfrac{1}{4}[11''.7 - \tfrac{1}{2}(21''.4) - \tfrac{1}{2}(-17''.0)] = 2.375$$

Substituting these values for C_1, C_2, C_3, and K in the four expressions previously given:

$$v_1 = v_2 = -\tfrac{1}{4}(-17) - 2.375 = -6''.6 \qquad v_5 = v_6 = \tfrac{1}{4}(21.4) + 2.375 = +7.7$$
$$v_3 = v_4 = -\tfrac{1}{4}(-17) + 2.375 = -1''.9 \qquad v_7 = v_8 = \tfrac{1}{4}(21.4) - 2.375 = +3.0$$

When these corrections have been applied to the values obtained in step I, the values obtained by this adjustment of step II are:

1 =	37° 47′ 14″.2	5 =	48° 46′ 01″.1
2 =	39° 57′ 13″.4	6 =	28° 58′ 26″.4
3 =	42° 55′ 12″.6	7 =	45° 29′ 54″.4
4 =	59° 20′ 19″.8	8 =	56° 45′ 38″.1
	180° 00 00 .0		180° 00 00 .0

The sum of the four angles in each of the four triangles *ABC*, *BCD*, *CDA*, and *DAB* should now equal 180°, and the sum of all eight angles should equal 360°. Further adjustment of the angles may be necessary, however, as explained in the next step.

Step III. Final Adjustment of Angles. (This is often called the *side adjustment.*) If the length of one side *AD* of the quadrilateral in Fig. 418(*b*) is known, the length of *BC*, the opposite side of the quadrilateral, may be computed through four different routes of two triangles each, and this may be done in such a way as to use *only* the eight angles which are being adjusted. (For purposes of adjustment, the use of any angle equal to the sum of two of the eight angles is avoided.) The values of the eight angles should be such that the computed length of any side is the same whichever route or combination of triangles is used. This condition may require a further adjustment of the values obtained in step II.

Assume that the length of *AD* is known and the length of *BC* is to be computed by the use of two different combinations of triangles, first by the triangles *ADB* and *ABC*, then by the use of *ADC* and *BCD*. Any discrepancy in the two results thus obtained indicates the errors in the angles.

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* For the derivation of these expressions see the book, *Adjustment of Observations*, by Wright and Hayford, D. Van Nostrand Company, Inc.

Since in any triangle the sines of the angles are proportional to the lengths of the sides opposite:

$$\text{From triangle } ADB, \frac{AB}{\sin 7} = \frac{AD}{\sin 2} \text{ and } AB = \frac{AD \sin 7}{\sin 2}$$

$$\text{From triangle } ABC, \frac{AB}{\sin 4} = \frac{BC}{\sin 1} \text{ and } BC = \frac{AB \sin 1}{\sin 4}$$

Substituting the value of AB indicated above,

$$BC = \frac{AD \sin 7 \sin 1}{\sin 2 \sin 4}$$

By a similar procedure applied to triangles ADC and BCD a second expression for BC may be derived, namely:

$$BC = \frac{AD \sin 8 \sin 6}{\sin 5 \sin 3}$$

Since the two values of BC should be equal,

$$\frac{AD \sin 7 \sin 1}{\sin 2 \sin 4} = \frac{AD \sin 8 \sin 6}{\sin 5 \sin 3}$$

$$\frac{\sin 1 \sin 3 \sin 5 \sin 7}{\sin 2 \sin 4 \sin 6 \sin 8} = 1$$

Expressed in logarithmic form,

$$(\log \sin 1 + \log \sin 3 + \log \sin 5 + \log \sin 7) - (\log \sin 2 + \log \sin 4 + \log \sin 6 + \log \sin 8) = 0$$

The values used for the angles in the above expressions should be the adjusted values found in step II. When the logarithmic sines in the two groups are added as indicated, the difference between the two sums should equal zero in order to satisfy the trigonometric condition stated in the first paragraph of step III. If this difference does not equal zero, it is the total of the small errors which remain in the eight angles after the adjustments in step I and step II have been made. It is not possible to make a further rigorous adjustment of the eight angles without disturbing the adjustments made in step II. Consequently, for the best possible results, the adjustments to satisfy the geometric condition in step II and those to satisfy the trigonometric condition in step III should be made simultaneously by the method of least squares. For most purposes, however, the following approximate but less complex method of adjustment in step III is sufficiently accurate.

Angle	*Log Sine*	*Dif. for 1″*	*Angle*	*Log Sine*	*Dif. for 1″*
1	9.7872703	27.1	2	9.8076491	25.1
3	9.8331335	22.7	4	9.9345985	12.5
5	9.8762381	18.4	6	9.6852155	38.0
7	9.8532305	20.7	8	9.9224075	13.8
Sum	9.3498724	88.9	Sum	9.3498706	89.4

(1) Prepare a table similar to that shown here. (2) Find the difference between the sums of the two colums of logarithmic sines, i.e., in this case 0.0000018. This is the total correction to be made in the log sines of the eight angles. (3) Find the sum of the eight tabular differences, i.e., 89.4 + 88.9 = 178.3. (4) Divide the total correction found in (2) by the sum found in (3), i.e., 18 ÷ 178.3 = 0.1″. This is the average correction in seconds of arc to be subtracted from each of the four angles the sum of whose log sines is the larger, and added to each of the four angles the sum of whose log sines is the smaller. It is sufficiently accurate to take the average correction to the nearest tenth of a second. Since the sums of the pairs of angles 2 + 3, 4 + 5, 6 + 7 and 8 + 1 remain the same as they were at the end of the adjustment in step II, the geometric condition for that adjustment remains satisfied; in this sense the adjustment in step II has not been disturbed.

When the correction $0''.1$ is subtracted from each of the values for 1, 3, 5, and 7 found in step 2 and added to each of the values for 2, 4, 6, 8, the final adjusted values are:

1 = 37° 47′ 14″.1	5 = 48° 46′ 01″.0
2 = 39° 57′ 13″.5	6 = 28° 58′ 26″.5
3 = 42° 55′ 12″.5	7 = 45° 29′ 54″.3
4 = 59° 20′ 19″.9	8 = 56° 45′ 38″.2

The sum of all eight of these final adjusted values is 360° 00′ 00″, as it should be for the sum of the interior angles of a quadrilateral. These values are practically the same as those that would have been obtained had the method of least squares been used in place of steps II and III.

Sum of log sines of final values of angles 1, 3, 5, and 7 = 9.3498715
Sum of log sines of final values of angles 2, 4, 6, and 8 = 9.3498715

The fact that the sums of the log sines are equal is one check on the computations.

SUMMARY OF ADJUSTED VALUES

Angle	*Step I*	*Step II*	*Step III*
1	37° 47′ 20″.8	37° 47′ 14″.2	37° 47′ 14″.1
2	39° 57′ 20″.0	39° 57′ 13″.4	39° 57′ 13″.5
3	42° 55′ 14″.5	42° 55′ 12″.6	42° 55′ 12″.5
4	59° 20′ 21″.7	59° 20′ 19″.8	59° 20′ 19″.9
5	48° 45′ 53″.4	48° 46′ 01″.1	48° 46′ 01″.0
6	28° 58′ 18″.7	28° 58′ 26″.4	28° 58′ 26″.5
7	45° 29′ 51″.4	45° 29′ 54″.4	45° 29′ 54″.3
8	56° 45′ 35″.1	56° 45′ 38″.1	56° 45′ 38″.2
	359° 59′ 55″.6	360° 00′ 00″.0	360° 00′ 00″.0

418(*c*). ADJUSTMENT OF A TRIANGULATION FIGURE BETWEEN TWO BASE LINES. When in triangulation a second base line is measured as a check (Art. 403(*h*)), there may be a discrepancy between the measured length and the computed length of that base line. If the discrepancy is not too great, it may be distributed by the adjustment of the triangulation figure between the two bases. In this adjustment an additional condition must be satisfied, namely, that the length of each side in the figure must be the same when computed from the second base line as when computed from the first. In geodetic surveying, the adjustment is made by the method of least squares. In triangulation of ordinary extent and precision approximate methods are sometimes used. In such triangulation, however, if the discrepancy in the measurements of the two base lines is sufficiently great to warrant an additional condition in the adjustment of angles, it is generally more satisfactory to repeat some or all of the field measurements until the discrepancy is sufficiently small to be within the required limits of precision. This is particularly true when the triangulation figure covers a comparatively small area.

CHAPTER 29

STADIA SURVEYING

In this chapter the principles of stadia surveying are explained; formulas for reducing inclined sights to horizontal and vertical distances are derived, methods of determining the stadia interval and of graduating rods are given, methods of procedure in field work are outlined, sources of error are pointed out, and limits of error are suggested.

419. Stadia, Stadia Hairs, Stadia Rod. The stadia method of measuring distances consists in observing how much of a rod held plumb at one end of a line appears to be included between two parallel cross hairs (or wires) as the observer looks through a telescope at the other end of the line. The farther the rod from the telescope, the more of the rod will be included between the two hairs. *Stadia hairs* in the telescope of a surveying instrument are horizontal, one above and one below the regular horizontal cross hair. In most telescopes they are a fixed distance apart; in some of the older instruments the distance can be changed, i.e., the stadia hairs are adjustable. A *stadia rod* is a self-reading rod on the face of which graduation marks are arranged to form a pattern that is designed especially for stadia readings. There is a great variety of stadia patterns in common use. Whatever may be the pattern, it should enable an observer, looking through a stadia telescope, to read easily and accurately the distance on the rod that appears to be included between the stadia hairs. For relatively short sights, not exceeding, say, 400 ft., a self-reading leveling rod can often be used in place of a stadia rod. Stadia rods are described in detail in Art. 719.

The term *stadia,* formerly used to denote the rod, is now used to include both the rod *and* the hairs. It also stands for the method itself: thus, for example, measuring with stadia, i.e., rod and hairs, and measuring by the stadia, i.e., by the stadia method. *Telemeter* is a term used in place of *stadia. Tachymeter* is sometimes used to denote a transit equipped with stadia hairs, vertical circle, and telescope level.

420. Use of the Stadia Method. When the telescope is directed at a stadia rod, two quantities may be observed, namely, (1) the space on the rod that appears to be intercepted between the two stadia hairs and (2) the vertical angle which the line of

sight makes with the horizontal. From these two quantities may be computed (1) the horizontal distance from the instrument to the point on which the rod is held and (2) the vertical distance from that point to the horizontal plane of sight. Thus both horizontal distances and elevations may be determined from the same observations and more quickly than by other methods. Measurements cannot be made with as high a degree of precision as can be attained by chaining and leveling; nevertheless, surprisingly accurate results may be obtained. In rough country, both horizontal and vertical measurements can often be made easily where it would be difficult to obtain them by other methods. When there are many measurements to be made with a relatively low degree of precision, as, for example, in locating contours and filling in detail in a topographic survey, the stadia method is usually the quickest and the best.

421. The Principle of the Stadia is based on the geometric principle that in two similar triangles corresponding sides and altitudes are proportional. In practice the distance of a point from a surveying instrument is found by holding a stadia rod in a vertical position on the point and by reading the portion of the rod which appears to be intercepted between the upper and lower stadia hairs. For example, if at 100 ft. away a *foot* of the rod is thus included, at 200 ft. away 2 ft. of the rod will be included, at 300 ft., 3 ft., and so on. This is not, however, strictly true, for a *constant,* which differs slightly for different telescopes, must be added to each reading, except when an internal focusing telescope is used.

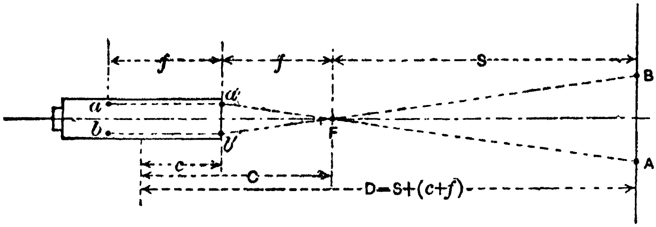


FIG. 421(*a*).

421(*a*). THEORY INVOLVED. Let *AB* in Fig. 421(*a*) be the portion of a stadia rod which appears to be exactly included between the upper and lower stadia hairs. From the theory of lenses (Art. 706), a ray of light from *A* which passes through a principal focus *F* will become parallel to the axis of the telescope after passing through the lens, i.e., it will lie in the line *aa'*. All rays from *A* are gathered by the lens and brought to a focus in a single point,

and since $Aa'a$ is one of these rays, the focus where all others meet must lie in the line $a'a$. Let a represent this focal *point;* it will represent also the position of the upper stadia hair. Similarly, the rays from B are gathered by the lens and brought to a focus at b; b also represents the position of the lower stadia hair.

Let a' and b' represent the points where the oblique rays through the prinicpal focus F are changed to parallel rays in the telescope. Let c be the distance from the center of the instrument to the center of the objective lens, a slightly variable quantity, dependent on the focusing, and let f represent the focal length. From the similar triangles $a'Fb'$ and AFB, $a'b' : f :: AB : S$, or $S = \frac{f}{a'b'} \times AB$. Since $a'b' (= ab)$ is the interval between the stadia hairs (denoted by i), and since $AB = R =$ the rod reading, then $S = \frac{f}{i} R$. The ratio f/i is called the *stadia interval factor* or simply the *interval factor,* and for any telescope with fixed stadia hairs, this factor is constant. The *stadia distance* S, therefore, is always equal to the rod reading multiplied by this ratio or stadia interval factor, no matter where the rod may be held. But the distance usually required is the distance from the rod to the *center* of the instrument. When the telescope is horizontal, this distance is: $D = S + (c + f)$ or $D = \frac{f}{i} R + (c + f)$. The quantity $(c + f)$ for all practical purposes is a constant C and is called the *stadia constant.* The process of obtaining the distance D is: (1) Observe the *rod reading AB;* (2) multiply this *rod reading* by the *stadia interval factor;* and (3) add the *stadia constant.* When the telescope is inclined, certain corrections must be applied as explained in Art. 425.

421(*b*). NOTE. The term *stadia distance* is sometimes used for the intercepted distance AB on the rod, sometimes for the distance D, which includes the constant C, and sometimes for the distance S, which does not include the constant. In this book it will be used for S as defined above. The term rod reading or stadia reading will be used for the distance AB. It will be shown later that in ordinary work the constant C may often be disregarded. (See Art. 425(*b*).)

421(*c*). TO FIND THE CONSTANT C. The quantities c and f may be found with much greater accuracy than is ordinarily necessary, as follows. (1) Focus on a point 100 ft. or more away. (2) Measure from the center of the objective lens to the center of the instrument to find c. (3) Measure from the center of the objective lens to the cross hairs to find f. (Since the cross hairs are approximately at one of the principal foci or as far behind the lens as the other principal focus F is in front of the lens.) (4) Add

these two measurements to find the total correction C. Notice that the constant C, once determined, is the same for all readings; it is *not* twice as much for 200 ft., three times as much for 300 ft., and so on.

421 (d). INTERNAL FOCUSING TELESCOPES. Many modern instruments have internal focusing telescopes so designed that the constant C is negligible, and the interval f/i is read *directly from the center of the telescope.* All explanations in this chapter which involve C should, accordingly, be modified when applied to such an instrument by disregarding C.

421 (e). REMARK. It makes little difference in determining the value of C whether the telescope is focused on a distant point or on a point a few feet away, since the extreme range of the objective lens slide is less than 1 in. for the average telescope. This extreme variation in the distance $(c + f)$ may be ignored. Moreover, in changing the focus from that for, say, 100 ft., to that for the longest distance ever read, the variation is only a small *fraction* of an inch.

The stadia constant C varies from about 0.7 ft. to about 1.4 ft. in different American transits, but in ordinary transit work it is usually close enough to consider it as an even foot. For longer telescopes, such as those used in wye levels, the stadia constant may be as great as 2.5 ft. The value of C for any particular instrument is usually determined by the maker and recorded on the inside of the instrument box.

421 (f). THE MOST COMMON VALUE FOR THE STADIA INTERVAL FACTOR f/i is 100. Hence, when 1 ft. of the stadia rod appears to be intercepted between the two stadia hairs, the rod is $(100 + C)$ feet from the center of the instrument; when 2 ft. of the rod are intercepted, it is $(200 + C)$ feet away, and so on. It will not do, however, to take for granted that the interval factor is exactly 100, even if it is supposed to be. When the stadia hairs are movable, they should be tested and adjusted frequently; when they are fixed, the true interval factor should be determined as explained later.

422. To Adjust the Stadia Hairs. In the better class of instruments stadia hairs are not adjustable. When, however, stadia hairs are movable, they may be adjusted as follows. (1) Set two targets on a leveling rod exactly 1 ft. apart, and hold the rod at a distance of $(100 + C)$ feet from the center of the instrument. (2) Sight at the rod, moving the telescope up or down with the slow-motion screw until the middle horizontal hair is exactly halfway between the two targets. (3) Make one stadia hair bisect one target and the other hair bisect the other target by turning the screws which control the stadia hairs. Watch the middle hair to see that it does not move off the middle point, and if it does bring it back again. (4) Hold the rod at $(200 + C)$ feet from the instrument, and test each stadia hair separately to see if it bisects

one target when the middle hair bisects the other. (5) Test both hairs at the same time by setting the targets 2 ft. apart. (6) Repeat the test for the wires at $(400 + C)$ feet, $(600 + C)$ feet, and so on.

Practical Suggestions. (1) The method just described makes the stadia hairs equal distances from the middle cross hair. This is not essential so long as the interval between the stadia hairs is correct, unless, as is sometimes the case, distances are measured by observing the space included between one stadia hair and the middle cross hair. (2) In place of the regulation targets it may be more convenient and more accurate to make targets of cardboard having well-defined ink lines at ½-ft. intervals, or three separate targets to be fastened to the rod at required intervals. (3) Notice that rods may be constructed and stadia hairs spaced so that units other than feet may be used as, for example, units in the metric system. (4) One advantage of the standard interval is that the ordinary leveling rod may be used as a stadia rod when desired. (5) The screws for adjusting the stadia hairs are usually two capstan-headed screws, close to the four screws for adjusting the cross-hair ring. (See Art. 707(a).) (6) The middle hair should never be disturbed in adjusting stadia hairs—it is supposed to be in adjustment. (7) When the telescope has internal focusing, disregard C.

423. Fixed Stadia Hairs. For the most accurate work stadia hairs should be *non-adjustable;* hence in most modern instruments stadia hairs are fixed. Instruments having such hairs are generally intended to be used with standard rods graduated to feet. A series of tests of the interval factor in an instrument will sometimes show, however, that, although the maker intended to fix the interval factor at $f/i = 100$, the actual interval varies slightly from this. The error may be too small to affect ordinary work, but must be taken into account in more accurate work. In such a case, either (1) the *actual* interval factor must be determined and applied to all rod readings or (2) a special rod must be graduated in such a way that it can be used just as if the interval factor were exactly 100.

423(a). TO DETERMINE THE INTERVAL FACTOR. Select a stretch of nearly level ground 600 ft. to 800 ft. long on which to carry out the tests. (1) Establish a point F a distance in front of the center of the instrument equal to the stadia constant C. (2) Establish points by careful chaining in a straight line at distances, *measured from F,* of 50 ft., 100 ft., 200 ft., and so on up to about 800 ft. (3) Hold a stadia rod at each of the points in succession, and at each point, determine to the nearest thousandth the stadia reading, i.e., the space intercepted by the stadia hairs. (4) Since the distances were all measured from the principal focus F in front of the instrument, each distance divided by the corresponding stadia reading will give a value for the stadia interval factor. The mean of these values may be taken as the probable value.

Note. When the telescope has internal focusing, distances are measured from the center of the instrument.

The method just explained may be extended as follows. At each point on which the rod is held observe three readings, namely, (1) between the middle and upper wires, (2) between the middle and lower wires, and (3) between the upper and lower wires. The *mean* interval factor for each of the three sets of observations may be obtained by the general method explained in the preceding paragraph.

Illustration. Suppose that the average intercept on a rod, as determined from a number of observations at different distances, repeated several times

is 1.0042 ft., then all distances as read would be too long and should be reduced by dividing by 1.0042. If the distances are inclined, this must be done before correcting for the inclination. If the interval is too short, say 0.9981, then all readings are divided by 0.9981.

Having determined the interval factor, the question arises as to whether or not the difference between it and the standard interval factor of 100 is great enough to affect the work in hand. This may be judged by noting errors for various lengths of sight. For example, if the interval factor is found to be 1.004 and the rod is held at a point 200 ft. + *C* from the center of the instrument, the reading for the stadia distance will be 200.8 ft. or 0.8 ft. more than the correct distance; at 400 ft. + *C*, it will be 401.6 ft. or 1.6 ft. more than the correct distance; at 1500 ft. + *C*, it will be 6 ft. too great. Obviously in rough work with relatively short sights, these errors would be negligible, whereas in the better class of work they would be too great to be ignored.

423(*b*). To graduate a special stadia rod. When it is desired to graduate a rod in such a way that it can be used as if the interval factor were exactly 100, establish points as explained in the preceding paragraph. (1) By holding the rod at each of the points in succession, determine as accurately as possible what unit of length measured on the rod corresponds to 100 ft. measured on the ground. The value of this unit is obtained for each point by dividing the corresponding stadia reading by the number of 100 ft. in the distance from *F* to the point. The mean of the values thus calculated may be taken as the probable value of the unit. (2) Mark the rod tentatively in such a way that the distance between two successive marks will be the unit of length determined in (1) instead of the even foot on the standard rod. (3) Test the rod again at 50 ft., 100 ft., 200 ft., and so on in order to determine whether or not the marks on the rod are correct. (4) When the main graduations on the rod satisfy the test in (3), the subdivisions may be painted on according to any desired pattern.

Note. When the telescope has internal focusing, distances are measured from the center of the instrument.

423(*c*). Practical suggestions. (1) In order to obtain accurate stadia readings in determining an interval factor or in graduating a rod, it is well to use a target equipped with a vernier or, better still, to use two such targets, one for each stadia hair. (2) In taking stadia readings at points, each of which is an even number of hundred feet from *F*, the instrument man may be influenced by the knowledge of what the reading in each case should be. It is well, therefore, to establish a few points at random distances from *F*, and, after the stadia readings on these points have been taken, to measure these distances. (3) For very accurate work the tests should be made not at any one hour of the day but during all hours of field work, so as to approximate as closely as possible the average conditions (Art. 435(7)). (4) Fixed stadia hairs are not likely to be displaced, and for this reason are preferable to adjustable wires, especially where results better than 1/600 are desired. The adjustable hairs are likely to change their distances apart with every change of temperature. (5) It is better in an extended survey to use a standard rod and to determine the actual interval factor for the instrument than to use a specially graduated rod for the following reasons. Subsequent tests of intervals can be made without repainting or regraduating the rod; rods can be interchanged among transits or used with new transits; rods can be used for leveling without computation; leveling rods can be used as stadia rods; rods can be used by observers having different personal equations.

424. Method of Reading the Rod. The reading of the rod is learned more easily from actual practice than from any printed directions. The main object is to determine as accurately as possible how much of the rod is included between the stadia hairs. In doing this, sight at a point on the rod about as far above the ground as the telescope is above the ground. Move the telescope slightly up or down until one of the stadia hairs, usually the lower one, coincides with the nearest whole foot mark on the rod, and then observe how far from that foot mark the other stadia hair

appears to be. When the interval factor is exactly 100, the graduations on the rod can be read *in terms of distance*. For example, if the lower hair is on the 3-ft. mark and the upper intersects the rod at 6.21, the stadia reading may be recorded as 621 − 300 or 321 ft. This is the distance from the rod to the principal focus in front of the instrument; the distance to the center of the instrument is 321 + *C*. Be careful not to mistake the regular horizontal cross hair for a stadia hair; do not use the former at all, except in special cases. In some instruments this danger is avoided by placing the stadia hairs so that when they are visible (in focus) the regular horizontal hair is invisible, and vice versa.

424(*a*). COMMON MISTAKES. (1) To read between the stadia hair and middle cross hair, thus getting *half* the correct distance. Avoided by habitually estimating distances by eye, and thus checking the stadia reading; also by purposely reading and entering the half distance in the notes, and then reading and entering the full distance independently. In many instruments diagonal wires are inserted to aid in avoiding the mistake just explained. (2) Omitting an even hundred feet or multiples of 10 ft., as 189 ft. in place of 289, or 129 in place of 139.

Practice. Before beginning the regular stadia survey, take a number of sights varying from 50 ft. or less to 300 ft. or more, and check each reading by measuring with a tape.

424(*b*). AN ALTERNATIVE METHOD OF READING THE ROD may be used to advantage under certain conditions, particularly in stadia leveling. A rod with target is required. The procedure is as follows. (1) Set the target at some convenient reading, say at the height of the supporting axle of the telescope above the ground, and note this reading. (2) When the rod is held for an observation, direct the telescope until the middle hair bisects the target. (3) Observe the readings for the upper and lower stadia hairs. If the middle hair is exactly halfway between the other two, this is equivalent to taking the readings for two intervals on the rod since the reading for the middle hair (target reading) is already known. Each interval should be equal to the other, and the sum of the two should be equal to the rod reading used in obtaining the stadia distance. If the observed values of the two intervals are not equal, some error has been made in the observations, and the chief value of the method lies in this check. The target may be kept at its original reading for an indefinite number of observations. For a method of procedure in which this method of reading the rod is used, see Art. 428.

425. Distance and Elevation When the Line of Sight Is Inclined. In the preceding articles it has been assumed that the line of sight in each case has been nearly horizontal. In practice, however, it is frequently necessary to incline the telescope. In such a case, in order to obtain a horizontal distance from a stadia rod

reading, two corrections must be applied: (1) Since the line of sight is inclined, the distance obtained from reading the stadia rod will correspond very nearly to the distance found by measuring along a slope, and, consequently, this distance must be reduced to a horizontal distance. (2) If the rod is held plumb, as it usually is, it will not be perpendicular to the line of sight. Hence more of the rod appears to be included between the hairs than would appear were the line of sight horizontal. This false reading must be reduced to the true reading. Similarly, two corrections must be made when determining vertical distances by means of stadia readings.

425(a). DERIVATION OF FORMULAS FOR INCLINED SIGHTS. In Fig. 425(a), let the line of sight be inclined at an angle a. If the rod were held *perpendicular* to the line of sight the stadia reading $b'e'$ would give the distance Fm, which, when added to the constant C, would be equal to the distance Bm. This could be reduced to the required distance BE by multiplying Bm by cos a. But it is not practicable to incline the rod until it is perpendicular to the line of sight, and even if it were complications would arise in getting elevations; hence the general practice is to hold the rod plumb. This introduces another error, since it makes the reading

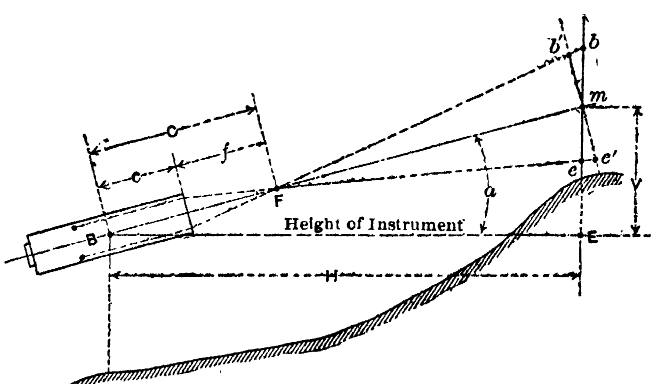


FIG. 425(a).

be longer than the true reading $b'e'$. The angle $bb'm$ is approximately a right angle, and $bmb' = a$; hence be cos $a = b'e'$ (very nearly). In other words, the distance actually read must be multiplied by cos a to get Fm. Hence

$$Bm = C + Fm = C + \frac{f}{i}(be \cos a) \quad (1)$$

But $BE = Bm \cos a$; hence

$$BE = C \cos a + \frac{f}{i} be \cos^2 a \tag{2}$$

Horizontal Distance. Let H represent the horizontal distance BE, and R the actual reading be multiplied by the interval factor f/i. Equation 2 may then be written:

$$H = R \cos^2 a + C \cos a = R - R \sin^2 a + C \cos a \tag{3}$$

Vertical Distance. In finding elevations, the vertical distance $V = Em$ is required. $Em = Bm \sin a$. Substituting the value of Bm from equation 1:

$$V = Em = \frac{f}{i} be (\cos a \sin a) + C \sin a$$

Since $\cos a \sin a = \frac{1}{2} \sin 2a$,

$$V = R (\cos a \sin a) + C \sin a = R \tfrac{1}{2} \sin 2a + C \sin a \tag{4}$$

425(b). APPROXIMATE FORMULAS. For the usual range of vertical angles in practice, the angle a is so small that $C \cos a$ is practically equal to C and $C \sin a$ is negligible; hence equations 3 and 4 may be reduced to the following approximate formulas:

$$H = R \cos^2 a + C = R - R \sin^2 a + C \tag{5}$$

$$V = R\tfrac{1}{2} \sin 2a \tag{6}$$

Equation 4 may also be modified as follows:

$$V = (R + C)\tfrac{1}{2} \sin 2a \tag{7}$$

Formula 7 is slightly more accurate than formula 6, and, for angles less than 25°, differences in elevations obtained by its use will be correct within 0.1 ft. Any of these formulas will be sufficiently accurate except in work of unusual precision in which sources of error are largely eliminated. *For internal focusing telescopes, the valve of C in all formulas is zero.*

In ordinary work certain sources of error, such as unequal refraction, are disregarded. Moreover, as shown by the table in Art. 436(a), the error that may be expected in measuring distances by the stadia method under ordinary conditions varies from approximately ±0.5 ft. for a distance of 100 ft. to ±1.0 ft. for a distance of 300 ft. Under these conditions, consistent accuracy justifies the omission of the constant C from formula 5. For all ordinary work, therefore, the following approximate formulas may be used:

$$H = R \cos^2 a \tag{8}$$

$$V = R\tfrac{1}{2} \sin 2a \tag{9}$$

When the telescope is constructed for internal focusing, these formulas are no longer approximate but are practically exact; hence they should be used under all conditions.

425(*c*). Stadia reductions. Only two field measurements are involved in any of the reduction formulas, namely, the vertical angle a and the stadia reading as it actually appears on a vertical rod. Unless the stadia reductions are made in the field, it is generally sufficient, in keeping field notes, to record, for each sight, these two measurements.

Regardless of whether the reductions are made in the field or in the office, the work involved in the application of formulas 3 and 4 (or of 5 and 6) is greatly facilitated by the use of a slide rule, or by stadia reduction tables, or by geometrical diagrams. The most accurate results will be obtained by the use of stadia tables. The various methods used in reducing stadia readings are explained in Art. 645.

426. To Measure Horizontal Distances with the Stadia. (*a*). On level ground. Keeping the telescope as nearly horizontal as possible, bring one of the stadia hairs to coincide with the nearest whole foot mark on the rod, and read the other wire *in terms of distance*, as directed in Art. 424.

426(*b*). On sloping ground. (1) Sight at a point on the rod about as far above the ground as the telescope is above the ground. (2) Bring a stadia hair on the nearest foot mark, and read and record the stadia distance. (3) Read and record the vertical angle which the line of sight makes with the horizontal. (4) From the values obtained in (2) and (3), the horizontal distance may be obtained by one of the methods based on formula 3, Art. 425(*a*) (or formula 5, Art. 425(*b*)).

426(*c*). Practical suggestions. (1) It is not essential to make one hair bisect a whole foot mark—it merely simplifies reading the rod. Clamp the telescope and use the slow-motion screw when the stadia hair is nearly on the mark. (2) If the stadia rod is not held plumb, an appreciable error may result (Art. 435(2)). (3) The length of an important line such as that between two stations may be checked by obtaining the stadia distance, first with the instrument at one end of the line and then with it at the other end. (4) When the distance between two stations is too great for accurate reading, assume a point about midway between the two, read the distance to this point from each station, and add the two readings. The objection to this method is that no check on the total distance can be had by two readings, one from each end of the line. This objection does not hold, however, if each of the distances from the midway point to the two stations is checked.

426(*d*). Practical questions.

1. *How accurately can distances be measured?* This depends upon the magnifying power of the telescope, the fineness of the stadia hairs, the graduations on the rod, the atmospheric conditions, and the length of sight.

With the average telescope and atmospheric conditions, the error in a 300-ft. sight should not exceed 1.0 ft. With high-grade instruments and favorable conditions, sights up to 700 ft. may be taken with an average error of less

than 0.5 ft. or 1/400. Even better results have been obtained. (See also "Limits of Error," Art. 436.)

2. *What should be the maximum length of sight?* This depends not only upon the conditions mentioned above, but also upon the accuracy required for the work in hand. In ordinary work best results are obtained when the length of sight does not exceed 300 ft. to 400 ft. At greater distances the stadia hairs are likely to appear coarse as compared with the rod graduations. Occasionally, however, when the map is to be plotted to small scale and the limits of error are comparatively large, very much longer sights may be taken, sights, for example, of 1000 ft. to 2000 ft.

3. *When can a slight inclination in the line of sight be ignored?* This depends upon the accuracy required for the work in hand. The longer the sight the greater the absolute but not the relative error. The following table gives, to the nearest tenth of a foot, values of the errors which will result from sights 100 ft. to 500 ft., if vertical angles 1° to 5° are ignored.

	100	200	300	400	500
1°	0.03	0.06	0.09	0.12	0.15
2°	0.1	0.2	0.4	0.5	0.6
3°	0.3	0.5	0.8	1.1	1.4
4°	0.5	1.0	1.5	2.0	2.5
5°	0.8	1.5	2.3	3.0	3.8

If, for example, 300 ft. is the longest sight taken, the maximum error involved by ignoring angles of 3° or less is 0.8, whereas for a 500-ft. sight it is 1.4. For sights of 300 ft. or less the maximum error involved by ignoring angles up to 5° is 2.3 ft. Hence, when lengths of sights do not exceed 300 ft., and the degree of precision required is not greater than 1/300, vertical angles up to 3° may usually be ignored in determining horizontal distances. In some work the limit is as high as 5°.

426(*e*). THE EFFECT OF ERRORS IN VERTICAL ANGLES UPON THE COMPUTATION OF HORIZONTAL DISTANCES is relatively small as compared with errors in stadia readings. For example, an error of 01′ corresponds to a ratio of precision of about 1/6000 for a vertical angle of 15°, and only of about 1/20,000 for an angle of 5°. This means that greater accuracy is required in observing stadia distances than in reading vertical angles.

On fairly level ground, vertical angles seldom exceed 5°, and even in hilly country, they are usually less than 14°. Consequently, in determining horizontal distances, it is generally sufficiently accurate to read vertical angles, by means of the vernier, to the nearest minute. In a good deal of surveying, it is not necessary to read vertical angles closer than 05′, and this may be done by estimating the angle without using the vernier. (See also "Limits of Error," Art. 436.)

426(*f*). USE OF THE GRADIENTER. When the ground is fairly level, the gradienter can be used to measure distances by a method similar to the stadia method.

427. To Obtain Vertical Distances or Elevations. There are two methods: One is used when the elevation of one point is required with reference to another point very near the instrument, as the station under the plumb bob; the other is used when the point of reference is not near the instrument, as some bench mark too far away for the first method to be used. The two methods correspond to those used in trigonometric leveling (Art. 180); they differ only in the method of setting the target at the start.

427(*a*). REFERENCE POINT VERY NEAR THE INSTRUMENT. (1) Stand the rod up beside the transit with its base on a level with the point of reference, and set the target at the height of the sup-

porting axis, or opposite the center of the objective lens when the telescope is level. (2) Hold the rod on the point of required elevation, direct the telescope at the target, and read the distance as usual, choosing that whole foot mark which brings the middle hair nearest to the target center. (3) Make the middle hair exactly bisect the target, signal to the rodman to move to the next point, read and record the vertical angle. (4) From the values obtained in (2) and (3), the vertical distance may be obtained by one of the methods based on formula 4, Art. 425(*a*) (or formula 6, Art. 425(*b*)).

427(*b*). Reference point not near the instrument. *First Method.* (1) Clamp the telescope in a horizontal position, bringing the bubble exactly to the center by the slow-motion screw. (2) Backsight on the bench mark or point of reference, setting the target so that it is bisected by the middle hair. (3) Without disturbing the target, proceed exactly as directed in steps 2 and 3 of the preceding paragraph.

Second Method. Let it be required to determine the difference in elevation between points *A* and *B*. Set up the instrument at

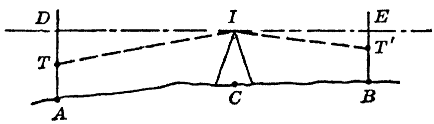


Fig. 427(*b*).

some convenient point *C* (not necessarily on line between *A* and *B*), hold the rod at *A*, and set the target at some convenient ***index mark*** *T* on the rod, usually a whole foot mark. Sight at the target *T*, measure the vertical angle *DIT*, and obtain by stadia the distance *IT*. Without disturbing the target, hold the rod on *B*, sight at the target, which is now at *T'*, measure the vertical angle *EIT'*, and obtain by stadia the distance *IT'*. The vertical distances *DT* and *ET'* may now be determined by one of the methods based on formula 4 (Art. 425(*a*)) or formula 6 (Art. 425(*b*)). Since *AT* and *BT'* are equal, the difference in the elevations of *A* and *B* is equal to $DT - ET'$; or, when *ET'* is above *E*, $DT + ET'$. This method is merely a modification of the first method, but it is more general and can often be used where it would not be convenient or possible to use the first method. (See Arts. 430, 431 for a practical application of the method in stadia leveling.)

When, in the method illustrated in Fig. 427(*b*), it is impracticable to use the same index mark at *B* that was used at *A*, choose some other convenient mark to sight at. The change, however,

should be made clear in the field notes in order that the difference in elevation between *A* and *B* may be corrected by applying the difference in rod readings between the index mark at *A* and that at *B*.

427(*c*). Practical suggestions.

1. Once the target has been set, any number of elevations may be taken, but it is essential to state definitely in the field notes what point is taken as a reference point; otherwise it may be uncertain which of the two methods was used.

2. It is not necessary, of course, to use a target in any of the methods of measuring vertical distances, but it is often more convenient to do so. When there is no target on the rod, a temporary one may easily be devised. When a target is not used, the index mark on the rod should be recorded. When it is necessary to obtain the elevations of several points from a single setup of the instrument, the index mark, chosen at the starting point, should be used if possible throughout.

3. For accurate results, a vernier reading to half minutes is desirable. The instrument should preferably be equipped with a sensitive control level by means of which the index error can be eliminated. Lacking a control level, the index error should be determined (Art. 271(*b*), *Note*) and the correction applied to the reading for each vertical angle.

4. Neither stadia diagrams nor slide rules will give results as accurate as those that may be obtained by means of stadia tables; consequently the latter should be used when a relatively high degree of precision is desired.

5. In ordinary work and for small vertical angles, such as those which generally occur in practice, the constant *C* may generally be disregarded. The approximate reduction formula in Art. 425(*a*) may then be used.

427(*d*). Practical questions.

1. *How accurately can elevations be obtained by this method?* This depends upon the accuracy with which the vertical angle is taken, the magnitude of that angle, the accuracy with which the inclined distance to the rod is measured, and the amount of that distance. The following tables give the errors in elevation due to angular errors of 2′, 6′, and 16′ for 100-ft. sights, and also the error due to an error of 1 ft. in distance for different angles. Angles of 1° and 26° are taken because they are the two extremes, vertical angles rarely exceeding 26° in practice; 14° is approximately halfway between the two. The extreme error in vertical angles is not likely to exceed 16′.

Errors in Elevation per 100 ft. Due to Errors of 2′, 6′, and 16′

	1°	14°	26°
2′	0.06	0.05	0.04
6′	0.18	0.16	0.11
16′	0.47	0.41	0.29

Errors in Elevation Due to an Error of 1 ft. in Distance

1°	14°	26°
0.017	0.235	0.394

Notice that the larger the angle the *less* the error due to angular error, and the greater the error due to linear error. Thus an error of 16′ results in an error of 0.47 ft. if the angle is 1°, whereas it is only 0.29 ft. if the angle is 26°. On the other hand, an error of 1 ft. in distance causes an error of only 0.017 ft. if the angle is 1°, but an error of 0.394 if the angle is 26°. From such comparisons, it is evident that when a vertical angle is small, the effect of an error in the measurement of the angle is approximately the same as that due to an error in the observed stadia reading. When, however, an angle is large, the effect of an error in the measurement of the angle is relatively small compared with the effect due to an error in the observed stadia distance. Consequently, in order to maintain a given degree of precision in measuring a vertical distance with a stadia, greater precautions should be taken in observing the stadia distance when a vertical angle is large than when it is small.

As vertical angles can be read easily to within 1′ and distances to within 0.5 ft., it follows that for a 100-ft. sight the maximum error need not exceed 0.2 ft. to 0.3 ft., and will usually be less. Thus, for errors of 2′ and of ½ ft., the total error for a 1° angle is 0.06 + 0.009 = 0.069 ft., whereas for a 26° angle it is 0.04 + 0.197 = 0.237 ft. (See also "Limits of Error," Art. 436.)

2. *How close should distances and vertical angles be measured?* If the vertical angle is estimated to the nearest 15′ the angular error will not exceed 6′ or 7′, and if the stadia is read to the nearest foot the linear error will not exceed ½ ft. Hence for a 100-ft. sight the total error of elevation due to errors of 6′ and ½ ft. is for a vertical angle of, say, 14°, 0.16 + 0.118 = 0.28 ft., or about 0.3 ft.; for 26° it is 0.11 + 0.19 = 0.3 ft. For errors of 6′ and 1 ft., the total error for 14° is 0.16 + 0.236 = 0.39, or about 0.4 ft. For a 300-ft. sight errors of, say, 6′ in a 14° angle and 1 ft. in distance would result in a total error in elevation of about 0.48 + 0.236 = 0.72 ft.

Caution. In the preceding paragraph, the linear error in each case is based on the assumption that the length of sight is short and hence the corresponding linear error is small. In estimating what the linear error may be for various lengths of sights, it should be remembered that it is more difficult to obtain accurate stadia distances under some conditions than under others and that there are certain sources of errors which must be taken into account. It is important, therefore, when judging the accuracy with which stadia distances must be measured to attain a certain degree of precision, to understand these sources of error and how to eliminate them, or how to make allowances for them. (See "Sources of Error," Art. 435.)

428. A Modified Technique for Measuring Either Horizontal or Vertical Distances. It is assumed that the stadia hairs in the instrument used are equidistant from the middle cross hair. The procedure, based on the method of reading two half intervals on the rod (Art. 424(*b*)), is as follows. (1) Set the target on the rod at the height of the supporting axle as explained in Art. 427(*a*) or at the index mark as explained in Art. 427(*b*), second method. Record the target reading. (2) Hold the rod on the point to which the distance is required, direct the telescope until the middle hair exactly bisects the target, and clamp the telescope in this position. Observe and record readings for the upper and lower stadia hairs. These readings may be recorded in the form of a fraction with the top-wire reading above, and the bottom-wire reading below. (3) Making sure that the telescope does not move in any way, signal the rodman to go to the next point to be sighted at, and while he is moving into position, read and record the vertical angle; also the horizontal angle or the azimuth if either of these angles is to be measured.

One advantage of this method is that the instrument wires are set only once for any one sight. Its chief advantage, however, is in the check on each sight obtained by observing whether or not the upper half interval and the lower half interval are equal. This check involves a little mental arithmetic, but it will, in the course of time, reveal many a mistake. Since for each sight there are only two wire readings taken through the telescope (the same number as that taken in the usual procedure), little if any time is lost in this method.

429. The Beaman Stadia Arc is a mechanical device by means of which an inclined stadia distance may be reduced to the corresponding horizontal and vertical distances without measuring di-

rectly the vertical angle. This arc, which in modified form is also called *stadia circle,* is attached to the vertical circle or to the vertical arc on transits and plane-table alidades. It has two scales and two corresponding index marks. One scale, marked "V," is for obtaining the vertical distance per 100 ft. of observed inclined stadia distance; the other scale, marked "H," is for obtaining the correction per 100 ft. of observed inclined stadia distance to be subtracted from that distance in order to obtain the corresponding horizontal distance. Thus the stadia arc is merely a device for simplifying the solution of formulas 5 and 6 in Art. 425(*b*). There are several different forms of this device, but they are all alike in principle.

The position of the V-index mark with respect to the V-scale is generally such that when the line of sight is horizontal the "50" mark on the V scale is exactly opposite the corresponding index mark. For a positive vertical angle the V reading will be greater than 50; for a negative vertical angle it will be less than 50.

429(*a*). To DETERMINE A HORIZONTAL DISTANCE. (1) Observe and record the inclined stadia distance by the method explained in the first two steps in Art. 426(*b*). (2) Observe and record the reading on the H scale of the stadia arc. (3) Multiply the reading observed in (2) by the number of 100 ft. in the inclined stadia reading observed in (1) (or by the interval on the rod between the stadia hairs). (4) From the inclined stadia distance subtract the product found in (3), and add the constant C. (For internal focusing $C = 0$.) The final result should be the same as that which would have been obtained for the value of H from the approximate formula 5, Art. 425(*b*), had the vertical angle been measured. The calculations in steps 3 and 4 are sometimes carried out in the field, sometimes in the office.

Illustration. Assume that the observed inclined stadia distance is 320 ft., that the reading on the H scale is 1.5, and that $C = 1.0$ ft. Then $1.5 \times 3.2 = 4.8$, and the horizontal distance H is $320 - 4.8 = 315.2$, or, say, 315 ft. $315 + C = 316$ ft.

429(*b*). To DETERMINE A VERTICAL DISTANCE. (1) Observe and record the inclined stadia distance. (2) Change the inclination of the telescope slightly until the reading on the V scale is the nearest whole number. Observe and record this reading and also the position of the middle hair on the rod. If it is assumed that the 50 mark and the index mark coincide when the telescope is horizontal, the difference d between 50 and the observed reading on the V scale is the value used in calculation. (3) Multiply the quantity d obtained in (2) by the number of hundred feet in the inclined stadia distance found in (1) (or by the interval on the rod between the stadia hairs). The result V is equal to the vertical distance between the horizontal plane of sight and the point on the rod intersected by the middle cross hair. It is assumed that $C \sin a$ is negligible; hence the final result should be the same as that which would have been obtained for the value of V from the approximate formula 6, Art. 425(*b*), had the vertical angle been measured.

Illustration. Assume that the observed inclined stadia distance is 320 ft., that the reading on the V scale is 62, and that the middle hair intersects the rod at 5.4. Then the reducing factor is $62 - 50 = +12$, and the distance V is $3.20 \times 12 = 38.4$ ft. Assume that the elevation of the plane of sight (or height of instrument) is 186.5 ft.; the elevation of the point on which the rod is held is $186.5 + 38.4 - 5.4 = 219.5$ ft.

The difference in elevation between the point on which the rod was held and a point on the ground near the instrument can be found by a method similar to the method in Art. 427(*a*); between *any* two points A and B, by a method similar to one of those explained in Art. 427(*b*).

429(*c*). COMMENTS. Precautions should be taken to eliminate sources of error similar to those taken in measuring vertical angles. For example, in accurate work any index error should be corrected either by setting the index at 50 when the telescope is leveled or by means of a vernier level.

The Beaman stadia arc simplifies the calculations in reducing observations but requires more time for these observations than when vertical angles are measured. It is doubtful if much is gained in ordinary work by the use of the stadia arc.

On some transits the horizontal stadia arc is attached on one side of the vertical circle and the vertical stadia arc on the opposite side.

430. Stadia Leveling. Under certain conditions, the stadia may be used to great advantage for differential leveling and for profile leveling. The instrument is set up between turning points as in spirit leveling. (See Fig. 427(*b*), in which *A* and *B* may be regarded as turning points.) The general method of procedure parallels closely that for spirit leveling except that the difference in elevation between two successive points is measured by one of the stadia methods explained in Art. 427(*b*). When the ground is nearly level, the first method in Art. 427(*b*) may be used, but generally it is necessary to use the second. The procedure for each setup is illustrated in Fig. 427(*b*). When the instrument is equipped with an ordinary telescope, the lengths of sights should preferably not exceed 400 ft. to 500 ft. Differences in elevations are generally calculated to the nearest tenth of a foot. It is not quite so important in stadia leveling as it is in spirit leveling to balance foresight and backsight, or to have the aggregates of the foresights and backsights equal. It is well, however, to balance foresight and backsight when this can be done conveniently.

430(*a*). A MORE ACCURATE METHOD. When greater accuracy is required than can be obtained by the method explained in the preceding paragraph, the procedure is as follows. Instead of setting up the instrument *between* turning points or stations, set it up at *each* point. The difference in elevation between two successive points may be obtained by the method of Art. 427(*a*). In this way two differences of elevation between two successive points are obtained, one when the instrument is at the first point, the other when it is at the second. The mean of the two differences may be taken unless the discrepancy between the two is too great. The permissible error in the difference in elevation between the two points should not exceed the sum of the two errors found from the following approximate equations:

E_1 = stadia distance multiplied by the sine of the permissible error in the vertical angle

E_2 = permissible error in the stadia distance multiplied by the sine of the vertical angle

The method of leveling just explained corresponds to the double-rodded or the double-turning-point methods of spirit leveling, explained in Art. 174. It provides a check on the distance as well as on the difference in elevation between two successive stations, and, at the same time, it makes it possible to attain a relatively high degree of precision.

431. Field Notes for Stadia Leveling. When all stations are turning points, the field notes may be recorded as shown in the accompanying form. This form of notes may easily be modified for other types of stadia leveling such as stadia profile leveling.

STADIA LEVELS ALONG C.C.C. ROAD TO STATE FOREST

Station	Backsight Readings	Vertical Distance	Foresight Readings	Vertical Distance	Elevation	
B.M.10	+8° 40′ 352	−52.4			654.71	U.S.G.S. B.M. at road intersection. B.S. sight at 6.0
T.P.1	+6° 48′ 228	−33.9	−2° 28′ 212	−9.1	593.2	All sights at 6.0
T.P.2	−5° 28′ 225	+21.3	+1° 50′ 365	+11.7	571.0	F.S. 6.0 on T.P.3 B.S. 10.0 on T.P.2
T.P.3	+2° 10′ 192	−7.3	+4° 18′ 310	+23.2	619.5	All sights at 6.0
B.M.11			−4° 50′ 318	−26.7	583.5	F.S. 8.0 on B.M.11

The Method of Procedure for each setup is the *second method* explained in Art. 427(*b*). The first backsight was on B.M.10, the elevation of which was known to be 654.71 ft. The middle cross wire was set at an index reading of 6.0 ft. as noted on the right-hand page of the field notebook. The readings for the backsight were vertical angle +8° 40′ and stadia reading 352 ft. (inclined distance). The plus sign for the angle indicates that the index mark is above the center of the instrument. From these two readings the vertical distance −52.4 ft. from the index mark to the horizontal plane through the center of the telescope may be computed, either in the field or in the office. (When reduction tables are not available, use formula 4 or 6, Arts. 425(*a*) and (*b*).) This vertical distance for the *backsight* is conceived as measured from the index mark *toward the horizontal plane* through the telescope, and since this direction is downward, the algebraic sign is minus. The foresight readings on T.P.1 were taken with the index point

on the rod at 6.0 ft., as indicated in the notes, and were recorded as $-2° 28'$ and 212 ft. The computed vertical distance for the foresight, i.e., -9.1 ft., is conceived as measured from the horizontal plane through the center of the telescope (where the vertical distance for the backsight ended) *toward the index mark;* and since this direction is downward, the algebraic sign is minus. The algebraic sum of the two vertical distances is $-52.4 + (-9.1) = -61.5$ ft. This result is the difference in elevation between B.M.$_{10}$ and T.P.$_1$; hence to obtain the elevation of T.P.$_1$ this difference in elevation is added algebraically to the elevation of B.M.$_{10}$, i.e., $654.7 + (-61.5) = 593.2$ ft.

It frequently is not practicable to use the same index reading for the foresight as for the backsight. In such a case, the index mark is chosen a convenient whole number of feet above or below the index reading used for the backsight, and this difference in index readings must be taken into account in computing the difference in elevation. This is illustrated by the notes for the setup between T.P.$_2$ and T.P.$_3$. For the backsight on T.P.$_2$ the index mark was 10.0 ft., as indicated in the notes, and the corresponding computed vertical distance was $+21.3$ ft. For the foresight on T.P.$_3$ the index mark was 6.0 ft., as indicated in the notes, and the corresponding computed vertical distance was $+23.2$. Had the index mark been 10.0 ft. as it was for the backsight, the computed vertical distance would have been greater by $10 - 6 = 4.0$ ft., i.e., $23.2 + 4.0 = +27.2$ ft. Hence the difference in elevation between T.P.$_2$ and T.P.$_3$ is $+21.3 + 27.2 = +48.5$ ft., and the elevation of T.P.$_3$ is $571.0 + (+48.5) = 619.5$ ft.

431(*a*). SUGGESTIONS. To avoid confusion in algebraic signs keep in mind: (1) An inclined sight upward is plus, downward minus. (2) For a *backsight,* conceive the computed vertical distance as measured from the index mark *toward* the horizontal plane through the center of the telescope, plus if upward, minus if downward. For a *foresight,* however, conceive the computed vertical distance as measured from the horizontal plane *toward* the index mark, plus if upward, minus if downward. (3) To determine the difference in elevation between two points, add *algebraically* the computed vertical distance for the backsight on one point and the computed vertical distance for the foresight on the other point. (4) To determine the elevation of a point on which a foresight is taken, add *algebraically* the result obtained in (3) to the elevation of the point on which the backsight was taken.

Until these procedures become more or less instinctive, it will be helpful at each setup to imagine a figure similar to Fig. 427(*b*), *Note.*

432. The Stepping Method, illustrated in Fig. 432, is sometimes used for determining vertical distances, but only where the slope of the ground is so small that an intercepted distance on the stadia rod is practically the same for an inclined sight as it is for a corresponding horizontal sight. This means that, as a general rule, vertical angles should not exceed, say 2°. In drawing the

figure, it was necessary to exaggerate the vertical scale so that vertical angles greatly exceed 2°, and this should be kept in mind.

Let it be required to determine the vertical distance between the horizontal plane of sight and a point *A*. (1) Observe the interval between stadia hairs when the rod is held on *A*. Assume that this interval is found to be 5.25 ft. This will be used as a unit for vertical measurement. (2) Find the point *B* where the horizontal line of sight intersects some prominent object and keep this point in mind. (3) Raise the telescope until the lower hair cuts *B* and observe the point *C* in which the upper hair intersects another

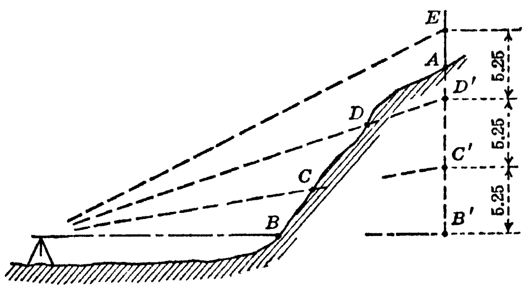


FIG. 432.

prominent object. Repeat the process by bringing the lower hair to *C* and finding *D* by the upper hair, then bringing the lower hair to *D* and finding the point *E* where the upper hair intersects the rod. Since the vertical angle for each line of sight is small, the lines to *B*, *C*, and *D*, when produced to points *B'*, *C'*, and *D'*, will intercept the vertical through *A* at intervals that are practically equal, i.e., *B'C'*, *C'D'*, and *DE* are each 5.25 ft. Hence *E* is $3 \times 5.25 = 15.75$ ft. above *B* in the horizontal plane of sight. Assume that the rod reading from *A* to *E* is 3.25 ft.; then the required vertical distance of *E* above the plane of sight is $15.75 - 3.25 = 12.50$ ft.

Comments. Points *B*, *C*, and *D* should be on objects easily recognizable through the telescope, and the instrument man should keep the point *C* in view while moving the lower cross hair from *B* to *C*, the point *D* in view while moving the lower cross hair from *C* to *D*, and so on. Usually the number of steps is limited to three.

When the points sighted at are below the horizontal plane of sight, the same general method of procedure holds good, except that the upper hair instead of the lower is set on successive points.

The method is used principally in plane-table work where it is usually necessary to determine elevations in the field as the work progresses. With reasonable care, surprisingly good results may be obtained.

433. Transit-Stadia Surveys are transit surveys in which the **stadia instead of the tape is used for horizontal linear measure-**

ments. The general methods of making a transit-stadia survey do not differ materially from those used in making an ordinary transit-tape survey. The same methods are used for running traverses (Chapter 21) and for locating details (Chapter 22) as in other kinds of surveys, and the suggestions for choosing stations, running transit lines, and carrying on field work given in Chapter 23 hold good. In addition, frequently it is necessary to use the stadia for determining elevations also. This is usually done at each transit station at the same time that horizontal measurements are made, particularly in topographic surveying. General suggestions pertinent to horizontal measurements in stadia surveying are given below. Additional suggestions for methods of procedure, not only in measuring horizontal distances, but also in determining elevations, will be found in Chapter 31, on topographic surveying.

433(*a*). PRACTICAL SUGGESTIONS. (1) Always read the distance between stations twice, once when the transit is at one station, a second time when the transit has been moved to the other station. *Record both readings, even if they agree.* (2) Important side shots may be checked by taking a half reading, using the middle hair and a stadia hair, and then a full reading, using the two stadia hairs. Enter both readings in the notes. (3) Since the stadia method is best adapted for comparatively rough work, and since such work is not usually done in the city, but outside in places where local attraction is not so likely to exist, the azimuth method (Arts. 303 and 351), in which azimuths may be roughly checked by magnetic bearings, is a favorite method for stadia surveying. (4) When taking azimuths or angles to important points, such as transit stations, hold the rod edgewise towards the transit. (5) Keep in mind the limits of error, and do not waste time on needless refinements in setting up, leveling up, and manipulating the transit. (6) Read the stadia first, then signal to the rodman to move to the next point, and read the horizontal angle (or azimuth) and the vertical angle while he is going there.

434. Forms of Field Notes. The forms of notes given in Chapter 24 may be easily modified to suit stadia surveying. The principal addition is a column for vertical angles, and frequently another column for corrected horizontal distances. When elevations are taken, two more columns may be added, one for vertical distances and one for elevations. The form of notes will depend largely upon whether the azimuth method, the direct-angle method, or some other method of field work is used. Some of the forms of tabulating notes are given below, and these forms will suggest others. The notes shown are those on the left-hand page of the notebook, and it is understood that these are supplemented by sketches and explanatory notes not shown here, but placed on the opposite page in the field book, as illustrated in Chapter 24.

The term *stadia distance* as used in Forms 1 and 2 means the distance obtained by multiplying the rod reading by the stadia interval factor, regardless of whether this distance is inclined or horizontal. Some engineers prefer to enter the rod reading instead

of the stadia distance, particularly when the stadia interval factor is not exactly 100.

Line	Azimuth	Stadia Distance	Vertical Angle	Corrected Distances	
				Horizontal	Vertical
	S 50°30′E				624.8
AB	129°28′	286	+8°30′	279.8	+41.8
					567.2
A2	184°54′	122	−7°30′	119.9	−15.8

FORM 1

434(*a*). FORM 1. This form is similar to Form 7, p. 356. Unless the notes are plotted in the field the last two columns may be worked up in the office. The bearing S 50° 30′ E checks the azimuth 129° 28′ as closely as the needle can be read. This bearing was recorded because *AB* is a line between stations, and it is well for the notes to indicate that the azimuths of important lines, such as transit lines, were checked by needle readings. The azimuth of *A*2 may have been checked also by glancing at the needle, but it was hardly worth while to record this bearing, as *A*2 is a side shot.

The elevation of the bench mark from which the rod was set (582.95 ft., see Form 3) should be entered in the notes, on the right-hand page if there is no place for it on the left-hand page. The distance 286 appears in the notes only once, but if the notes were continued to include those taken at station *B* the check value would be recorded as *BA*, since the stadia reading is taken from *B*. The elevations 624.8 and 567.2 for *B* and 2 respectively are frequently entered in a separate column as shown in Form 3.

Angle	Value	Line	Stadia Distance	Vertical Angle	Corrected Distance
	2 | 101°20′				
DAB	50°40′	AB	286	+8°30′	279.8
DA2	106°06′	A 2	122	−7°30′	119.9

FORM 2

434(*b*). FORM 2. This form is the preceding form modified to suit the direct-angle method (Arts. 301 and 349). Vertical dis-

tances are not shown in the notes. When these are desired, another column, corresponding to the last column in the preceding form, may be added. Space for this column may be gained by omitting the third column, provided it is understood that the last two letters in the first column indicate the line measured.

Point Sighted	Azimuth	Vertical Angle	Distance	Vertical Distance	Elevation
Transit at Station A. held on B.M. No. 24			Target set for rod Elevation 582.95		
	S50°30'E		279.8		
B	129°28'	+8°30'	286	+41.8	624.8
			119.9		
2	184°54'	-7°30'	122	-15.8	567.2

FORM 3

434(c). FORM 3. In this form the magnetic bearing is entered directly above the corresponding azimuth, and the corrected distance directly above the stadia distance. This leaves the last two columns for vertical distances and elevations. In topographic surveying, where a large number of side shots are taken in locating contours, this is an excellent form. The station occupied and the elevation of the bench mark or other reference point are carefully noted at the head of the page. A sketch is put on the opposite page with points numbered (usually in the order in which the side shots are taken), and the names of party, explanatory notes, and all data which may be of future use complete the notes. (See illustrations of field notes, pp. 357–376.)

As a rule, the azimuth to a transit station will be the first entry after setting up. (Why? See (2) in Art. 347.)

434(d). MODIFICATIONS. Corrected horizontal distances, vertical distances, and elevations are frequently calculated in the office. When they are, the corresponding columns may be entirely omitted from the field notes, particularly when it is desired to use these columns for notes which pertain strictly to the field work. For example, if in Form 3, it is desired to enter the magnetic bearing and the corrected horizontal distance each in a separate column, it would be necessary to have the left-hand page ruled in eight columns instead of six, or to transfer the two columns for vertical

distance and for elevation to the right-hand page, or to omit these two columns altogether.

The values of $(f + c)$ and of f/i for the instrument used may be entered in the notes once for all, say at the beginning of the field work.

The forms just shown may be modified in various ways to suit the requirements of different types of stadia surveys or the preferences of different surveyors. As a general rule, however, it is desirable to record field observations across the page in the order in which they are observed and, preferably, in such a way that two quantities closely related are in adjacent columns. For example, the point sighted at, the value of the azimuth, the stadia distance, and the vertical angle may be entered in that order across the page if that is the order in which the observations are made. When, however, a magnetic bearing is observed, it is well to enter it either directly above the corresponding azimuth or in a column adjacent to the azimuth column so that the two may be easily compared. Likewise, if there are columns for the corrected horizontal and corrected vertical distances, it is desirable that these columns should be adjacent to that for the vertical angle.

434(*c*). Sketches. The sketches are fully as important in stadia surveying as in other kinds of surveying, because, as a rule, when the stadia is used there are often a large number of points to be located from each station, and the relative position of these points should be clearly shown. Suggestions for making sketches have been given in Arts. 366–372. In some kinds of work, notably topographic surveying, notes are often plotted in the field.

435. Sources of Error in Stadia Surveying. In addition to the errors in ordinary transit work (Chapter 20), stadia work is subject to many of the errors discussed in connection with leveling (Chapter 12). For example, it is important that the rod should be plumb, that the rod and wires should be in focus, and that the rod should be correctly graduated. Most of the errors discussed here are those peculiar to stadia work.

1. *Errors in rod graduations. Constant.* Usually negligible in ordinary work, but for more accurate results rods should be standardized and corrections if any applied.

2. *Error due to inclination of the rod.* Errors due to this source may be negligible when vertical angles are small but important when angles are large. Best avoided by the use of a plumbing level on the rod.

3. *Mistakes or blunders in reading the rod.* Large blunders, particularly an error of an even 100 ft., may often be avoided by checking roughly by eye the distance from the instrument to the rod. This method of checking should become so much a habit as to become automatic. Smaller blunders may be detected by observing the distance between the middle wire and one of the other wires to see if it is approximately one-half the distance between the upper and lower hairs. A common mistake is to read the wrong stadia hair.

4. *Error in observing the space on the rod intercepted by the stadia hairs. Accidental.* This error is affected by the magnifying power of the telescope, the fineness of the stadia hairs, the character of the rod graduations, the length of sight, and atmospheric conditions. For very long sights a rod with two targets is better. When cross hairs appear coarse, better results may be obtained by sighting across the top of each hair instead of attempting to estimate where the center of each cuts the rods. Parallax should be eliminated (Art. 103(*a*)). Important distances should be determined twice, first from the transit station at one end and then from the station at the other end.

5. *Error in the measurement of a vertical angle.* The effect of errors in vertical angles has been discussed in Art. 427(*d*). Such errors may be avoided by the usual precautions observed in reading angles and either by observing and applying the index correction or by eliminating this error by the use of a control level (Art. 271(*b*)).

6. *Error due to the use of an incorrect stadia interval factor. Constant.* This is one of the most important sources of error because it is cumulative; avoided by determining and using the true stadia interval (Art. 423(*a*)). As explained in the next two paragraphs, atmospheric conditions affect the accuracy with which the stadia interval may be determined.

7. *Errors due to unequal refraction* are of little consequence in ordinary stadia surveying but may be too great to be ignored in the better class of work. Experiments have shown that unequal refraction due to varying densities in different strata of air is a source of cumulative error in stadia work. The stratum of air within 3 or 4 ft. of the ground is of greater density than the air above. Between the hours of 7.00 A.M. and 9.00 A.M., and between the hours of 2.30 P.M. and 7.00 P.M., this difference in density is comparatively slight, and does not seriously affect stadia readings; but during the midday hours of 9.00 A.M. to 2.30 P.M. the density of the lower stratum is increased *upward* by the radiation of the earth's heat. The depth of this stratum varies for different hours and seasons, but is never over 4 ft. A ray of light above this stratum is not greatly affected, but a ray of light passing through it is bent upward. Hence the space on a rod included between the upper and lower rays is less at noon than in the morning, and the error is enough to affect stadia work seriously where results of, say, 1/1000 or better are desired.

Unequal refraction may be a source of error not only in observing the intercepted space on a rod when measuring distances but also when determining an interval factor; hence the following *practical suggestions.* (1) A stadia interval factor determined during midday hours will give distances too *long* when these distances are read during early or late hours, and, vice versa, if the interval factor is determined during early or late hours of the day; distances read during midday hours will be too short. (Why?) Hence do not attempt accurate work during hours which do not correspond to your interval determination. (2) If work is to be carried on all day, determine the stadia interval factor by tests made at different hours throughout the day. (3) During midday hours do not take sights which require the line of sight to pass nearer than 3 ft. to the ground. For long sights use a half reading on the upper portion of the rod. (4) Errors due to unequal refraction may be reduced by working between the hours of 7:00 A.M. and 9:00 A.M., or between 2:30 P.M. and 7:00 P.M., or when it is cloudy.

436. Limits of Error. Two questions arise in discussing limits of error in stadia surveying, viz.: (1) With what degree of precision can a single horizontal distance be measured or a single difference in elevation be found? (2) What error may be expected in running a series of connected lines where the accidental errors from different setups are likely to be compensating?

The degree of precision attainable in any survey will depend to a considerable extent upon whether conditions are favorable or unfavorable. Unfavorable conditions include: inferior instruments; terrain so hilly that large vertical angles are necessary; conditions that require long sights; unequal refraction or other unfavorable atmospheric conditions; unskilled observer.

436(*a*). THE ERROR IN A SINGLE HORIZONTAL DISTANCE, such as the distance between two successive stations, or the length of a side shot, may be due wholly to the error in obtaining the stadia distance, or, if the sight is inclined, partly to that error and partly to the error in the observed vertical angle. An error in the vertical angle does not seriously affect the determination of a horizontal distance when the angle is small.

From a series of tests made by 125 of the author's students it would appear that the probable error of sighting is not likely to exceed 1 ft. up to a distance of 400 ft. The mean of ten probable errors *of a single observation* for each of four distances is given below. Each probable error was found from at least 25 readings by different students, several transits being used. The mean of the probable errors involves therefore approximately 250 readings for each distance.

Distance	90–125 ft.	185–210 ft.	285–300 ft.	415–435 ft.
Mean of probable errors	0.51 ft.	0.72 ft.	0.83 ft.	0.93 ft.

These results do not indicate the *discrepancies* to be expected between tape measurements and stadia measurements, since such discrepancies will involve constant errors, which are not involved in the mere act of reading the rod, such, for example, as errors in the wire interval and errors in rod graduations. In order, therefore, to determine what the discrepancies really were, each of the distances was measured with a tape. Each quantity in the following table is the mean of approximately 50 discrepancies, i.e., the readings of approximately 50 students at each transit and for each distance were compared with the tape measurement for that distance and the mean of the 50 discrepancies entered in the table.

Mean Discrepancies between Stadia and Tape Measurements

Distance	90–125	185–210	285–300	415–435
Transit 1	0.58	0.66	1.37	1.77
" 2	0.65	1.30	1.09	1.30
" 3	0.57	0.65	0.73	1.87
" 4	0.44	0.63	0.93	0.99
Mean	0.56	0.81	1.03	1.48
Minimum	0.44	0.63	0.73	0.99
Maximum	0.65	1.30	1.37	1.87

The above results were obtained under somewhat *unfavorable* conditions. With reasonable care the discrepancies in practice should rarely exceed the mean discrepancies of 0.5, 0.8, 1.0, and 1.5 ft. respectively, and will frequently be less than the minimum values of 0.4, 0.6, 0.7, and 1.0 ft.

436(*b*). The error in the measurement of a single vertical distance above or below the horizontal plane of sight is due to two principal errors, viz., (1) the error in the observed stadia distance and (2) the error in the observed vertical angle. The error in the vertical angle seldom exceeds 5′ of arc, and if necessary it can easily be reduced to less than 1′. The mean discrepancy in distance may be taken from the preceding table as 0.6 ft. for a 100-ft. sight, 1.0 ft. for a 300-ft. sight, and 1.5 ft. for a 400-ft. sight. The total errors given below are based on these discrepancies, combined with errors in vertical angles of 1′ and 5′ respectively. The total error in each case is the maximum error (expressed in feet) in the computed vertical distance from the point sighted at to the horizontal plane of sight. The table is useful in judging the error that is likely to occur in the measurement of a single vertical distance under ordinary conditions and with an ordinary instrument in good adjustment.

	3°		5°		10°		15°		25°	
Distance	1′	5′	1′	5′	1′	5′	1′	5′	1′	5′
100 ± 0.6	0.05	0.15	0.08	0.20	0.15	0.25	0.20	0.25	0.25	0.35
300 ± 1.0	0.15	0.45	0.20	0.55	0.25	0.60	0.35	0.70	0.50	0.80
400 ± 1.5	0.20	0.60	0.25	0.75	0.35	0.85	0.45	0.90	0.70	1.05

The use of the table may be illustrated as follows. If the vertical angle does not exceed 3°, and the stadia distance does not exceed 300 ft., the total error in the vertical distance due to a maximum error of 1.0 ft. in the stadia distance and a maximum error of 1′ in the vertical angle should not exceed 0.15 ft. For the same conditions except that the maximum error in the vertical angle is 5′, the total error should not exceed 0.45 ft. For a maximum vertical angle of 15°, a maximum stadia distance of 400 ft., a maximum error in this distance of 1.5 ft., and a maximum error in the vertical angle of 5′, the total error in the vertical distance should not exceed 0.9 ft.

436(*c*). Limits of error in a series of connected lines. When a straight line such as a base line is so long that it must be measured in sections from successive setups, the error in the total length may or may not be in proportion to that length. Likewise, when a series of straight lines form angles as in a traverse, the total error due to errors in stadia measurements and errors in angles may or may not be in proportion to the total length.

If precautions are taken to eliminate sources of constant errors, the total error may be expected to vary as the square root of the total distance, provided the number of setups is sufficiently large to permit the balancing effect of accidental errors; but if such precautions are not taken, the total error may vary more nearly as the total distance. It is somewhat easier to eliminate sources of constant errors in transit stadia surveys than in those made with transit and tape. Consequently, in extensive surveys, the degree of precision that can be attained by the stadia method under favorable conditions is often comparable to that attained with transit and tape.

Since conditions under which different surveys are made vary so greatly, it is impossible to state what degree of precision may be expected in any particular survey, except in a most general way. Consequently, the degrees of precision that are suggested in the paragraphs that follow can, at best, serve merely as a rough guide.

1. Conditions: a straight line; relatively few setups; ordinary conditions; ordinary precautions to eliminate errors. A ratio of precision of 1/500 to 1/700 may easily be attained.

2. Conditions: a straight line; a relatively large number of setups; ordinary conditions; ordinary precautions. Ratio of precision may be as great as 1/1200.

3. Conditions: a straight line; ordinary conditions; more than ordinary care taken to eliminate sources of constant error. Comparatively high degrees of precision that can be attained as indicated by the following example.

From experiments in measuring a base line nearly 1.5 miles long, the ratio of precision attained for the mean of 13 measurements was 1/2250, the measurements being taken under ordinary field conditions and at different hours of the day. The ratio of precision of the mean of five afternoon measurements was 1/3180. These results are excellent. More common results under similar conditions but with a smaller number of measurements run from 1/1200 to 1/2000.

4. For ordinary transit-stadia traverses the principal part of the error of closure will be due to errors in stadia observations and not to errors in horizontal angles. Hence the degrees of precision that may be expected, judged by the error of closure, will be 1/500 to 1/1200, according to whether conditions correspond to those in (1) or to those in (2). When a traverse is long enough to require a large number of setups, accidental errors are more likely to balance; consequently, a ratio of precision of 1/2000 or even better may be attained. In such an extensive traverse, however, part of the field work may be done under favorable conditions and part under unfavorable; this must be taken into account in applying any criterion of precision. Limits of error for given conditions may be determined after a short experience in the field, and this is often the best course to pursue if the survey is extensive enough to warrant it.

436(*d*). Limits of error in leveling. Errors in the measurement of a single vertical distance have been discussed in Art. 427(*d*). In rapid work, under unfavorable conditions, the error in the difference of elevation obtained from a single sight, such as a side shot, may be as great as 2 or 3 ft. per 1000 ft. of horizontal distance; under favorable conditions, with ordinary precautions to eliminate errors, it may be less than 0.5 ft. per 1000 ft.

For a series of lines such as the lines of a traverse, the maximum total error in the difference in elevation may be expected to vary about as follows.

1. Conditions: vertical angles up to 15° or 20°; sights limited to 1000 ft.; vertical angles measured to the nearest minute; no special precautions to eliminate constant errors; maximum error 2 to 3 ft. per mile.

2. Conditions: vertical angles up to 15°; sights up to 1000 or to 1500 ft.; angles measured to nearest minute; precautions taken to eliminate constant errors; maximum error ± 0.5 ft. $\sqrt{D}$ to ± 1.0 ft. $\sqrt{D}$, in which D is the distance in miles.

3. Conditions: similar to those in (2) except that the country is so nearly flat that vertical angles are small. Maximum error ± 0.3 ft. $\sqrt{D}$ to ± 0.4 ft. $\sqrt{D}$.

437. Practical Suggestions. Summary.

1. Determine the constant $C = c + f$ and do not forget to add it to each stadia distance unless an internal focusing telescope is used.
2. Stadia hairs should preferably be fixed, and they usually are in the better class of modern instruments. The true interval factor should be determined. When stadia hairs are adjustable, they should be tested and adjusted as often as the requirements of the survey may warrant.
3. As a rule, sight at a point about as far from the bottom of the rod as the telescope is above the ground, and then bring either the upper or lower hair to coincide with a whole foot mark. When the stadia hairs are coarse, lower the telescope until the whole foot mark can just be seen over one hair, and then look just above the other hair, thus getting the distance more exactly than if the marks were entirely covered by the hairs.
4. In reading the rod do not mistake the middle hair for a stadia hair; do not omit even 100 ft. or multiples of 10 ft., as 189 in place of 289, and 129 in place of 139. Form the habit of estimating distances by eye to guard against mistakes of an even 100 ft. and the mistake of reading the middle cross hair.
5. If the stadia rod is not held plumb, an appreciable error may result. In the better class of work, use a plumbing level on the rod.
6. The best results for single sights are obtained when the distance does not exceed 300 to 400 ft.; it should seldom be greater than 1000 to 1500 ft.
7. When the distance between two stations is too great for a single sight, assume a point about midway between, read the stadia distance to this point from each station, and add the two readings. If a check is desired, set up at the intermediate point and take two check readings. When the length of a long straight line is found from several successive setups, the total error may be less if relatively long sights are taken with a corresponding reduction in the number of setups.
8. When only a small portion of the rod can be seen, take a half reading and multiply by 2. This cannot be done unless the middle hair is halfway between the other two.
9. Be familiar with the effect of errors of 1° to 5° in a vertical angle on computed distances, both horizontal and vertical, in order to know how accurately they must be measured and when a vertical angle may be ignored. Do not forget to read and record the vertical angle (Arts. 426(*e*), 427(*d*)). Observe precautions for eliminating errors in the measurement of such angles.
10. Make it a rule to read distances between stations *twice*, once from each station, and enter both readings in the notes.
11. Important sideshots may be checked by taking a half reading and then a whole reading. If the middle hair is not halfway between the other two, take three readings, two of which added together should equal the third. Some engineers prefer habitually to follow the procedure of taking two half readings as outlined in Art. 428.

When Elevations Are Required

12. Assuming that the target has been set, either by holding the rod beside the supporting axis of the telescope or by backsighting on some point of known elevation, observe the following order of procedure. (1) Read the stadia distance. (2) Bring the middle hair to the target center. (3) Signal the rodman to move to the next point to be sighted on. (4) Read the vertical angle.
13. It is not necessary to use a target, and frequently it is better to sight directly on some index mark on the rod. When a target is desired and none is on the rod, a rubber band with white paper as a background, or some other simple device, may be attached temporarily to the rod to mark the point to be sighted at.
14. Don't fail to make clear in the field notes what bench mark or reference point was used in obtaining the H.I. at the first setup, and don't fail to record its elevation. The same suggestions hold good for a bench mark used for checking the work, or for a turning point.
15. In obtaining elevations, the longer the sight the more carefully the vertical angle should be read. The larger this vertical angle, the less important are small angular errors and the more important are small errors in distance. In work where elevations are required to 0.5 ft. and the lengths of

sight do not exceed 200 or 300 ft., read the distance to the nearest foot and the vertical angle to the nearest 5′ without using the vernier.

16. Any error in the adjustment of the telescope bubble affects materially the measurement of vertical angles; hence special attention should be paid to this adjustment.

General Suggestions

17. When a traverse or a line of levels can conveniently be run in either direction, it is well to choose the direction which will avoid as much as possible sighting toward the sun. Since the telescope must be inclined upward for many sights, looking toward the sun not only is annoying, but also renders the reading of the rod difficult. If one starts sighting toward the sun in the morning, in running a closed traverse, one may be sighting toward it all day, for as the direction of the traverse changes toward the west, the position of the sun also moves into the west. Incidentally, when the instrument man is sighting away from the sun, the face of the rod is toward the sun, and, consequently, the graduations are clearer than they would be if they were in shadow.

18. Follow a systematic method of procedure in observing stadia distances, horizontal angles, and vertical angles. If the same general procedure is consistently followed, a minimum amount of time will be consumed and a minimum number of mistakes will be made.

19. A thorough knowledge of the sources of error in stadia observations and how best to eliminate these sources is part of the stock in trade of a good surveyor.

20. Finally, in all stadia work keep the limits of error in mind and do not waste time on needless refinements in setting up, leveling up, and manipulating the instrument, or in reading the stadia distances. Use approximate reduction formulas (Art. 425(*b*)) for ordinary work. (For the use of reduction tables, stadia diagrams, and stadia slide rules see Art. 645.)

CHAPTER 30

PLANE-TABLE SURVEYING

In this chapter five methods of plane-table surveying are explained, practical suggestions for the use of each method are given, and the different methods are compared. General suggestions for the use of the plane table are made, practical questions of field work are discussed, and methods of solving the three-point problem and the two-point problem are explained.

438. The Plane Table consists of two parts: (1) the *table* or *board* supported by a tripod and (2) the *alidade* with telescope attached. The edge of the alidade and the axis of the telescope are parallel.

The best alidades are provided with a circular level and a compass box, both attached to the base. When an alidade is not equipped with these, a declinator, which consists of a square brass plate carrying a compass box and two bubble tubes, may be used. When an alidade is equipped with a circular level but not with a compass box, a declinator without bubble tubes will answer the purpose. *The traverse board* is a rough form of plane table, with upright peep sights in place of the telescope. The plane table is described more in detail in Art. 731.

439. Plane-Table Surveying is peculiar in this respect: the field work and the plotting are carried on simultaneously. The board, with drawing paper stretched upon it, is brought into a horizontal plane and kept there. The alidade, resting on the board, may be moved from place to place on the paper. When the telescope is pointed to any object, the *direction* of the line of sight is transferred to the paper by drawing a line along the edge of the alidade. Linear measurements are laid off to any desired scale. Thus, as the field work progresses, it is reproduced in map form on the board of the plane table, step by step, without the use of other instruments, and often without any direct measurements, angular or linear.

An important difference between plane-table surveying and transit surveying should be noted. In transit surveying, the angles are measured in degrees and minutes and later may be plotted; in plane-table surveying, angles are not measured but are plotted directly in the field by transferring to the paper the directions of the two corresponding lines of sight.

Linear measurements may be obtained as in transit surveying, either by chaining or by the stadia method. Vertical measurements may be obtained either by spirit leveling or by the stadia method; the plane table can be used in either method. Frequently, however, a separate spirit level is used for determining elevations of points located by means of the plane table. This is an example of how a plane table can be used in combination with other surveying instruments.

All seven methods of locating a point explained in Art. 10 may be used in plane-table surveying. However, the method of locating a point by determining the positions of two straight lines intersecting at the point (triangulation) involves no calculations in plane-table surveying and therefore is more easily applied than in transit surveying. The frequent use of this method in plane-table surveying greatly reduces the number of linear measurements as compared with the number in transit surveying. Moreover, in rough country where it is difficult to make linear measurements, plane-table triangulation may be used to great advantage.

Many of the general suggestions given in Chapter 23, "Transit Surveying—Field Work," such as those for choosing stations and planning the survey, methods of running traverses, and methods of locating details, hold good in plane-table surveying, and it is helpful constantly to keep in mind the similarity, not only in methods of field work, but also in methods of plotting.

The plane table is not intended for extremely accurate work, yet surprisingly good results may be obtained with it. It is especially useful for plotting indefinite details, such as roads, streams, and edges of woods, while at the same time more definite details, such as buildings, bridges, and fences, can be located accurately enough for most purposes. Thus many surveys may be made wholly with the plane table, as surveys of parks and private estates, but its chief use is for filling in the details and contour lines of large topographic surveys where the primary stations have been established by more accurate methods.*

439(*a*). Advantages of the plane-table method. (1) It dispenses with nearly all field notes (since the survey is plotted as fast as it progresses), and hence mistakes and omissions in reading, recording, and plotting measurements are less likely to occur. (2) It eliminates a great deal of the office work. (3) Since the map is made in the field, with the area surveyed in plain view, it is easy to see what data are necessary and whether or not any have been omitted which should be obtained. (4) By means of *check lines* (see sugges-

* A more comprehensive treatment of plane-table surveying than can be given in this chapter will be found in *Plane Table Manual,* by D. B. Wainwright, U.S. Coast and Geodetic Survey; *Topographic Instructions,* Special Publication 85, the U.S. Geological Survey; *Topographic Manual,* by O. W. Swainson, Special Publication 144, U.S. Coast and Geodetic Survey.

tion for *orienting,* Art. 440(a)) errors in measurements or in plotting are readily detected. (5) Points may be easily located by triangulation methods without measuring angles and without calculations, and thus linear measurements may be reduced to a minimum. (6) Inaccessible points are easily located by triangulation. (7) The position of a point with respect to three known points may be quickly determined (method 7, Art. 10, three-point problem). (8) Under conditions which render linear measurements difficult, more ground can be covered in a given time than by almost any other method, and with reasonable accuracy.

"The special advantages of the plane table as a mapping instrument are due to the rapidity with which it obtains results by the method of graphic triangulation and to the facility it affords the topographer in determining his position at an unknown point by the graphic solution of the three-point problem.

"When the latter method is applicable—that is, when the country is open and signals can be easily seen—its superiority over a system of traverse lines is manifest. The topographer is then at liberty to choose his ground without reference to his last station or to the one succeeding. He is not tied down to a backsight nor restricted by the conditions imposed by a foresight. He need not set up his instrument on an area barren of detail, nor cut his way through obstacles (bushes, hedges, trees) to establish a station at a commanding point of view." (Wainwright in *Plane Table Manual.*)

439(b). Disadvantages of the plane-table method. (1) The plane table is cumbersome, awkward to carry, and requires many accessories for plotting. (2) Under certain conditions of climate and weather, the work of plotting is disagreeable if not impossible. (3) The use of the plane table is more or less limited to relatively open country. (4) For certain types of surveys and under certain conditions more time is required for the field work than is needed for transit surveying. (5) Numerical values of angles are not known and can be obtained only by scaling them from the plane-table map; hence they are sometimes of doubtful value for the calculation of areas or for other office computations. On the other hand, when great accuracy is not required, areas may be determined from the map by the use of the planimeter or by calculations in which the angular and linear measurements used are scaled.

440. Setting Up and Orienting the Plane Table. (1) Set up the table nearly over the station with the board approximately level and at a convenient height. (2) Shift the instrument until the point on the map corresponding to the station occupied is approximately over that station and then level up the board. (3) Orient the table as follows. Set the edge of the alidade in such a position that it passes through the point on the map that represents the station occupied and also through a second point that represents some other station or some other object that can be used as a backsight; rotate the board until the line of sight is directed to the backsight and clamp the board in this position. (4) If now the station on the map is not directly over the station on the ground. it may be necessary to repeat steps 2 and 3 until it is. Frequently, however, the plotted position of the station need not be exactly over the station.

440(a). Practical suggestions.

1. *In setting up the plane table,* loosen the tripod-leg screws and grasp one leg of the tripod in each hand. Push the third leg out with the knee and rest the table on this leg. Spread the other two legs and draw them away from the third leg until the table is at the correct height. In this operation the board should be kept approximately level by sighting across its top at the

horizon or estimated horizon. The tendency is to set the board too high; bring it a little below the elbow, low enough so that all parts of the board may be reached easily, and yet high enough so that the work may be carried on with comfort. A low table is very tiring. When the table is in position over the station, press the legs firmly into the ground before leveling up. If the setup site is rocky, make sure that the metal tripod shoes are tight and that they rest in niches so that they will not slip. In some cases it may be necessary to chisel small niches in the rock. Finally, tighten the tripod-leg screws so that the table will be firm.

2. *The method of leveling the table* depends on whether it is equipped with leveling screws (usually three) or with some type of cup joint (ball and socket). On some tables with leveling screws, the plate resting on the screws is secured to the base by a central bolt. A handwheel on the lower end of this bolt is loosened before leveling the board and is then tightened after leveling in order to hold the plate firmly against the leveling screws. Assuming that such a table has been set up with its board approximately level, the procedure for leveling the board more exactly is as follows. Loosen the horizontal motion clamp and the central handwheel under the tripod head. Place the alidade on the board so that the fiducial edge of the base is parallel to two of the leveling screws. Turn these screws simultaneously (both inward or both outward) until the bubble of the level on the base of the alidade is in line with the center of the level and the third leveling screw. Turn the third screw to bring the bubble to the center of the level. (One of the advantages in having three screws instead of four is that one screw can be turned, within limits, without disturbing the others.) When there is no level attached to the base of the alidade, the telescope bubble may be used, or a separate bubble tube may be placed on the board. A declinator with two level bulbs can be used to advantage instead of a single tube. When the table has a cup joint (ball and socket), it is leveled by moving the board with the hands; when it is nearly level, it may be moved slightly by tapping with the finger.

Theoretically, the bubble in the level attached to the base of the alidade should remain in the center regardless of the position of the alidade on the board but it is impracticable to keep the board exactly level, and any error that results from a slight inclination of the board is usually negligible.

3. *Plumbing.* Time may be wasted in setting the table more accurately over a station than conditions require. It is true that when the map is to a large scale, say 50 ft. to the inch or greater, it is important that the point on the map representing the station occupied should be plumbed accurately over the station on the ground; in the U.S. Coast and Geodetic Survey this is done if the scale is $\frac{1}{5000}$ or larger. It is true also that when stations are near together, say 200 yd. or less, it is necessary to plumb more accurately than when they are far apart. When accurate plumbing is required, a device for suspending a plumb bob from any point on the board may be used. When, however, the scale is small, say 100 ft. to the inch or smaller, such accurate plumbing is generally unnecessary. Indeed in a great deal of plane-table work the scale is so small that the whole board may be considered as a point. The accuracy with which the plumbing should be done depends mainly, therefore, on the limit of error adopted and the scale used in mapping.

4. *Orienting.* The line used in orienting the table is more often than not the backsight to the preceding station, just as it is in orienting the transit when running traverses by the azimuth mehod. It is well, however, to take a *check sight* to some other point, such as another station or some prominent landmark, the position of which has already been plotted, in order to see if the edge of the alidade passes through the corresponding point on the map. The plane table may be oriented at the first station by one of several methods which correspond to the methods explained in Art. 303(*b*). When only a relatively low degree of precision is required and the scale is small, the table may often be oriented at each station with sufficient accuracy by means of a magnetic compass, or declinator, but the limitations and errors of compass surveying (Arts. 216, 217) must be kept in mind. When the table is over a station, the position of which has not been plotted, it may be oriented by the three-point or by the two-point method, as will be explained later.

5. *In taking up the plane table* the procedure is the reverse of that for setting up. The tripod-leg screws are loosened, two of the legs are grasped, and the table is supported on the third leg until the two or all three legs are together. In carrying the table between setups, some table men rest the tripod on one shoulder whereas others prefer to spread one tripod leg and carry the table astride the neck. For short distances where the footing is poor, the table should be carried in such a manner that it will not fall violently if the table man slips.

441. Five Fundamental Methods used in plane-table surveying are explained in the following pages. These methods are:

1. *Radiation.* Method of angles and distances (Art. 319(*a*)).
2. *Progression or traversing.* Corresponds to the azimuth method (Art. 303).
3. *Radio progression.* A combination of (1) and (2).
4. *Intersection.* Corresponds to triangulation (Art. 320).
5. *Resection.* A modification of (4).

The only linear measurement required in either of the last two methods is the length of one base line.

All five methods are illustrated by the same simple survey of the boundary lines of a house lot. Not all five are equally well adapted to such a survey, but by using the same illustration for each, an opportunity is afforded for a comparison of the different methods. In each case only one method is used for the entire survey, but in an extensive survey, two or more methods are frequently combined. In all but one of the illustrations, the boundary corners are used as stations. The location of the house illustrates methods of locating details. It is to be understood that the linear measurements required in plotting are made by one of the usual methods. In all five figures the actual boundaries are represented by the large polygon, and these boundaries are shown drawn to smaller scale on rectangles representing the plane table.

In studying these figures bear in mind that the size of the plane table is very greatly exaggerated. Thus, for example, it appears in Fig. 441(I) to be larger than the house. It is customary to represent a point on the ground by a capital letter and its plotted position on the plane table by the corresponding small letter.

441(I). Radiation. *Method.* Set up the plane table in a position from which all points to be located can be seen. Clamp the table so that it cannot revolve. Stick a pin in the board at any convenient point, and let this pin represent the point on the ground at which the plane table stands. With the edge of the alidade against the pin, draw radial lines to all points to be located, and mark off the distances to these points to scale.

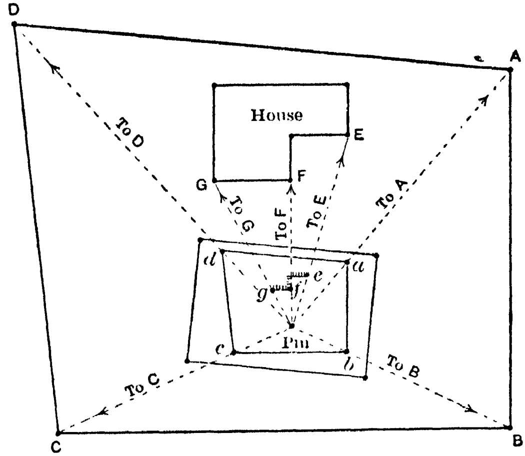


FIG. 441(I).

Illustration. In Fig. 441(I) the radial lines are drawn to the four corners of the lot and to three corners of the house. The distances to these points as fast as they were measured were laid off to scale on the corresponding radial lines.

Practical Suggestions. (1) Choose such a position for the pin that no point to be plotted will fall outside the limits of the paper. (2) In the illustration shown it would be well to measure the lengths of the boundary lines and scale the corresponding distances *ab, bc, cd,* and *da* to see if they agree. (3) The house is, of course, measured up to locate the remaining three corners.

Comments. This method is employed chiefly in combination with some other method; otherwise its use would be restricted to a comparatively small area, i.e., the ground which can be covered from a single setup. It is a common procedure for locating details from stations established by one of the other plane-table methods or by transit surveying.

441(II). PROGRESSION OR TRAVERSING. In this method the plane table is moved from one station to another. After setting up over one station, it is necessary to *orient* the table. This is done by making the edge of the alidade pass through the plotted positions of two stations, *one of which must correspond to the station at which the plane table stands.* The table is then revolved until the line of sight is directed to the other station. This usually amounts to backsighting on the preceding station, as in transit surveying; hence this method corresponds to the azimuth method (Art. 303).

Illustration. Required to plot the boundary lines *ABCD* in Fig. 441(II). Set up at any corner as *C*. Choose a corresponding point *c* on the board, and draw *cb* with the alidade directed to *B*. Measure the distance from *C* to *B* and lay it off to scale, thus establishing *b*. Set up at *B*, make the edge of the alidade pass through *b* and *c*, turn the table until the line of sight strikes *C*, and clamp. The table is now oriented. Sight at *A*, draw *ba*, measure *BA*, lay off the distance to *a*. Set up at *A*, orient as before by sighting along *ab* to *B*, draw *ad*, measure *AD*, and lay off the distance to *d*. Thus any number of stations may be plotted one after the other. In Fig. 441(II), the lines on the board are numbered in the order in which they were drawn.

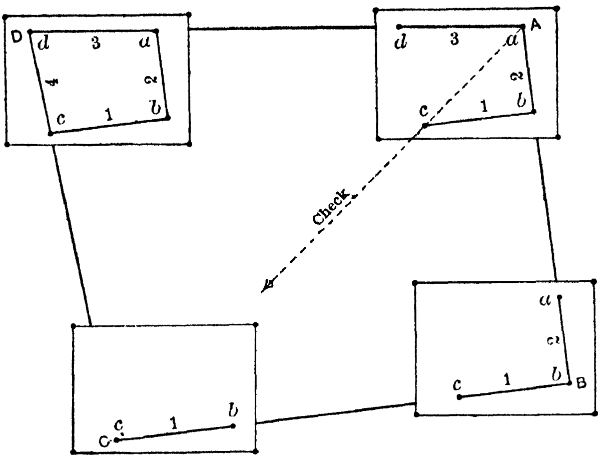


Fig. 441(II).

Practical Suggestions. (1) When the area covered is small and the scale adopted is large, it may be necessary to set up so that the station at which the plane table stands is exactly under the corresponding point on the board (Art. 440(*a*),3). (2) Choose the first line *cb* with due regard to the limits of the paper. (3) After orienting at *A*, if the alidade is made to pass through *ac*, the line of sight should strike *C*. Such checks as this should be used constantly. (4) It is not necessary to set up at the last station *D* except as a check. (5) The house or any other detail could be located from the most convenient station by the preceding method of *radiation*, or by the next method of *intersection*. (6) If it is impracticable to set up over boundary corners (as is usually the case), establish stations and corresponding traverse lines inside or outside the boundaries, as in transit surveying. (See Arts. 326, 345(*a*).)

Question. How could the work have been started at *C* so as to plot the four corners in two setups, counting the one at *C*? How could the positions of *d* and *a* be checked in this case?

Comments. This method requires all stations to be accessible. It is used for the main lines of a survey (corresponding to transit lines), the details being located in some other way. It is especially adapted to surveys of roads, streams, etc., the stations being chosen at the bends, whereas the sides of the road or banks of the stream are located by offsets—buildings, fence-corners, etc., by radiation. Frequently offsets may be sketched in by the eye. The method is often used in topographic surveying in connection with a system of horizontal control the primary stations of which have been previously established by a more accurate method such as traversing or triangulation. Each plane-table traverse begins at a primary station and closes on another. This is a valuable check on the plane-table work just as it is in transit traversing.

441(III). Radio progression. A combination of the first two methods just explained. Characteristic features: (1) The table is always set up with its center directly over the station occupied. (2) For all sights at all stations the edge of the alidade passes through the same point on the paper, i.e., a point over the axis around which the table revolves. (3) The direction of any line of sight is transferred from the center of the board to the point

representing the station occupied, by drawing through this point a line parallel to the edge of the alidade.

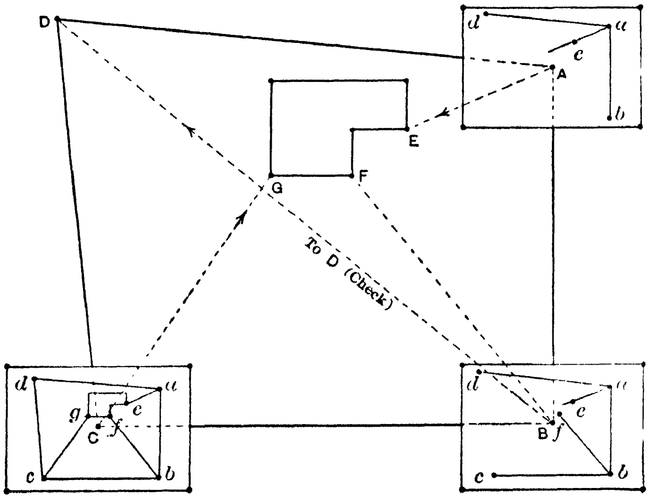


FIG. 441(III).

Illustration. In Fig. 441(III) lines of sight are represented by broken lines—the lines actually drawn on the table by full lines. Set up at *A*, with the center of the table over the station. Stick a pin in at the center of the table, and with the edge of the alidade against the pin, sight at *D*. Assume a point *a* and draw the line *ad* parallel to the edge of the alidade. In like manner sight on *B* and *E* and draw the corresponding parallel lines *ab* and *ae*. The distances *AD, AE*, and *AB* are measured, and the corresponding distances *ad, ae*, and *ab* are laid off to scale. Move the board to *B*, orient by sighting at *A* (edge of alidade parallel to *ba*). As a check, sight at *D* and see if a line *parallel to the edge of the alidade* passes through both *b* and *d*. Determine points *c* and *f* by drawing from *b* lines parallel to the corresponding lines of sight in the usual manner. It is not necessary to go beyond this station, except to check the work, but in this case the plane table was again set up at *C* and oriented, the location of *d* checked by sighting at *D* and by measuring the distance *CD*, and a third corner *g* of the house was plotted.

Practical Suggestions. (1) Parallel lines may be drawn by placing one edge of a triangle in perfect contact with an edge of the alidade, and then sliding this triangle along the edge of a second triangle, as in ordinary drafting. If the two edges of the alidade are parallel, place the triangle against either edge as may be most convenient. (2) In the illustration shown, the house could have been located wholly from *A*, the most convenient station. Two or three corners having been located, the others are found by measuring the sides of the house. The lengths of the boundary lines should be measured also. (3) If in orienting at *B* it is necessary to make the edge of the alidade parallel to *ba*, extra work is involved; moreover, *ba* may be too short for accuracy. It is better to draw a long line through the *center* of the table before leaving *A*, by pointing the alidade to *B*, and use this line for orienting at *B*. This means that a number of such lines will eventually be drawn through the center of the table for purposes of orienting it; hence it is better to omit the center portion of each line, drawing the two ends only.

Comments. The radio progression method is sometimes used for mapping small areas to large scale, i.e., when one must be particular to have a given point on the table directly over the station occupied. When, however, many points are to be plotted from any one station, much time is wasted in drawing parallel lines; it is better to set the station on the board directly over the station on the ground and use the first two methods.

441(IV). Intersection. This method corresponds to triangulation (see Art. 320); one distance being known, no further linear measurements are required. The first thing to do after setting up the plane table it to orient it as in the preceding methods. The work then consists in drawing radial lines to the points which are to be located. The distances will not be determined until these lines are intersected by corresponding lines to the same points, drawn at some other station.

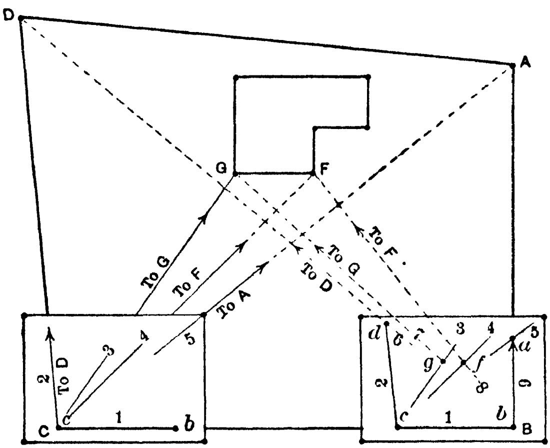


Fig. 441(IV).

Illustration. Figure 441(IV) is practically self-explanatory. The table was first set up at *C*, the length of *CB* measured, and the line *cb* plotted. Lines 2, 3, 4, and 5 of indefinite length were then drawn, the table moved to *B*, oriented, and lines 6, 7, 8, and 9 drawn to determine the points of intersection *d*, *g*, *f*, and *a*. Had it been necessary to locate other points not visible from *C*, but visible from both *B* and *A*, the line *ab* could have been used as a base line in exactly the same way. Thus the plane table can be moved from station to station until all the ground is covered.

Suggestions. (1) As in all triangulation, stations should be chosen so that intersections are definite and not too oblique. Angles of intersection less than 30° or more than 150° should be avoided. (2) As this method is usually used for large areas plotted to small scale, it is seldom necessary to set up so that the point on the table is exactly over the station on the ground. (3) If a point cannot be located by a definite intersection from two given stations, consider if it cannot be located more definitely from two other stations. This suggests a method of check lines, i.e., locate points occasionally from at least three stations. (4) It is often desirable to sight on a church steeple, the top of a tree, or some other prominent landmark merely as a check, even though the location of the point itself is of no consequence. Having oriented a table, it is very reassuring to direct the line of sight at a distant object and to find that the edge of the alidade passes through the corresponding point already plotted from other stations. (5) Theoretically the measurement of the base line is the only linear measurement required, but in an extensive survey with a relatively large number of stations, it is usually desirable to obtain at least one check measurement by chaining the distance between two stations more or less remote from the base line as is done in extensive triangulation.

Comments. This method is especially useful for plotting *inaccessible* points, and for mapping banks of streams, edges of woods, and other indefinite topo-

graphic features. The ease with which points may be located by intersection is one of the great advantages of the plane-table method. The method is especially valuable as an occasional check on distant points.

441(*a*). Graphic triangulation. The method of intersection can be used for establishing the stations of a triangulation network over a large area. These stations may then be used as control points for plane-table traverses. The traverse lines, in turn, may be used as lines of reference in filling in the details of a survey.

441(V). Resection. This method is a modification of the method of intersection. Its chief characteristic is that the point determined by the intersecting lines is *the station occupied by the plane table* at the time those lines are drawn. This means that a plane table can be set up anywhere and the corresponding point on the map can be found by *resection*, provided three points already plotted are visible from the plane table. This method requires one linear measurement, a base line.

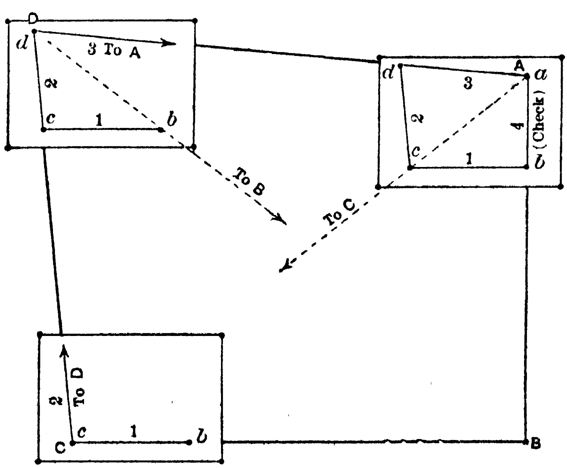


Fig. 441(V).

Illustration. In Fig. 441(V) set up at *C*, assume *c*, direct the alidade to *B*, measure *CB*, lay off *cb* to the scale adopted, and draw an indefinite line 2 towards *D*. Set up at *D*, orient the table by sighting at *C* with the edge of the alidade along the line 2. Then move the alidade to *b*, sight at *B*; where the edge of the alidade cuts, i.e., resects, the line 2 will be the point *d*. Draw an indefinite line 3 towards *A*. Set up at *A*, orient the table, center the alidade at *c*, sight toward *C*, and find the intersection *a* with the line 3. As a check, center the alidade at *a*, sight to *B*, and see if the edge of the alidade cuts *b*.

Practical Suggestions. (1) In setting up the plane table, estimate about where the station to be plotted will come, and set this point approximately over the point on the ground. (2) When there are more than four sides to the survey a station may be plotted by sighting *across* the enclosure to *any* station previously plotted. This often gives a choice of two or more sights. It is well to use one for locating the point on the paper, and the other for a

check sight. (3) Avoid too oblique intersections. (4) Details are best located by radiation or intersection.

Comments. The method of resection is not a very useful method except in cases where it becomes necessary first to occupy a station and then to locate it on the map. The well-known *three*-point problem and the *two*-point problem are problems in *resection*. (See Arts. 447 and 448.)

441(*b*). THE FIVE METHODS COMPARED. SUMMARY.

1. *Radiation.* An accurate method, but restricted to one setup. Used chiefly in combination with other methods.

2. *Progression.* Requires all stations to be accessible. Used only for locating *stations.* Corresponds to transit traversing. Useful for establishing main lines of reference, i.e., the skeleton of a survey.

3. *Radio Progression.* Adapted to plotting small areas to large scale, since the center of the table is always over the station. Not much used if many points are to be plotted from each station.

4. *Intersection.* Requires no linear measurements except a base line, and is especially useful in plotting inaccessible points. Method is inaccurate when intersections are too oblique. Method used chiefly in combination with other methods, especially for check sights.

5. *Resection.* Used chiefly when it becomes necessary to plot the point that will represent the station occupied.

442. Measurement of Vertical Distances. For the measurement of vertical distances, there should be attached to the telescope of the alidade either a level tube similar to that used on the transit or a striding level. Although the plane table can be used for direct leveling or for trigonometric leveling, it is used mainly for stadia leveling. In finding the elevation of a distant point by trigonometric leveling, the position of the point on the map may be determined by the method of intersection. The distance between this point and the point representing the station occupied can then be scaled and used in calculating the elevation.

The precautions taken in transit work to eliminate errors in measuring vertical angles are even more necessary in plane-table work since the table cannot be maintained in a truly horizontal position with anything like the stability of the horizontal plate of the transit. Consequently, in order to obtain accurate work either there should be a control bubble on the vernier arm, or the index error should be determined by leveling the telescope after each sight (Arts. 271(*a*) and (*b*)). Fortunately, in a great deal of plane-table work, such as locating contours, the required degree of precision is such that these precautions are unnecessary. Elevations of important points in a plane-table survey, such as important

stations and points of control, are often determined not by means of the plane table but by some more accurate method.

443. Systems of Control. In a plane-table survey of a small area, details are usually located as the survey progresses from station to station. When, however, the area is large and traverses are run for considerable distances before closing, it is often better to complete the plane-table traverse before beginning the location of the details. Then if the error of closure indicates a serious error in the traverse, that error can be corrected before the location of the details. When a relatively high degree of precision is required, the traverse can be run with transit and tape or with transit and stadia. The stations can then be plotted as control points on the plane-table sheet before beginning the plane-table survey.

In extensive surveys, control points established by triangulation can be used in running traverses either with the transit or with the plane table. In this case, each traverse begins at one control point and ends at another (Chapter 56).

Vertical control can be established in much the same way as horizontal control. If a plane-table traverse is run before the details are located, it may be sufficiently accurate to determine the elevations of the stations at the same time by the plane-table method. If the traverse is run with a transit, elevations of stations may be determined by spirit leveling or by stadia leveling. When control points are established by triangulation, the usual accurate methods of determining the elevations of the triangulation stations may be employed if the required degree of precision warrants such methods.

444. Practical Suggestions for Plane-Table Work. (*a*). THE DRAWING PAPER. (1) Only the best paper should be used, paper which has a good surface for drawing hair lines and is least affected by atmospheric changes. For accurate work, a single sheet may be mounted on muslin; or two sheets with the grain of one across the grain of the other may be pasted together with muslin cloth between. This double-sheet mounting, however, cannot be rolled; consequently the size of the sheet cannot be larger than the size of the board. The large plane-table sheet of the U.S. Coast and Geodetic Survey consists of Whatman's hot-pressed, handmade, antiquarian paper 53 by 31 in. It is backed with muslin which extends about 1 in. beyond the edge of the paper to protect it from fraying. Drawing paper mounted on both sides of a thin aluminum sheet will not change its dimensions with changes of humidity; it may be purchased in sheets to fit the board. In localities of excessive rainfall, celluloid sheets are sometimes used in order to carry on work in light rains. Tinted paper is less trying to the eyes than pure white. (2) If the paper is flexible enough to be rolled, it can be kept in a metal cylinder or carrying case of diameter not less than 3 or 4 in. It is preferable, however, when sheets are the size of the board, to keep them *flat* in a leather or canvas carrying case. (3) Paper should be fastened to the board firmly and smoothly so that neither wind nor friction of the alidade will disturb it. For this purpose, thumb tacks, clamp screws, or spring clamps are used. The better plane tables have clamp screws. When the sheet of paper is larger than the

board, spring clamps are generally used. The width of the sheet is usually equal to the length of the long side of the board; consequently, the sheet is put on with its edges even with or parallel to the edges of the short side. The excess length is rolled up, turned underneath the long edges of the table, and fastened with metal spring clamps which press on top of the table and on the inside of the roll beneath. One clamp at each end of the roll serves to hold the roll ends securely. Shorter clamps are sometimes used to hold the edges of the sheet, particularly when it is necessary for the alidade to extend over the edges and the longer clamps would interfere with its movement. When the table is carried with the sheet on it, care should be taken not to crush the rolled-up portion of the sheet. (4) A waterproof cover for protecting the paper on the board in case of sudden rain is almost indispensable. If caught without it, turn the board upside down.

444(*b*). Manipulation of the plane table. (1) If there is no slow-motion screw for turning the table *after* it is clamped, it may be turned slowly *before* it is clamped by tapping a corner sidewise with the finger. After clamping, look again to see that the cross hairs still cover the point sighted at. (2) In turning the alidade, steady the standard with the left hand, while the right swings the rear end of the ruler. A fine needle may be stuck at the point on the paper representing the station, and the edge of the alidade pivoted against it. (3) It is difficult to keep the plane-table board in a horizontal position and to prevent it from rotating slightly after it has been clamped. Be careful not to disturb the board after it is oriented by leaning heavily on it, or by contact with its edge. The less the weight of the alidade rests near the edge of the board the better.

444(*c*). Plotting. (1) The pencil used should be *hard*, sharpened at both ends (round point and chisel point). A smooth stone may be used in place of a sandpaper pad, but it is better to have the latter always at hand, attached to the board by a string. A needle for pricking points and a pencil eraser are also needed. A triangular scale is recommended, and distances are laid off by laying the edge of the scale along the line. The method of transferring distances by dividers can sometimes be used to advantage, but generally speaking the use of the scale is better. (2) *Draw lines as fine as possible, as few as possible, and as short as possible.* Coarse lines make accurate work out of the question; when the scale of the map is small, a coarse line may be several feet in width. Avoid unnecessary lines; the paper soon becomes covered with a network of lines which is bad enough when the number is reduced to a minimum. Instead of drawing a line the full length of the alidade, draw a short line approximately where the point is to be plotted. Points on contours and similar points often can be plotted without drawing lines, by marking off distances along the edge of the alidade while it is yet in position. An important line, however, such as one to be used in orienting, should be drawn long enough to indicate accurately its direction even though the distance laid off on it to scale may be short. It is not necessary to draw a long line throughout its length; it is often better to draw only a portion at each end. In plotting an intersection the point is simply marked or pricked where the second line crosses the first; the second line is not drawn. (3) Several points of intersection on the same straight line may be plotted quickly by first drawing the line and then by simply pricking the points at which the edge of the alidade crosses the line as sights are taken to successive points. (4) Draw all lines close to the straightedge, so that each is hardly visible until the straightedge is moved away. If, however, any part of the edge is raised off the paper, be careful that the pencil does not run under it, thus deviating from a straight line. (5) Above all, *have some good system of keeping track of points and lines,* so that those plotted at one station may be easily identified when the table has been moved to some other station. This is especially important in the method of intersections. (6) When more than one sheet is used, it is sufficient in ordinary work simply to transfer to one sheet at least two points from the preceding sheet, say the last two stations occupied. In addition it is well to transfer such other points as are likely to be needed for check-sights. Points may be pricked through from one sheet to another, or if greater accuracy is required they may be transferred by any of the well-known methods used in drafting.

444(*d*). General suggestions. (1) Colored or sun glasses relieve the strain upon the eyes. (2) Rules for cleanliness: Keep the leavings of the pencil eraser and dust of all kinds off the paper. Keep the triangles and the bottom of the alidade clean. Do not slide the alidade unnecessarily over the

paper; lift it up and set it down again to avoid rubbing. When a large amount of plotting is to be done on the same sheet of paper, a temporary cover of tough paper, portions of which can be removed as the work progresses, will help to keep the drawing paper clean. Cloth large enough to cover the whole sheet is often used for the same purpose, particularly when the sheet is longer than the board. It is folded back to expose only that portion on which work is being done and should be rolled up with the sheet of paper to keep the back of the paper, which is usually soiled, from rubbing the face. (3) When it is desired to keep magnetic bearings in a survey, a north-south line may be drawn at the first setup by means of a compass or declinator. This line can be checked by the compass at subsequent setups. (4) The first thing to do after setting up and orienting the table is to take a check sight to some distant point which has been previously plotted; the last thing to do before leaving the station, and at other times when not employed, is to sight at the same point to see if the table has been displaced. One of the most important precautions in all plane-table work is to take check sights frequently.

445. Practical Questions Relating to Field Work. (*a*). CHOICE OF SCALE. If the plane-table work is carried on in connection with a transit survey, it may be desirable to adopt the same scale as that used in the office for plotting the transit notes. This is not essential, however, as the plane-table map can be transferred easily to a map of different scale by means of the pantograph or by some other method. Assuming, however, that the choice of scale does not depend upon other work, the following suggestions are made:

1. Choose a scale in common use. For ordinary work some multiple of 10, as 10, 20, 30, 40, 50, 60, 80, 100, or 200 ft. to the inch. In some cases, as in a survey for an architect, it is often better to use a scale that corresponds to that used in working drawings, such as 16, 8, or 4 ft. to the inch. For government work standard scales are adopted, such as $1/10{,}000$ and $1/20{,}000$.
2. If the size of the map is limited, for example, to the size of the board, estimate the extreme dimensions; this will still further limit the choice of scale.
3. Choose a scale consistent with the limit of error. Thus, for example, if measurements to the nearest foot are wanted, 80 or 100 ft. to the inch is as small a scale as should be used; 50 ft. to the inch would be better, since it is rather difficult to plot to $1/100$ in.
4. For other suggestions pertaining to the choice of scale see Arts. 652–653.

445(*b*). CHOICE OF STATIONS. The suggestions given in Arts. 339 and 345 for choosing stations hold good, remembering, however, that in plane-table work it is well to have each station visible from at least three others, though this is not essential. When a plane-table survey is made in connection with a transit survey, it should begin at one of the main stations and be "tied in" with other transit stations as often and as completely as may be necessary for definite checks.

445(*c*). METHODS OF LOCATING DETAILS explained in Chapter 22 may be used, modified, of course, to suit plane-table work. Thus, for example, offsets often may be paced, and details sketched in by eye.

445(*d*). WHAT DETAILS TO TAKE. In a large survey it should be clearly understood what details are to be taken by the plane-table party, and what are to be left for other parties.

445(*e*). LIMITS OF ERROR depending upon the nature of the work in hand should be kept constantly in mind.

446. Sources of Error. Some of the sources of error in plane-table surveying are the same as corresponding sources in leveling and in transit work, such as errors due to the poor adjustment of the instrument, to lack of skill in manipulating the instrument, to mistakes in reading vertical angles, and to such natural causes as atmospheric conditions. The principal errors peculiar to plane-table work are as follows.

446(*a*). ERRORS IN THE POSITION OF THE TABLE. (1) The plotted position of the occupied station not exactly over the station on the ground. This error is not likely to be serious unless the scale is large, say 1 in. equals 50 ft. or

greater, in which case a plumb bob is used. (2) Table not horizontal. This error affects chiefly the measurement of vertical angles, eliminated as explained in Art. 442. (3) Table displaced from its initial position. There is likely to be a slight rotation or other displacement of the table, no matter how carefully the work may be carried on. Consequently, frequent check sights should be taken to detect and correct any displacement.

446(*b*). ERRORS IN PLOTTING include many of those common to ordinary drafting. Some of those most common in plane-table work are drawing too coarse lines, drawing a line which does not coincide with the edge of the alidade or is not parallel to it, careless pricking of points of intersection, and mistakes in laying off distances to scale.

446(*c*). ERRORS DUE TO THE CONTRACTION AND EXPANSION OF THE DRAWING PAPER may be serious unless precautions such as those outlined in Art. 444(*a*) are taken.

447. The Three-Point Problem. *Given: Three control points, A, B, and C, visible from a station P occupied by a plane table; also a, b, and c, the plotted positions of the three visible points. Required: To determine p, the plotted position of the station P.*

Remark. This is a special case of the method of resection where the position of the station occupied is the point required, but, unlike the general case (Art. 441(V)), no line known to pass through *p*, the required point, has been drawn at some previous setup. Hence there is no direct method of orienting the table as is the general case. The problem, then, amounts to finding some point *p* such that when the edge of the alidade is made to pass through *p* and any one of the points *a*, *b*, or *c*, the table may be correctly oriented by sighting at the corresponding point *A*, *B*, or *C*. Thus the solution of the three-point problem determines the location on the sheet of the station occupied and orients the table simultaneously. As in all forms of triangulation, the definite location of the point depends upon well-proportioned triangles and sharp intersections.

Solutions. (*a*) Mechanical; (*b*) by trial; (*c*) graphic.

447(*a*). TRACING-CLOTH METHOD. (Mechanical solution.) (1) Fasten a piece of tracing cloth to the board, and assume a point *p* to represent the station occupied. (Assume it as near its true position as can be estimated by eye.) (2) Center alidade on *p*, and sighting successively on the three visible points *A*, *B*, and *C*, draw lines of indefinite length along the edge of the alidade. (3) Shift the tracing cloth until the three lines pass through the points *a*, *b*, and *c* previously plotted from other setups, and prick the point *p* through to the paper. (4) With the edge of the alidade passing through *p* and *a*, orient the table by sighting at *A*. (5) Take check sights through *b* and *c* to *B* and *C* respectively. If, because of little inaccuracies, the lines do not meet at *p*, the triangle of error will usually be so small that a slight movement of the board will correctly orient the table.

This method is quick, and, under favorable conditions, easy to use and sufficiently accurate for most purposes.

447(*b*). LEHMANN'S METHOD OR THE COAST-SURVEY METHOD. Methods by trial are those in which the position of the required point is alternately assumed and corrected until its proper position

is determined. The most satisfactory method of procedure, known as Lehmann's method or the coast-survey method, is as follows. (1) After the table has been set up at P, orient it as nearly as possible by eye or by the compass; clamp it in this position. (2) Through a, b, and c, the plotted positions of the three control points, draw lines by sighting to these control points A, B, and C, respectively. (3) If the table has been oriented correctly (which is unlikely) these three lines will meet in the required point p; otherwise the intersection will be in the form of a small triangle called the *triangle of error.* (4) Guided by the triangle of error and by the rules given later, estimate the position of p, the plotted position of the station occupied. (5) Reorient the table by pivoting the alidade on the point p, whose position was estimated in (4), and then by sighting at the most distant station. Clamp the table in this position. (6) By pivoting on each of the plotted positions of the two nearer stations and by sighting to these stations, draw the two corresponding lines of resection. (7) If the position of the point p was not estimated accurately, a second, but smaller, triangle of error will result, and the operation must be repeated until the correct position of the table is determined. The accuracy of the first estimation will increase with practice.

The method by trial just described is obviously a series of approximations, the triangle of error becoming smaller and smaller as the orientation becomes more nearly correct. A skillful instrument man seldom requires more than three trials. The method is used extensively, especially by the government topographers, and is quick and accurate, more so than would at first appear.

447(*c*). Rules for guidance in estimating the position of p, the point sought, as given in *Topographic Manual* * by O. W. Swainson, are as follows.

Rule I. The point sought is on the same side of all resection lines, that is, either to the right or left (as the observer faces the station from which the line is obtained) of all lines.

Rule II. The distance of the point sought from each resection line is proportional to the distance of the point occupied from the station from which the line is drawn. In other words, the distance from the lines is proportional to the length of the lines if prolonged from the point to each station.

Rule III. If the table is located outside the triangle formed by the three control stations, its position on the sheet will lie outside the triangle of error; if it is located inside the triangle, its position will lie inside the triangle of error.

Illustration. In Fig. 447(*c*), the application of the rules is illustrated by two general cases. The positions of the control stations A, B, and C are not shown, but the resection line from each of these stations is indicated by the line drawn through the corresponding plotted position of the station. These resection lines form a triangle of error which is indicated in each case by

* U.S. Coast and Geodetic Survey, Special Publication 144.

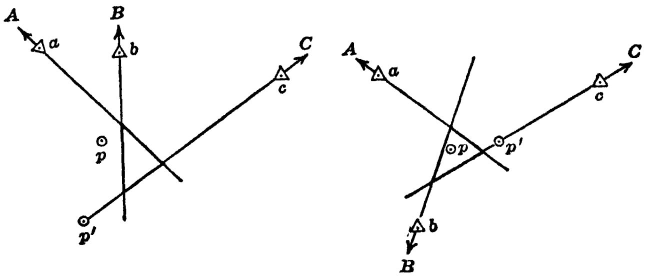


FIG. 447(c).

heavier lines. The position of the table, assumed for the first orientation, is indicated in each case by p'.

In case 1, the plotted position (p) of the table must, in conformity with rule I, lie either in sector 3, to the right of all resection lines, or in sector 6, to the left of all lines. By rule II, it must lie in sector 6 in order to be nearer to the line from the nearest station (B) than to the lines from the other stations. It is accordingly plotted in sector 6 at the proper proportionate distances from the three lines.

In case 2, the point p must lie inside the triangle of error in order to satisfy the rules.

447(*d*). BESSEL'S METHOD. There are several other graphic methods but only Bessel's method will be explained. In Fig. 447(*d*), the table is represented in three positions *over the same station.* (The three positions correspond to the three principal steps.) Let A, B, and C represent the three visible points or control stations, and a, b, and c their plotted positions.

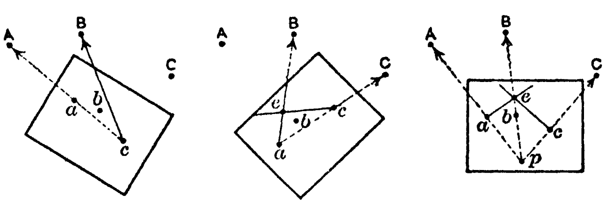


FIG. 447(*d*).

First Step. (1) With the edge of the alidade through c and a, sight at A and clamp the board. (2) Sight at B and draw a line of indefinite length.

Second Step. (1) Unclamp; with the edge of the alidade through a and c, sight at C and clamp. (2) Sight at B and intersect the first line drawn, thus obtaining the point e.

Third Step. Draw eb, and unclamp; with the edge of the alidade through b and e, sight at B and clamp. Pivot the alidade on a, sight at A, and intersect eb at p.

Check. Pivot on p and see if c and C are on line.

Comments. The objection to Bessel's method is that the points *e* and *b* frequently fall so near together that the direction of the line between them is practically indeterminate; or else the point of intersection *c* comes off the board altogether. Often this difficulty may be overcome by expanding or contracting the triangle formed by the three fixed points as follows. (1) Draw a line parallel to *ac*, and find its intersection with each of the lines *ab* and *bc*. (2) Through the two points thus found, draw lines parallel to *ae* and *ce*. These lines will intersect in a *third* point of *eb*.

Bessel's solution is based on the geometric principle that *in any inscribed quadrilateral the angle made by one of the sides with one of the diagonals is equal to the angle made by the opposite side with the other diagonal.* The board of the plane table *is treated as a point,* i.e., compared with the area covered by the survey it has no dimensions.

447(*c*). CHOICE OF CONTROL STATIONS. The strength of the position and the orientation of the plane table as obtained by any of the methods used to solve the three-point problem depend chiefly on the relative positions of the three stations used. The principal considerations to be remembered in selecting suitable control stations are as follows.

1. A strong position will be obtained when the position sought is inside the triangle formed by the three control stations, or when the latter are nearly in line, or the center station is nearer than the others, with no angle less than 30°.
2. A weak position will generally result from using two stations so situated that the angle between them is small. An exception to this occurs when two stations, a considerable distance apart, are in range or nearly in range with the nearer station, the center one, provided the angle to the third station is not less than 30°. When the more distant of the two stations is in the center, however, such a position may be very weak.
3. A position is indeterminate when the point sought lies on or near the circle passing through the three stations; the position weakens as the point approaches this circle.

448. The Two-Point Problem.* *Given: Two control points, A and B, visible from a station P to be occupied by a plane table; also a and b, the plotted positions of the two control points. Required: To determine the point p, the plotted position of the station P.*

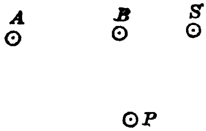


FIG. 448(*a*),1.

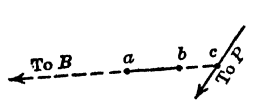


FIG. 448(*a*),2.

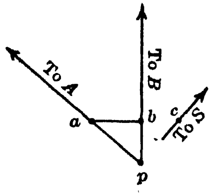


FIG. 448(*a*),3.

448(*a*). SPECIAL SOLUTION. When it is practicable to establish a temporary auxiliary station *S* in range with the two control points *A* and *B*, the plotted position *p* of a station *P* may be determined as follows.

1. Set up at *S* (Fig. 448(*a*),1). Orient by placing the alidade along the line *ab* and then sighting at *A* or *B*. Select any convenient point *c* on the line *ab* produced. Pivot the alidade on this point *c* (Fig. 448(*a*),2), sight at *P*, and draw the corresponding line, indefinite in length.
2. Set up at *P* (Fig. 448(*a*),3), place the alidade along the line drawn in (2), and orient by backsighting on *S*. Find *p* by resection, pivoting on *a* and sighting to *A*, then pivoting on *b* and sighting to *B*. Obviously the same pro-

* The explanation given here follows quite closely that given in the *Topographic Manual,* Special Publication 144, U.S. Coast and Geodetic Survey.

cedure could be followed if S were between A and B or to the left of A, provided it is on the range line through A and B.

448(*b*). GENERAL SOLUTION.

1. Choose a point C (Fig. 448(*b*),1) to be used as a temporary auxiliary station. (The considerations that govern this choice will be given later.)

2. Set up at C (Fig. 448(*b*),1), and orient the table as closely as possible by eye or by compass. By pivoting on a and sighting to A, then pivoting on b and sighting to B, find by resection the point c', the plotted position of the station C. Because of inexact orientation, c' will probably not be the exact plotted position.

3. With the table still at C, sight to P, and draw the corresponding line. Estimate the distance from C to P, lay off this distance to scale on the line just drawn, and thus determine the point p', the tentative position of the required point p.

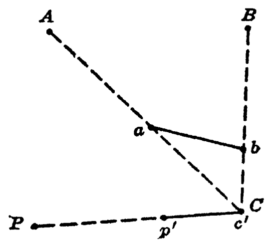


FIG. 448(*b*),1.

4. Set up at P (Fig. 448(*b*),2), place the alidade along the line $p'c'$, and orient by backsighting on C. The error in orientation is now the same in extent and direction as it was when the table was at C. Pivot on p', sight on A, and draw the line until it intersects the line $c'a$ at a point a'. Similarly, pivot on p', sight on B, and determine the point of intersection b' where the line intersects the line through c' and b.

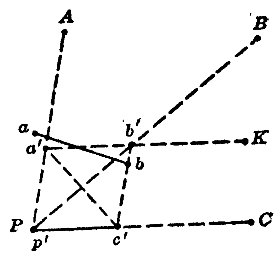


FIG. 448(*b*),2.

5. The quadrilateral $a'b'c'p'$ has the same shape as the quadrilateral $ABCP$, but it is erroneous in size and position. Nevertheless, the line $a'b'$ is parallel to AB; hence the error in orientation is indicated by the angle between $a'b'$ and ab. To orient the table correctly, place the alidade along the line $a'b'$ and sight at some distinctive and distant object K (Fig. 448(*b*),2). Then place the alidade along ab, unclamp, and rotate the table until the line of sight is on the same object K (Fig. 448(*b*),3). Leave the table clamped in this truly oriented position. When the table is rotated, the point p', the tentative position of the required point p, will move off the line from P to C, as will also the point c'. The relative positions of the points are not to scale; consequently the change in position of any point is not what it appears to be in the figure.

6. Find p, the true position on the board for the station P, by resection from sights on A and B through a and b respectively (Fig. 448(*b*),3).

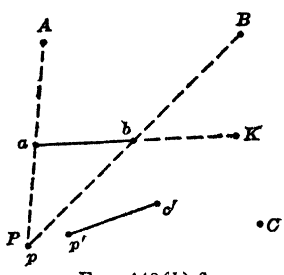


FIG. 448(*b*),3.

448(*c*). CHOICE OF THE AUXILIARY STATION C. The auxiliary station C should be so situated with respect to the two control points A and B that the resection lines from A and B in step 2 will form a strong intersection. Moreover, the station C should be so situated with respect to P that, in determining a' and b' in step 4, the lines $p'a'$, $p'b'$, $c'a'$, and $c'b'$ will give good intersections.

Comment. In the regular procedure for the two-point method as just outlined, there is no check on the plotted position of p as there is in the procedure for the three-point method. The two-point method can be checked by using an additional auxiliary station and also by a special geometric construction on the board. Such checks, however, are not always definite, hence it is important to carry through the regular procedure with great care. When it is possible, the three-point method should be used in preference to the two-point method.

CHAPTER 31

TOPOGRAPHIC SURVEYING

In topographic surveying, methods of leveling are combined with methods of horizontal location. So many combinations are possible that the subject cannot be comprehensively treated in a book of this kind. The aim, therefore, is to explain thoroughly fundamental principles, and then to discuss briefly the most important questions which arise in ordinary topographic surveying. Practically every surveying instrument may be used to advantage at one time or another in topographic work, and hence a thorough knowledge is required of all of the methods of horizontal and of vertical measurements explained in preceding chapters.

449. Contours and Contour Intervals. A *contour* is an imaginary line on the ground passing through points of equal elevation. A *contour line* is a line on a map representing a contour. Since there is an indefinitely large number of elevations there can be an indefinitely large number of contours. It is customary in surveying, however, to work with contours which differ in elevation by whole-foot intervals. This difference in elevation between two successive contours may be 1 ft., 2 ft., 5 ft., 10 ft., or any other fixed interval adopted to meet the requirements of the work. The interval adopted for any survey is the *contour interval* for that survey. In general, the same interval is used throughout the survey, but occasionally one interval is adopted for one portion of a survey and another interval for another portion.

Illustration. Imagine the earth's surface to be partly covered with water. Suppose that the surface of the water is lowered 5 ft. at a time, leaving contact marks at the end of each interval. These marks correspond to *5-ft.* contours. If the surface of the water is lowered 1 ft. at a time, the contact marks correspond to *1-ft.* contours.

Suggestion. Select a small plot of ground with an irregular surface. By means of the level find a number of points of the same elevation 30 to 50 ft. apart. Mark each point with a stake or a chain pin and imagine a contour passing through these points. This is an excellent way for the beginner to see how a contour winds in and out along the surface of the ground. Ultimately he must learn to follow with the eye the approximate windings of a contour which has *not* been staked out. He will be helped in this by imagining the ground intercepted by a water surface, as explained in the preceding illustration.

450. Contour Maps. A map without contour lines shows only the relative position of points regardless of their elevations, but when contour lines are added, the approximate elevations of all points are at once evident from inspection. Such a map, called a *contour map* or a *topographic map,* shows the elevations and de-

pressions of the surface of the ground, i.e., the configuration of the earth's surface.

Illustration. In the upper portion of Fig. 450 the approximate elevations of the points *A, B,* and *C* are shown by the contour lines. Thus it is seen that the elevation of *A* is somewhere between 80 and 85 ft., *B* is somewhere between 95 and 100 ft., and *C* is 90 ft.

The lower portion of the figure is a vertical section at *EE*. There is nothing shown by this section which is not evident from a study of the contour lines. On the contrary the contour lines show the exact shape of the hill better than the section. From the latter, one might suppose the ground to run in the form of a ridge, but from the contours it is seen to be a cone-shaped hill.

Question. If the hill were a right cone, would the contour lines be perfect circles with a common center?

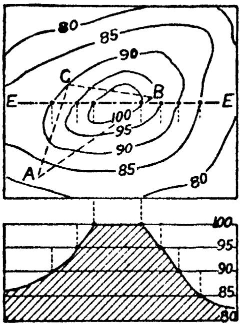


FIG. 450.

There are other methods of representing elevations on a map which give a general idea of the configuration of the surface of the ground, but, on the whole, a contour map is the most advantageous method of representing topography.

If a line, straight or curved, is drawn in any direction across a contour map, the elevation of every point where it crosses a contour is known, and the distances between these points can be scaled from the map. From these data, a profile of the surface along the line can be constructed. In Fig. 450 is shown the profile of the surface along the line *EE*, but profiles along *AB, BC, CA*, or any other line could be constructed equally well. Profiles may be used for determining gradients, for obtaining data used in computing earthwork quantities, and for other purposes already explained in the chapters on spirit leveling.

To one accustomed to reading contours, a topographic map presents a clear picture of the surface of the ground. A map of a small area shows where the ground is nearly level, where it is sloping, where slopes are steep, and where they are gradual. A map of an extensive area pictures the flat country, the hills and valleys, the lakes and water courses, and other topographic features.

450(*a*). USE OF CONTOUR MAPS. A simple illustration of the use of a contour map of a small area is the following. A residence is to be built on a fairly large plot of land. A contour map of the plot, drawn to large scale, with a small contour interval, can be used in selecting the exact sites for the house and garage, in locating such features as driveways and walks, in preparing plans for grading the land, and in estimating volumes of earthwork for

grading. Simple as this example is, it illustrates three of the most common uses of a contour map, namely, as an aid in location, in the determination of gradients, and in the computation of volumes of earthwork.

In general, contour maps are used in preliminary studies for engineering projects and in obtaining many of the data used in design and construction. Some of the most common examples of the use of topographic maps are: (1) for the location of structures such as buildings and bridges, and various masonry structures; (2) for the location or relocation of highways, railways, canals, pipe lines, and other route projects in which consideration must be given to both line and grade; (3) for determining the location of the dam for a reservoir, the maximum flood line, the drainage area, and the submerged area; (4) for estimating volumes such as the volume of cut and fill in grading an airport, or the volume of water to be impounded in a reservoir. From these examples, it is obvious that contour maps are practically indispensable as an aid in the design of a great variety of engineering projects.

450(*b*). Geology and topography. There is an intimate relation between geology and topography. Consequently, a knowledge of geology is often of great value to a topographer, and, conversely, a topographic map is often a valuable aid to the student of geology.

451. Ridge Lines and Valley Lines indicate largely the general character of the surface of the country.

Ridge Lines or Watershed Lines. Lines along the tops of ridges dividing watersheds, i.e., on one side of a ridge line water will flow in one direction, on the other side in the opposite direction.

Valley Lines or Thalwegs. Lines along the bottom of valleys where the two sides of the valley meet. They are often indicated by streams and rivers which follow them.

452. Fundamental Characteristics of Contours and Contour Lines.

1. *All points on the same contour line have the same elevation.*

2. *Two contour lines of different elevations cannot cross each other.* If they did, the point of their intersection would have two different elevations, which is absurd. (How can an overhanging cliff or a cave be an exception to this rule?) Contour lines of different elevations sometimes appear to merge in one line as in the case of a vertical cliff.

3. *Two different contour lines of the same elevation cannot merge and continue as one line.* For the single line would indicate a knife-edge ridge or depression, something which does not occur

in nature. Different contours of the same elevation, however, often approach very near to each other.

4. *Contour lines close together indicate a steep slope; far apart a gentle slope; equally spaced a uniform slope.* It is evident that the horizontal distance between successive contour lines indicates the rate of slope.

5. *Contours on a plane surface are straight, parallel, and uniformly spaced; on a rough surface, they are irregular.*

6. *A contour passing through any point is perpendicular to the line of steepest slope at that point.* This agrees with (4), since the perpendicular distance between contour lines is the shortest distance.

7. *Ridge lines and valley lines cross contours at right angles.* The sharpest bends or curves in contour lines occur at points on ridge lines and valley lines, the convex side of the curve being toward the lower ground for ridges, toward the higher ground for valleys. (See Figs. 450 and 453(*b*).)

8. *A closed contour line with one or more higher ones inside it indicates a hill (Fig.* 450) ; *one or more lower ones inside it indicates a depression without an outlet* (Fig. 453(*b*)).

9. *A contour line cannot have an end within the limits of the map. It must either close upon itself, or else it must begin at some point on the edge of the map and extend continuously until it disappears at some other point on the edge of the map.*

10. *No single contour line can intervene between two higher or two lower contour lines.*

11. *The contours at the top of a ridge or at the bottom of a valley either close or go in pairs.* For the highest horizontal plane that cuts a ridge must cut it on both sides; the same is true of the lowest horizontal plane that cuts a valley.

12. *Contours that cross a valley run up the valley on one side, and, turning at the valley line, run back on the other side.* If there is a stream, the contour on either side, turning upstream, may disappear in coincidence with the edge of the stream and cross underneath the water surface.

453. General Methods of Locating Contours. Contours are located with respect to traverse lines or other reference lines which form the framework of a survey. The general procedure at any station is to select points on the surface of the ground which, when plotted, will serve as guide points for sketching in the contours. Each of these guide points must be located by horizontal measurements, and its elevation, if not known, must be determined. There are two basic methods of locating contours,

namely, the direct method and the indirect method. Two other methods commonly used are the cross-section method, a special application of the direct method, and the coordinate method, a special application of the indirect method. Only a brief description of each of the four methods will be given here; the field procedure for each will be given later.

453(*a*). Direct method. In this method, each contour is located by determining the positions of a series of guide points *through which* the contour passes. This procedure is often called tracing out or running out contours.

Illustration. In Fig. 453(*a*) the dots represent guide points located in the field. These points were determined first on one contour and then on the next. Thus a number of guide points at irregular intervals, as 1, 2, 3, 4, 5, and 6, each having an elevation of 100 ft., were found by leveling and each was located with respect to stations *E* and *F*. The 100-ft. contour line was sketched through these points. Then a number of other points at irregular intervals, each having an elevation of 105 ft., were located, and the 105-ft. contour was sketched in. In a similar manner the other contours were successively established by locating the guide points indicated by the dots. The routine of the field work will be further explained in Arts. 462(*a*), 467(*a*).

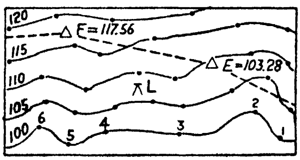


Fig. 453(*a*).

453(*b*). Indirect method. In this method the surface guide points are selected wherever there are appreciable changes in the slope of the surface and each is located horizontally by any convenient method. These guide points, however, are not, except by coincidence, points on the contours to be located. On the contrary, the elevation of each point must be determined. The slope between any two adjacent guide points is assumed to be uniform, and the point at which a contour cuts the line between two such guide points is plotted by interpolation. This method is also called the controlling-point method, and the procedure is known as interpolating contours.

Illustration. The dots (Fig. 453(*b*)) represent selected guide points. Each point was located with respect to stations *M* and *N*, and its elevation determined. These guide points were then plotted and used to locate the points on contour lines (indicated by circles). The latter were found by interpolation. Occasionally, by chance, a point located in the field happens to be a guide point on a contour line. It will be noticed that a point indicated by a dot can be used for locating more than one contour point. Thus, for example, circle 2 was interpolated between dots 1 and 4; similarly, circle 3 was interpolated between dots 1 and 5. The elevations from the field notes were (1) = 63.5, (4) = 67.3, and (5) = 57.5. For the methods of interpolating contours see Art. 660. For routine field work see Arts. 462(*b*), 467(*b*).

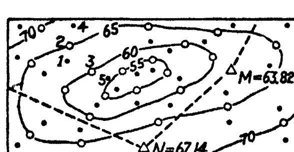


Fig. 453(*b*).

453(*c*). THE CROSS-SECTION OR CROSS-LEVELING METHOD. This method is a modification of the direct method. As in that method, all guide points are on contours to be located. The work corresponds somewhat to cross leveling (Art. 176) and is used mainly in route surveying, in which stations are usually established every 50 or 100 ft. along the center line. The difference between this method and the first is merely in the procedure followed in locating the guide points.

At each station, the guide points are determined on a cross line or offset line passing through the station at right angles to the center line. Each guide point is on one of the contour lines to be located, and different guide points are where different contours cross the offset line. The guide points are determined first on one cross line and then on another instead of first on one contour and then on another, as in the direct method. This method is equivalent to determining a profile along each cross line, and the points at which the profile changes are points on contours to be located. If there are breaks in the surface between the cross lines through the regular stations, additional guide points are located on intermediate cross lines. Cross lines which are not perpendicular to the center line may be used wherever it is advantageous to do so.

Illustration. In Fig. 453(*c*) is shown a portion of a topographic survey extending each side of a center line between station 30 and station 33. Contours at 5-ft. intervals were located as follows. Starting from a known elevation at station 30, guide points were determined on an offset line through

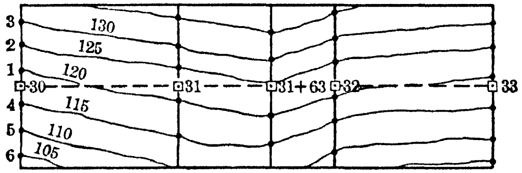


FIG. 453(*c*).

that station. First a point was found whose elevation was 120; then a point whose elevation was 125; then a point whose elevation was 130; and so on for all six points. As each point was found, the offset distance to it from station 30 was measured. By a similar procedure, guide points, represented by dots, were located on offset lines through the regular stations 31, 32, and 33. Guide points on an offset line through the intermediate station, 31 + 63, were also determined in order to locate the somewhat abrupt changes in the directions of the contours. After the guide points had been plotted, the contour lines were sketched through them. The location of contours may be extended to any desired distance on either side of the center line. (See Arts. 462(*h*), 467(*e*) for field routine.)

453(*d*). THE COORDINATE METHOD. This is a modification of the second method. As in that method, contours are interpolated between guide points, but the method of locating these points is quite

different. A coordinate system of squares or rectangles is laid out on the ground in practically the same way as in preliminary leveling for grading (Art. 177); the surface elevations at the corners of the squares are determined, and these corners are then used as guide points. The points where contours cut the sides of the squares or rectangles are plotted by interpolation. When there are appreciable breaks in the surface between corners, guide points in addition to those at the corners may be necessary. This coordinate method is equivalent to determining a profile along each coordinate line.

In any given case the squares or rectangles should be as large as practicable yet small enough to conform to the inequalities of the ground and to the accuracy required. In rough surveys over extensive flat areas, squares several hundred feet on a side may often be used.

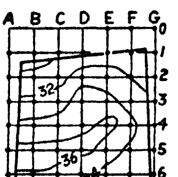


FIG. 453(*d*).

Illustration. A building lot about 160 ft. by 190 ft. is shown in Fig. 453(*d*). A rectangle 160 ft. by 200 ft., subdivided into 20-ft. squares, was staked out as a convenient method of locating guide points, i.e., the points at the corners of the squares, indicated by dots. The elevation of each corner was determined, and contour lines were interpolated. Each corner of a square is known by the letter and number corresponding to the intersecting lines. Thus the dot indicated by the circle is *C*8. (See Arts. 462(*i*), 467(*d*) for field routine.)

453(*e*). THE SELECTION OF GUIDE POINTS. When tracing out contours by the direct method, guide points should be chosen near enough together so that no appreciable error will exist when the contour lines are sketched through them. This will depend to a large extent upon (1) the regular or irregular nature of the ground and (2) upon the scale adopted for the map. The critical points of contours are usually apparent to the eye, and these points should always be taken. Thus, guide points should be close together at sharp curves or abrupt changes in direction and farthest apart where contours are nearly straight. In uniform slopes the highest and lowest contours having been run in, the others can frequently be interpolated without any field work, or perhaps every fifth contour may be located and the others interpolated.

When contours are interpolated, as in the indirect method, the preceding suggestions still hold good, the guide points, in general, being taken close together for uneven ground or steep slopes and farther apart for even ground or gentle slopes. In this method, guide points should be selected on ridge lines, on valley lines, and on lines wherever there are changes of slope.

In whatever method is used for locating contours, it is a waste of time to take guide points too close together when the scale of the map is small. On the other hand, it leads to inaccuracy to take them too far apart when the scale is large. In general, an imaginary straight line between two adjacent guide points should not pass above or below the ground surface by more than one contour interval. There is a tendency to locate too few points and with too great accuracy. It is better to locate a greater number of points with less accuracy.

453(*f*). Methods compared. The direct method can be used to advantage when the scale of the map is large, the contour interval small, and a high degree of precision in locating contours is required. The method is quite generally used for accurate surveys of small plots, particularly when the map is to be used for the exact location of structures and other objects. It may also be used in larger surveys where the surface of the ground is relatively flat or there are gentle sloping hills and valleys. Under such conditions there are relatively few contours, and consequently rod readings over a fairly large area can be taken from a single setup. When, however, the ground surface is quite irregular and there are many contours, this method is slow and expensive, and it is better to use the indirect method of interpolation. The cross-leveling method is quite generally used for route surveys where the topography is taken for a relatively short distance on either side of the center line and where data are required for earthwork computations. The coordinate method is quite commonly used for large-scale mapping when the required degree of precision is intermediate. It is sometimes advantageous to use in the same survey two or more methods of locating contours. For example, the direct method could be used for that portion of a survey in which the ground is comparatively flat and the slopes uniform, and the indirect method for that portion in which the ground is hilly and the topography quite irregular.

454. General Methods of Sketching Contours. When a topographic survey covers a small area and the surface of the ground is such that contours are more or less regular and easily drawn in, the work of sketching contours may be done in the office from guide points located in the field. This method can be used to advantage when it is desired to save time in the field work. In general, however, it is much better to sketch the contours in the field while the terrain is in view. This, of course, is always done when a plane table is used; indeed it is one of the advantages of the plane-table method. When a transit is used, guide points are

plotted as they are located and contours sketched. For this purpose some form of sketch board is needed. Boards vary from the simple form that can be held on the lap to the more elaborate traverse-table drawing board supported on a tripod. When stations of the traverse or other control system have been previously located in the field, the positions of these stations can be plotted in the office before the field work of contour location is started.

When the guide points have been located by the direct method or by the cross-section method, the work of sketching contours is largely mechanical since contours pass through the guide points. When, however, the indirect method or the coordinate method is used, contour points must first be found by interpolating between guide points; the contours are then sketched through these contour points. A contour point may be interpolated between two adjacent guide points by estimation, by calculation, or by a graphic method. For the graphic method some mechanical device is often used (Art. 660(*d*)).

455. Planning the Survey. Before going into the field, it is usually necessary to decide the following questions: Degree of precision required? Scale of the map? Magnitude of the contour interval? System or systems of control for traverses or other reference lines which form the skeleton of the survey? General method of running traverses or other reference lines? General method of locating contours and of locating details? General method of keeping field notes and of sketching contours? Surveying instruments and equipment to be used? Organization of the surveying party or parties?

456. Degree of Precision. The degree of precision adopted for a topographic survey depends largely on the use to be made of the map. For example, the degrees of precision required for topographic maps to be used in the design of a reservoir may differ widely for different portions of the area surveyed. For determining the general location of the dam, for estimating the drainage area, for judging the character of the runoff, and for estimating the volume of the water impounded, the degree of precision required for the map is ordinarily low; for the exact location of the dam, the degree of precision for the map of the relatively small area covered by the dam is generally high; for determining the location of the maximum flood line, the degree of precision for the map of that portion of the area where the flood line will be is usually intermediate, though in some cases it may be rather high.

Not all field operations in a survey are necessarily carried on with the same degree of precision. In an extensive survey, for

example, the degree of precision for triangulation is usually higher than that for traversing, and it is higher for traversing than for locating contours and details. Definite lines, such as boundary lines and the sides of structures, are located more accurately than indefinite objects such as edges of streams or sides of gravel roads. The customary degrees of precision for different field operations will be discussed in subsequent articles.

The degree of precision adopted for field operations is generally greater than the scale of the map would seem to require, since errors in field measurements if added to errors in plotting may cause needlessly large errors on the map. It is generally easy to attain a degree of precision in field measurements three or four times as great as that required for plotting, and such a ratio is usually large enough to render field errors negligible on the map.

457. Choice of Scale. Many of the considerations that govern the degree of precision with which a topographic survey should be made govern also the choice of scale for the map. The greater the degree of precision with which it is desired to scale distances on the map, the greater should be the scale. For example, if it is desired to scale to the nearest foot, the minimum scale should be at least 1 in. = 100 ft.; a larger scale would be better. To scale to the nearest 10 ft., the minimum scale should be 1 in. = 1000 ft., or greater.

In choosing the scale, it is sometimes necessary to consider unavoidable or permissible errors in plotting. If it is assumed that errors in field measurements are negligible, average errors in plotting may be determined by scaling distances between selected points and comparing these distances with corresponding field measurements. In many cases, errors in plotting will be negligible, provided the usual precautions for accurate drafting are taken.

The scale of the map should be great enough to permit details to be shown clearly and contour lines to be plotted without becoming crowded. On the other hand, there is sometimes a tendency to adopt too large a scale, particularly if one is accustomed to large-scale working drawings. The purpose for which a map is to be used should be kept in mind in choosing the scale. When different portions of an area are surveyed with different degrees of precision, the scale of the map for each portion should usually be consistent with the degree of precision of the field work for that portion, i.e., a large scale for a high degree, and a small scale for a low degree of precision.

It is often desirable that the scale of a map should correspond to the architect's scale instead of to the engineer's, as, for example, 1 in. = 8 ft., or 1 in. = 16 ft.

457(*a*). A CLASSIFICATION OF MAP SCALES. The following classification is quite generally recognized. *Large scale:* 1 in. = 10 ft. to 1 in. = 100 ft., or 1/120 to 1/1200. *Intermediate scale:* 1 in. = 100 ft. to 1 in. = 1000 ft., or 1/1200 to 1/12,000. *Small scale:* 1 in. = 1000 ft. to 1 in. = 20,000 ft., or 1/12,000 to 1/240,000.

458. Choice of Contour Interval. The first consideration is the accuracy with which it is desired to obtain elevations from the map. Obviously if two maps are equally accurate but one has a smaller contour interval than the other, elevations may be obtained with greater accuracy from the map with the smaller contour interval.

The second consideration is the character of the ground surface. A contour interval that is satisfactory for the map of a terrain with gentle uniform slopes and regular topography may be too large to represent satisfactorily a terrain with steep slopes and irregular topography. On the other hand, if the contour interval for steep slopes is too small, contour lines will be too close together. For ordinary conditions, the following table of values will serve as a guide in choosing contour intervals for the three classes of scales given in the preceding article and for three types of ground surface.

	Large Scale	*Intermediate Scale*	*Small Scale*
Flat surface	½ or 1 ft.	1 or 2 ft.	2, 5, and 10 ft.
Rolling surface	1 or 2 ft.	2 or 5 ft.	10 or 20 ft.
Hilly surface	2 or 5 ft.	5 or 10 ft.	20 or 50 ft.

The third consideration is the purpose of the survey. Thus, for example, in the survey of a city lot, a 2-ft. interval or even a 1-ft. interval may be necessary; for the preliminary survey in railway location a 5-ft. interval is often adopted; for more extended surveys, such as the U.S. Geological Survey, 20-ft., 50-ft., and even 200-ft. intervals are used.

Note. The degree of precision required for field work, the choice of scale, and the choice of contour interval are interrelated questions, no one of which should be decided without considering the other two.

459. Systems of Control. The stations from which topographic details are located horizontally may be considered points of horizontal control, and the bench marks from which elevations are obtained may be considered points of vertical control. It is desirable to check the positions of such stations by some system of horizontal control, and the elevations of bench marks by some system of vertical control. In many localities there are permanent control points and bench marks, the positions or elevations of which have been accurately determined in previous surveys. In some locali-

ties, a large number of these control points and bench marks are relatively close together, particularly in those states in which plane, rectangular-coordinate systems have been established. It is an advantage to have a local survey tied in permanently with a state-wide system of both horizontal and vertical control (Chapter 56). The use of permanent control points as a means of checking traverses is explained in Art. 309.

When there are no permanent control points and bench marks which can be conveniently used in making a topographic survey, it becomes necessary to establish systems of control as explained in succeeding paragraphs.

460. Horizontal Control. For a small area, a single closed traverse or a series of closed traverses may be used for horizontal control. Field operations for running these traverses and for locating contours and details may be carried on simultaneously, but it is frequently better to establish the traverse stations first and then to locate details and contours afterwards. When it is necessary to run an open traverse for any considerable distance, directions or azimuths may be checked by astronomical observations. The different methods of running traverses have been explained in Chapter 21. The azimuth method is in many respects the best for topographic surveys (Art. 303). The field work for transit traverses has been described in Chapter 23. Specifications for first-order, second-order, and third-order traverses will be found in Art. 360. Methods of running plane-table traverses are explained in Chapter 30.

In an extensive survey, the first step is to establish the most important or *primary* stations. This is usually done by triangulation, and in Chapter 28 suggestions are given for locating the base line, choosing stations, building signals, and carrying on the work of triangulation. In the same chapter are given specifications and requirements for the measurement of triangulation angles, and in Chapter 8 are given the methods of measuring base lines with different degrees of precision. Ordinarily, the triangulation net does not need to be elaborate. Starting from any triangulation station, traverses may be run to establish *secondary* stations, each traverse closing on some primary station, though not necessarily upon the one from which it started. In this way, the lines between triangulation stations form a framework upon which the survey is built, and this framework is further developed by traverse lines, until every portion of the ground is covered. Thus, in extensive surveys, it is not unusual to establish primary stations by triangulation, secondary stations by running traverse lines, and still other

stations by the plane-table method or by the compass method (see seventh illustration, Art. 467(*g*)). In wooded areas where triangulation may be difficult and expensive, it may be better to establish primary stations by the traverse method.

460(*a*). REFERENCE MERIDIAN. In extensive topographic surveys, it is well to use a true meridian as a reference meridian. If the survey is started from a permanent control point, the true azimuth of some line to that point may be known from a previous survey, and the instrument may be oriented by sighting along this line. If the survey is not tied in to a system of permanent control points, a true meridian may be established at any station by astronomical observations (Chapters 34, 35).

When a survey is started from control points such as those of a state plane-coordinate system, it is, of course, oriented from those points (Arts. 768, 773).

460(*b*). CHOICE OF STATIONS FOR HORIZONTAL CONTROL. When stations for horizontal control are established by triangulation, the accuracy of the work will depend, to a great extent, on the proper proportions of each triangle (Art. 403(*f*)). When traverses are run, the suggestions for choosing stations given in Art. 345 should be followed. In general, contours are most easily controlled from stations along ridge lines and valley lines. Stations should be located also near prominent topographic features, such as deep gullies, high cliffs, ponds, lakes, and watercourses. (See Art. 409 for other suggestions concerning the choice of primary stations.)

461. Vertical Control. (*a*). PRIMARY STATIONS. The first step is to ascertain the elevations of the principal stations, or to establish bench marks near them, so that a party can start from any station to locate contours, or several parties can start simultaneously from different stations. When the principal stations are located by triangulation, it involves very little additional field work to ascertain their elevations by *trigonometric* leveling—simply measure the vertical angles. (See "Triangulation and Trigonometric Leveling," Art. 416.) Spirit leveling, however, is usually preferable to trigonometric leveling except in rough country. Customary limits of error in spirit leveling are given in Art. 170.

461(*b*). ELEVATIONS OF TRAVERSE STATIONS or near-by bench marks are usually determined by spirit leveling, the leveling party following close behind the traverse party. Sometimes stadia leveling (Art. 430) is sufficiently accurate. For plane-table traverses, plane-table leveling (Art. 442) is sometimes used when the required degree of precision is not too high; otherwise elevations are determined by spirit leveling. For compass traverses the aneroid

barometer will sometimes give close enough results (Arts. 182–184). Permanent bench marks should be placed near stations and in other places where they will be of use in contour work.

462. Location of Contours—General Methods of Field Work. (*a*). The direct method (Art. 453(*a*)). It will be assumed for the present that a transit and a spirit level are the principal instruments used. The general procedure for tracing out contours at any traverse station or other point of horizontal control is as follows. The level party finds by trial a guide point on one of the contours to be located. For example, if the height of instrument is 124.7 and it is desired to locate points on the 120-ft. contour, the target is set at the grade-rod reading of 4.7 ft. and the rod is moved uphill or downhill until a point is found whose elevation is 120 ft. The horizontal location of this point is immediately determined by the transit party, and this location is either plotted in the field or entered in a notebook to be plotted in the office. The level party then finds another point on the contour, and this point is then located and plotted. Thus the procedure is repeated until as many guide points on that contour have been located as may be necessary or practicable from that station. The target is then set at the grade-rod reading for another contour, and this contour is traced out in a similar manner. When as much of the topography as practicable has been located from a station, the instruments are moved to the next station, and the work is continued. On hilly ground it is often better to locate guide points on a portion of one contour and then on a portion of the next higher and the next lower contours, instead of locating all the points first on one contour and then on another. In this way a greater number of points may be located from any one setup of the level. Frequently for each setup of the transit, there will be several setups of the level.

462(*b*). The indirect method (Art. 453(*b*)). The field work is similar to that for the direct method except that the work of the level party consists in finding the elevations of selected, critical guide points instead of determining guide points at given contour elevations. These guide points are located horizontally by the transit party, just as in the direct method, but after the guide points have been plotted, the contour points must be interpolated.

462(*c*). Horizontal measurements. As far as horizontal measurements are concerned, it makes no difference whether the guide points to be located are on the contours themselves or whether they are for the purpose of interpolating contours. Any guide point may be located by one of the seven methods given in

Art. 10, but probably the method most used for transit work is the angle-and-distance method. For angular measurements, the azimuth method is used perhaps more frequently than the direct-angle method, because of the ease with which readings may be checked by the needle. For linear measurements, the stadia method is more useful than chaining, since it is quicker, and the accuracy required for side shots is not great.

462(*d*). READING HORIZONTAL ANGLES TO CONTOUR POINTS. Side shots are rarely longer than 500 ft. An error of 10′ amounts to a displacement of not more than 1½ ft. at 500 ft., and of only about 0.3 ft. at 100 ft. from the transit (Art. 283(*d*)). As it usually makes little difference whether a contour point is 1 or 2 ft. one way or the other and as limb readings can be estimated easily to less than 10′, it follows that, as a rule, *it is not necessary to read the vernier in locating contour points horizontally; simply estimate the reading from the limb.*

462(*e*). VERTICAL MEASUREMENTS. When the direct method of locating contours is used, guide points at the elevation of the contour are usually determined by spirit leveling, either an ordinary level or a hand level being used. When the indirect method is used and slopes are steep, not many guide points can be taken without moving the level, a decided disadvantage. It is often better, in such cases, to use the stadia-vertical-angle method (explained in Art. 427), so that points up and down the slope can be taken from one setup. A third method used in rough work is to find the elevation of points by the aneroid barometer (Art. 184).

462(*f*). READING VERTICAL ANGLES TO CONTOUR POINTS. The directions for measuring horizontal angles (Art. 462(*d*)) hold good for vertical angles also. For a sight 100 to 200 ft. long, an error of 10′ causes an error of 0.3 ft. to 0.6 ft. in elevation. This is no more than the difference in height between the top and bottom of a grass hummock; hence the absurdity of reading vertical angles to the nearest minute in obtaining elevations for contours in ordinary work. Occasionally, on long sights, it may be necessary to consult the vernier, but *most of the angles should be read directly from the vertical arc, estimating the minutes.*

462(*g*). WHEN THE PLANE TABLE IS USED INSTEAD OF THE TRANSIT, the field methods just explained for locating guide points are only slightly modified. (1) Horizontal angles are plotted directly by lines drawn along the alidade. (2) More use is made of the method of locating a point by intersection. The plane-table-stadia method, in which both the horizontal and the vertical measurements required in locating guide points are made by means of

the stadia, is one of the most common methods of locating contours; it has many advantages (Art. 439(*a*)).

462(*h*). THE CROSS-LEVELING METHOD (Art. 453(*c*)). When this method is used, stations along a center line are usually established and their elevations determined by profile leveling. The field work of locating contours by the cross-leveling method differs from that of tracing out contours by the direct method in two respects. (1) Instead of tracing out first one contour and then another, guide points on *different* contours are located first on one cross line and then on another. (2) Guide points are located horizontally merely by measuring offset distances from the center line. The field work is so simple that it can be done by a party of three, two levelmen and a topographer. The topographer generally plots the contours as the work progresses, but, in some cases, he may simply record the offset distance and the elevation of each guide point in the form of a fraction, the numerator of which is the contour elevation, and the denominator the offset distance or vice versa. The hand level is frequently used for measuring vertical distances; in this case the party may consist only of two levelmen, one of whom also acts as topographer. The routine field work is explained more in detail in the fifth illustration, Art. 467(*e*).

462(*i*). THE COORDINATE METHOD OF LOCATING CONTOURS (Art. 453(*d*)). The field work of laying out the coordinate system of squares and of determining the elevations of guide points is practically identical with that described in Art. 177, "Preliminary Leveling for Grading." Another system of laying out coordinates which can often be used to better advantage is described in "Leveling for a Borrow Pit," Art. 178.

462(*j*). THE ACCURACY REQUIRED IN LOCATING GUIDE POINTS for contours should be consistent with the scale of the map and the magnitude of the contour interval. A common rule for guidance is: The error in the plotted position of a guide point should not exceed one-fifth the map distance between adjacent contours near that point, and the error in the elevation of the point should not exceed one-fifth the contour interval. A general idea of the accuracy with which horizontal and vertical angles should be measured has been given in Arts. 462(*d*) and (*f*). In general, the sources of error in the order of their importance are: error in the vertical angle, error in the measured distance, and error in the horizontal angle or azimuth. Thus, for example, an error in the measurement of a vertical angle is more serious than one in the measurement of a horizontal angle. On gentle slopes an error in the measurement of a vertical angle is more serious than one in

the stadia distance, but, for steep slopes, an error in distance becomes more serious than one in the vertical angle.

462(*k*). THE FIELD ROUTINE in the application of the general methods of locating contours varies to such a great extent under different conditions and requirements that illustrations of various methods of procedure are given in Art. 467 near the end of this chapter.

462(*l*). PRACTICAL SUGGESTIONS. (1) Whatever method of locating contours is used, the results will depend to a large extent on how well guide points are selected (Art. 453(*c*)). (2) When a portion of a slope is a plane, a contour near the top and one near the bottom may be located and the other contours between these two may be sketched in by interpolation. (3) For large-scale maps and small contour intervals, spirit leveling is preferable to trigonometric leveling. (4) It is sometimes desirable to establish a station which is not one of the stations of a traverse or other control system. Such a substation may be located by any of the usual methods, but with a degree of precision consistent with that required for regular stations since the correct location of all guide points will depend upon the correct location of the station itself. In plane-table work a substation may often be located by the method of intersection or by the three-point method.

463. Location of Details. The first step is to decide what details other than contours should, in general, be located. Keeping the purpose of the survey in mind, it is necessary to ask continually as the work progresses whether this detail or that is essential. There is sometimes a tendency to locate details which are not essential.

In general, the methods of locating details in transit surveying (explained in Chapter 22) and in plane-table surveying (explained in Chapter 30) hold good in topographic surveying. In any given case, the simplest method that is consistent with the requirements should be used. Thus, for example, roads, streams, fences, and even buildings may often be located wholly by offsets. For indefinite lines, such as edges of woods, banks of rivers, and irregular roads, the method of intersection is rapid and sufficiently accurate, particularly if a plane table is used. Keeping in mind methods based on angles and distances, offsets, range lines, tie lines, intersecting lines, and dimensions, the surveyor, exercising foresight and ingenuity, can plan the location of each detail in such a way as to save time.

The accuracy with which details should be located is determined largely by the same considerations that determine the accuracy with which guide points should be located. In general, details which have clear-cut, definite outlines are located with greater accuracy than objects with indefinite lines. When measurements are taken solely for the purpose of plotting a detail on the map, there is no advantage in a greater degree of precision than can be

used in plotting and scaling. Ordinarily, in the location of any point of a detail, an error corresponding to a distance on the map of one-fiftieth of an inch or less is negligible, and, in most cases, the permissible error may be considerably greater.

464. The Choice of Methods and Instruments depends upon such considerations as the purpose of the survey, the degree of precision required, the character of the ground covered, the permissible cost, and the amount of time available. It is often impossible to choose methods and instruments which will meet all the requirements suggested by these considerations; the problem then becomes one of best adjusting these requirements and of selecting the best combination of methods and instruments. Practically every method and every instrument may be used to advantage at one time or another in topographic surveying. Even in the same survey various parts of the work may call for the use of different methods and different instruments. Thus, for example, in an extensive survey less accurate methods and instruments may be used for running secondary traverses than for establishing the primary stations, whereas the contours and topographic details may be filled in by the use of still less accurate methods and instruments.

The instruments and methods chosen for vertical control should correspond in accuracy to those chosen for horizontal control. For example, if rough instruments are used for locating points, rough instruments are used for finding their elevations. There are, of course, exceptions to this rule. Consistent accuracy should be maintained between linear and angular measurement. Thus, for example, when directions are obtained with a compass it is inconsistent to measure distances with great accuracy; in some cases they may even be paced.

The number of possible combinations of instruments and of methods is too large to be given in detail, but some of the most common combinations will be found in the illustrative examples of field routine given at the end of this chapter (Art. 467).

465. Field Notes. When conditions are such that it is advantageous to plot a topographic map in the office instead of in the field, the field notes may be kept by any of the common methods explained in Chaper 24. The combination method of notes is often used. A sketch is made on a right-hand page, and such measurements as cannot be conveniently placed on the sketch are recorded on the opposite page in tabulated form. The points on the sketch should be carefully numbered to conform to some well-understood system. Thus, for example, all guide points on the 100-ft. contour may be numbered 100, regardless of location; all

points on the 105-ft. contour may be numbered 105, and so on, the measurements for the location of the points being given in the tabulated form on the left-hand page. When contours are located by the indirect method, it is better to give each guide point used in interpolating contours a separate number and to note its elevation in a column set apart for that purpose. When the contours are regular and easily drawn in, no great care is necessary in representing them on the sketch, but when they are irregular, they should be sketched approximately to scale. A very good book for this purpose has the page divided by radiating lines and concentric circles, so that a point can be plotted approximately by azimuth and distance without the use of instruments. A common mistake is to record in the tabulation the azimuth to one point and the distance to another point, as if they were taken to the same point. Notes for stations and details are kept by the usual methods.

466. Plotting the Map in the Field. When the topography is irregular, results, as a rule, will be far more satisfactory if the map is plotted in the field. This can be done in two ways: (1) by the use of the plane table and (2) by the use of ordinary plotting methods. The plane table is especially useful for filling in details from stations which have been established previously by more accurate instruments. Often the entire survey can be made with it. Ordinary plotting methods are used when angles are measured directly as in a transit topographic survey. Angles and distances are plotted as they are measured, and contours are sketched in as the work progresses. In this way errors and omissions are likely to be detected and doubtful points settled. Thus, with the land actually before him, the topographer can secure a more truthful representation of it than he could otherwise, and this is true also of plane-table work. Various devices are used to facilitate the plotting, such as sketching boards, various forms of protractors, and specially designed cross-section paper.

466(*a*). SKETCHING. The general method of sketching contours has already been explained in Art. 454, but these methods should not be applied too mechanically. In order to sketch topography successfully in the field, one must have a faculty for selecting the most important topographic features and transmitting them to paper in their proper relations. This sense of topography comes only from long experience. In ordinary class work the beginner first plots guide points of known elevation as they are located. Guided by these points, he then sketches in each contour. His imagination may be aided, perhaps, by the conception of a line on the ground at the edge of a water surface. Successive points on

the same contour should be joined as soon as they have been plotted, by an irregular line which corresponds as nearly as possible to the actual contour which it represents. If contours are interpolated, they should be carried along in a similar manner. It answers the purpose in many cases simply to interpolate contour points by eye; for more accurate methods of interpolation see Art. 660. Each contour should be marked in one or more places with its elevation. It is customary to emphasize in some way, as by a heavy line, every fifth contour. When the slope is uniform, some of the contours may be sketched in by interpolation without actually locating them by field measurements.

466(*b*). Practical suggestions. (1) When notes are plotted in the field, the drawing board may be mounted on a tripod and set up near the transit. Instead of using a protractor it is sometimes more convenient to plot the notes on paper or Bristol board which has printed upon it a full circle graduated to quarter degrees. This paper comes in sheets and may be obtained from dealers in such supplies. (2) Keep track of points as fast as they are plotted by marking each point with its elevation. This is especially important when points are taken at random and contours are interpolated; otherwise confusion will result. (3) A disadvantage common to all plotting operations in the field, including the plane table, is the change in the paper due to moisture. This matter has been treated in Art. 444(*a*).

467. Field Routine. In addition to a knowledge of the *general* methods of establishing control systems and of locating contours, a surveyor should also know the best field routine for the application of any of these methods. This knowledge is best gained through actual field experience. Such experience will be less costly, however, if one gains as soon as possible a general knowledge of field routine in topographic surveying. An attempt is made in the remainder of this chapter to give such a general knowledge by explaining some of the most common methods of procedure. Illustrations of several different types of surveys are given. It is assumed in each illustration that the stations are chosen in accordance with Art. 460(*b*), that the contour interval has been adopted to suit the purpose of the survey (Art. 455), that all details, other than contours, are located by the usual methods (Chapter 22), and that the guide points of contours are chosen according to Art. 453(*e*). The main points to be brought out are: (1) How traverse stations or other control points are established. (2) How the work at each station is started. (3) How guide points are located and (4) how their elevations are determined. (5) How the levelman, the transitman, and other members of the party may work together efficiently.

Remark. The methods of procedure given in the following illustrations may be modified in many ways to suit conditions. Thus, for example, it is often advantageous to use the direct-angle method in place of the azimuth

method, especially if stations cannot be occupied in regular order (Art. 306). The illustrations suggest how to proceed for different combinations of instruments and of methods—this is the first thing to learn; then the student should begin a study of the advantages and disadvantages of the different combinations, a full knowledge of which can be gained only by extended experience in the field. Special attention is called to the last illustration in Art. 467(*g*), since it combines in one survey nearly all the methods of topographic surveying and, hence, serves as a summary of this chapter.

467(*a*). FIRST ILLUSTRATION. *Conditions:* comparatively small area, fair degree of accuracy, medium speed, 5-ft. contour interval. *Instruments:* transit, level, and leveling rod (used also as a stadia rod), drawing board mounted on a tripod, with accessories for plotting, including protractor, scale, and triangles. *Methods:* azimuth method (Art. 351) and stadia method (Art. 426) for horizontal measurements; spirit leveling for vertical measurements; method of running in contours by ascertaining guide points through which they pass (Art. 462(*a*)). *Party:* transitman, notekeeper and draftsman, levelman, and rodman.

Method of Procedure for the Levelman. (1) Set up where a backsight can be taken on some bench near by, and, at the same time, near enough to the transit station so that any point which can be seen from the transit will be visible from the level also. (It is assumed that benches have already been established near stations.) (2) Backsight, find H.I., figure the grade rod for one of the contours. (See Art. 136 for grade rod.) Trace out the contour by the direct method (Art. 462(*a*)). It is well to begin with the highest contour that can be run in from the first setup of the level and work downhill, or else begin with the lowest and work uphill. (3) When as much of the first contour has been run in as can be located conveniently from the transit station, run in two more in the same way, moving the target 5 ft. for each contour. By this time it will be necessary to move the level; hence establish a turning point as far downhill or uphill as practicable, set up the level as far downhill or uphill as the turning point will permit, find the new H.I., and the new grade rod for the next contour, and proceed as for the first setup. (4) Continue the work, moving the level as often as may be necessary until all the contour points that can be located advantageously from the transit station have been taken. If the work has been carried on downhill from the station, it may be necessary to start again at the bench mark and work uphill, or vice versa. When benches have not been established near the stations, it may be necessary to take extra turning points in carrying the work from one station to the next; otherwise it is better to start anew at each station by backsighting on the bench mark near it.

Method of Procedure for the Rodman. (1) Check the grade rod for the first contour as figured by the levelman, set the target at the grade-rod reading; find a guide point on the contour by moving the rod uphill or downhill until the levelman signals that the line of sight strikes the target, and then hold the rod for the transitman to take the azimuth and stadia readings. (The levelman should not split hairs in sighting. Often it will be close enough if the line of sight strikes anywhere on the target.) (2) Repeat for other guide points, selecting only critical points along the contour. (See Art. 453(*e*).) (3) Check the grade rod for the next contour, set the target, and run in this contour by selecting critical points as before. Repeat for all the contours, establishing turning points whenever it becomes necessary to move the level uphill or downhill.

Method of Procedure for the Transitman. (1) Set up and orient the transit (Art. 303(*a*)). Assuming that transit stations have not been established, it will be necessary at the first station to adopt a meridian and to determine the corresponding azimuth for the first transit line (Art. 303(*b*)). At other stations the azimuths of lines must be measured with reference to this meridian (Art. 303(*c*)). (2) Take the azimuths to contour points as fast as these points are established by the level party, reading the corresponding stadia

distances and vertical angles. (It is not usually necessary to read the vernier for vertical angles in ordinary work when such angles are used merely for reducing stadia distances to horizontal distances—estimate the minutes.) (See Art. 462(*f*).) Vertical angles less than 3° to 5° may be ignored. (3) Observe the method of procedure for the azimuth method (given in Art. 351), in running traverse lines, and read the stadia distance between stations twice, once from each end of the line. Check the azimuths of all traverse lines by the needle, and check at least every fifth side shot in the same way.

Method of Procedure for the Draftsman. (It is assumed that the notes are plotted in the field as fast as they are taken.) (1) Set up the plotting board near the transit. If it is the first station, draw a long line near the center of the board to represent the meridian. Azimuths may be plotted directly from this line or from parallel lines through the stations. Use a protractor for ordinary work or else some form of plotting paper. (See Art. 466.) (2) Plot the azimuth to each guide point as soon as it is taken, and then the stadia distance, reducing the latter to a horizontal distance when the vertical angle is greater than, say, 3° (Art. 645). Keep track of guide points by marking near each one the elevation of the contour which passes through it, and sketch in each contour line as fast as it is determined by the plotted guide points; try to reproduce each contour as it actually appears on the ground. When the guide points have been taken too far apart and the sketch shows the need of more points, indicate to the level party where the additional points are needed.

When the notes are not plotted in the field, use some form of notes similar to Form 3, Art. 431(*c*), but make the sketches on the right-hand pages represent the actual contours as nearly as possible.

467(*b*). SECOND ILLUSTRATION. *Conditions:* same as in the preceding illustration, except that guide points are not on the contours but are selected as "critical" points between which contours will be interpolated (indirect method, Art. 453(*b*)). *Methods:* The only change in routine is in the leveling and in the plotting. The rodman selects critical points (Art. 453(*e*)), and the elevation of each point as determined by the level party is given to the draftsman. The latter plots these points from azimuths and distances as in the preceding illustration, and marks the elevation of each point; but instead of sketching contour lines through the points, he interpolates them as fast as the plotted points permit him to do so.

467(*c*). THIRD ILLUSTRATION. *Conditions:* an area of considerable extent; medium accuracy, rather high speed. *Instruments:* transit and stadia rod or leveling rod, board mounted on a tripod, with accessories for plotting, including protractor, scale, and triangles. *Methods:* azimuth method and stadia method for horizontal measurements, vertical angle and stadia for vertical measurements (Art. 427); points chosen for the interpolation of contours (Arts. 462(*b*), and 453(*e*)). *Party:* transitman, rodman, a computer, and a draftsman.

Method of Procedure for the Transitman. (1) Set up and orient the transit as in the first illustration. (2) Cooperate with the rodman in setting the target in order to determine elevations as explained in Arts. 180(*a*) and (*b*). Take the azimuth, the stadia reading, and the vertical angle to each point where the rod is held. (3) Unless control stations have already been established, follow the procedure for running traverse lines outlined for the transitman in the first illustration. Usually it is better to establish bench marks near stations by spirit leveling, but if this has not been done, it will be neces-

sary for the transitman to determine elevations from station to station by vertical angles (stadia leveling, Art. 430).

Method of Procedure for the Rodman. (1) Set the target (or the rubber band on the rod) from the nearest bench, or in some cases from the transit station (Arts. 180(*a*) and (*b*)). (2) Hold the rod on ridge lines, valley lines, wherever the slope changes, and at other critical points (Art. 453(*e*)). Co-operate with the draftsman in determining what points are needed.

Method of Procedure for the Computer. Compute the horizontal distance from the station to a point as soon as the transitman has given the stadia reading and the vertical angle. While the draftsman is plotting the point, find the corresponding vertical distance, and from that ascertain the elevation of the point with respect to the datum used for the survey. See that the elevation of each point is marked near its plotted position. Use the table in the back of the book for reducing stadia readings, or a slide rule.

Method of Procedure for the Draftsman. Plot the azimuth of the line to each point as soon as the transitman reads it (see first illustration); plot the distance as soon as the computer has ascertained it; mark the elevation near each point as soon as the computer has calculated it. Be careful to keep track of the azimuth, distance, and elevation of each point until there is no likelihood of using any one of these three data in connection with some other point. Sketch in the contours as fast as they can be interpolated from the points plotted, always with due regard to the ground as it appears to the eye. When in doubt call for other points to be taken near the doubtful places.

467(*d*). FOURTH ILLUSTRATION. *Conditions:* small area, high degree of accuracy, low speed. *Instruments:* transit, tape, leveling rod, and level (or the transit may be used as a level). *Method:* area subdivided into squares or rectangles for horizontal control, elevations of the corners of the squares or rectangles taken for vertical control, contours interpolated (coordinate method, Art. 453(*d*)).

Routine of the Field Work. Decide how to lay out the squares in order to cover the ground to the best advantage, what size to make them, and in what direction to run the lines. Stake out the squares with the transit and tape. (Two or three setups on the outside corners will usually suffice, the interior stakes being lined in by eye.) Backsight on a bench near by, or establish one if none exists. Find the elevation of the ground at each stake and in between if there is a sudden change in elevation, and thus determine points from which the contours can be interpolated. The level notes should include sketches on the right-hand pages, carefully lettered and numbered according to some well-understood system (Fig. 453(*d*)), and showing all dimensions and measurements necessary for plotting.

Note. This same method may be modified for very rough work by using the compass for running out the lines, and a clinometer level or some other rough instrument for ascertaining the elevations. Where the contour interval is large, elevations may be determined by barometric leveling, particularly if an altimeter is used. (See "Barometric Leveling," Arts. 182–184.) If bench marks have been established throughout the area, the barometer reading may be checked from time to time by means of these bench marks. With proper precautions, contour lines can thus be located in a fairly satisfactory manner. Rough methods are frequently combined with more accurate systems of control where other methods would be difficult and expensive as in the topographic survey of a large wooded area.

467(*e*). FIFTH ILLUSTRATION. *Conditions:* the survey progresses in one general direction as in railroad work; the main transit lines have been established, and the elevations of points at frequent intervals, say at every 100-ft. station, have been found by the method of profile leveling; fair degree of accuracy, fair speed, and a 5-ft.

contour interval. *Instruments*: two hand levels, two 5-ft. contour rods, a tape, a sketch board, and plotting paper. *Methods*: contour points are located on cross lines or offset lines and contour lines are sketched through these points (cross-leveling method, Arts. 453(*c*), 176). Horizontal distances measured with a tape; vertical distances with hand levels and contour rods. *Party:* two men, who do practically the same work, except that only one of them plots the contours.

Routine. Levelman 1 holds his rod on a point of known elevation, say at a 100-ft. station. Levelman 2, resting his hand level on the top of his rod, moves up- or downhill along the cross line through the 100-ft. station until the base of his rod is on a point of some 5-ft. contour. He finds this contour point by the method explained in the illustration in Art. 453(*c*). The offset distance from the station to the contour point is measured with the tape, or, in rough work, it may be paced. Levelman 2 now holds his rod on the point just located and, resting his level on top of the rod, directs levelman 1 to a point on the cross line where the line of sight cuts the ground (assuming the slope is uphill). This point must be on another 5-ft. contour, and its position is located by measuring the offset distance or else the distance from the preceding contour point on the offset line. Levelman 1, holding his rod on the contour point just located and resting his level on top of the rod, directs levelman 2 to the point where the next 5-ft. contour crosses the offset line. Thus the two men alternate in sighting in points where different 5-ft. contours cross the offset line, proceeding as far uphill as may be desired. They then continue the work in a similar manner along the cross line on the downhill side of the station, assuming that the slope is downhill. (How will the procedure downhill differ slightly from that just planned?) All the points on one offset line having been plotted, the party repeats the work at the next 100-ft. station. Thus the work is carried on, contour lines being sketched by joining points on the same elevation. Offset lines are taken between stations where irregularities in the surface of the ground render it necessary.

When the contour lines are plotted in the office instead of in the field, the measurements for each contour point may be recorded in the field notes in the form of a fraction as explained in Art. 462(*h*).

467(*f*). Sixth illustration. Since a plane table with a good alidade can be used for both horizontal and vertical measurements, the field work in each of the five illustrative surveys just outlined could be done partly or wholly with that instrument. In the first and second illustrations it could be used in place of the transit and plotting board, each guide point being located and plotted almost simultaneously on the plane-table board. In the third illustration the entire work could be done with the plane table and a stadia rod or a leveling rod. In the fourth illustration it could be used, but not to advantage. In the fifth illustration it could be used in connection with the hand level; in fact the combination of the traverse board (a rough plane table) and the hand level is a common one.

When the horizontal distances and the elevations of points are reduced from stadia readings and vertical angles, the man who runs the plane table may be assisted by a computer, whose duties are those described in the second illustration.

467(*g*). Seventh illustration, summary of methods. The following outline indicates the work done by a class in making a topographic survey of a rather extended area. It was desired to illustrate as many methods as possible, and to furnish work for a large number of parties, each party going from one kind of work to another until it had been given practice in all the different methods. This illustration will serve to summarize the methods of this chapter and to show how different methods may be combined in the same survey.

1. Made a reconnaissance with the class, with the following objects in view: to get an idea of the drainage of the country; to ascertain what areas were wooded and what were comparatively open; to find favorable locations for triangulation stations and to plan the triangulation net (Art. 409); to select a place for a base line (Art. 410(*a*)), and decide upon its approximate length (Art. 410(*b*)); to decide tentatively where to run traverses and what instruments and methods to use.

2. Secured from a railroad company the elevation of the nearest bench, used this elevation in adopting a datum for leveling, and started parties from this bench to run lines of levels to the two ends of the base line, where the first benches of the survey were to be established.

3. Built signals at points selected for triangulation stations (Art. 407).

4. Set a hub for each end of the base line, referenced each hub, set parties at work measuring the base line (Art. 410(*c*)).

5. Set parties at work measuring the angles of the triangulation net (Art. 411), and the vertical angles to triangulation stations required for trigonometric leveling (Art. 416).

6. Established a true meridian through one end of the base line by observation on Polaris (Chapter 34), found the true bearing of the base line by measuring the angle between it and the meridian, checked this bearing by observations on Polaris at the other end of the base line.

7. Started transit parties to establish secondary stations by running traverses (direct-angle method, angles doubled), each party to begin at one primary (triangulation) station and to close on the same or some other primary station, all secondary stations to be established with due regard to their future usefulness (Art. 345).

8. Started level parties from the benches near the base line to establish benches near secondary stations, checking whenever possible on primary stations, the elevations of which were found by trigonometric leveling in (5).

9. Set plane-table parties at work in open country, some of the parties to start from primary or secondary stations, and, establishing their own tertiary stations by traversing (Art. 441(11)), to close on some primary or secondary station. Other parties are to set up in some commanding position from which three stations previously plotted are visible, and orient by the three-point problem (Art. 447). In either case contours are to be sketched in by stadia and vertical-angle method as in the third illustration (Art. 467(*c*)).

10. Set transit parties at work in wooded areas unfavorable for triangulation, each party to start from a primary or secondary station, establishing tertiary stations if necessary, and to close on a primary or secondary station. Method of procedure for locating contours exactly the same as in the third illustration (Art. 467(*c*)).

11. Set transit parties at work to run a preliminary survey for constructing a new road, tying this survey whenever possible to the main survey, and running in contours with the level and hand level as in the fifth illustration (Art. 467(*e*)).

12. Detailed compass parties to locate high cliffs and deep gorges, differences in elevations between top and bottom lines or other controlling lines to be taken with an aneroid barometer, and less steep slopes with the clinometer level, intervening contours to be interpolated. Compass lines are to be tied to primary or secondary stations, and aneroid readings to be compared with elevations at benches whenever possible.

468. Use of Aerial Photographs in Topographic Surveying. Aerial photographs are used extensively in mapping relatively large areas. Maps which show relief by means of contour lines can be made from such photographs. Consequently aerial photographic surveying has become a common method of obtaining topographic maps. There are many cases where the method can be used to advantage even when the area mapped is relatively small. Aerial photographic surveying is treated in Chapter 60.

CHAPTER 32

VERTICAL CURVES

In this chapter are given methods of computing the elevations used in establishing grade points on a vertical parabola.

469. Vertical Curves. When a profile changes from one rate of grade to another, the angle at the intersection of the two grade lines is rounded off by a connecting vertical curve. On railroads and highways this vertical curve is commonly a parabola. For a pipe line, the vertical curve is often the arc of a circle, particularly where the pipe is of riveted steel and it is desired to use standard steel plates throughout the length of the curve.

On railroads and highways, a vertical curve should be of sufficient length to obtain an easy and safe transition from one rate of grade to another. The proper length of curve will be discussed later.

For ordinary flat vertical curves, there is little difference between circular and parabolic arcs. Elevations, however, can be computed more easily for points on a parabolic arc than for points on the arc of a circle; moreover, for a relatively abrupt change in the rate of grade the parabola is the better connecting curve.

Once the elevations of points on a vertical curve have been determined, the field work of setting grade stakes does not differ from that explained in Chapter 14. The remainder of this chapter will therefore be devoted mainly to methods of determining elevations.

470. Vertical Parabola—General Problem. In Fig. 470 let AB and BD represent two grade lines or tangents intersecting at B. Assume that the gradient of AB is different from that of BD. The point A, the beginning (P.V.C.) of the curve, and the point D, the end (P.V.T.) of the curve, are generally so selected that the horizontal distance of each from the vertical axis through B is equal to $\frac{1}{2}L$. Since the rates of grade for AB and BD differ, the elevations of points A and D differ; hence the long chord AD is not horizontal.

For vertical parabolas on highways and railroads, the point B, the P.V.I. of the curve, is generally made to come at a regular or full station, not at an intermediate station. Moreover, the length

½L of each of the tangents is often made to equal some multiple of the normal interval between regular stations in order to cause the P.V.C. and the P.V.T. of the curve to coincide with regular stations. This somewhat arbitrary location of the points P.V.C., P.V.I., and P.V.T. greatly simplifies the computations without materially changing the position of the curve. This is particullarly true of the location of P.V.C. and P.V.T. at equal horizontal distances from P.V.I. Such a parabola is known as a parabola with equal tangents or as a *symmetrical parabola.* Vertical parabolas with unequal tangents will not be treated in this book.

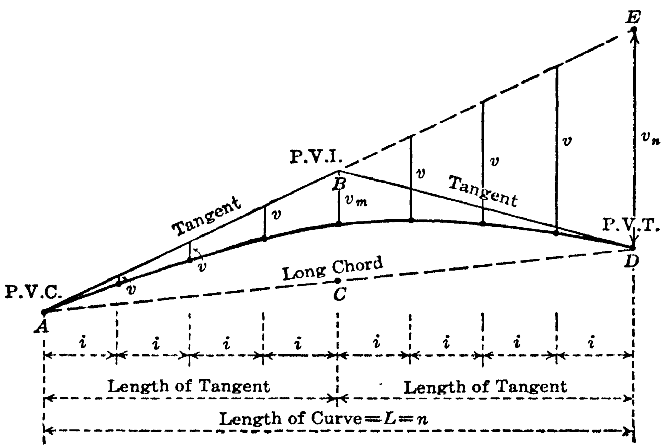


FIG. 470.

When a vertical curve is at a summit, the tangent through P.V.C. is ascending and that through P.V.T. is descending, as shown in Fig. 470. When the curve is in a "sag" connecting the bottom of one grade with the bottom of another, the reverse is true. (This may be seen by turning Fig. 470 upside down.) Other cases include both tangents ascending, both descending, one level, the other ascending, and one level, the other descending.

471. Fundamental Concepts. In studying Fig. 470 and the other figures in this chapter, all distances should be conceived as they appear when the page is held in a vertical position. All lengths, such as the length of a tangent, the length of the curve, the length of the long chord, or the length between any two points of the curve, are measured *horizontally* as, of course, they are in the field. Vertical distances, however, represent differences in elevations.

The length of the curve is generally divided into an even number of station intervals so that the number of intervals on one side of the vertical through P.V.I. is equal to that on the other. The number of intervals chosen depends upon the length of the curve and upon how close together it is desired to establish points on the curve. Common lengths of the interval are 100 ft., 50 ft., and 25 ft.; in some cases the interval is much less than 25 ft. The number of intervals are represented by n. The length of the curve may be expressed in feet or in terms of n. For example, if in Fig. 470 the length i of a station interval is 100 ft., $L = 800$ ft., or $L = n = 8$ (intervals).

Vertical distances are generally *theoretical* distances used in computing elevations of points which are to be established later by leveling. The vertical distance from any point on a tangent to a point on the parabola is called a *tangent offset* or a *vertical offset*. In Fig. 470 each of the vertical lines marked v is a vertical offset from the tangent through P.V.C. The middle offset v_m extends from P.V.I. to the parabola; the final offset v_n is a distance n from P.V.C. and extends from E, on the tangent produced, to P.V.T. The points on the tangent from which the offsets extend are theoretical points used only in computation. The elevations of these points and the lengths of the offsets are calculated, and the results are used in computing the elevations of the corresponding points on the parabola.

471(*a*). SUGGESTIONS. The term *offset* is sometimes misleading because it is so often used in horizontal measurement. For this reason, the symbol v is used in this book to represent a *vertical* tangent offset. Other terms used in connection with vertical curves are sometimes confused with similar terms used in connection with horizontal curves. For example, "the length of the tangent" to a horizontal curve means the actual length; but when the term is used for the tangent to a vertical curve, it means the *horizontal projection* of the actual length.

To avoid confusion it is well to keep in mind that: (1) A vertical parabola is a portion of a *profile.* (2) The tangents are *grade lines.* (3) Intervals between stations and all other lengths are measured *horizontally,* as in the field. (4) A tangent offset is a *vertical* offset and is equivalent to the *difference in elevation* between a point on the tangent and a point on the parabola.

472. Properties of a Vertical Parabola. In order to compute the elevations of points on a vertical parabola one should have a working knowledge of the properties of a vertical parabola.

In Fig. 472, points 1, 2, 3, and 4 are taken at random on the parabola in order that the problem may be perfectly general. The lines v_1, v_2, v_3, v_4, and v_5 are vertical offsets from the tangent through P.V.C., and x_1, x_2, x_3, x_4, and x_5 are the corresponding abscissas measured from the point P.V.C. as an origin.

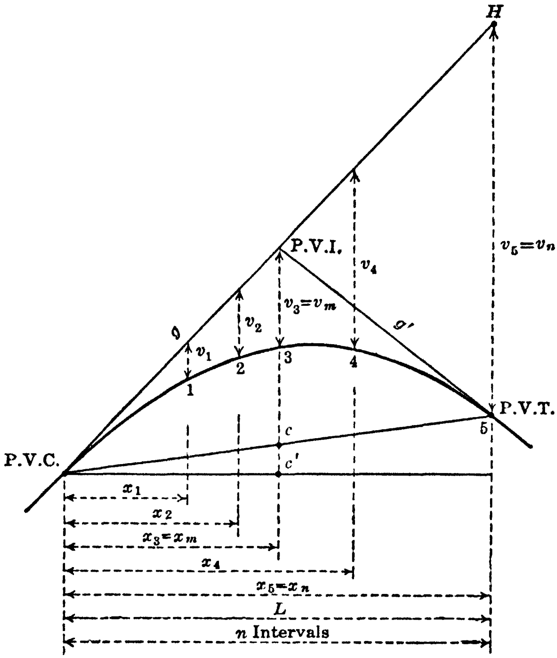


FIG. 472.

From the equation of the parabola may be derived the general equation for determining the value of a tangent offset v, to any point on the parabola:

$$v = ax^2 \tag{I}$$

In this equation a is a constant to be determined for any given parabola: as explained later. The equations for the vertical offsets in the figure are,

$$\begin{gathered} v_1 = ax_1^2; \quad v_2 = ax_2^2; \quad v_3 = ax_3^2 \quad \text{or} \quad v_m = ax_m^2; \\ v_4 = ax_4^2; \quad v_5 = ax_5^2 \quad \text{or} \quad v_n = ax_n^2 \end{gathered} \tag{II}$$

The following relations are among those that may be derived from the equations in (II):

$$\frac{v_1}{v_2} = \frac{x_1^2}{x_2^2}; \quad \frac{v_2}{v_3} = \frac{x_2^2}{x_3^2}; \quad \frac{v_3}{v_4} = \frac{x_3^2}{x_4^2}; \quad \frac{v_4}{v_5} = \frac{x_4^2}{x_5^2} \tag{III}$$

$$\frac{v_1}{v_3} = \frac{x_1^2}{x_3^2} \quad \text{or} \quad \frac{v_1}{v_m} = \frac{x_1^2}{x_m^2}; \quad \frac{v_1}{v_5} = \frac{x^2}{x_5^2} \quad \text{or} \quad \frac{v_1}{v_n} = \frac{x_1^2}{x_n^2} \tag{IV}$$

472(*a*). *Vertical offsets from a tangent to any two points on a vertical parabola are to each other as the squares of the corresponding horizontal distances from the point of tangency to the vertical offsets.*

This holds true regardless of which of the two tangents is used and regardless of whether the tangents are equal or unequal.

From equations in (II):

$$a = \frac{v_m}{x_m^2} = \frac{v_n}{x_n^2} \tag{V}$$

$$a = \frac{v_m}{(\frac{1}{2}L)^2} = \frac{v_n}{L^2} \tag{VI}$$

The value of v_m, the offset from P.V.I., or v_n, the offset to P.V.T., can easily be determined independently of the general equation $v = ax^2$; hence the value of a may be calculated from one of the expressions in (V) and (VI).

From similar triangles in Fig. 472, the vertical distance from P.V.I. to c, the midpoint of the long chord, is one-half the vertical offset v_n. From (VI), the offset v_m is one-fourth of v_n. Hence v_m = one-half the distance from P.V.I. to c.

472(*b*). *The vertical parabola cuts the vertical through V.P.I. halfway between the point P.V.I. and the midpoint of the long chord.*

The simple fundamental equation $v = ax^2$ represents the only property of the parabola that is absolutely needed in computing the elevations of points on a vertical parabola. However, other principles, properties, and methods, based on this fundamental property and given in the remainder of this article, are useful in simplifying and checking computations.

472(*c*). *When two points on the parabola are on opposite sides of and equidistant from the vertical through V.P.I. the vertical offset to one point from the nearer tangent is equal to the vertical offset to the other point from the other tangent.*

472(*d*). *When points on a parabola are equal distances apart, measured horizontally, the second differences between the elevations of consecutive points are equal.* This principle is used mainly in checking computations (Art. 475(*b*)).

472(*e*). *When points on a parabola are equal distances apart, measured horizontally, the elevation of each point differs from the elevation of the preceding point by an amount equal to twice the constant a in the equation* $v = ax^2$.

472(*f*). Value of a in the Equation $v = ax^2$. In Fig. 472 let n represent the number of station intervals in the length L of the

parabola. Let g and g' represent, respectively, the rise or fall *per station interval* of the tangents from P.V.C. to P.V.I. and from P.V.I. to P.V.T. in Fig. 472. The plus sign is generally given to an ascending grade, and a minus sign to a descending grade (in Fig. 472, $+g$ and $-g'$). The tangent produced to H rises $\frac{1}{2}Lg = \frac{1}{2}ng$ in the distance from P.V.I. to H; the tangent from P.V.I. to P.V.T. falls $\frac{1}{2}ng'$. The *algebraic* difference of these two vertical distances is equal to the vertical offset to P.V.T., or $v_n = \frac{1}{2}n(g - g')$. Obviously this equation holds true only when the proper algebraic sign is given to both g and g', or, in this case, $v_n = \frac{1}{2}n(g-(-g)) = \frac{1}{2}n(g+g)$. From (VI), Art. 472(*a*), $a = \frac{v_n}{L^2}$; hence $a = \frac{v_n}{n^2}$. Substituting in this equation the general value of $v_n = \frac{1}{2}n(g - g')$,

$$a = \frac{g - g'}{2n} \qquad \text{(VII)}$$

This is a general expression for the value of a in the fundamental equation $v = ax^2$, *provided the horizontal projections of the two tangents are equal.* If proper algebraic signs are given to g and g', the equation holds true for each of the combination of grades in Art. 470.

472(*g*). Caution. It should always be kept in mind that each of the symbols g and g' represents the rise or fall *per station interval.* When this interval is 100 ft., each symbol represents the gradient as usually expressed, i.e., feet per 100 ft., or some percentage, as 3 per cent or 0.03. But if the station interval is less than 100 ft., the gradient must be reduced accordingly. For example, a gradient of 0.08 as usually expressed becomes 0.04 for a 50-ft. station interval, and 0.02 for a 25-ft. station interval. For a given length L of curve, the value of n for 50-ft. station intervals and for 25-ft. intervals, is, respectively, twice as great and four times as great as it is for 100-ft. intervals.

472(*h*). Number of station intervals. From equation VII

$$n = \frac{g - g'}{2a} \qquad \text{(VIII)}$$

The proper algebraic signs should be given to g and g' (Art. 472(*f*)).

472(*i*). The sharpness of a vertical curve depends upon the angle of intersection of the tangents and upon the length of curve. In formula VIII, $g - g'$ is a linear measurement of the angle and n a linear measurement of the length of curve; hence the sharpness of the curve may be judged by the value of a. In fact, when stations are spaced at regular intervals:

The gradient of a chord between any two successive stations on a parabola differs from the gradient of the preceding chord between stations by an amount equal to 2a. (See Art. 472(*e*).)

472(*j*). RATE OF CHANGE is defined as:

$$r = \frac{g - g'}{L} = \frac{g - g'}{n} \tag{IX}$$

Note that this is equal to $2a$. Proper algebraic signs should be given to g and g' (Art. 472(*f*)).

472(*k*). *The tangent offset from the point of intersection P.V.I. to the midstation on the parabola is equal to one-fourth vertical offset to the point P.V.T.* Let v_m represent this middle vertical offset. Substituting in $v_n = an^2$ the value of a from $a = \frac{g - g'}{2n}$, the value of v_n becomes $v_n = \frac{n(g - g')}{2}$. Since v_m is one-fourth of v_n,

$$v_m = \frac{n(g - g')}{8} \tag{X}$$

Note that this offset corresponds to the *external distance* in a horizontal curve.

473. To Compute the Value of *a* in $v = ax^2$. There are three methods.

1. Compute the value of a from $a = \frac{g - g'}{2n}$. (See VII, Art. 472(*f*).) Proper algebraic signs must be given to g and g'.

2. From the elevation of P.V.I. and the gradient of the tangent through P.V.C., compute the elevation of the point in which that tangent intersects the vertical through P.V.T. (point E in Fig. 470). From the elevation of P.V.I. and the gradient of the tangent through P.V.T., compute the elevation of P.V.T. The difference between the two elevations is v_n, the vertical offset to P.V.T. From (VI), Art. 472(*a*), $a = \frac{v_n}{L^2}$. If x in the fundamental equation $v = ax^2$ is expressed in terms of n, then $L = n$; if x is in feet, then L is in feet.

3. Compute the elevation of P.V.T. as in (2). Compute the elevation of the midpoint of the long chord, i.e., elevation of C in Fig. 470 (equal to the mean of the elevations of P.V.C. and P.V.T.). Compute the difference in elevation between the midpoint and P.V.I., i.e., between C and B in Fig. 470. (One-half this difference is equal to the tangent offset v_m at P.V.I. (Art. 472(*b*)).

From (VI), Art. 472(a), $a = \frac{v_m}{\frac{1}{2}L^2}$. The length L is expressed in feet or it is equal to n, according to whether x in $v = ax^2$ is expressed in feet or in terms of n. The value of v_m may be checked from $v_m = \frac{n(g - g')}{8}$. (See (X), Art. 472(k).)

One of these three methods can be used to check the value of a obtained by one of the other methods. Note that the method in (2) is the same in principle as that in (1).

474. Computation of Tangent Offsets. The computation of tangent offsets is involved directly or indirectly in all the methods generally used in determining the elevations of points on a vertical parabola.

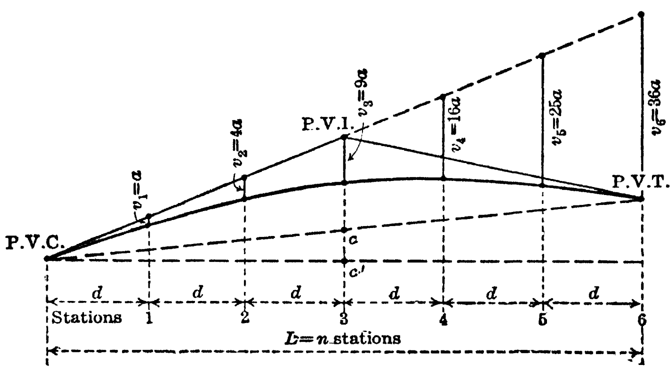


FIG. 474.

Let one half of the parabola in Fig. 474 be divided into any number of station intervals, say three, and let the other half be divided into the same number. The tangent offset at any station on the curve may be calculated from $v = ax^2$, in which x is expressed in terms of the number of station intervals and v in feet. Taking P.V.C. as the origin, the values of the offsets are:

$$v_1 = a(1)^2; \quad v_2 = a(2)^2; \quad v_3 = a(3)^2$$

$$v_4 = a(4)^2; \quad v_5 = a(5)^2; \quad v_6 = v_n = a(6)^2 = a(n)^2$$

$$v_1 = a; \quad v_2 = 4a; \quad v_3 = 9a; \quad v_4 = 16a; \quad v_5 = 25a; \quad v_6 = 36a$$

The value of the constant a is calculated by one of the methods explained in the preceding article. The station interval may be any distance d; it is usually 100 ft., or 50 ft., or 25 ft. On short parabolas, such as the surface line of the cross section of a pavement, the interval may be as short as 1 ft.

474(*a*). THE VALUE OF x FOR ANY INTERMEDIATE STATION involves a fractional or decimal part of a station interval. For example, if an intermediate station is halfway between 2 and 3, the value of x is 2.5; if it is ¼*d* from 2, $x = 2.25$.

475. Computation of Elevations—Method of Offsets. The general method is: (1) Determine the elevations of all points on the tangent through P.V.C. that are directly above or below stations on the curve. (2) Compute the tangent offsets for all stations (Art. 474). (3) From the results obtained in (1) and (2) compute the elevations of all stations. If the curve is at a summit the offsets are subtracted from the corresponding tangent elevations; if the curve is in a sag, offsets are added. This method is given more in detail in the following example.

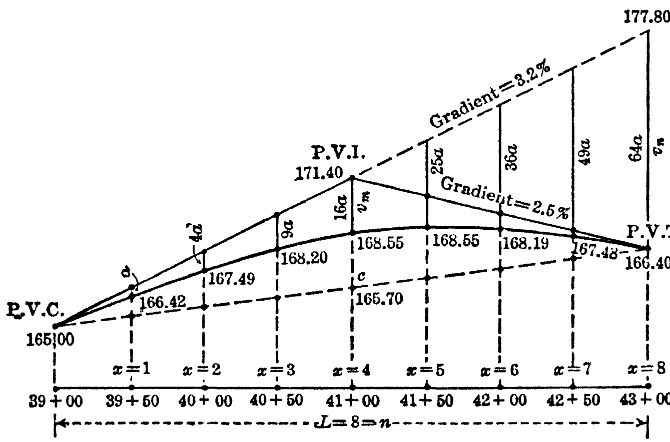


FIG. 475(*a*).

475(*a*). ILLUSTRATIVE EXAMPLE. In Fig. 475(*a*) a grade line ascending at the rate of 3.2 ft. per 100 ft. meets a grade line that is descending at the rate of 2.5 ft. per 100 ft. The point of intersection P.V.I. is at station 41 + 00. The vertical parabola chosen is 400 ft. long, measured horizontally. It is required to compute the elevations of stations on the curve at 50-ft. intervals.

Note. In this illustrative example, the point P.V.T. is taken at a regular 100-ft. station; consequently the points P.V.C. and P.V.T. are at regular 100-ft. stations. If, for any reason, P.V.T. is taken at a plus station, all the stations on the curve established by the general method will be plus stations, but regular 100-ft. stations on the curve can easily be established, if desired.

The point P.V.C. will be at station 39 + 00, and the point P.V.T. at station 43 + 00. The elevation of P.V.C. is known to be 165.00 ft.

1. Compute the elevations of all points on the tangent through P.V.C. that are directly above the stations on the curve. The elevation of each point is obtained by adding *algebraically* to the preceding station the change in elevation per station interval, which in this case is 3.2 ft. per 100 ft., or 1.6 ft. per station interval of 50 ft. (See column 2 in the accompanying tabulation.)

2. Starting with the elevation of P.V.I., compute the elevation of P.V.T., using the gradient of the tangent through P.V.T., which in this case is −2.5 ft. per 100 ft. = −1.25 ft. per station interval.

$$\text{Elevation of P.V.T.} = 171.40 - (1.25 \times 4) = 166.40$$

3. Compute the elevation of the middle point of the long chord, i.e., the elevation of $c = \frac{1}{2}(165.00 + 166.40) = 165.70$.
4. Compute the elevation of the midstation on the curve, i.e., station 41 + 00, halfway between P.V.I. and c (Art. 472(*b*)).

$$\text{Elevation of station } 41 + 00 = \tfrac{1}{2}(171.40 + 165.70) = 168.55$$

5. Calculate the value of a in ax^2 by one of the methods of Art. 473.

$$v_n = v_8 = 177.80 - 166.40 = 11.40 = a(8)^2, \quad \text{and} \quad a = 0.1781$$

$$(\text{Check})\ a = \frac{g - g'}{2n} = \frac{1.60 - (-1.25)}{2 \times 8} = \frac{1.60 + 1.25}{16} = 0.1781$$

Note. Unless the value of a is to be obtained by the third method explained in Art. 473, steps 2 and 3 may be omitted. However, they afford a quick method of obtaining an independent value for the elevation of station 41 + 00 that later can be used as a check. The value of a may be obtained by the third method as follows. Elevation of P.V.I. − elevation of midstation = tangent offset from P.V.I. = v_m.

$$v_m = 171.40 - 168.55 = 2.85$$

$$v_m = \frac{n(g - g')}{8} = 1.60 - (-1.25) = 2.85 \text{ (Check)} \quad \text{(Art. 472}(k)\text{.)}$$

$$v_m = 2.85 = a \times (4)^2, \quad \text{and} \quad a = 0.1781$$

6. Calculate the offsets as indicated in column 3. Each offset is given a minus sign since it is to be subtracted from the elevation of the corresponding point on the tangent.
7. Calculate the elevations of stations on the parabola by subtracting from the elevations in column 2 the offsets in column 3. Results are in column 4.

Station (1)	*Elevation on Tangent* (2)	*Offset* = ax^2 (3)		*Elevation on Curve* (4)	*Differences in Elevations* (5) *First*	*Second*
P.V.C. 39 + 00	165.00			165.00		
	1.60	$0.178 \times (1)^2$			1.42	
39 + 50	166.60	−0.178	=	166.42		0.35
	1.60	$0.178 \times (2)^2$			1.07	
40 + 00	168.20	−0.712	=	167.49		0.36
	1.60	$0.178 \times (3)^2$			0.71	
40 + 50	169.80	−1.602	=	168.20		0.36
	1.60	$0.178 \times (4)^2$			0.35	
P.V.I. 41 + 00	171.40	−2.848	=	168.55		0.35
	1.60	$0.178 \times (5)^2$			0.00	
41 + 50	173.00	−4.450	=	168.55		0.36
	1.60	$0.178 \times (6)^2$			0.36	
42 + 00	174.60	−6.408	=	168.19		0.35
	1.60	$0.178 \times (7)^2$			0.71	
42 + 50	176.20	−8.722	=	167.48		0.36
	1.60	$0.178 \times (8)^2$			1.07	
P.V.T. 43 + 00	177.80	−11.392	=	166.41		

475(*b*). CHECKS. The computations may be checked by second differences as shown in column 5 of the tabulation. See also Art. 545(*m*). The slight variations between the second differences would disappear if computations were carried to more decimal places. Note, as another partial check, that the

elevations of P.V.T. and station 41 + 00 in the tabulation are the same or nearly the same as those computed in steps 2 and 4.

Remark. The computations and the resulting elevations in the tabulation indicate a greater degree of precision than is often required for the field work. (Values for elevations have purposely been carried to two decimal places to illustrate the method of checking by second differences.)

475(*c*). MODIFIED METHOD OF OFFSETS. In this method the elevations of stations from P.V.C. to the middle station are computed by vertical offsets from the tangent through P.V.C. as already explained, but the elevations of stations on the other half of the curve are computed by vertical offsets from the tangent through P.V.T. These latter offsets are not computed, but, instead, the offset to each station is taken as equal to the offset to the corresponding station on the other half of the curve (from the principle stated in Art. 472(*c*)). However, the elevations of offset points on the tangent through P.V.T. must be determined. This is done by starting from the elevation of P.V.I., previously calculated, and using the gradient of the tangent through P.V.T. From the elevations of the offset points and from the vertical offsets the elevations of stations on the corresponding half of the curve are computed.

475(*d*). ILLUSTRATIVE EXAMPLE. The use of the modified method in computing the elevations of stations on the right-hand half of the curve in Fig. 475(*a*) is shown in the accompanying tabulation. Note that in calculating the elevations of offset points on the tangent through P.V.T. (second column), a gradient of 2½ per cent, or 1.25 ft. in 100 ft., was used. Note also that each vertical offset in the third column is the same as that calculated in the preceding illustrative example for the corresponding station on the other half of the curve.

Station	*Elevations on Tangent*	*Offset*	*Elevations on Curve*
41 + 00	171.40		168.55
	− 1.25		
		(Sta. 40 + 50)	
41 + 50	170.15	−1.602	168.55
	− 1.25		
		(Sta. 40 + 00)	
42 + 00	168.90	−0.712	168.19
	− 1.25		
		(Sta. 39 + 50)	
42 + 50	167.65	−0.178	167.47

475(*e*). ALTERNATIVE METHOD OF COMPUTING OFFSETS. Let v_m represent the tangent offset from P.V.I. to the curve, and x_m the corresponding horizontal distance from P.V.C. Let v and x represent, respectively, the *tangent offset* and the horizontal distance from P.V.C. to any other point on the parabola. From Art. 472(*a*):

$$\frac{v}{v_m} = \frac{x^2}{x_m{}^2} \quad \text{or} \quad v = v_m \frac{x^2}{x_m{}^2}$$

If x_m is taken as unity and x is taken as a *fraction of* x_m,

$$v = v_m x^2$$

This method of computing tangent offsets may be illustrated by applying it to the vertical parabola in Fig. 475(a). (1) Calculate v_m, the tangent offset from P.V.I. (Equal to one-half the distance from P.V.I. to c, the midpoint of the long chord from P.V.C. to P.V.T. (Art. 472(b).) $v_m = 2.85$ ft. (2) Substituting this value, the equation becomes

$v = 2.85x^2$ in which x is a fraction of x_m when x_m is unity

x_1 for station 39 + 50 = ¼; x_2 for station 40 = ½; x_3 for station 40 + 50 = ¾

The tangent offsets are:

Station 39 + 50 = $v_1 = 2.85 \times (1/4)^2 = 2.85 \times 1/16 = 0.178$ ft.

Station 40 + 00 = $v_2 = 2.85 \times (1/2)^2 = 2.85 \times 1/4 = 0.712$ ft.

Station 40 + 50 = $v_3 = 2.85 \times (3/4)^2 = 2.85 \times 9/16 = 1.602$ ft.

These values agree with those obtained in Art. 475(a). Note that this method can be applied to the right-hand portion of the curve, by using P.V.T. as the point from which to measure x distances. Many surveyors prefer this simple method of computing offsets.

476. Computation of Elevations—Method of Chord Gradients. This method consists (1) in computing the differences in elevation between successive stations on the curve and then (2) in computing by means of these differences the elevation of each station from the elevation of the preceding station. The difference in elevation between two successive stations divided by the horizontal distance between stations is equal to the gradient of the chord between the two stations; hence this method is known as the *method of chord gradients.*

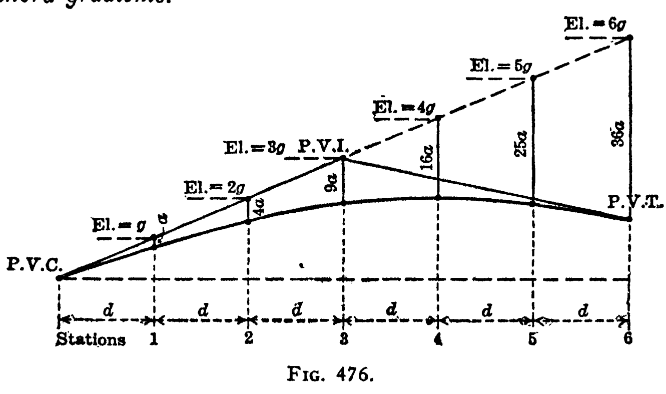


FIG. 476.

If in Fig. 476, the elevation of P.V.C is assumed to be zero, the elevations of points on the tangent through P.V.C. for stations 1, 2, 3, . . . 6 are g, $2g$, $3g$ . . . $6g$, in which g is the distance the tangent rises per station interval d. The elevations of the stations on the parabola are $g - a$, $2g - 4a$, $3g - 9a$ . . . $6g - 36a$. The

differences in elevation between successive stations are $(2g - 4a) - (g - a)$, $(3g - 9a) - (2g - 4a)$, and so on, as shown in the tabulation. (The constant a in $v = ax^2$ must be computed for any given parabola.) The numbers 1, 2, 3, etc., represent the first, second, third, etc., stations on *any* vertical parabola beginning with the first station on the curve beyond P.V.C.

Station P.V.C. (1)	*Elevation* (2)	*Between Stations* (3)	*Difference in Elevation* (4)
		.... P.V.C. and 1	$(g - a)$
1	$g - a$		
		.... 1 and 2	$g - 3a$
2	$2g - 4a$		
		.... 2 and 3	$g - 5a$
3	$3g - 9a$		
		.... 3 and 4	$g - 7a$
4	$4g - 16a$		
		.... 4 and 5	$g - 9a$
5	$5g - 25a$		
		.... 5 and 6	$g - 11a$
6	$6g - 36a$		
		.... 6 and 7	$g - 13a$
7	$7g - 49a$		
		.... 7 and 8	$g - 15a$
8	$8g - 64a$		
		..............$(N - 1)$ and N	$g - (2N - 1)a$
Nth	$ng - n^2a$		

Strict attention should be paid to algebraic signs of g (Art. 472(*f*)). *When the station interval is less than 100 ft. the value of g should be expressed in terms of that distance.* (See *Caution*, Art. 472(*g*).) *Note that the successive differences in elevation change by an amount equal to 2a.* (See Art. 472(*e*).)

476(*a*). Method of chord gradients illustrated. The given data are the same as in Art. 475(*a*), i.e., the gradients are +0.032, or 1.60 ft. per 50-ft. station interval, and −0.025, or −1.25 ft. per 50 ft. The elevation of station 39 + 00, the P.V.C , is 165.00 ft. The length of the parabola is 400 ft.

Figure 475(*a*) will be helpful in studying the following procedure.

1. Compute the elevation of P.V.I. = 165.000 + 4 × 1.60 = 171.40 ft.
2. Compute the elevation of station 43 + 00, the P.V.T., = 166.40 ft. (See step 2, Art. 475(*a*))
3. Compute the elevation of the midpoint of the long chord = 165.70 ft. (See step 3, Art. 475(*a*).)
4. Compute the elevation of the midstation on the curve, i.e., station 41 + 00 = 168.55 ft. (See step 4, Art. 475(*a*).)
5. Compute the value of a in ax^2. Use the tangent offset v_m at station 41 + 00 (elevation of P.V.I. minus elevation of the midstation) = 171.40 − 168.55 = 2.85 ft.

$$a = v_m \div \tfrac{1}{2}L^2 = 2.85 \div (4)^2 = 0.1781 \quad \text{(See Art. 473,3.)}$$

$$\text{Check.} \quad a = \frac{g - g'}{2n} = \frac{1.60 - (-1.25)}{2 \times 8} = \frac{1.60 + 1.25}{16} = 0.1781.$$

6. Constant difference in successive gradients = $2a$ = 0.3562 (Art. 472(*e*)).
7. Compute the gradients between successive stations. The gradient from the P.V.C. to the first station on the curve, i.e., from station 39 + 00 to station 39 + 50, is $g - a$ = 1.600 − 0.178 = 1.422. Each of the other gradients may be obtained by subtracting *algebraically* from the preceding gradient the quantity $2a$ = 0.356. (See column 2.)

8. Compute the elevation of each station by adding *algebraically* to the elevation of the preceding station the difference in elevation (equal to the gradient) between the two stations. (See column 3.)

Station (1)	*Gradient* (2)	*Elevation* (3)
39 + 00	$g = 1.600$	165.000
	$a = 0.178$	+ 1.422
39 + 50	$g - a = +1.422$	166.422
	$2a = 0.356$	+ 1.066
40 + 00	$g - 3a = +1.066$	167.488
	$2a = 0.356$	+ 0.710
40 + 50	$g - 5a = +0.710$	168.198
	$2a = 0.356$	+ 0.354
41 + 00	$g - 7a = +0.354$	168.552
	$2a = 0.356$	− 0.002
41 + 50	$g - 9a = -0.002$	168.550
	$2a = 0.356$	− 0.354
42 + 00	$g - 11a = -0.354$	168.196
	$2a = 0.356$	− 0.710
42 + 50	$g - 13a = -0.710$	167.486
	$2a = 0.356$	− 1.066
43 + 00	$g - 15a = -1.066$	166.420

Checks. From step 4, station 41 + 00 = 168.55. From step 2, station 43 + 00 = 166.40. All elevations may be checked by second differences. (See Art. 475(*b*).)

476(*b*). SUGGESTIONS. (1) In the method of gradients, errors due to any error in the value of a are cumulative; consequently the value of a should usually be carried out to enough decimal places to secure the desired degree of precision. (2) In computing gradients and elevations it is important to keep in mind whether, in proceeding from one station to the next, the curve is ascending or descending. This is the surest way of avoiding mistakes in algebraic signs and in corresponding values of elevations.

476(*c*). EXERCISE. Assume the same data as those given in Art. 475(*a*) and in the example just given, except that $g = -1.60$ ft. per 50 ft., and $g' = +1.25$ ft. per 50 ft., thus making the curve one with a sag, as if Fig. 475(*a*) were turned bottom side up. Calculate the elevations for all stations on the curve, first by the method of tangent offsets and then by the method of chord gradients. Tabulate the results and compare with the tabulations in Arts. 475(*a*) and 476(*a*).

477. Comparison of Methods. The method of *gradients* involves somewhat less arithmetical work than the method of tangent offsets. Moreover, when the elevation of P.V.T. computed by gradients is found to be the same as its elevation computed directly from the elevation of P.V.I., it indicates that the elevations of all the points on the curve have been determined correctly, a useful check. A disadvantage of the method is that any error in the value of the quantity a results in cumulative errors in the computed elevations. The method is quite generally used in railroad work where the station interval is usually 100 ft. The method of tangent offsets is preferred by many engineers when the station

interval is less than 100 ft., as is frequently true on highways and on short parabolas which form the surface crowns of pavements.

The example in Art. 479(*e*), in which the computations for the tangent-offset method are placed side by side with those for the gradient method, affords an opportunity for the comparison of the two methods.

478. Proper Length of Vertical Curve.* The main purpose of a vertical curve is to provide a safe as well as an easy transition from one grade to another. The sharpness of curvature depends upon the gradients of the tangents and upon the length of the curve; it may be measured by the rate of change:

$$r = \frac{g - g'}{L} = \frac{g - g'}{n} = 2a \quad \text{(See (IX), Art. 472}(j)\text{.)}$$

478(*a*). RAILROADS. It is recommended in the *American Railway Engineering Association Manual* of 1937: "For high-speed, main tracks the rate of change (in gradient) should not be more than 0.054 foot per station (of 100 feet) in sags and not more than 0.10 foot per station on summits. For secondary main tracks, the rates of change may be twice those for high-speed main tracks." Gradients may be adjusted slightly, but the rate of grade depends mainly on the length of curve. This length n, once the gradients g and g' are fixed, may be computed by substituting in the formula just given, the values of g and g' and the permissible value of r; the equation can then be solved for n.

478(*b*). HIGHWAYS. The length of curve at the summit of a highway is governed largely by the sight distance S at which two vehicles, moving in opposite directions at maximum permissible speed, may be seen from each other in time for safe and unstartled passing. The minimum value of S for high-speed highways is about 800 ft., and for low-speed, 300 to 500. Longer distances are desirable when practicable. A common formula is:

$$L = n = \frac{S^2(g - g')}{8h} \qquad \text{(XI)}$$

The quantity $g - g'$ is the *algebraic* difference between the gradients of the intersecting tangents expressed in per cent. (See Art. 472(*f*).) The symbol h represents the height of the driver's eye above the roadway.

Curves in sags may be much shorter than those at summits. The amount of earthwork involved is often a factor in determining the length of curve, either at a summit or at a sag. In the design of the better class of highways, the length of vertical curves in sags may be determined from the relationship between headlight-sight distance and minimum safe-stopping distance.† The former, on sag vertical curves, can be calculated.‡ It is assumed that the headlights are 2.5 ft. above the pavement and that the beam diverges upward from the longitudinal car axis at an angle of 1°. Two cases follow:

$$\text{Case 1.}\quad \text{When } S < L, \quad L = \frac{AS^2}{5 + 0.035S}$$

$$\text{Case 2.}\quad \text{When } S > L, \quad L = 2S - \frac{5 + 0.035S}{A}$$

S = headlight-sight distance in feet; L = length of vertical curve in feet; A = algebraic difference in grades.

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* The design of vertical curves is not within the scope of this book. A more complete treatment will be found in treatises on railroad and on highway engineering.

† See "Design Features—Pennsylvania Turnpike," by Charles M. Noble, *Civil Engineering,* July 1940, p. 437. Published by American Society of Civil Engineers.

‡ See "Sight Distance on Sag Vertical Curves," by Donald Thompson, *Civil Engineering,* January 1944, p. 22.

It is assumed that the headlight distance S should be equal to the safe-stopping distance. The latter, depending upon the speed in miles per hour for which the curve is designed, may be assumed to vary as follows: 600 ft. for 70 m.p.h.; 475 ft. for 60 m.p.h.; 350 ft. for 50 m.p.h.; 275 ft. for 40 m.p.h.; 200 ft. for 30 m.p.h. The lengths of sag-vertical curves determined from either of the two formulas will be considerably greater than the lengths of curves designed only from the standpoint of comfortable riding.

479. Special Problems. (*a*). PLOTTING VERTICAL CURVES. Some of the special problems in vertical curves can be solved graphically by plotting the curve on profile paper, such as Plate A profile paper. The vertical scale is usually much greater than the horizontal scale—large enough to permit vertical distances to be scaled to, say, one-tenth of a foot, or to whatever may be the permissible error in setting grade stakes. The graphic method is particularly useful in determining elevations of points on short irregular vertical curves, but the limitations of the method are obvious.

479(*b*). *To determine the lowest or highest point of a vertical parabola.* At either point the grade is level, that is $g - rx = 0$, in which r, the rate of change, is equal to $(g - g') \div n$ ((IX), Art. 472(*j*)). Substituting this value for r:

$$x = \frac{ng}{g - g'} \qquad \text{(XII)}$$

in which x is the distance, measured in station intervals, from the beginning of the curve to the lowest or highest point.

479(*c*). *Given the two gradients g and g' and the rate of change r to find n, the number of stations on the curve.*

$$n = \frac{g - g'}{r} \quad \text{(From (IX), Art. 472}(j))$$

479(*d*). *To make the parabola connecting two given grades pass through a point at a given distance below the point of intersection P.V.I. at a summit, or above P.V.I. at a sag.*

From (X), Art. 472(*k*), the tangent offset at P.V.I. is $v_m = n(g - g') \div 8$. Substitute for v_m the amount the curve must pass above or below P.V.I., and solve for n.

479(*e*). ILLUSTRATION. The elevation of P.V.C. of a curve is 98.6 ft. Beginning at P.V.C. a grade descending at the rate of 2.0 ft. per 100 ft. is to be connected by a vertical parabola with a grade ascending at the rate of 3.6 ft. per 100 ft. The curve must pass through a point 1.4 ft. above P.V.I. Stakes must be set at 25-ft. intervals. $g = -2$ ft. per 100 ft. $= -0.5$ ft. per 25 ft. $g' = +3.6$ ft. per 100 ft. $= +0.9$ ft. per 25 ft.

By the conditions $v_m = 1.4$; hence $1.4 = \frac{n(-0.5 - 0.9)}{8}$, and $n = 8$. Hence from (VII), (Art. 472(*f*)), $a = \frac{-0.5 - 0.9}{16} = -0.0875$. In the following tabulations are shown the computations of the elevations of stations on the curve at 25-ft. intervals

first by the method of tangent offsets (Art. 475), and then by the method of gradients (Art. 476). All elevations are given in feet.

By Tangents Offsets				By Gradients		
Station	*Elevations on Tangent*	$y = ax^2$	*Corrected Elevation*	*Line*	*Gradient*	*Elevation*
P.V.C.	98.6	0.0	98.6	− to P.V.C.	−0.5	98.6
	− 0.5				−0.0875	− 0.4125
1	98.1 −	−0.0875 =	98.1875	P.V.C. to 1	−0.4125	98.1875
	− 0.5				−0.1750	− 0.2375
2	97.6 −	−0.35 =	97.95	1 to 2	−0.2375	97.9500
	− 0.5				−0.1750	− 0.0625
3	97.1 −	−0.7875 =	97.8875	2 to 3	−0.0625	97.8875
	− 0.5				−0.1750	+ 0.1125
4	96.6 −	−1.4 =	98.0	3 to 4	+0.1125	98.0000
	− 0.5				−0.1750	+ 0.2875
5	96.1 −	−2.1875 =	98.2875	4 to 5	+0.2875	98.2875
	− 0.5				−0.1750	+ 0.4625
6	95.6 −	−3.15 =	98.75	5 to 6	+0.4625	98.7500
	− 0.5				−0.1750	+ 0.6375
7	95.1 −	−4.2875 =	99.3875	6 to 7	+0.6375	99.3875
	− 0.5				−0.1750	+ 0.8125
P.V.T.	94.6 −	−5.6 =	100.2	7 to 8	+0.8125	100.2000

Check. Elevation of P.V.I. = 98.6 + 4(−0.5) = 96.6. Elevation of P.V.T. = 96.6 + 4 × 0.9 = 100.2 ft. (check). Elevation of midpoint on the long chord is ½(98.6 + 100.2) = 99.4 ft. Station 4 is 1.4 ft. above P.V.I.; hence, from Art. 472(*b*), it must be 1.4 ft. below the midpoint. The elevation of station 4 is, therefore, 99.4 − 1.4 = 98.0 ft. (check). (See also the check on all the elevations by second differences as computed in Art. 545(*m*).

479(*f*). Vertical arcs of circles. When the vertical curve is the arc of a circle, elevations of points on the curve may be easily computed from geometric and trigonometric principles pertaining to the circle. For example, *vertical* offsets from the long chord between P.V.C. and P.V.T. can be computed by methods similar to those explained in Art. 494 for horizontal offsets.

CHAPTER 33

HORIZONTAL CURVES

In this chapter are given the methods of computing the angles and distances used in laying out horizontal curves of various types and the corresponding methods of laying out these curves in the field. There is also given a brief discussion of the most important elements to be considered in the design of horizontal curves.

480. Introduction. *Horizontal Curves.* It is often required in surveying to lay out a horizontal curve. This is particularly true in route surveys, such as those for highways, railroads, and pipe lines, in which the alignment consists of straight lines, called *tangents,* connected by horizontal curves. These curves are computed and laid out as if they were in a horizontal plane even when they are on grades. The main portion of a horizontal curve is generally an arc of a circle; on highway and railroads, spiral easement or transition curves are quite generally used at the two ends of a circular curve.

480(*a*). NOTE. A *simple curve* is the arc of a circle which connects two tangents. A *compound curve* is a combination of two or more simple curves in the same general direction of curvature, with a common tangent at every point at which one curve ends and another begins. A *reversed curve* is a combination of two simple curves that turn in opposite directions and have a common tangent at the point of junction.

480(*b*). LOCATING CURVES. When the radius of a simple, horizontal curve is small, as, for example, 100 ft., and ground conditions permit, points on the curve may be located by using a tape as a swinging radius with one end held at the center of the circle. For curves of large radius, however, other methods, to be explained later, must be employed.

481. Degree of Curve. The rate of curvature of a simple curve may be expressed by stating the length of radius, usually in multiples of 100 ft. A more common method is to state the *degree of curvature.* Two quite different definitions of degree of curve are in general use, one based on a *chord* of the curve, the other based on an *arc* of the curve. The engineer must be careful not to use one definition when he should use the other, and to keep in mind the one on which his computations and field work are based.

The *degree of curve, chord basis,* is expressed as the angle at the center subtended by a *chord* of the curve 100 ft. in length.

The *degree of curve, arc basis,* is expressed as the angle at the center subtended by an *arc* of the curve 100 ft. in length.

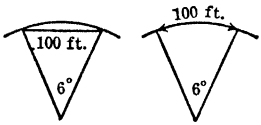


FIG. 481.

Illustrations. If a curve is so computed and laid out that the angle at the center subtended by a 100-ft. *chord* is 6°, the curve on the chord basis is a 6° curve; if the angle at the center subtended by a 100-ft. arc is 6°, the curve on the arc basis is a 6° curve (Fig. 481).

In flat curves there is little difference between the length of a 100-ft. arc and its chord and therefore little difference in the results obtained by the use of the two definitions. In sharp curves, however, the difference is considerable. The chord definition is the oldest; it is generally used in railroad engineering as the best suited to the accurate alignment of the flat curves of a railroad. The arc definition is quite generally used in highway engineering as best suited to the alignment of the curves of a highway, many of which are relatively sharp.

Note. Degree of curve is sometimes expressed as the angle at the center subtended by an arc of the curve 100.007 ft. in length. Based on this definition, the radius of a 1° curve is 5730 ft.; of a 2° curve it is 5730 ÷ 2 = 2865 ft.; of a 3° curve it is 5730 ÷ 3 = 1910 ft., and so on.

In selecting a circular curve to connect two tangents, it simplifies computations and the use of tables to select one in which the degree of curve is a *whole* number. Few railroad curves are less than 1° or greater than 6°. Some highway curves are as great as 20°, but the trend on the better highways is toward the flatter curves used by railroads.

482. Definitions and Notations. In Fig. 482 two tangents intersecting at point P.I. are connected by a simple, horizontal curve. Definitions and notations commonly used are:

482(*a*). *Degree of Curve* is denoted by *D*.

482(*b*). *Radius of Curve* is denoted by *R*.

482(*c*). *Point of Intersection*—P.I.—the point in which the two tangents intersect.

482(*d*). *Intersection Angle—I*—the *deflection angle* between the two tangents.

482(*e*). *Point of Curve*—P.C.—the point where the alignment changes from a tangent to a curve, i.e., the beginning of the curve.

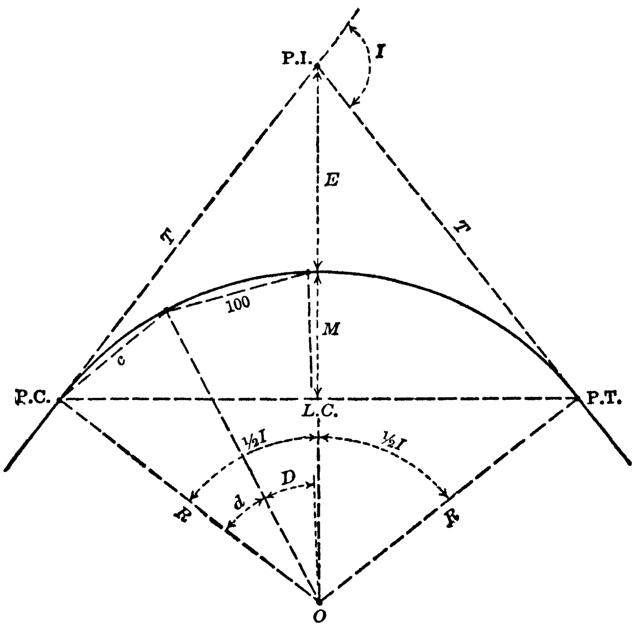
482(*f*). *Point of Tangent*—P.T.—the point where the alignment changes from a curve to a tangent, i.e., the end of the curve.

Note. Other terms in common use for the beginning and the end of a curve are: *Beginning of Curve* (B.C.) and *End of Curve* (E.C.); *Tangent to Curve* (T.C.) and *Curve to Tangent* (C.T.). The latter pair of terms is recommended by the American Railway Engineering Association. The terms denoted by P.C. and P.T. are, however, the oldest, and they have been used in railroad engineering for a great many years.

482(*g*). *Back Tangent*—the tangent that passes through P.C.

482(*h*). *Forward Tangent*—the tangent that passes through P.T.

482(*i*). *Tangent Distance*—*T*—the distance from P.C. to P.I.; also the distance from P.T. to P.I.



I=Intersection Angle. *E*=External Distance.
R= Radius. *D*=Degree of Curve.
T=Tangent Distance. *L.C.*=Length of Long Chord from P.C. to P.T.
M=Length of Middle Ordinate. c=Length of Subchord.
M_c=Length of Middle Ordinate to a Subchord.
d=The angle at the center subtended by a subchord.
L=The length of curve. *N*=The number of normal chords in *L*.

FIG. 482.

482(*j*). *External Distance*—*E*—the distance from the midpoint of the curve to P.I.

482(*k*). *Long Chord*—*L.C.*—the chord from the beginning to the end of the curve, i.e., from P.C. to P.T.

482(*l*). *Middle Ordinate*—*M*—the ordinate from the midpoint of the long chord to the midpoint of the curve.

482(*m*). *Length of Curve*—*L*—the total length of the curve from P.C. to P.T. When the *chord* definition of *degree of curve* is used, this length is the sum of the lengths of the 100-ft. chords.

plus the lengths of the shorter chords at the ends of the curve when there are such chords. When the *arc* definition is used for degree of curve, the length of curve is the actual length of the arc from P.C. to P.T.

482(*n*). *The Deflection Angle*—D.A.—to any point on the curve is the deflection angle at P.C. between the back tangent and the chord from P.C. to the point on the curve, i.e., the acute angle between the line from P.C. to P.I. and the chord from P.C. to the point on the curve.

482(*o*). *Normal Chord*—a chord between two successive regular stations on a curve. When the chord definition of degree of curve is used, the length of a normal chord is generally understood to be 100 ft. unless otherwise stated. (For shorter lengths see Art. 489(*e*).)

482(*p*). *Subchord*—*c*—any chord shorter than the normal chord; it usually extends between a regular (full) station and a substation.

482(*q*). *Middle Ordinate of a Subchord*—M_c—the ordinate from the midpoint of the subchord to the curve.

482(*r*). *The angle at the center subtended by a subchord* is denoted by *d*.

482(*s*). *The Number of Chords*—*N*—the number of *normal* chords in the length of the curve.

483. Geometric Principles used in calculating and laying out circular curves are:

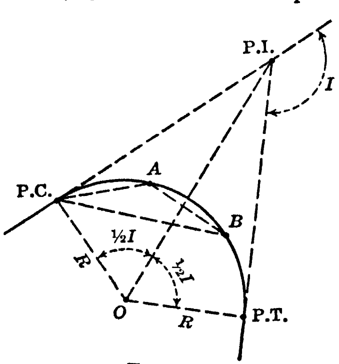


FIG. 483.

483(*a*). *Each tangent is perpendicular to the radius at the point of tangency.*

483(*b*). *The tangent distances from P.C. to P.I. and from P.T. to P.I. (Fig. 483) are equal.*

483(*c*). *The angle at the center O between the radius to P.C. and the radius to P.T. is equal to the angle of intersection I, and this angle is bisected by the line from P.I. to O.*

483(*d*). *A deflection angle at P.C. to any point on the curve is exactly equal to one-half the angle at the center O between the radius to P.C. and the radius to the point on the curve, i.e., it is measured by one-half the arc from P.C. to the point.* For example, the deflection angle at P.C. between the tangent at P.C. and the chord from P.C. to *B* is equal to one-half the angle between the radius to P.C. and the radius to *B*, i.e., it is measured by one-half the arc from P.C. to *B*.

483(e). *The deflection angle at P.C. between the tangent at P.C. and the long chord to P.T. is ½I.* This is a special case of (d).

483(f). *An inscribed angle is measured by one-half its intercepted arc.* For example, the angle at P.C. between the chords from P.C. to A and from P.C. to B is measured by one-half the arc from A to B.

Note. Intelligent work in computing and staking circular curves depends largely upon keeping these few simple principles clearly in mind.

484. Formulas—Chord Basis.* The formulas given in this article may be easily derived from simple geometric and trigonometric relations, and by reference to Fig. 482. Note from that figure that the angle at the center between the radii to P.C. and to P.T. is equal to I. All formulas except the four in brackets may be used regardless of whether the degree of curve is defined on the chord basis or on the arc basis. For other formulas on the arc basis see Art. 497.

FORMULAS FOR CIRCULAR CURVES

	Required	
I	R	$\left[R = \dfrac{50}{\sin \frac{1}{2}D}\right]$
II	R	$R = \dfrac{c}{2 \sin \frac{1}{2}d} = \dfrac{L.C.}{2 \sin \frac{1}{2}I}$
III	D	$\left[\sin \frac{1}{2}D = \dfrac{50}{R}\right]$
IV	T	$T = R \tan \frac{1}{2}I = \dfrac{L.C.}{2 \cos \frac{1}{2}I}$
V	$L.C.$	$L.C. = 2R \sin \frac{1}{2}I$
VI	E	$E = R(\sec \frac{1}{2}I - 1) = R \text{ exsec } \frac{1}{2}I$
VII	M	$M = R(1 - \cos \frac{1}{2}I) = R \text{ vers } \frac{1}{2}I$
VIII	M_c	$M_c = R(1 - \cos \frac{1}{2}d) = R \text{ vers } \frac{1}{2}d$
IX	c	$c = 2R \sin \frac{1}{2}d$
X	c	$\left[c = 100 \dfrac{d}{D}\right]$ (Approximately)
XI	d	$\sin \frac{1}{2}d = \dfrac{c}{2R}$
XII	D	$D = 100 \dfrac{I}{L}$
XIII	I	$I = \dfrac{DL}{100}$
XIV	L	$L = 100 \dfrac{I}{D}$
XV	N	$\left[N = \dfrac{I}{D}\right]$

Note. Formulas I, III, X, and XV, enclosed in brackets, are not used where D is on the arc basis. (See Art. 497.)

Note. When the normal chord is other than 100 ft. in length, formulas X, XII, XIV, and XV must be modified. (See Art. 489(f).)

(XII–XIV) These three formulas are *approximately* correct. When $I \div D$ is a whole number, i.e., when all chords are 100 ft. in length with no subchords at the ends, the formulas are exactly correct.

(XV) When $I \div D$ is not a whole number, the value of N is equal to the whole number next below the quotient.

R = Radius of curve. D = Degree of curve. I = Intersection angle.
c = Length of subchord. $L.C.$ = Length of long chord.
d = Angle at the center subtended by a subchord.
T = Tangent distance. E = External distance.
M = Length of the middle ordinate.
M_c = Length of the middle ordinate to a subchord.
L = Length of the curve. N = Number of normal chords in L.

(See Art. 482 and Fig. 482 for definitions and explanations of terms.)

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* Only formulas frequently used are given; others will be found in handbooks on railroad engineering, highway engineering, and route surveying.

484(*a*). Relation between the radius R and the degree of curve D. From formula I the value of R for a one-degree curve is 5729.65 ft. For any other degree of curve, an approximate value of R may be obtained by dividing 5729.65 by the degree of the curve. For example, if the degree of curve is 6, then $5729.65 \div 6 = 954.94$ ft. (approximately); the true value as computed from formula I is 955.37 ft. For flat curves, the discrepancy between the approximate and the true values for R is usually negligible. This holds true, also, when the value 5730 is used in place of 5729.65; one of these values should be memorized by anyone who has frequently to compute values for curves. (See Art. 497 for relation of R and D on arc basis.)

485. General Method by Deflection Angles (Chord Basis). A common method of locating points on a circular curve is by

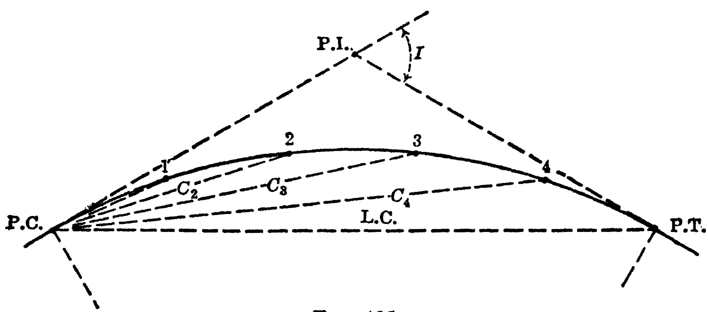


Fig. 485.

means of *deflection angles,* each of which is measured at P.C. from the back tangent through P.C. (See Art. 482(*n*).) Note that the value of the deflection angle to any point on a curve, measured from a backsight along the tangent *behind* P.C., is the same as that of a direct angle to the point measured from a backsight on P.I. For example, the deflection angle to point 3 in Fig. 485 is equal to the direct angle to that point measured from a backsight on P.I.

In route surveying it is desirable to locate points which are whole-number stations. Since the tangent distance is seldom such as to bring P.C. at a whole-number station, it is customary to make the chord from P.C. to the first point on the curve of such a length that it will end at a whole-number station. For example, if P.C. falls at the plus station $40 + 20.4$, the chord from P.C. to point 1 on the curve will be 79.6 ft. in length. Point 1 on the curve will then be station 41, point 2 will be station 42, and so on. The last chord, such as that from station 4 to P.T., must be of such a

length as to extend from the last full station to P.T., and this length is generally less than 100 ft. The curve as staked out is, therefore, a series of chords, all of which are 100 ft. in length, except the chords at the two ends. The 100-ft. chords are often called *normal chords,* and the end chords are called *subchords* or *short chords.*

485(*a*). GENERAL METHOD (100-FT. CHORD BASIS). Each point on the curve is located by means of the deflection angle to that point and the chord distance from the preceding point on the curve. A transit is set up at P.C. and all deflection angles are turned off from a backsight along the back tangent. To locate point 1, for example, the line of sight through 1 is established by turning off the deflection angle for 1 (previously calculated). The chord distance from P.C. to 1 is laid off by holding the zero end of the tape at P.C. and then lining in the graduation mark on the tape that corresponds to the chord distance from P.C. to 1. To locate point 2, the deflection angle to 2 is turned off, the zero end of the tape is held at 1, and the 100-ft. graduation on the tape is then lined in. Succeeding points on the curve are located in a similar manner.

486. Deflection Angles and Inscribed Angles. (General Principles.) (*a*). From the principle in Art. 483(*d*) : *A deflection angle to any point on a circular curve is measured by one-half the arc from P.C. to that point.* For example, in Fig. 485 the deflection angle to point 3 is measured by one-half the arc from P.C. to 3.

486(*b*). From the principle in Art. 483(*f*) : *The inscribed angle at P.C. between two chords from P.C. to any two points on a circular curve is measured by one-half the arc between the two points. If the chord of this arc is 100 ft in length, the arc is equal to the degree of curve D; hence the corresponding inscribed angle at P.C. is equal to ½D.* For example, in Fig. 485, the inscribed angle at P.C. between the chords C_1 and C_2, or between C_2 and C_3, or C_3 and C_4 is equal in each case to one-half the arc between the two points; and if each of the corresponding chords is 100 ft. in length, the angle is equal to ½*D.*

486(*c*). *The deflection angle to any point on a circular curve is obviously equal to the sum of the inscribed angles at P.C. that lie between the tangent at P.C. and the chord to the point on the curve.* For example, in Fig. 485 the deflection angle to point 3 is equal to the angle at P.C. between the tangent and the chord to point 1, plus the inscribed angle between the chord to point 1 and chord C_2, plus the inscribed angle between the chords C_2 and C_3.

486(*d*). Subdeflection angle to the first point on a curve. The chord c_1 from P.C. to the first point on a curve is usually a subchord. The corresponding deflection angle $D.A._1$ is measured by one-half the arc from P.C. to the point. The value of this arc (or of the corresponding angle d_1 at the center) may be obtained from $\sin \frac{1}{2}d_1 = c_1 \div 2R$ (Art. 484, formula XI). Substituting $D.A._1$ for $\frac{1}{2}d_1$:

$$\text{Sine of the subdeflection angle D.A.}_1 = \frac{c_1}{2R} \qquad \text{(XVI)}$$

When the degree of the curve is small it is usually sufficiently accurate to obtain an approximate value for $D.A._1$ from equation XVII or XVIII derived as follows:

$$\frac{c_1}{100} = \frac{\sin \frac{1}{2}d_1}{\sin \frac{1}{2}D} = \frac{\frac{1}{2}d_1}{\frac{1}{2}D} \quad \text{(Approximately)}$$

The last term is based on the fact that the sines of small angles are nearly proportional to the angles themselves. Substituting $D.A._1$ for $\frac{1}{2}d_1$,

$$\text{Subdeflection angle, D.A.}_1 \text{ (in degrees)} = \frac{D}{2} \times \frac{c_1}{100} \quad \text{(approximately)} \qquad \text{(XVII)}$$

$$\text{Subdeflection angle, D.A.}_1 \text{ (in minutes)} = 0.3c_1D \quad \text{(approximately)} \qquad \text{(XVIII)}$$

One or the other of these two formulas is almost always used for flat curves. Deflection angles are usually calculated to the nearest minute of arc.

Illustration. In Fig. 485 assume that the chord from P.C. to 1 is 79.6 ft. in length and that the degree of curve is 6. The deflection angle $D.A._1$ is approximately $0.3 \times 79.6 \times 6 = 143.28' = 2° 23'$.

486(*e*). The increment angle for a substation. When it is required to locate a point on a curve at a distance *less* than 100 ft. from the preceding full station, the deflection angle is obtained by adding to the deflection angle to the full station an increment angle determined from one of the following formulas:

$$\text{Sine of the increment angle} = \frac{c}{2R} \qquad \text{(XIX)}$$

$$\text{Increment angle in degrees} = \frac{D}{2} \times \frac{c}{100} \quad \text{(approximately)} \qquad \text{(XX)}$$

$$\text{Increment angle in minutes} = 0.3cD \quad \text{(approximately)} \qquad \text{(XXI)}$$

In these formulas, which are practically identical with formulas XVI, XVII, and XVIII, *c* represents the length of the subchord from the substation to the preceding full station. This length added to the length of the chord from the substation to the next full station will exceed slightly 100 ft., the distance between the two full stations, but this discrepancy may usually be ignored in flat curves. Note that the increment angle which corresponds to the subchord from the last full station on to curve to the station at P.T. may be computed from one of these formulas.

486(*f*). TO FIND THE LENGTH OF A SUBCHORD FROM A GIVEN POINT ON A CURVE TO THE PRECEDING FULL STATION. If the deflection angle to the given point is not known, it can be either computed or measured. The difference in value between this deflection angle and the deflection angle to the preceding full station is the increment angle. From formulas XX and XIX above:

Length of subchord c = 100(increment angle $\div \frac{1}{2}D$) (approximately) (XXII)

Length of subchord c = Sine of increment angle $\times 2R$ (exact) (XXIII)

One of these formulas may be used for checking the measured length of any subchord.

487. Calculation of Deflection Angles. (1) Calculate the deflection angle to the first station on the curve. This is usually a subdeflection angle and is calculated as explained in Art. 486(*d*). (2) Calculate each succeeding deflection angle to a full station by adding to the deflection angle to the preceding station an *increment angle* equal to one-half the degree of curve ($\frac{1}{2}D$).

487(*a*). ILLUSTRATION. Let $D.A._1$, $D.A._2$, $D.A._3$, and $D.A._4$ represent the deflection angles to points 1, 2, 3, and 4, respectively, on the curve in Fig. 485. Assume that the chord from P.C. to point 1 is 79.6 ft. in length and that the degree of curve is 6. The corresponding deflection angles are $D.A._1 = 2° 23'$. (See calculation in the example in Art. 486(*d*).) $D.A._2 = 2° 23' + 3° = 5° 23'$; $D.A._3 = 5° 23' + 3° = 8° 23'$; $D.A._4 = 8° 23' + 3° = 11° 23'$.

487(*b*). CHECK. A check on the calculation of all deflection angles may be obtained as follows: (1) From formula XIV, Art. 484, calculate the length of curve L. (2) Determine the length of the subchord from the last full station to P.T. This length is equal to the length of the curve L minus the length of the curve from P.C. to the last full station, measured along the chords. (3) From the length of the subchord, determine the increment angle at P.C. between the chord from P.C. to the last full station and the long

chord from P.C. to P.T. (See Art. 486(*e*).) (4) Add the angle found in step 3 to the deflection angle to the last full station; the result is the deflection angle to P.T. and should be equal to one-half the intersection angle *I*.

487(*c*). REMARK. *Deflection angle* is sometimes defined as the angle at P.C. between two chords from P.C. to two full stations on the curve. Since the chord between the stations is 100 ft. in length, a deflection angle, thus defined, is always equal to one-half the degree of curve, i.e., $\frac{1}{2}D$. This inscribed angle is the normal increment angle. The deflection angle measured from the tangent to any point on the curve is sometimes called the *total deflection angle;* in this book it will be called simply *deflection angle* as defined in Art. 482(*n*).

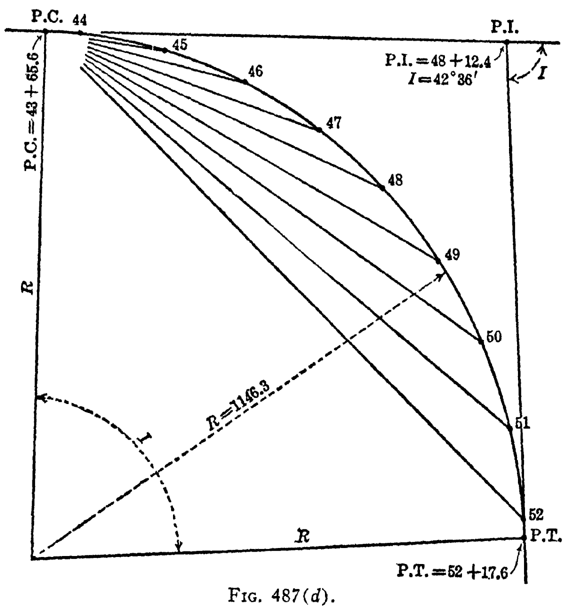


FIG. 487(*d*).

487(*d*). EXAMPLE OF COMPUTATIONS FOR CURVE NOTES. (Fig. 487(*d*).) The point of intersection P.I. is found, by chaining, to be temporary station 48 + 12.4. The angle of intersection *I* as measured with a transit at P.I. is 42° 36′. The degree of curve adopted is $D = 5°$.

(1). The radius $R = 50$ ft. $\div \sin \frac{1}{2}D = 1146.3$ ft. (Art. 484, formula I)
(2). The tangent distance $T = R \tan \frac{1}{2}I = 446.8$ ft. (Art. 484, formula IV)
(3). P.C. = (station 48 + 12.4) − (4 + 46.8) = station 43 + 65.6
(4). Length of first chord to station 44 = 100 − 65.6 = 34.4 ft.
(5). Length of curve $L = 100(I \div D) = 100(42° 36' \div 5) = 852.0$ ft. (Art. 484, formula XIV)
(6). P.T. = (43 + 65.6) + (8 + 52) = station 52 + 17.6
(7). Length of last chord from station 52 to P.T. = 17.6 ft.
(8). Deflection angle to station 44 = $0.3c_1D = 0.3 \times 34.4 \times 5 = 0° 52'$ (Art. 486(*d*), formula XVIII)
(9). Normal increment angle = $\frac{1}{2}D = 2° 30'$.
(10). Increment angle for the last chord = $0.3 \times 17.6 \times 5 = 0° 26'$ (Art. 486(*e*), formula XXI)

Station	Deflection Angle
44	0° 52′
	2° 30′
45	3° 22′
	2° 30′
46	5° 52′
	2° 30′
47	8° 22′
	2° 30′
48	10° 52′
	2° 30′
49	13° 22′
	2° 30′
50	15° 52′
	2° 30′
51	18° 22′
	2° 30′
52	20° 52′
	0° 26′
52 + 17.6 = P.T. =	21° 18′
Check: $\frac{1}{2}I$ =	21° 18′

487(*e*). THE USE OF TABLES. Tables found in handbooks on railroad engineering, highway engineering, and route surveying greatly simplify the computations in determining such values as lengths of arc, lengths of chords, midordinates, external distances, tangent distances, radii, and deflection angles.

488. Field Notes. The results obtained in the preceding article may be tabulated for use in the field in a form similar to the following:

Station	Points	Deflection Angle	Curve Data	Bearings	
				Magnetic	Calculated
52 + 17.6	P.T.	(Right) 21° 18′	I = 42° 36′ $\frac{1}{2}I$ = 21° 18′ Check	N 4° 30′ E	N 4° 24′ E
52		20° 52′			
51		18° 22′	D = 5°		
50		15° 52′	R = 1146.3		
49		13° 22′	T = 446.8		
48	P.I. = 48 + 12.4	10° 52′	L = 852.0		
47		8° 22′			
46		5° 52′			
45		3° 22′			
44		0° 52′			
43 + 65.6	P.C.	0° 00′		N 38° 15′ W	N 38° 12′ W

Tabulated field notes such as those on page 535 are entered on the left-hand pages of the notebook. Corresponding sketches, if any, are on the opposite pages. In sketching, the book is held with its top toward the stations in advance; for this reason the tabulated notes read upward from the bottom of the page. Frequently the straight vertical line through the center of the right-hand page represents the alignment line through the stations; it is desirable that a point representing a station on this line should be approximately opposite the corresponding station on the left-hand page.

489. Field Work.

1. Prolong the back tangent, carrying forward the regular stationing, until it extends slightly beyond the point where, judged by the eye, the point of intersection P.I. will be.

2. Establish at least two points on the forward tangent. (One point each side of the P.I. to be found.)

3. Establish the point of intersection of the two tangents.

4. Measure the intersection angle I. (It is well to double it.)

5. Make the necessary computations for the curve values similar to those in Art. 487(*d*), and from the results prepare the field notes.

6. From the computed tangent distance establish the points P.C. and P.T. The point P.C. may be found by measuring from the nearest whole-number station on the back tangent; the point P.T. may be established by laying off from P.I. the tangent distance along the forward tangent.

Note. When a curve is long, it may be desirable, before moving the transit from P.I., to establish the midpoint of the curve. This is done by bisecting the angle between the tangents at I in order to establish a line which passes from P.I. to the center of the curve, and then by laying off on this line a distance from P.I. equal to the external distance E as computed from formula VI, Art. 484. The point thus established is used as a check when running in the curve.

7. Set up at P.C., clamp the plates at 0° 00′, and backsight on the point P.I. If the station previously established at P.T. can be seen from P.C., turn off the deflection angle to P.T. equal to $\frac{1}{2}I$; the line of sight should pass through P.T. This check should always be applied when practicable, not only before establishing stations on the curve but also after the last station on the curve has been set. If a permanent sight is established at P.T. a check sight may be taken as often as may be desirable.

8. Turn off the subdeflection angle to the first full station on the curve, and line in a point that is a distance from P.C. equal to the length of the subchord to that station. Set the first full station at this point.

9. Keeping the backsight on P.I., turn off the deflection angle to the second full station, hold the zero mark of the tape at the preceding full station, and swing the tape until the 100-ft. mark is in the line of sight. Set the second full station at this point. Repeat this procedure for each full station on the curve.

Note. Instead of backsighting on P.I. in step 7, the backsight may be taken along the tangent backward from P.C. The telescope is then plunged before turning off deflection angles in steps 8 and 9.

10. *Checks.* (1) Measure the distance from the last full station on the curve to P.T.; it should be equal to the computed length of the corresponding subchord (formulas XXII or XXIII, Art. 486(*f*)). (2) When a curve is long, it may be desirable to check on the midpoint established when the transit was at P.I. (See note, step 6.) The deflection angle for the midpoint is ¼*I*. The approximate length of the subchord from the preceding full station to the midpoint may be found by subtracting from one-half the length of curve (measured along the chords) the sum of the chords from P.C. to the full station immediately preceding the midpoint. The correct length of the subchord may be obtained from formula XXIII, Art. 486(*f*), in which the *increment angle* is equal to ¼*I* minus the deflection angle to the preceding full station.

Note. The two tangents and the chords of a curve form a closed polygon, and the linear distance between the position of P.T. as determined by laying off the chords from P.C. to P.T. and the position of P.T. as determined by measuring the tangent distance from P.I. along the forward tangent is the linear error of closure.

11. When it is required to establish an intermediate station on the curve, the increment angle is obtained as explained in Art. 486(*c*). The corresponding subchord, used in swinging the tape, is the distance from the preceding full station to the intermediate or "plus" station.

489(*a*). MODIFICATIONS OF THE GENERAL METHOD. Theoretically, a curve may be staked just as well with the transit at P.T. as at P.C. Indeed, when the curve is long, errors may be reduced by setting some of the stations with the transit at P.C. and the remaining stations with the transit at P.T. Frequently it is impossible to sight to all the stations on a curve from either P.C. or P.T.; it may then be necessary or desirable to set some of the stations with the transit at an intermediate point on the curve, preferably at a full station. Regardless of where the transit may be set up in laying out a curve, all computations for curve notes are made and the notes are recorded as if the transit were at P.C. In the following paragraphs it will be understood, therefore, that all

references to deflection angles are to deflection angles computed for the transit at P.C.

489(*b*). Transit at P.T. (1) Set the plates at 0° 00′ and, with the telescope normal, backsight along the long chord to P.C. (2) When the whole curve is to be staked from the setup at P.T., the procedure is the same as if the transit were at P.C. Beginning with the first station on the curve from P.C., stations are established in order along the curve *toward* P.T. The reading used in setting the plates for the purpose of establishing any station will now be the deflection angle recorded for that station in the field notes, i.e., the same reading that would be used if the transit were at P.C. (Why?) The telescope is kept normal throughout.

When only a portion of the curve is to be laid out from P.T., stations are established backward along the curve, beginning with the station nearest P.T., but the readings used in setting the plates are the same as before, provided it is possible to backsight on P.C. When P.C. is not visible, set the plates at ½*I* and backsight on P.I. The deflection angle recorded for any point may now be used in setting the plates for the line of sight to that point, just as if the transit were at P.C. (Why?)

489(*c*). Transit at an intermediate point. In this case, the stations up to and including that at which the transit stands will usually have been located from P.C. (1) Set the plates at 0° 00′, reverse the telescope, and backsight on P.C. (2) Plunge the telescope. The reading used in setting the plates for the purpose of establishing any station will now be the deflection angle recorded for that station in the field notes, just as if the transit were at P.C. (Why?)

When it is necessary to set up at *another intermediate station* farther on, the procedure is this. (1) Set the plates at the angular value for the deflection angle (given in the notes) for the preceding transit station on the curve, reverse the telescope, and backsight on that station. (2) Plunge the telescope. The reading used in setting the plates for the purpose of establishing any station will now be the deflection angle for that station given in the notes. (Why?)

489(*d*). General rules. When the procedures that have just been outlined for various setups of the transit are followed, readings may be checked by the following general rules:

1. *Whenever the line of sight is directed at any station on the curve, including P.C. and P.T., the plates should read the deflection angle for that station as recorded in the notes.*

2. *When the plate reading is the deflection angle for the station over which the transit is set, the line of sight should be tangent to the curve at that station.*

These rules hold good regardless of whether the transit is at P.C. or at P.T. or at an intermediate station.

Suggestions. (1) Keep continually in mind the principle that the deflection angle at P.C. to any point on the curve is equal to one-half the angle at the center subtended by the arc from P.C. to the point. (2) No one can be proficient in the computations and field work for curves unless he is familiar with the geometric principles of the circle, can derive quickly any formula involved, and keeps continually in mind such useful principles and rules as those just given in Art. 489(*d*) and in suggestion 1. (3) Both in computations and in field work, limits of error should be adopted that are consistent with construction. Often approximate methods are better than too accurate methods. On the other hand, precautions should be taken to avoid unnecessary inaccuracies. For example, in turning off angles, an error of 1 in. or 0.08 ft. can easily occur in sighting to a point 500 ft. from the transit (see Art. 283(*d*); hence too great a length of sight should be avoided. Again, in locating a point by swinging one end of a tape into a line of sight, the acute angle at the intersection of the line of sight and the tape should be large enough to make definite the point of intersection, say about 30°. (4) In setting a 100-ft. station, the head chainman can save time by moving forward 100 ft. on line with the preceding chord and then by moving toward the center of the curve a distance in feet equal approximately to 1¾ times the degree of curve. (5) When conditions are difficult for chaining, points on a curve may sometimes be determined by the method of intersection, with one transit at P.C. and another at P.T. For some points, however, the angles of intersection will be so small as to render the method relatively inaccurate for those points. (6) There are many variations in curve problems. For example, it may be required to run a curve of a given radius instead of a given degree; or the point of intersection may be inaccessible; or one or more stations on the curve may be inaccessible. Oftentimes a little thought and ingenuity will solve such problems. Handbooks on curves will be found helpful in solving certain special problems, and the tables in such handbooks will save enough time, even in ordinary work, to repay many times the cost of the books.

489(*e*). NORMAL CHORDS OF 50 FT., 25 FT., OR OF SOME OTHER LENGTH LESS THAN 100 FT. are often used in establishing stations on a sharp curve. The increment angle for a 50-ft. chord is $D/4$; for 25 ft., it is $D/8$; for 10 ft., it is $D/20$. In these values the degree of curve D is based on a 100-ft. chord. If two full stations on a curve are 100 ft. apart, measured on the chord between them, they will be less than 100 ft. apart, measured in the same way, if one has been located from the other through two 50-ft. chords, or four 25-ft. chords, or through any other series of short chords. The discrepancy may be neglected in many cases, particularly if the curve is flat, but it increases as the degree of the curve increases and may become too great to be permissible. In the latter case, the length of each subchord should be increased until it is the true length for its increment angle as determined from formula XXII or XXIII, Art. 486(*f*), or from a table of "True Subchords" to be found in some of the handbooks.

Illustration. When there are two equal subchords between two full stations, the correct length of each for a 10° curve is 50.048 ft.; for a 20° curve it is 50.191 ft. When there are four chords, the correct lengths are 25.030 ft. for a 10° curve, and 25.119 ft. for a 20° curve.

489(*f*). CAUTION. When normal chords shorter than 100 ft. are used the length of curve, measured along the chords, will be different than when measured along 100-ft. chords. In using formulas based on 100-ft. chords it is essential to note carefully whether or not they should be modified when normal chords are not 100 ft. in length. For example, formulas X, XII, XIII, XIV, and XV in Art. 484 are obviously incorrect unless normal chords are 100 ft. Formula XIV should be $L = c_n \frac{I}{d_n}$ and XV should be $N = \frac{I}{d_n}$, in which c_n represents the length of the normal chords and d_n the central angle subtended by a normal chord. (Should formulas I and III be changed?)

490. To Lay Out a Circular Curve with Tapes. Points on a circular curve can be located by using only tapes. There are three general methods: (1) By offsets from a tangent to the curve. (2) By offsets from a chord produced. (3) By ordinates from a chord. Greater accuracy can be attained in each of these methods by the use of a transit as well as a tape, but it is often convenient or necessary to locate points on a curve by linear measurements only. The degree of precision that can be attained with tapes alone is sufficiently great for many purposes. Each of the methods can be used also in plotting a curve to scale when its radius is too large for the use of a beam compass.

491. Method of Tangent Offsets (with Tapes). This method consists in locating each station on the curve by finding the intersection of two arcs. The radius of one arc is the length of the chord from the preceding station swung from that station as a center; the radius of the other arc is the offset distance from the tangent through P.C. to the station that is being located, swung from the point on the tangent from which the offset distance is measured. For example, in Fig. 491 station 2 is located at the intersection of two arcs, one swung from station 1 as a center with a radius of 100 ft., the other from the point 2′ as a center with a radius equal to the computed offset distance from 2′ to station 2. Station 3 and the remaining stations on the curve are located in a similar manner. In order to make station 1 a whole-number

station, it is usually necessary to make the chord from P.C. to station 1 a subchord. Station 1 will then be at the intersection of two arcs, one swung from P.C. as a center with the length of the subchord from P.C. to 1 as a radius, the other arc swung from 1′ as a center with the computed offset from 1′ to 1 as a radius.

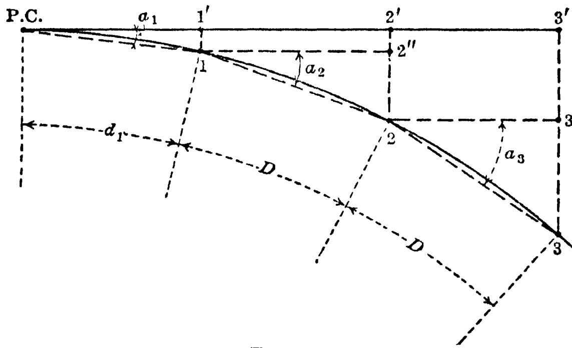


FIG. 491.

The distances from P.C. to each offset are determined by computing first the distances from P.C. to 1′, 1 to 2″, 2 to 3″, and so on. The lengths of the tangent offsets are determined by computing first the distances 1′ to 1, 2″ to 2, 3″ to 3, and so on.

Let a_1 represent the angle 1′–P.C.–1 that the first chord makes with the tangent through P.C., a_2 the angle 2″–1–2 that the second chord makes with the same tangent, and a_3 the angle 3″–2–3 between the third chord and the tangent through P.C. Then:

Distance from P.C. to Offset	*Length of Offset*	
P.C. to 1′ = P.C. to 1 × cos a_1	1′ to 1 = P.C. to 1 × sin a_1	(XXIV)
P.C. to 2′ = P.C. to 1′ + 100 cos a_2	2′ to 2 = 1 to 1′ + 100 sin a_2	
P.C. to 3′ = P.C. to 2′ + 100 cos a_3	3′ to 3 = 2′ to 2 + 100 sin a_3	

The angle which any chord makes with the tangent through P.C. is equal to the central angle between the radial line to P.C. and the radial line to the beginning of the chord plus one-half the central angle subtended by the chord. (Prove.)

$$a_1 = 0 + \tfrac{1}{2}d_1; \quad a_2 = d_1 + \tfrac{1}{2}D; \quad a_3 = d_1 + D + \tfrac{1}{2}D;$$

$$a_4 = d_1 + 2D + \tfrac{1}{2}D \qquad \text{(XXV)}$$

$$a_1 = \tfrac{1}{2}d_1; \quad a_2 = d_1 + \tfrac{1}{2}D; \quad a_3 = d_1 + 1\tfrac{1}{2}D;$$

$$a_4 = d_1 + 2\tfrac{1}{2}D; \quad a_5 = d_1 + 3\tfrac{1}{2}D, \text{ and so on.}$$

Note that beginning with the *third* chord, the angle that each chord makes with the tangent through P.C. is greater than the corresponding angle for the preceding chord by an amount equal to D. This, however, does not hold true for the chord that ends at P.T. since this chord is usually less than 100 ft. Assume, for example, that the last 100-ft. chord is from station 3 to station 4; then, from the rule, the angle a_5 for the subchord from station 4 to P.T. is $a_5 = d_1 + 3D + \frac{1}{2}d_5$, in which d_5 is the central angle subtended by the subchord.

The central angle d_1 for the chord from P.C. to station 1 may be computed from $\sin \frac{1}{2}d_1 = \frac{c_1}{2R}$ (exact), or from $d_1 = \frac{c_1 D}{100}$ (approximate). (See formulas XI and X, Art. 484.) The central angle for the subchord at the end of the curve could be computed in a similar manner, but it may be found more directly by subtracting from the intersection angle I at P.I. the sum of all the other central angles.

When the normal chord is less than 100 ft., say 25 or 50 ft., the central angle d_n subtended by the chord may be computed from $\sin \frac{1}{2}d_n = \frac{c}{2R}$ (Art. 484, formula XI). This value should be used instead of D in determining the values of a_2, a_3, and so on in the method of formula XXV, and the length of the normal chord should be substituted for 100 ft. in formulas XXIV. Note also other changes due to short normal chords, as explained in Art. 489(*f*).

491(*a*). CHECK. The distance along the tangent from P.C. to the offset for P.T. should be equal to $R \sin I$, and the length of the offset to P.T. should be equal to R vers I. (Prove these relations.) These two conditions provide a check not merely on the computations for *one* distance along the tangent and for *one* offset distance but also for *all* tangent and offset distances. (See application of check in Art. 491(*c*).)

491(*b*). ILLUSTRATIVE EXAMPLE. Given P.C. = 17 + 36.2; I = 28° 50′; c = 100 ft.; D = 6° 30′.

$N = I \div D$ = 28° 50′ ÷ 6°.5 = 4.436 (from formula XV, Art. 484)

P.T. = (17 + 36.2) + (4 + 43.6) = station 21 + 79.8

$R = 50 \div \sin \frac{1}{2}D$ = 50 ÷ sin 3° 15′ = 881.95 ft. (from formula I, Art. 484)

First chord c_1 = (18 + 00) − (17 + 36.2) = 63.8 ft. Last chord c_f = 79.8 ft.

$\sin \frac{1}{2}d_1 = c_1 \div 2R$ = 63.8 ÷ 2 × 881.95 (from formula XI, Art. 484). $\frac{1}{2}d_1$ = 2° 04′.4

$\sin \frac{1}{2}d_f = c_f \div 2R$ = 79.8 ÷ 2 × 881.95; hence $\frac{1}{2}d_f$ = 2° 35′.6

$a_1 = \frac{1}{2}d_1$ = 2° 04′.4

$a_2 = d_1 + \frac{1}{2}D$ = 4° 08′.8 + 3° 15′ = 7° 23′.8

$a_3 = d_1 + 1\frac{1}{2}D$ = 4° 08′.8 + 9° 45′ = 13° 53′.8

$a_4 = d_1 + 2\frac{1}{2}D$ = 4° 08′.8 + 16° 15′ = 20° 23′.8

$a_5 = d_1 + 3D + \frac{1}{2}d_f$ = 4° 08′.8 + 19° 30′ + 2° 35′.6 = 26° 14′.4

From these results, the computations to determine the distances from P.C. to the various tangent offsets and the lengths of the offsets may be tabulated as follows:

$D = 6° 30'$	Checks on P.T.	Short chords and angles
Given: P.C. = 17 + 36.2	$y = R$ vers I = 109.33	$c_1 = 63.8$ $\frac{1}{2}d_1 = 2° 04'.4$
$I = 28° 50'$	$t = R \sin I$ = 425.30	$c_f = 79.8$ $\frac{1}{2}d_f = 2° 35'.6$
$c = 100.0$		

Station on Curve (1)	Angle between Chord and Tangent (2)	Natural Sine (3)	Natural Cosine (4)	$c \cos a$ (5)	Distance from P.C. on Tangent (6)	Station on Tangent (7)	$c \sin a$ (8)	Total Offset (9)
21 + 79.8 P.T	26° 14'.4	0.44213	0.89695	71.58	425.3	21 + 61.5	35.28	109.4
21	20° 23'.8	0.34857	0.93728	93.70	353.7	20 + 89.9	34.86	74.1
20	13° 53'.8	0.24023	0.97072	97.10	260.0	19 + 96.2	24.00	39.2
19	7° 23'.8	0.12880	0.99167	99.17	162.9	18 + 99.1	12.90	15.2
18	2° 04'.4	0.03619	0.99935	63.70	63.7	17 + 99.9	2.31	2.3
17 + 36.2 P.C.	0° 00'					17 + 36.2	0.00	0.0
				425.25			109.35	

491(*c*). Checks. Note that the summation of the fifth column (425.25) is equal to the distance from P.C. on the tangent to the offset to P.T.; consequently all values in the fifth column must be correct. Likewise the summation of the eighth column (109.35) is equal to the tangent offset for P.T.; consequently all values in the eighth column must be correct. Since values in the sixth column are obtained by adding progressively values in the fifth column and since values in the last column are obtained by adding progressively values in the eighth column, values in the sixth and in the last column must be correct.

491(*d*). Alternative method of tangent offsets. The general method used in Art. 491(*a*) as a check can be used for computing the distance from P.C. to the tangent offset and the length of the offset for every station on the curve. Let A represent the central angle subtended by the chord from P.C. to any station on the curve, y the distance along the tangent through P.C. to the tangent offset for that station, and t the length of the offset. Then:

$$y = R \sin A \qquad \text{(XXVI)}$$

$$t = R \text{ vers } A \qquad \text{(XXVII)}$$

When the central angle A is small, this method is not so accurate as the first. Moreover, there is no check on results that is comparable to the checks explained in Art. 491(*c*). For these reasons the first method is generally preferable.

491(*e*). Suggestions. (1) Equations in XXIV are best solved by the use of natural sines and cosines; formulas XXVI and XXVII by logarithmic sines and versines. (2) It is often best to locate some of the points of a curve by offsets from the back tangent through P.C. and the remaining points

by offsets from the forward tangent through P.T. (3) The method of tangent offsets is often used for establishing two or three stations near each end of the curve and the method of chord offsets for locating the remaining stations. (See example in Art. 492(*a*).) (4) The points on the tangent from which tangent offsets are swung may be lined in by eye. (5) If it is necessary to measure the intersection angle *I* at P.I. with the tape, it may be done by the method explained in Art. 89. (6) The intersection of two arcs may be determined more quickly by the use of two tapes. (7) The use of tables greatly facilitates computations. (8) The field work may be checked by comparing actual measurements with computed measurements. (See (10) in Art. 489.) (9) On sharp curves it may be desirable to locate points on the curve by the use of chords shorter than 100 ft. The same general methods can be used, but keep in mind what formulas are affected by the change from 100-ft. chords. (See Art. 489(*f*).)

492. Method of Offsets from Chords Produced (with Tapes). This method is illustrated in Fig. 492. It is required to locate station 46. The steps in the field work are: (1) Locate 44

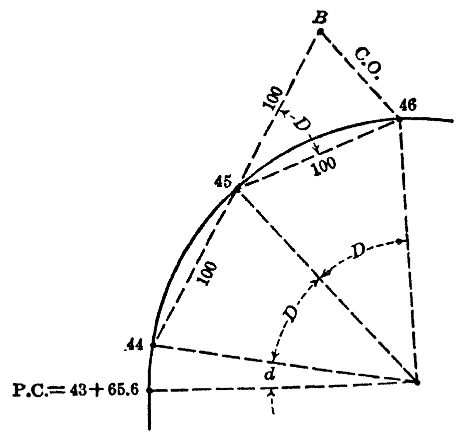


FIG. 492.

and 45 by tangent offsets. (2) Produce the chord through 44 and 45 100 ft. to a temporary point *B*. (3) Locate 46 at the intersection of two arcs, one swung from 45 as a center with a radius of 100 ft., the other swung from *B* as a center with a radius equal to the calculated distance from *B* to 46. Station 47 can be located in a similar manner from a temporary point on the chord through 45 and 46 produced, and so on around the curve.

$$\text{The chord offset C.O.} = 2 \times 100 \sin \tfrac{1}{2}D = \frac{100^2}{R} \quad \text{(XXVIII)}$$

(The second expression may be derived from similar triangles, i.e., $R : 100 :: 100 : \text{C.O.}$)

492(*a*). ILLUSTRATIVE EXAMPLE. (Fig. 492.) Assume the curve to be the same as that in the example in Art. 487(*d*). Degree of curve is 5. From the calculations in that

example: $R = 1146.3$ ft.; deflection angle to station 44 $= \frac{1}{2}d = 0° 52'$; the length of chord from P.C. to 44 is 34.4 ft. The calculations for locating 44 and 45 by tangent offsets are:

$$\text{Distance along tangent} = y = c \cos a_1 \quad \text{(Art. 491, formula XXIV)}$$

For station 44, $y_1 = 34.4 \cos 0° 52' = 34.39$ ft.

For station 45, $y_2 = y_1 + 100 \cos a_2 = y_1 + 100 \cos (d + \frac{1}{2}D)$

$$= 34.39 + 100 \cos 4° 14' = 34.39 + 99.73 = 134.12$$

Tangent offset $= t$. (See computations for offsets in Art. 491, XXIV.)

$$\text{For 44, } t_1 = 34.4 \times \sin 0° 52' = 0.52 \text{ ft.}$$

$$\text{For 45, } t_2 = 0.52 + 100 \sin 4° 14' = 7.90 \text{ ft.}$$

$$\text{Chord offset} = \text{C.O.} = 200 \sin \tfrac{1}{2}D \quad \text{(Art. 492, formula XXVIII)}$$

$$\text{For 46, C.O.} = 200 \sin \tfrac{1}{2}5° = 200 \sin 2° 30' = 8.72 \text{ ft.}$$

Station 44 is located by the intersection of two arcs, one swung from P.C. as a center with a radius of 34.4 ft., the other swung from a center on the back tangent 34.39 ft. from P.C. with a radius of 0.52 ft. Station 45 is located by the intersection of two arcs, one swung from station 44 as a center with a radius of 100 ft., the other swung from a center on the back tangent 134.12 ft. from P.C. with a radius of 7.90 ft. Station 46 is located by the intersection of two arcs, one swung from station 45 as a center with a radius of 100 ft., the other swung from B as a center with a radius of 8.72 ft.

492(*b*). SUGGESTIONS. (1) Some of the suggestions in Art. 491(*e*) hold good for the method of offsets from chords produced. (2) Instead of locating station 45 by the tangent-offset method, it is sometimes more convenient to proceed as follows. Locate a temporary station 43 as if it were on the curve extended backward through P.C. This is done by the tangent-offset method as explained in the method of middle ordinates. (See next article.) The 100-ft. chord from temporary station 43 to station 44 may then be produced and the method of chord offsets used in locating station 45. (3) On sharp curves chords less than 100 ft. in length may be used.

493. Method of Middle Ordinates (with Tapes). This method is illustrated in Fig. 493. As a preliminary step, two full stations are established on the curve by the method of tangent offsets. Both these stations may be beyond P.C. on the actual curve as was the case in the explanation of the method of chord ordinates, or one may be ahead of P.C. and the other back of P.C., as suggested in the preceding paragraph. The latter procedure is followed here in explaining the general method. The steps in the field work are:

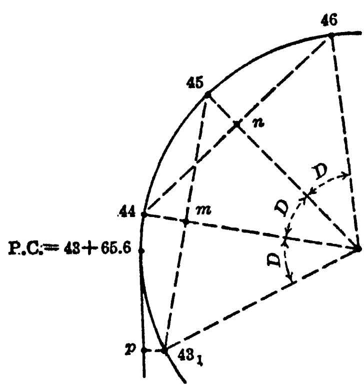


FIG. 493.

1. Locate station 44 by its tangent offset as explained in Art. 491. Locate the temporary station 43_1 as if it were on the curve. To do this, first establish the point p on the tangent by measuring from P.C. a distance P.C. to $p = c \cos \frac{1}{2}d$ (Art. 491, formula XXIV), in which c is the length of the chord from P.C. to station

43_1 and d the central angle subtended by that chord, as found from $\sin \frac{1}{2}d = \frac{c}{2R}$ (Art. 484, formula XI). From p lay off the tangent offset to station 43_1 as found from $t = c \sin \frac{1}{2}d$. The chord from station 43_1 to station 44 will then be 100 ft. in length.

2. The point m is on the radius to 44 a distance from 44 equal to the middle ordinate of the 200-ft. chord from station 43_1 to station 45. This distance may be computed from one of the following formulas:

$$M = R - R\cos D = R \text{ vers } D \quad \text{(from formula VIII, Art. 484)} \qquad \text{(XXIX)}$$

$$M = \frac{c^2}{8R} = \frac{200^2}{8R} \quad \text{(approximately)} \qquad \text{(XXX)}$$

To locate m, find the intersection of two arcs, one swung from 44 as a center with a radius equal to the middle ordinate M, the other swung from 43_1 as a center with a radius of 100 ft. (The distance from 43_1 to m is slightly less than 100 ft., but the error is usually negligible.)

3. To locate station 45 hold one end of the 100-ft. tape at 44 and swing the forward end until it comes into the line from 43_1 to m (produced).

To locate station 46, first establish n, the midpoint of the chord from 44 to 46, at the intersection of two arcs, one swung from 45 as a center with a radius equal to the middle ordinate M, the other swung from 44 as a center with a radius of 100 ft. Establish station 46 at the point in which an arc swung from 45 as a *center* with a radius of 100 ft. intersects the line from 44 to n (produced). Thus the procedure may be repeated until all stations on the curve are located.

More accurate results may be obtained, particularly on sharp curves, by the use of 100-ft. chords instead of 200-ft. chords. The middle ordinate M to a 100-ft. chord is: $M = R \text{ vers } \frac{1}{2}D$ (exact) or $M = 100^2 \div 8R$ (approximate). The procedure applied to the curve in Fig. 493 is as follows: (1) Establish stations 44 and 44 + 50 by the method of tangent offsets (Art. 491). (2) Establish the midpoint m' of the chord from 44 to 45 at the intersection of two arcs, one swung from 44 + 50 with a radius equal to M, the other swung from 44 with a radius of 50 ft. (3) Establish 45 by swinging from 44 + 50 as a center an arc of 50-ft. radius until it intersects the line through 44 and m' (produced). Repeat the procedure until all stations on the curve are located. Obviously twice as many points are located by the use of 100-ft.

chords as by the use of 200-ft. chords. Still shorter chords may be used for sharp curves.

493(*a*). SUGGESTIONS. (1) Middle ordinates are shorter than offsets from chords produced, and this is one advantage of the middle-ordinate method. (2) If the chord is 62 ft. instead of 100 ft. or 200 ft., the middle ordinate in inches is equal approximately to the degree of the curve. ($M = 1.006 \times D$.) From this relation the degree of a curve may be quickly determined. (3) A string equal in length to the chord is often used in place of a tape in swinging chord lengths. The string should be of a kind in which there is little stretch such, for example, as a silk fishline. (4) The method of middle chords is the basis of the "string-lining method" often used in realigning railroad track curves. (See handbooks on railroad curves or on route surveying.)

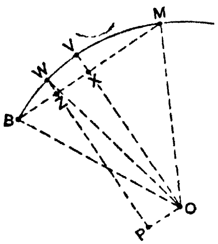


FIG. 494.

494. Method of Offsets from a Single Chord (with Tapes). A circular curve of known degree between two given points B and M may be located by ordinates perpendicular to the chord BM. (1) Measure the length of chord BM. (2) Calculate the radius R from: $R = 50 \text{ ft.} \div \frac{1}{2} \sin D$ (formula I, Art. 484). (3) Calculate the angle d at the center subtended by the chord BM from: $\sin \frac{1}{2}d = BM \div 2R$ (formula XI, Art. 484). (4) Calculate the middle ordinate VX by one of the following formulas:

$$VX = M_c = R \text{ vers } \tfrac{1}{2}d \quad \text{(formula VIII, Art. 484)} \qquad \text{(XXXI)}$$

$$VX = R - \sqrt{R^2 - \overline{BX}^2} \qquad \text{(XXXII)}$$

When the chord is 100 ft. in length, the angle d at the center is D. (5) The length of any other ordinate WZ a distance XZ from the middle ordinate is:

$$WZ = WP - ZP = \sqrt{\overline{WO}^2 - \overline{PO}^2} - XO = \sqrt{R^2 - \overline{ZX}^2} - \sqrt{R^2 - \overline{BX}^2} \qquad \text{(XXXIII)}$$

When the length of any arc from B or V is known as, for example, arc BW, the corresponding ordinate WZ and its distance from B may be calculated from:

$$WZ = (R - R\cos BOV) - (R - R\cos WOV) = R\,(\text{vers } BOV - \text{vers } WOV) \qquad \text{(XXXIV)}$$

$$BZ = R\sin BOV - R\sin WOV = R\,(\sin BOV - \sin WOV) \qquad \text{(XXXV)}$$

495. Graphic Method for Curves. In this method lengths of ordinates or offsets are scaled from a drawing. (1) Plot the curve to a scale large enough so that distances can be scaled as closely as will be required for measurements in the field. (2) Scale ordi-

nates or offsets to some assumed base line, and enter these distances on a map or a specially prepared sketch for use in the field. (3) Lay out the base line in the field, and lay off the measurements as indicated on the sketch.

This is a very useful method for work in which great accuracy is not required, such, for example, as landscape work. Measurements required in any one of the methods of staking a curve with the tape may be obtained graphically in a similar manner. Sometimes a special problem arises which can be solved by a cut-and-try method on a drawing made to scale. Distances in the field, however, are so large compared with those on a drawing, particularly in curves of large radii, that frequently the graphic method is not sufficiently accurate.

The graphic method for curves is often combined with geometric and trigonometric computations. A good example of the use of such a combination in planning subdivisions of land with curved streets is to be found in Art. 593*A*, "Subdivisions Involving Curved Boundaries."

496. Horizontal Curves—Arc Basis. When the degree of curve is defined as the angle at the center subtended by an *arc* of the curve 100 ft. in length, some slight modifications are necessary in the computations and field work. These modifications are due to the fact that there is no normal chord of 100 ft. or of any other length; on the contrary, the lengths of chords must be computed for different degrees of curvature.

497. Formulas—Arc Basis. Formulas in Art. 484, which must be changed or modified when the degree of curve D is defined on the arc basis, are enclosed in brackets. All other formulas may be used regardless of whether the degree of curve is defined on the chord basis or on the arc basis. In some highway departments, the sharpness of the curve is not designated by stating the degree of curve, but simply by giving the radius, usually in some round number, such as 2000 ft., or 400 ft., or the like. This is, however, really designating the curve on an arc basis.

From the definition of degree of curve *arc* basis:

$$D : 360^\circ :: 100 : 2\pi R$$

$$R = \frac{5729.58}{D} \qquad \text{(XXXVI)}$$

$$D = \frac{5729.58}{R} \qquad \text{(XXXVII)}$$

Obviously, the radius varies *inversely* with the degree of curve.

The following formulas, which are slightly approximate on the chord basis, are exactly correct on the arc basis:

In Art. 484: (XIV) $L = 100\frac{I}{D}$; (XII) $D = 100\frac{I}{L}$; (XIII) $I = \frac{DL}{100}$.

(L is measured around the arc instead of along the chords.)

In Art. 486(*d*): (XVII) Deflection angle $= D/2 \times \frac{a}{100}$ degrees (a = length of arc); (XVIII) Deflection angle $= 0.3aD$ min. (a = length of arc).

Practically all formulas from XXIV, Art. 491, to XXXV, Art. 494, used when curves are laid out with tapes, are the same for the arc basis as for the chord basis.

It should be kept in mind that although most of the formulas in Art. 484 remain unchanged *in form*, any formula that involves the value of R does not give the same result on the arc basis that it does on the chord basis. This is because the value of R for a given degree of curve is slightly different on the chord basis from the value on the arc basis. For flat curves this difference may be so slight as to be negligible, but as the degree of curve increases, with a corresponding decrease in the length of R, the difference will become more marked. For example, the tangent distance, computed from $T = R \tan \frac{1}{2}I$, will differ according to whether R is calculated on the chord basis or on the arc basis. This means that the points P.C. and P.T. located on the chord basis will not be in exactly the same positions as they would be if located on the arc basis. These differences in position may or may not be negligible, depending upon the sharpness of the curve and upon the degree of precision required.

498. The General Method of Deflection Angles—Arc Basis does not differ essentially from that used in staking out curves on the chord basis. (See Art. 485.) The normal chord, however, is not 100 ft. but depends on the normal arc. There are two procedures. (1) When it is desired to keep the full stations 100 ft. apart (or at some other normal distance measured on the curve) the first *arc* will be of such a length, measured from P.C., that the first station on the curve will be a full station. The remaining stations are located so that the distance between two consecutive stations, measured along the curve, is equal to the normal arc. The arc between the last station and P.T. is usually shorter than the normal arc. (2) A second or alternative procedure is to divide the curve from P.C. to P.T. into a number of equal parts, thus making all chords equal. In either procedure, the lengths of the

chords must be computed; in flat curves, however, the difference in length between an arc and its chord may be negligible.

499. Computations—Arc Basis are similar to those explained in Art. 487 and illustrated in Art. 487(*d*). Assuming that P.I. has been established and that I has been measured, the steps may be summarized as follows:

1. If the degree of curve is given, compute R from formula XXXVI, Art. 497; if R is given, compute D from formula XXXVII.

2. From $T = R \tan \frac{1}{2}I$ (formula IV, Art. 484) compute the tangent distance and from that distance determine the position of P.C. (usually a plus station).

3. From $L = 100\ I/D$ (formula XIV, Art. 484), compute the length of curve measured around the arc from P.C. to P.T.

4. From the station number of P.C. and the length L compute the station number of P.T.

5. From the station number of P.C. and the length of the normal arc determine the length of the subarc to the first full station on the curve; from this length of arc and L determine the length of the subarc from the last full station to P.T.

6. From formula XVII or XVIII, Art. 486(*d*), compute the subdeflection angle to the first full station on the curve, the normal *increment angle* for the normal length of arc (Art. 487), and the increment angle from the last full station on the curve to P.T. (Art. 486(*e*)).

7. As a check: Multiply the normal increment angle by the number of arcs of normal length and add to this product the subdeflection angle to the first full station and the increment angle from the last full station to P.T. The sum should equal $\frac{1}{2}I$.

8. Beginning with the second full station, compute the deflection angle to each full station by adding the normal increment angle to the deflection angle of the preceding full station, as illustrated in Art. 487(*a*).

9. From $c = 2R \sin \frac{1}{2}d$ (formula IX, Art. 484) compute the length of chord for the normal arc, and for each of the shorter arcs, one at either end of the curve.

499(*a*). Problem. Given P.I. = station 48 + 12.4; $I = 42° 36'$; $D = 5°$. Check the following results and compare them with those obtained in Art. 487(*d*). (1) R = 1145.91 ft. (2) T = 446.77 ft. P.C. = station 43 + 65.6. (3) L = 852.0 ft. (4) P.T. = station 52 + 17.6. (5) Length of first subarc = 34.4 ft. Length of last subarc = 17.6 ft. (6) First subdeflection angle = 0° 51'.6. Normal increment angle = 2° 30'. Last subincrement angle = 0° 26'.4. (7) Sum of angles = 0° 51'.6 + 8 × 2° 30' + 0° 26'.4 = 21° 18' = ½I (check). (8) Deflection angles = (see field notes, Art. 500).

When the curve is divided into a number of equal parts, there will be only one length of chord; consequently computations are even simpler. In any case, the use of tables greatly facilitates the work of computation. (See Art. 487(*c*).)

On sharp curves it is frequently desirable to locate points nearer together than 100 ft., i.e., to use shorter arcs, say 10 to 50 ft. The general method is the same for any length of the normal arc.

500. Field Notes—Arc Basis. One convenient form of field notes may be illustrated by the following arrangement of the results obtained in the preceding article:

Form of Field Notes (Arc Definition of Degree of Curve)

Station	Point	Deflection Angle	Chord	Arc	Alignment Notes
52 + 17.6	P.T.	21° 18′.0			P.I. = 48 + 12.4
			17.60	17.60	I = 42° 36′
52		20° 51′.6			D = 5°
			99.97	100.00	R = 1145.91
51		18° 21′.6			L = 852.0
			99.97	100.00	T = 446.77
50		15° 51′.6			
			99.97	100.00	
49		13° 21′.6			
			99.97	100.00	
48		10° 51′.6			
			99.97	100.00	
47		8° 21′.6			
			99.97	100.00	
46		5° 51.′6			
			99.97	100.00	
45		3° 21′.6			
			99.97	100.00	
44		0° 51′.6			
			34.40	34.40	
43 + 65.6	P.C.	0° 00′.0			

501. The Field Work of staking out a curve by means of deflection angles is the same regardless of whether the field notes have been computed on the chord basis or on the arc basis. On the arc basis, distances between stations are on the arc, but the distance between two successive stations is actually determined by measuring along the chord between the two stations, the length of the chord having been calculated. Consequently, the procedure in the field work, step by step, is identical with that outlined in Art. 489.

501(*a*). The methods of staking out curves with tapes, explained in Arts. 490 to 494, may be used regardless of whether the degree of curvature D is defined on the chord basis or the arc basis.

502. Compound Curves. In Fig. 502 is shown a compound curve consisting of the arcs of two circular curves of different radii which meet at P.C.C. (point of compound curvature). AB is the tangent common to both curves. The center P of the shorter curve lies on the radius of the longer curve. To design a compound curve, points P.C., P.T., and P.C.C. are selected as well as the radii (or degrees) of the component curves. This may be done

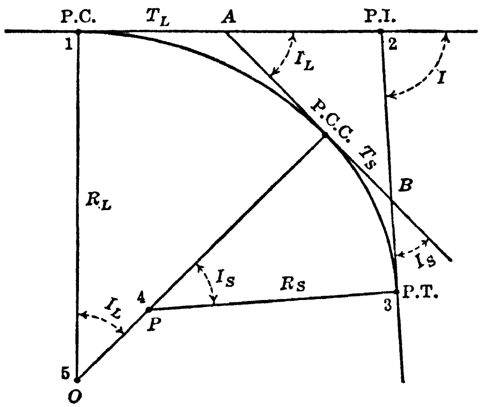


FIG. 502.

either in the field or on a map. Angles I_L and I_S are first measured. Tangent lengths T_L and T_S can then be computed for each curve.

For the curve of longer radius: $T_L = R_L \tan \frac{1}{2}I_L$; for the curve of shorter radius: $T_S = R_S \tan \frac{1}{2}I_S$. Also $I = I_L + I_S$. Computations required to stake out the compound curve are made separately for each curve. These methods have already been described in Art. 487 or Art. 499. Distances from P.I. to A and B can be computed by solving the triangle A, B, P.I., *in* which the side $AB = T_L + T_S$ and all the angles are known.

502(a). FIELD WORK. To lay out the curves by means of deflection angles the transit is set at P.C., sighted along the tangent, and the curve run as far as P.C.C. The transit is then moved to P.C.C., sighted along the common tangent AB, and the second curve laid out. The general method of field work for each curve is explained in Art. 489.

502(b). PRACTICAL SUGGESTIONS. (1) Compound curves are used in laying out railroads and highways in situations where a single curve cannot be made to fit points on the ground which the engineer wishes to have on the curved alignment. It is sometimes necessary to compute and stake several trial curves before a satisfactory layout is obtained. In some cases, property boundaries are composed, in part, of compound curves. (2) If it is desired to plot the controlling points of a compound curve by means of coordinates, it should be noted that the principal tangents and radii constitute a

closed traverse. Thus in Fig. 502 points 1–2–3–4–5 indicate the boundaries of a five-sided traverse, in which all lengths are known. Bearings or azimuths can be found from the angles I, I_L, and I_S. (3) A compound curve consists sometimes of three or more consecutive circular arcs, but such curves are unusual.

503. Reversed Curves. A reversed curve is composed of two adjacent simple curves which turn in opposite directions, as shown in Fig. 503. The point of reversed curvature P.R.C. is common to both curves, being the P.T. of the first curve and the P.C. of the second curve. The line from P.I.1 to P.I.2 is, therefore, tangent to both curves. Points P.C., P.R.C., and P.T. are selected in

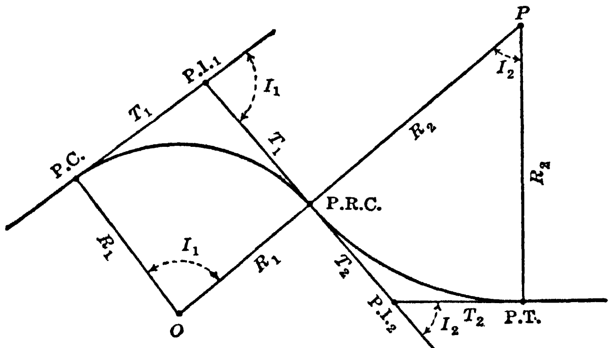


FIG. 503.

the field in the approximate positions they are to occupy, and the angles I_1 and I_2 are measured. Radii R_1 and R_2 are chosen to fit the field conditions. From these data the values T_1 and T_2 can be computed from $T_1 = R_1 \tan \frac{1}{2}I_1$ and $T_2 = R_2 \tan \frac{1}{2}I_2$.

In case the length of the common tangent is fixed, one radius, as for example R_1, can be assumed and T_1 computed. This value of T_1 subtracted from the length of the common tangent gives T_2, from which R_2, the radius of the second curve, can be computed, i.e., $R_2 = T_2 \cot \frac{1}{2}I_2$. Reversed curves find their greatest field of application in railroad yards where speeds are low. They are not adapted to use on highways.

504. Easement Curves. In railroad track construction it has been the practice for many years to elevate the outer rail on curves. To avoid the shock which would result if the entire superelevation were introduced at the junction of tangent and curve, an additional curve of varying radius called an easement curve or transition curve is introduced into the alignment between the tangent and the circular curve. The length of the easement curve is usually several hundred feet; its length depends mainly upon the speed assumed in the design and upon the radius of the central curve. The superelevation is distributed uniformly throughout the length of the easement curve. Radii of successive chords of the easement curve are designed to vary from a minimum length equal to that of the central curve itself to a maximum length near the section adjacent to the tangent. The use of radii of varying lengths along the curve has caused the easement curve to be referred to as a *spiral-easement* or *spiral-transition* curve. The gradual change in radii

throughout the length of the easement curve is of great advantage in promoting the smooth passage of a rapidly moving vehicle along its path from tangent to curve and from curve to tangent. In recent years the increasingly higher speeds attained by motor vehicles have resulted in the almost universal application of transition curves to the design of highway curves.

The position of the easement curves with respect to the tangents and to the central curve are shown in Fig. 504. Controlling points such as tangent-to-spiral, spiral-to-curve, curve-to-spiral, and spiral-to-tangent are shown by their respective abbreviations, T.S., S.C., C.S., and S.T. S_c is called the

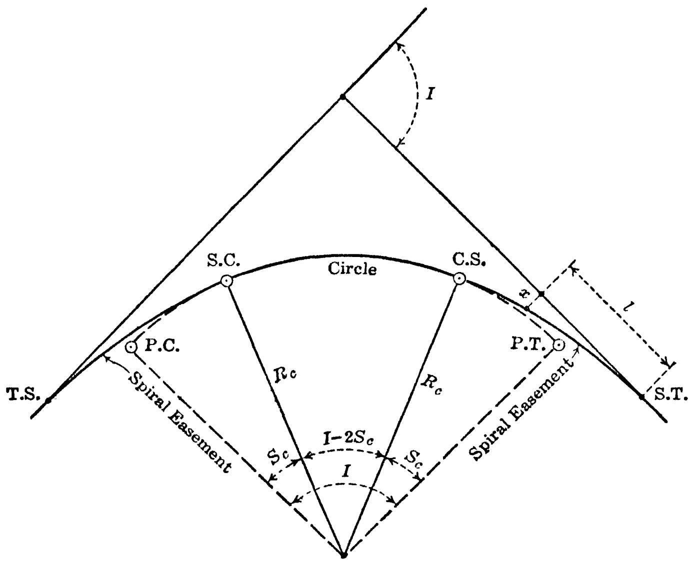


Fig. 504.

spiral angle. It will be noted that the introduction of the transition curves requires the central curve to be shortened somewhat and to be moved inward a short distance; or, stated in another way, the easement curve replaces part of the original tangent to the circular curve and also replaces a portion of the unspiraled central curve itself. On sharp curves for highways, spirals are sometimes chosen of such length that they meet at the center, thus eliminating the circular curve altogether.

The equation of the spiral easement curve, usually called the "cubic spiral," is $x = l^3 \div 6R_c l_c$, in which x is the tangent offset at any point on the spiral at a distance l from the beginning of the spiral (T.S.), R_c is the radius, in feet, of the central curve, and l_c is the total length of the spiral, in feet, measured along the curve.

The easement curve most used by railroad engineers is that known as the *American Railway Engineering Association* (*A.R.E.A.*) *spiral.* Tables for laying out this curve by deflection angles and by offsets from the tangent are given in books on railroad curves and on route surveys. The length of the A.R.E.A. spiral is measured by ten equal chords. The methods and tables used for laying out easement curves for railroads can also be used for laying out easement curves for highways.*

* An excellent set of tables for use in laying out spiral easement curves for highways are those in *Transition Curves for Highways* by Joseph Barnett, Senior Highway Design Engineer, Public Roads Administration. This book, published in 1938, contains many valuable suggestions for the design of highway curves. It may be obtained from the Superintendent of Documents, Government Printing Office, Washington, D. C.

505. The Design of Horizontal Curves is a problem that arises most frequently either (1) in route surveys, such as those for highways, railroads, and pipe lines, or (2) in surveys to establish boundaries, particularly boundaries of subdivisions of land with curved streets. (See Art. 593*A*.)

The design of safe and economical alignment for railroads and highways is treated in books on railroad engineering and on highway engineering. Only the most general guiding principles will be given here.

When the intersection angle I between two tangents is large, a curve of relatively short radius may generally be used; but when the intersection angle is small, a curve of longer radius is required.

A curve of short radius is never used on railroad track designed for high speed and is objectionable on highways for two reasons. (1) A sharp curve may dangerously restrict the distance in which it is possible for the driver in one vehicle to see another vehicle approaching in the opposite direction. (2) The tendency for a vehicle to leave a sharp curve may be too great for safety.

In highway design the element of safety should control the selection of curves. Property damage and construction cost are generally of secondary importance. Highway curves must have long, unobstructed sight lines. The introduction of long easements is also desirable. These can be obtained only from the use of long tangents; hence the curve radii must also be long since $T = R \tan \frac{1}{2}I$, in which the angle I remains constant for the particular tangents which have been selected for study.

On both railroads and highways, the tendency for a train or a vehicle to leave the curve because of centrifugal effects may largely be overcome by raising or superelevating the outer rail of the track or by "banking" the surface of the roadway. Since the theoretical amount of superelevation required depends largely on speed and since speeds are likely to vary through a wide range, the only workable solution is to base the amount of superelevation upon an assumed reasonable speed and then to post appropriate signs to indicate the safe speed for each curve under normal weather conditions. In northern latitudes, where packed snow and ice may accumulate upon the road surface, the amount of superelevation on highways is strictly limited because of the danger that a stalled or a slow-moving vehicle may slide toward the inner side of the curve if the banking is greater than the slope determined by the coefficient of friction. Under favorable conditions of weather and road surface, a speed in excess of that used in designing the banking may be used safely since increased friction between tires and road surface can be depended upon to aid in holding the vehicle upon the curving roadway and thus to counteract centrifugal effects. At roadway intersections in towns and cities where curves of very short radii must be used and banking cannot be introduced, the only alternatives available to the designer are to select curves with radii somewhat longer than the minimum required for turning ordinary commercial vehicles and to erect signs indicating the safe speed at those points. In open country, where topographic conditions often permit the use of curves having radii of five thousand feet or more, superelevation on both railroad and highway curves is generally unnecessary.

In selecting a curve to fit given conditions, it sometimes happens that there is a definite point through which the curve should pass or some object which it should avoid, and frequently the point or the object is either on or near the bisector of the inner angle between the two tangents. The distance from the P.I. to this controlling point constitutes the external distance or simply the "external." Once this external is fixed, the radius of the curve is likewise fixed, since $E = R$ exsecant $\frac{1}{2}I$. The tangent lengths are also fixed, since $T = R \tan \frac{1}{2}I$. The method of designing curves by means of external distances is so common that textbooks on curves include tables of both natural and logarithmic functions of external secants.

The method of approach to curve design in most common use consists primarily of a plat showing the proposed tangents laid out to scale. Adjacent buildings, trees, walls, streams, culverts, embankments, or other objects which may either influence or be affected by the design are also shown to scale on the plat. Curved templates of wood or celluloid with varying radii are then used to join the tangents already platted. When the template is found which best fits the conditions, its radius (in inches) is noted. This value multiplied by the scale of the map gives the radius of the curve to be staked upon the ground. For example, suppose a template marked 24 in. is found to fit best the conditions shown on the map whose scale is 1 in. = 50 ft. Then the radius

of the curve which will fit the conditions in the field will be 24 × 50 = 1200 ft., thus making the degree of curve approximately 5°. Beam compasses as well as ordinary compasses can be used in curve design, but they are not so satisfactory as curved templates except perhaps in subdivisional work where limited block lengths make small curves necessary.

Situations sometimes occur in which a single curve cannot be fitted satisfactorily to the desired alignment. In such cases it may be necessary to select a combination of two or more simple curves forming a compound curve, as explained in Art. 502. Such curves are not objectionable from the standpoint of highway safety, provided the difference in the radii (i.e., degree of curvature) is not too great.

Under certain conditions, a *horizontal parabola* can be used to advantage instead of the arc of a circle or of a combination of simple curves.

506. To Lay Out a Horizontal Parabola with Tapes. Two general methods of laying out a horizontal parabola by the use of tapes will be explained. In studying these methods, certain fundamental properties of the parabola, given in Art. 472, should be kept in mind.

506(*a*). To LAY OUT A HORIZONTAL PARABOLA BY OFFSETS FROM TANGENTS. It is assumed that the tangents have been staked on the

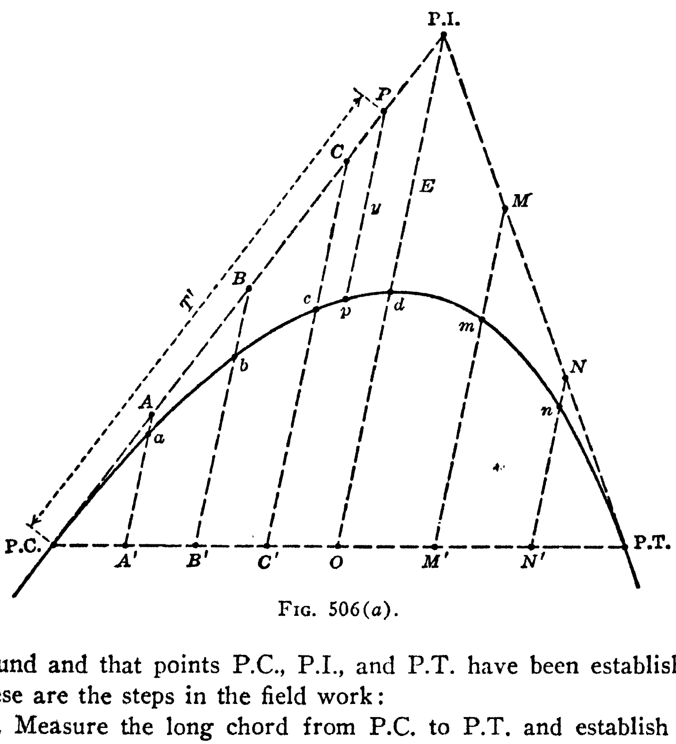


FIG. 506(*a*).

ground and that points P.C., P.I., and P.T. have been established. These are the steps in the field work:

1. Measure the long chord from P.C. to P.T. and establish the midpoint *O*.

2. Measure the distance from P.I. to *O* and establish the mid-

point d. The parabola will pass through d (Art. 472(*b*)), and the distance from P.I. to d is the *external distance* E.

3. Subdivide the distance from P.C. to O into any desired number of equal parts—in this case four—and mark points A', B', and C'.

4. Subdivide the tangent from P.C. to P.I. into the same number of equal parts and mark points A, B, and C.

5. Set points a, b, and c on the curve by laying off the oblique tangent offsets Aa, Bb, and Cc. These offsets are computed by using the principle that the tangent offsets to any two points are to each other as the squares of the corresponding tangent distances from P.C. This is a modification of the principle in Art. 472(*a*). Note that one of these offsets is the external distance E. The method is practically the same as that explained in Art. 475(*e*). $Aa = (1/4)^2E = 1/16E$; $Bb = (1/2)^2E = 1/4E$; $Cc = (3/4)^2E = 9/16E$.

6. Subdivide the distance from O to P.T. into any number of equal parts (not necessarily the same number as used in the other portion of the parabola), and find points on the curve by the method just explained. For purposes of illustration, the distance from O to P.T. was divided into three equal parts instead of four. $Nn = (1/3)^2E = 1/9E$; $Mm = (2/3)^2E = 4/9E$.

Suggestions. (1) When a considerable number of points on the curve is required, the computed values of the offsets may be checked by means of second differences. (2) When an obstacle prevents the setting of a point on the curve by the general method, or when it is desired to establish an intermediate point, proceed as follows. Let p represent the point, T' the tangent distance from P.C. to the oblique offset through p, y the length of the offset, and T the length of the tangent from P.C. to P.I. Then $y = (T'/T)^2E$.

506(*b*). TO LAY OUT A HORIZONTAL PARABOLA BY MIDDLE ORDINATES. It is assumed that the tangents have been staked on the ground and that P.C., P.I., and P.T. have been established. These are the steps in the field work:

1. Establish points O and d as in the preceding method.

2. Subdivide each tangent into four equal parts and mark the points A, B, C, L, M, and N.

3. Line in B' the midpoint of the chord from P.C. to d, and M' the midpoint of the chord from d to P.T.

4. Lay off the middle oblique ordinate $B'b$ on the line $B'B$, and the middle oblique ordinate $M'm$ on the line $M'M$. The length of each of these middle ordinates is $1/4E$. Points b and m are on the curve.

5. Set the points A' and C' at the midpoints of the chords from P.C. to b and from b to d; set points L' and N' at the midpoints of the chords from d to m and from m to P.T.

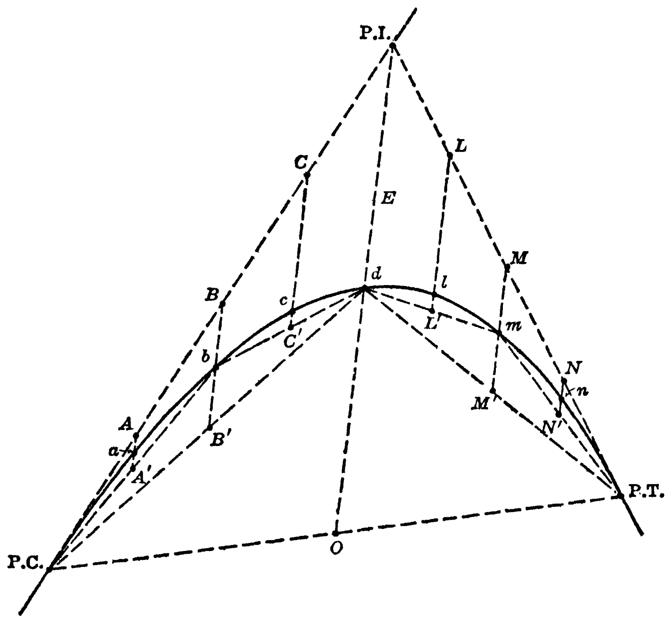


FIG. 506(*b*).

6. Lay off the middle ordinates $A'a$, $C'c$, $L'l$, and $N'n$ as in step 4, and thus locate points a, c, l, and n on the curve. Each of these ordinates is $\frac{1}{16}E$ or one-fourth the length used for middle ordinates in step 2.

7. The procedure may be repeated until the desired number of points on the curve has been located. Each middle ordinate used in one step will be one-fourth the length of each middle ordinate used in the corresponding preceding step.

It is well to note that when there are trees or underbrush, the chord method generally requires less cutting and clearing than does the offset method.

Exercise. Derive the values $\frac{1}{4}E$ and $\frac{1}{16}E$. Suggestions: (1) Express the values of the oblique offsets from the tangent to points on the curve in terms of E. (Use the principle given in step 5 of the preceding method of offsets.) (2) Express the values of the oblique offsets from the tangent to the midpoints of the chords in terms of E. (Use proportions derived from similar triangles.)

506(*c*). ADVANTAGES OF A HORIZONTAL PARABOLA. (1) It can easily be laid out with a tape and calculations are simple. Points may be lined in by eye unless tangents are long or obstructed. (2) It is not necessary to know the angle of intersection at P.I. (3) It is nearly as easy to lay out a parabola when the lengths of the tangents are unequal as when they are equal. Unequal tangents often facilitate fitting the curve to the ground and thus securing a more satisfactory alignment. (As, for example, in laying out park drives.)

506(*d*). Disadvantages of a horizontal parabola. (1) Unless the ground is comparatively free from obstructions, there may be difficulty in measuring and lining in points. (2) The length of the parabola or any part of it is not readily computed. It can, however, be measured after it has been staked out. (3) The parabola is not the type of curve that can most easily be staked out with transit and tape when accurate measurements are required. (4) Because of difficulties in description, the parabola is not well adapted for use as a curved property boundary.

CHAPTER 34

DETERMINATION OF A MERIDIAN

BY OBSERVATIONS ON POLARIS

THE simplest and most direct method of determining a true meridian is by observations on Polaris. This method is easily understood, and easily carried out in the field. It involves almost no computations, and the few astronomical data required are easily obtained.*

507. Meridian. The two points in which the axis of the earth, if indefinitely produced, pierces the celestial sphere are the *celestial poles.* A vertical line through any point on the earth's surface pierces the upper portion of the celestial sphere at the *zenith* of that point, and the lower portion of the sphere at the *nadir.* The *celestial meridian* for any given point on the earth's surface is a great circle of the celestial sphere passing through the zenith of that point and the celestial poles; the corresponding *meridian plane* intersects the earth's surface in a great circle or *meridian* through the given point. In surveying, this meridian is often called the *true meridian* to distinguish it from a magnetic meridian. Since the meridian through a point on the earth's surface and the corresponding celestial meridian lie in the same vertical plane, there is little need in practical work of making any distinction between the two; each is referred to simply as "the meridian." Similarly a celestial pole may be referred to simply as the "pole." The *longitude* of a point on the earth's surface is the angular distance measured along the equator between a reference meridian, called the *primary* meridian, and the meridian circle through the point. The primary meridian generally used passes through Greenwich, England. Longitude is measured eastward or westward, 0° to 180°. Longitude may be expressed either in units of angular measurement or in units of time. (One hour of time is equivalent to 15° of longitude.)

508. Why Polaris Is Chosen for Observations. All heavenly bodies *appear* to revolve once a day about the earth, describing

* Other methods most used by surveyors for determining a true meridian depend upon observations on the sun. They are somewhat difficult to understand and to carry out in the field, more or less computation is involved, and, for accurate results, astronomical data are required which are not always at hand. These methods are explained in Chapter 35.

circles on the celestial sphere. If the distance of a star from the pole is less than the altitude of the pole, the circle which the star apparently describes will not meet the horizon at all. The star will, therefore, neither rise nor set, but will appear to revolve about the pole as a center. If there were any star exactly at the pole it would seem to remain stationary, and to determine a true meridian it would be necessary simply to bring the cross hairs of the telescope upon it. Unfortunately for the purposes of the surveyor, there is no such star. Contrary to a general impression, Polaris (North Star) is not exactly at the pole, but like all the other stars, it appears to revolve around it. The radius of its diurnal orbit is so small, however, that of all the stars, Polaris is the most convenient for determining a meridian. Any other star near the pole can be used, provided that certain data concerning it are known. The inconvenience of night observations on Polaris is counterbalanced by the simplicity of this method of determining a meridian.

508(*a*). Latitudes too high or too low for observations on Polaris. At any place of observation, the vertical angle which the line of sight to Polaris makes with the horizontal is nearly equal to the latitude of the place. When this angle is so large that the observer cannot sight through the telescope because of its steep inclination, the latitude is obviously too high for direct observation on Polaris. On the other hand, the latitude of a place may be so low that Polaris is too near the horizon. When observation on Polaris is difficult, inconvenient, or impossible, a meridian or an azimuth may be determined by observation on some other star or on the sun, as explained in Chapter 35. (Note particularly Art. 538, "Determination of the Meridian in High Latitudes," at the end of Chapter 35.)

Although Polaris may be observed at places of high latitudes by means of a reflecting eyepiece, there are various obstacles and sources of error which render such observations difficult and inaccurate.

Note. Since meridians converge toward the poles, the meridian through one station of a survey is *not* parallel to the meridian through another station. When a survey covers a small area, the convergence of the meridians is usually negligible. When, however, the distance between two stations, measured east and west, is comparatively great, as it often is, for example, in route surveying, it may be necessary to take into account this convergence. (See Art. 813 for convergence of meridians.)

509. Positions of Polaris.

1. A simple way of finding Polaris is to find first the "Great Dipper" (Ursa Major), and then to follow the line through the two "Pointers," i.e., the two stars farthest from the handle. This line passes very near to the pole. (See Fig. 509(*a*).)

2. Polaris is the end star in the handle of the "Little Dipper" (Ursa Minor) and is bright enough to be seen easily. Approximately 15° from the pole is another star about as bright as Polaris, and between these two is a curved row of three stars too faint to be easily observed, which form the handle of the "Little Dipper."

3. As Polaris, appearing to revolve counterclockwise about the celestial pole, reaches its highest point, U.C. (Fig. 509(*b*)), it is said to be at its *upper culmination;* when it reaches its lowest point, L.C., it is at its *lower culmination.* In either case *it is in the true meridian plane.*

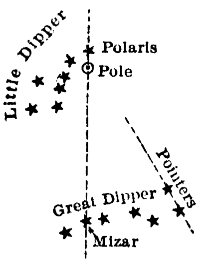


FIG. 509(*a*).

4. When Polaris reaches the most easterly point of its orbit it is at its *eastern elongation,* E.E.; likewise at the most westerly point it is at its *western elongation,* W.E.

5. The earth turns on its axis once in about 23 hr. 56.1 min. (*the sidereal day*); hence Polaris reaches any point in its orbit, as, for example, its culmination or elongation, about 4 min. earlier each day than it did the preceding day.

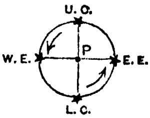


FIG. 509(*b*).

6. The radius of the circle in which Polaris appears to revolve is about one degree, but it is constantly changing by a very small amount per year. It will continue to grow less until the star is about 30′ from the celestial pole when it will begin to increase.

7. The *hour circle* for Polaris at any instant is a great circle passing through Polaris and the celestial poles. The *hour angle* for Polaris at the same instant is the angular distance measured westward along the celestial equator from the local or observer's meridian to the hour circle of Polaris. It may be measured in units of angular measurement or in units of time.

8. The *declination* of Polaris at any instant is the angular distance between the celestial equator and Polaris, measured along the hour circle. The *polar distance at the same instant* is equal to 90° minus the declination.

9. When Polaris is at its upper or lower culmination a line of sight directed to it from any point on the earth's surface will be in a meridian plane through the point. This occurs twice every twenty-four hours.

10. If Polaris were exactly at the pole, the vertical angle which a line of sight to the star makes with the horizontal, corrected for refraction, would be equal to the latitude of the point where the transit stands. The latitude of a place can, therefore, be determined by measuring the vertical angle to Polaris when it is at either upper or lower culmination and then subtracting from or adding to this vertical angle the star's polar distance. (See Art. 537 for methods of determining latitude.)

510. The Azimuth of Polaris for any place on the earth's surface and at any instant is the angular distance, measured along the horizon, from the meridian through the given place to the vertical circle through Polaris at the given instant. It may be expressed in units of angular measurement or in units of time. If Polaris is east of the meridian, its azimuth is positive; if it is west of the meridian, its azimuth is negative.

The value of the azimuth is a maximum when Polaris is at either the eastern or the western elongation; it is zero when the star is at either the upper or the lower culmination. In latitudes below 50° N, the value of the azimuth at elongation is never much greater than 1° 30′. When Polaris is near either culmination, its apparent motion is comparatively rapid and the value of its azimuth changes rapidly whereas, when Polaris is near either elongation, its apparent motion is slow and the value of the azimuth changes slowly. At elongation, the star seems to remain stationary for several minutes, and there is little appreciable change in azimuth during that period of time.

The values of the azimuth at elongation that are given in tables are generally those for eastern elongation. The same values considered as negative may be used for western elongation.

510(*a*). NOTE. In astronomical work, azimuths are generally measured from the south point, but for observations on Polaris many engineers find it more convenient to measure them from the north point.

510(*b*). AZIMUTH AT ELONGATION. If an observer at the equator points the telescope of his transit at Polaris when at its *eastern* elongation, the line of sight will be east of the true north by a horizontal angular distance equal to the azimuth at elongation. If at the same time an observer at a latitude *north* of the equator points the telescope of his transit at Polaris, his line of sight will also be east of the true north by a horizontal angular distance equal to the azimuth at elongation *for the latitude of the place of observation. This angle will be greater,* however, than that at the equator, and the farther north the observer the greater the difference between the two angles. For example, the value of the angle if the observer is at latitude 32° N is about 10′ greater than it would be if he were at the equator, and about 10′ less than it would be if he were at latitude 42° N. In other words, the azimuth at elongation at a given instant is different for observations taken at different latitudes. Moreover, the value of the azimuth at elongation changes slightly from year to year and from month to month. Consequently, in order to obtain the value of an azimuth at elongation for use in connection with an observation on Polaris, both the

latitude of the observation station and the time (year, month, and possibly the day and hour) at which the observation is made must be taken into account. This holds true not only for an azimuth at elongation but also for the azimuth when Polaris is at any other point.

510(*c*). NOTE. *The spherical triangle PZS* having for its vertices the pole (*P*), the zenith (*Z*), and a heavenly body *S* is an *astronomical triangle*. (See Art. 523.)

The *azimuth* of any star *at elongation* is equal to the angle *PZS*, but since *Z* changes for different points of the earth's surface the *azimuth PZS* must change also; but the *polar distance PS* at any given time is, of course, the same, for it is in no way connected with the position of the observer.

If the observer is at the equator his zenith must be 90° from the pole, i.e., $PZ = 90°$, and if the star is at its elongation *ZS* must equal 90° also; hence the angle *PZS* must equal the arc *PS*, i.e., the azimuth equals the polar distance. As the observer goes north, however, his zenith moves towards *P*, i.e., *PZ* becomes less than 90°, *for PZ must always equal* 90° *minus the latitude of the observer*. The farther north the observer goes the shorter become the two sides *PZ* and *SZ* of the triangle *PZS*, but since the third side *PS* remains the same the angle *PZS*, opposite to *PS*, must increase, i.e., the greater the latitude of the observer the greater the azimuth of the star.

510(*d*). THREE METHODS OF OBSERVING POLARIS WITH A TRANSIT.

1. By sighting at Polaris when it is at either its upper or its lower *culmination*. The line of sight will then be directed to the true north and will lie in the plane of the meridian through the transit station. Previous to the observation, the *exact time of culmination* must be determined.

2. By sighting at Polaris at either its eastern or its western *elongation*. The line of sight will then be an angular distance east or west of the true north equal to the azimuth of Polaris at its elongation. Previous to the observation, the *approximate time* at which the star reaches its elongation must be determined. Either before or after the observation, the *azimuth of Polaris at elongation* for the latitude of the place of observation must also be determined.

From tables in the back of this book, the time of culmination (first method) or the time of elongation (second method) may easily be determined (Arts. 512, 513). Likewise from these tables, the azimuth of Polaris at elongation for the year of observation and for the latitude of the place of observation may easily be determined (Art. 514). However, since the time of culmination or of elongation given in the tables is the time *at the meridian of Greenwich*, it is necessary to understand clearly how to obtain *local civil time* from *Greenwich civil time* and from *standard time* or *watch time* (Art. 511).

Both the field work of observation (Arts. 516, 517) and the calculations involved in either the first or the second methods are very simple and require little knowledge of field astronomy.

3. In a third method of observation, the sight on Polaris is taken at *any* convenient time (Art. 518). In this method the azimuth of Polaris at the time of observation and for the latitude of the place of observation must be determined, and this involves the solution of the astronomical triangle *ZPS* (Arts. 510(*c*), 515). The third method is used when it is inexpedient to observe Polaris at the time of its culmination, or of its elongation.

511. Time. (*a*). THE APPARENT SOLAR DAY at any place is the period of time that elapses between two successive lower transits of the sun's center across the meridian of that place. Because of the elliptical shape of the earth's orbit about the sun, the length of the *apparent solar day* varies slightly from day to day throughout the year. In order to make our solar days of uniform length, an imaginary *mean* sun is assumed to cross the same meridian every twenty-four hours. This *mean solar day* is the unit of *mean solar time,* the kind of time with which we are most familiar. (See standard time.) The difference between apparent solar time and mean solar time is given for each day of the year in *The American Ephemeris and Nautical Almanac* and is called the *equation of time. Apparent time* and *mean time* have the same meanings, respectively, as *apparent solar time* and *mean solar time.*

511(*b*). CIVIL TIME is the same as mean time and is generally measured from the primary meridian through Greenwich, England. The difference between *local civil time* for any place and Greenwich civil time is equal to 1 hr. per 15° of difference in longitude between the place and Greenwich, or 4 min. per degree of difference in longitude. Civil time begins at midnight and the hours are numbered consecutively from 0 to 24. Any hour greater than 12 is, therefore, past noon. The designations A.M. and P.M. are not used in astronomical work.

511(*c*). STANDARD TIME. Standard time is based on mean solar time. Standard meridians are meridians at intervals of 15° (one hour), beginning with zero at the Greenwich meridian. The standard meridians within the continental United States and the corresponding standard time zones are: 75° W = 5 hr. (eastern); 90° W = 6 hr. (central); 105° W = 7 hr. (mountain); and 120° W = 8 hr. (Pacific). The *standard time* and the *local civil time* at any standard meridian are the same, but for any place not on a standard meridian the local civil time differs from the standard time. Time pieces within a certain time zone extending on each side of a standard meridian are regulated to keep the local civil time (or standard time) of that meridian, but they will not give the local civil time at any place of observation in the zone unless

that place happens to be on the meridian. In determining the standard time of an observation, it is not necessary that the observer should know where the boundaries of his time zone are. All that he needs to know are (1) the longitude of the place where the observation is taken and (2) that his watch is keeping correctly the time of one of the standard meridians.

511(*d*). TIME ZONES. The boundary lines between the time zones run quite irregularly in a general northerly and southerly direction. Within the continental United States, the eastern zone extends from the Atlantic Ocean to a boundary line that meanders between longitudes 83° and 85° west; the central zone extends from the line just named to an irregular line between longitudes 99° and 105°; the mountain zone extends from the last-named line to an irregular line between longitudes 111° and 117°; the Pacific zone extends from the last-named line to the Pacific Ocean.

511(*e*). STANDARD TIMES FOR ZONES. The standard time in the *eastern* zone is that of the 75th meridian, i.e., it is noon (standard time) in this zone 5 hr. after it is noon at Greenwich; in the *central* zone the time is that of the 90th meridian, or 6 hr. slow; in the *mountain* zone the time is that of the 105th meridian, or 7 hr. slow; in the *western* zone the time is that of the 120th meridian, or 8 hr. slow. The *local civil* time at any place will be faster or slower than standard (or watch) time according to whether the place is east or west of its standard time meridian. Thus the true local time at Boston, Massachusetts, is about 16 min. faster than standard time, whereas at Buffalo, New York, it is about 16 min. slower, the 75th meridian being approximately halfway between those two cities. Local civil time and standard time nearly agree at Denver, Colorado, as Denver is almost on the 105th meridian.

511(*f*). WATCH TIME is a common expression used to denote the time as given by the timepiece used during an observation. It should be identical with the standard time for the zone in which the observation is taken except when the timepiece is keeping daylight-saving time.

511(*g*). RADIO TIME SIGNALS. Standard time is broadcast by signals to radio and other stations by the U.S. Naval Observatory. Radio time signals given hourly by the larger broadcasting systems are usually correct within a fraction of a second. Shortly before the period of observations, either the error in the timepiece should be noted or the timepiece should be set by means of these time signals or by comparison with another timepiece which is known to be correct.

511(*h*). TO OBTAIN THE DIFFERENCE BETWEEN STANDARD TIME AND LOCAL CIVIL TIME at any place. Find the difference in longitude between the place and the standard meridian for the time zone in which the place is situated. Multiply this difference by four and call the product of the degrees *minutes*, and of the minutes, *seconds* of time. For example, if the longitude of an observation station is 70° 20′ 24″: Difference in longitude = 75° − 70° 20′ 24″ = 4° 39′ 36″; difference in time = (4° 39′ 36″) × 4 = 16 min. 156 sec. + (144 ÷ 60) sec. = 18 min. 38.4 sec. (This method is based on the fact that 15° of longitude is equivalent to 1 hr. of time.) When the longitude of a place is expressed in units of time, the difference in time is found by simple subtraction. Using the illustration just given, the longitude of the station, expressed in units of time, is (70° 20′ 24″) ÷ 15 = 4 hr. 41 min. 21.6 sec. Difference between standard time and local civil time at the observation station is 5

hr. – (4 hr. 41 min. 21.6 sec.) = 18 min. 38.4 sec. (See Table I in the back of this book for conversion of degrees to time, and vice versa.)

511(*i*). TO DETERMINE THE STANDARD TIME OR WATCH TIME FROM THE LOCAL CIVIL TIME AT ANY PLACE. (1) Find the difference between the standard time and the local civil time as just explained. (2) Add this difference to the local civil time if the place is west of the standard time meridian for the time zone in which the place is situated; subtract the difference if the place is east of the time meridian.

511(*j*). SIDEREAL TIME. An equinox is either of the two points where the celestial equator and the elliptic (sun's apparent path) intersect; the *vernal* equinox is the one at which the sun apparently crosses the celestial equator from south to north. (About March 21, each year.) Although the vernal equinox is an imaginary point, it is helpful to imagine it as if it were a star in a fixed position on the equator of the celestial sphere.

The sidereal day is the time that elapses during one apparent revolution of the vernal equinox; it is divided into 24 *sidereal hours* each of 60 *sidereal minutes*. At any place of observation, the sidereal day begins at the instant of the upper transit of the vernal equinox across the meridian of that place and ends at the instant of the next upper transit of the vernal equinox across the meridian 24 sidereal hours later. (A little less than 24 hours later by watch [mean solar] time.)

The sidereal day is shorter than the mean solar day of 24 hr. by 3 min. 55.909 sec. mean solar time or civil time, or by 3 min. 56.555 sec. sidereal time. Hence sidereal time is faster than mean time at the rate of approximately 4 min. per 24 hr., or 1 min. per 6 hr., or 10 sec. per hr.

To change a given sidereal-time interval to the equivalent mean-time interval, subtract a quantity equal to the *sidereal-time interval*, expressed in hours, multiplied by 10 sec. To change a given mean-time interval to the equivalent sidereal-time interval, add a quantity equal to the mean-time interval, expressed in hours, multiplied by 10 sec. For example, 4.5 hr. (mean time) = 4.5 hr. + (10 × 4.5) sec. = 4 hr. 30 min. 45 sec. (sidereal time). In *The American Nautical Almanac* and in *The American Ephemeris and Nautical Almanac* there are convenient conversion tables for changing sidereal time to mean time and the reverse.

The sidereal day is a unit of time much used in astronomy. For example, the hour angle of Polaris at any instant of observation is measured (in units of time) by the period of time between the in-

stant of upper culmination and the time of observation. If this time of observation is mean time, it must be changed to sidereal time in order to obtain the true hour angle of the star that would be needed in the solution of an astronomical spherical triangle. Likewise, a change from mean time to sidereal time, or vice versa, is often necessary in the use of various tables such as those referred to in the next article.

512. To Determine the Time of Culmination of Polaris. The time of the upper culmination of Polaris may easily be determined by means of Tables IV, IV*A*, and IV*B* in the back of this book.* The lower culmination precedes or follows upper culmination by 11 hr. 58.0 min. If the observation is to be taken at upper or at lower culmination, the time of culmination should be determined as accurately as practicable. If, however, the observation is to be taken at eastern or at western elongation and the time of culmination is determined as the first step in computing the time of elongation, an error in time of a minute or two should not affect the result of the observation. The procedure in accurately determining the time of upper culmination is as follows.

Step 1. Ascertain the longitude of the place of observation if it is not already known to the observer. If the observation is to be at either elongation, the latitude will be needed also.

Remark. A geographic difference of 1 mile in an east-west direction is equivalent to approximately 1′ of arc or 4 sec. of time. Consequently the accurate determination of both longitude and time is important, particularly if the observation is to be made at culmination.

Step 2. Find in Table IV the date of observation, or the nearest preceding date. Note opposite this tabular date the local civil time of upper culmination for the meridian of Greenwich for the year given in the heading at the top of the table. Note also (1) the variation per day for the tabular date and (2) if the tabular date is not that of the observation, note also the number of days elapsed between the tabular date and the date of observation.

* *Other tables.* Data pertaining to Polaris similar to the data in Tables IV, IV*A*, IV*B*, and IV*C* but given more in detail and with a greater degree of precision are contained in tables in the following government publications: (1) *The American Ephemeris and Nautical Almanac;* (2) *The American Nautical Almanac;* and (3) *The Ephemeris.* Any one of these publications can be purchased from the Superintendent of Documents, Government Printing Office, Washington, D.C. For ordinary surveying, *The American Nautical Almanac,* commonly called "Nautical Almanac" is recommended. (See Art. 522.) It is more concise and less expensive than *The American Ephemeris and Nautical Almanac,* a publication widely used by astronomers. (See Art. 522.) Some of the makers of engineering instruments issue annually excellent booklets in the form of an ephemeris which contain not only data pertaining to the sun and Polaris but also explanations of methods of computations together with illustrative examples.

Step 3. Apply to the time of upper culmination that was obtained from Table IV in step 2 the correction in Table IV*A* for the year (and in some cases the portion of the year) in which the observation is to be made. If the year is the same as the one for which Table IV was computed, there is, obviously, no correction.

Step 4. In Table IV*B*, find, in the first column, the number of "days elapsed" that is the same as the number noted in step 2. Opposite this number of days, find the total variation in time that is in the column headed by the same variation per day as the variation per day noted in step 2. Subtract this total variation from the corrected time found in step 3. The result is the local civil time of upper culmination for the meridian of Greenwich on the date of the observation.

Remark. The "variation per day" is due to the fact that the relative positions of Polaris and the sun are continually changing. The rate of change is not exactly the same from day to day; it is given in *The Nautical Almanac* for each day. In general, it is a few seconds less than 4 min.

Step 5. From the result obtained in step 4 subtract 10 sec. of time for each 15° (1 hr.) of longitude that the place of observation is west of Greenwich, or add 10 sec. for each 15° that the place is east of Greenwich. The result will be the local civil time (not the standard or watch time) of upper culmination for the meridian at the place of observation. If the time of lower culmination is desired, subtract 11 hr. 58.0 min. if it is the preceding lower culmination, or add 11 hr. 58.0 min. if it is the succeeding lower culmination.

Remark. The "variation per hour" is due to the same cause as the "variation per day" and is one twenty-fourth as great. The total correction obtained in step 5 is for the time that elapses while the earth turns sufficiently to bring Polaris from the meridian of Greenwich to the meridian of the place of observation. This total correction varies from about 45 sec. on the Atlantic coast of the United States to about 1 min. 22 sec. on the Pacific coast. In determining the time for an observation at *elongation* it is not essential that this correction be made.

Step 6. Determine the standard or watch time that corresponds to the local civil time of upper culmination found in step 5. (See Art. 511(*i*).)

Suggestions. (1) If the longitude and latitude of the place of observation are unknown, they can easily be scaled from a topographic atlas sheet issued by the U.S. Geological Survey or from some other equally complete map. When no good map is available, a request for the longitude and latitude may be addressed to the Director of the U.S. Coast and Geodetic Survey. Note that the latitude may be determined from the vertical angle to Polaris at culmination (Art. 537(*a*).) Ordinarily, the value of latitude is required only to the nearest 5′ of arc, or approximately 5 miles on the earth's surface. (2) Table VI in the back of the book will be found useful in converting longitude in units of arc to longitude in units of time, or vice versa.

512(*a*). EXAMPLE. To find the standard or watch time of upper culmination of Polaris on Sept. 21, 1945, at a place whose latitude is N 41° 25′ and longitude W 72° 30′. (The longitude expressed in time is 4 hr. 50 min. See Table VI.)

Step 1. Observer's local coordinates are: Lat. N 41° 25′; Long. W $4^h\ 50^m$		
Step 2. From Table IV the mean time of upper culmination at Greenwich meridian on September 18 is		$1^h\ 58^m.6$
Step 3. From Table IV*A* the correction for the year 1945 is	+	$1^m.7$
Greenwich mean time of upper culmination on Sept. 18, 1945		$2^h\ 00^m.3$
Step 4. From Table IV*B* the correction for 3 days is		$11^m.7$
Greenwich mean time of upper culmination, Sept. 21, 1945		$1^h\ 48^m.6$
Step 5. Correction for difference in longitude (expressed in hours) between Greenwich and the place of observation 4.8 hr. at 10 sec. ± per hr.	subtract	$0^m.8$
Local civil time of upper culmination at the place of observation on Sept. 21, 1945		$1^h\ 47^m.8$
Step 6. Correction for standard (i.e., watch time). The place of observation is 2° 30′ east of the standard meridian for the place of observation. For 2.5° at 4 min. per degree	subtract	$10^m.0$
Watch time, upper culmination at place of observation on September 21		$1^h\ 37^m.8$
If daylight-saving time is desired, add one hour		$2^h\ 37^m.8$

513. To Determine the Time of Elongation of Polaris. (1) Calculate the time of upper culmination for the time and the place of observation as just explained in Art. 512. (2) From Table IV*C*, obtain the time interval (by interpolation if necessary) which corresponds to the *latitude* of the place of observation. (3) To obtain the time of eastern elongation, subtract from the time found in (1) the time interval obtained in (2); to obtain the time of western elongation add to the time found in (1) the time interval obtained in (2).

Note. If the *eastern* elongation of Polaris occurs during the interval between 6:00 P.M. (approximately) and midnight of any day, the upper culmination, which follows about 6 hours *later*, will occur after midnight in the early hours of the *next* day. Consequently, in determining the time of eastern elongation for an observation during the evening of a certain day, it is convenient to begin with the time of the upper culmination on the following day. Thus in the example which follows, the observation is on September 20; the time of culmination used is that of September 21. If a *western* elongation occurs between midnight and 6:00 A.M. on any day, the corresponding upper culmination will occur about 6 hours *earlier* or in the late hours of the *preceding* day.

513(*a*). EXAMPLE. To find the standard or watch time of eastern elongation of Polaris on the night of Sept. 20, 1945, at a place whose latitude is N 41° 25′ and longitude 72° 30′ west.

Step 1.	Standard time (watch time) of upper culmination at the place of observation previously calculated for September 21. (See Art. 512(*a*).)	$1^h\,37^m.8$
Step 2.	Mean time interval for latitude N 41° 25′ from interpolation, Table IVC	subtract $5^h\,55^m.5$
		$19^h\,42^m.3$
	For P.M. time	subtract 12^h
	Standard time of eastern elongation at place of observation on September 20	$7^h\,42^m.3$
	If the observer's watch is set to daylight-saving time	$8^h\,42^m.3$

513(*b*). APPROXIMATE TIME OF ELONGATION. In the example just given, the results obtained are more accurate than is generally required. At elongation, Polaris appears to move neither east nor west for several minutes; hence, as will be explained later, it is not necessary to know the exact time of elongation. Indeed an observation can be taken with little regard to time by keeping the vertical hair on Polaris until the star apparently ceases to move. This cannot be done, of course, in observing Polaris at culmination; hence for such an observation the time of culmination should be determined accurately.

Note. If the time of elongation at any place of observation is known for a given day, it can be obtained *approximately* for any other day by allowing for a change of 4 min. per day, or, more accurately, the change may be taken from Table IV*B*. This allowance should be subtracted when the desired time is on a later day, added when it is on an earlier day. When there is an interval as great as 30 days between the given day and the day for which the time is desired, the error in the time obtained by using the approximate variation of 4 min. per day could vary from 1½ to 2½ min.; but for an interval of 10 days it could not exceed 30 to 50 sec., and would be correspondingly less for a shorter interval. For the purpose of checking bearings or azimuths in an extended survey, it is advantageous to prepare a table of times of elongation for a period of several days.

513(*c*). THE TIME OF ELONGATION CAN BE CALCULATED by use of the formula:

$$\text{Cosine hour angle} = \text{tangent latitude} \times \text{cotangent declination}$$

Methods of calculation, based on the spherical triangle, are similar to those in Chapter 35.

514. To Determine the Azimuth of Polaris at Elongation. (1) From Table V, in the back of this book, find the azimuth, by interpolation if necessary, that corresponds to the latitude of the place of observation and to the year of observation. If the azimuth at western elongation is desired, the azimuth angle may be considered negative (Art. 510). (2) Since, in computing Table V, the mean declination of Polaris for the *beginning* of each year was used, a more accurate result may be obtained, if desired, by applying to the result obtained in (1) a correction for the month in which the observation is taken, as given in supplementary Table V*A*.

514(*a*). Example. It is required to determine the azimuth of Polaris at eastern elongation for an observation taken in September 1945, in a place whose latitude is N 41° 25′.

Azimuth for 1945, latitude N 41° 25′ = 1° 19′.3 + $^{25}/_{60}$ × 1.3 =	1° 19′.8
Correction for September	− 00′.1
Required azimuth	1° 19′.7

The same angle substracted from 360° will give the azimuth of Polaris at western elongation for the same latitude, year, and month

Note. The most accurate published values of azimuths of Polaris are those at elongation. They are given in *The American Ephemeris and Nautical Almanac* to a degree of precision of 0.1″ of arc. Declinations are tabulated to each 10″.0 and latitudes to each 10′.0, with a supplementary correction table for variations of 1′ of latitude. In the use of these tables of azimuths, it is necessary to know the hour angle of Polaris in terms of sidereal time.

514(*b*). The azimuth of Polaris at elongation can be calculated by spherical trigonometry from:

$$\text{Sine of azimuth at elongation} = \frac{\text{sine polar distance}}{\text{cosine latitude}}$$

Note that the polar distance of Polaris at any instant is equal to 90° minus the declination of Polaris at the same instant (Art. 509(8)).

515. Azimuth of Polaris at any Time. In *The American Nautical Almanac* and also in *The Ephemeris*, tables of hour angles of Polaris are given. By means of these tables the azimuth of Polaris for an observation at any time can be obtained quickly with but little calculation. To find azimuth by the use of the tables it is necessary to know only the time of observation and the latitude and longitude of the place. The method will be explained by means of an illustrative example.

515(*a*). Example. Find the azimuth of Polaris at 8.20 p.m., eastern standard time, on October 20, 1941, at a place whose latitude is N 41° 23′.2, longitude W 72° 15′.3.

The corresponding Greenwich mean time is $25^h 20^m$ on October 20 or $1^h 20^m$ on October 21. From the table in *The American Nautical Almanac* entitled "Greenwich Hour Angle of Polaris," the Greenwich hour angle of Polaris (see Art. 509(7)) at zero hours on October 21 is found to be 2° 43′.2. To this value must be added the correction to hour angle for $1^h 20^m$. This correction is found in the table entitled "Correction to Greenwich Hour Angle of Stars" to be 20° 03′.3. The sum 22° 46′.5 is the Greenwich hour angle of Polaris at the instant of observation. Since both the Greenwich hour angle and the longitude of the place of observation are measured from the meridian of Greenwich and in units of arc on the great circle of the equator, the required hour angle of Polaris at the place of observation can be obtained by subtracting the longitude of the place (W 72° 15′.3) from the hour angle of Polaris at Greenwich, i.e., (22° 46′.5) − (72° 15′.3) = −49° 28′.8. The minus sign in this hour angle simply indicates that Polaris is *west* of the meridian of the place of observation. This hour angle is the quantity required to obtain the azimuth of Polaris from Table IV of *The American Nautical Almanac* entitled "Azimuth of Polaris; Argument, Hour Angle." (The table gives azimuths to tenths of degrees.) By interpolation, the azimuth of Polaris is found to be −1°.0 or 1° west.

Note. Greenwich hour angles of Polaris are not given in *The American Ephemeris and Nautical Almanac.* In that publication, hour angles of Polaris are given in terms of sidereal time only.

The azimuth of Polaris at any time may be determined more accurately from *The Ephemeris* issued by the General Land Office, which gives azimuths of Polaris at hour angle to tenths of minutes of arc for all declinations. To use the latter table, however, it is necessary to convert hour angle into *units of time.* In the present illustration, 49° 28′.8 corresponds to $3^h 17^m.9$. Using this value of the hour angle and the observer's latitude (N 41° 23′.2), the azimuth of Polaris is found by interpolation (first for time and then for latitude) to be 1° 02′.4 (east). Since the declination of Polaris during the

month of October is very closely that used in compiling the table of azimuths in *The Ephemeris*, no further correction is necessary. For other months in which the declination differs considerably from the mean, it may be necessary, when close results are desired, to apply a correction to the azimuth. The correction will be found in a supplementary table in *The Ephemeris*. To enter the correction table, it will first be necessary to select from the ephemeris of Polaris its declination for the month and day of observation. The algebraic sign of the correction for each declination and hour angle is given in the table.

515(*b*). AZIMUTH OF POLARIS DETERMINED BY THE SOLUTION OF AN ASTRONOMICAL TRIANGLE. The azimuth of Polaris for an observation at any given time may be calculated by means of the following formula:

$$\tan Z = \frac{\sin t}{\cos\phi \tan d - \sin\phi \cos t}$$

Z the azimuth angle; t the hour angle of Polaris (in units of arc) at the place of observation; ϕ the latitude of the place of observation; d the declination of Polaris at the time of observation. In the use of this formula, proper algebraic signs should be given to trigonometric functions. Note that the hour angle is measured westward from 0° to 360° (Art. 509(7)).

The hour angle t at the place of observation must first be determined as follows:

Step 1. Calculate the Greenwich *sidereal* time of observation from the Greenwich mean time. The procedure is: (1) Add to the Greenwich mean time of observation a small correction, approximately 10 sec. per hr. of that mean time, in order to obtain the Greenwich sidereal interval (Art. 511(*j*)). This small correction, which is required to convert a mean solar interval into a sidereal interval, can be taken from the table entitled "Mean Solar into Sidereal Time" in *The American Ephemeris and Nautical Almanac*. The sum is the sidereal interval since 0^h of mean time at Greenwich. (2) To the Greenwich sidereal interval found in (1) add the sidereal time that corresponds to 0^h of Greenwich civil time on the date of observation. This value can be taken from the column in *The American Ephemeris and Nautical Almanac* for the month and day of observation, under the heading "Sidereal Time (Right Ascension of Mean Sun + 12 hr)." The sum is the Greenwich sidereal time of observation.

Note. Right ascension is the angular distance of a heavenly body measured eastward along the celestial equator from the vernal equinox.

Step 2. Subtract the west longitude of the place of observation from the Greenwich sidereal time in order to obtain the local sidereal time of observation.

Step 3. From the local sidereal time found in step 2, subtract the right ascension of Polaris for the month and day (at Greenwich). The result is the desired hour angle of Polaris at the place of observation. Convert this hour angle into terms of arc to obtain the value of t for use in the equation for azimuth. The right ascension of Polaris will be found in the table in *The American Ephemeris and Nautical Almanac*, headed "Apparent Places of Circumpolar Stars."

515(*c*). ILLUSTRATIVE PROBLEM. Determine the azimuth of Polaris at 8:20 P.M., eastern standard time, on Oct. 20, 1941, at a place whose latitude is N 41° 23′.2 and longitude W 72° 15′.3 (or, in time 4 hr. 49 min. 01.2 sec.).

Eastern standard time at place of observation Oct. 20, 1941, is	$20^h\ 20^m$
Standard time, W longitude	$5^h\ 0^m$
Greenwich mean time October 20	$25^h\ 20^m$
October 21	$1^h\ 20^m\ 00^s$
Step 1. (1) Correction to sidereal interval (for $1^h\ 20^m$)	$0^m\ 13.1^s$
Sidereal interval since 0^h at Greenwich	$1^h\ 20^m\ 13^s.1$
(2) Sidereal time of 0^h at Greenwich on October 21 (from *The Ephemeris*)	$1^h\ 56^m\ 11^s.1$
Greenwich sidereal time of observation	$3^h\ 16^m\ 24^s.2$
Step 2. W. longitude of place of observation	$-\ 4^h\ 49^m\ 01^s.2$
Local sidereal time of observation at the place of observation	$22^h\ 27^m\ 23^s$
Step 3. Right ascension of Polaris on October 21 (from *The Ephemeris*)	$1^h\ 45^m\ 18^s.2$
Hour angle of Polaris at place of observation	$20^h\ 42^m\ 4^s.8$
Hour angle, in arc = t	310° 31′ 02″

Substituting in equation for azimuth and solving for Z gives 1° 02′.2 or N 1° 02′.2 E as the true azimuth of Polaris at the time of observation. For accurate results seven-place tables of logarithms reading to seconds are recommended.

When computing machine and tables of natural funcations are used, the following formula is recommended:

$$Z = p \sin t \sec (\phi + p \cos t)$$

p represents polar distance in terms of arc. The remaining symbols are the same as those in the formula for azimuth used in the preceding example.

516. Observation at Elongation. (1) Before the night of the observation, ascertain the time of elongation (Art. 513). It is desirable but not essential to know the time within a minute or two. (2) Select some fixed point as an observation station and set up the transit over this station at least 10 or 15 min. before the time of elongation. Sight on Polaris and, keeping the vertical cross hair exactly on the star, follow it in its apparent movement until it reaches its elongation, i.e., when it appears to move neither east nor west. Make sure that all plates are clamped before taking the next step. (3) Depress the telescope and set a point on line at least 400 ft. from the transit. The line between this point and the observation station makes an angle with the true meridian through the station equal to the azimuth of Polaris at elongation. The remainder of the work may be done more conveniently in the daytime. (4) Before continuing the work the next day, determine the azimuth of Polaris for the time of the observation and for the latitude of the observation station (Art. 514). (5) Set up over the observation station occupied the night before, backsight on the point established by observation on Polaris in (3), and turn off an angle equal to the azimuth of Polaris determined in (4), clockwise if the observation was taken at western elongation, counterclockwise if taken at the eastern elongation. The line of sight should now be in the meridian plane through the station. A point may be set on line, thus establishing a true meridian between the observation station and the point.

516(*a*). ALTERNATIVE METHOD. Instead of establishing a meridian, it is frequently better to determine the true azimuth of a line that extends from the observation station to a selected fixed point of reference. If a meridian is desired, it can then be established at any convenient time from this line of reference. The procedure for this second method is as follows. (1) Ascertain the time of elongation. (2) Select an observation station and also a fixed point of reference at least 400 ft. from the station. Set up over the station at least 10 or 15 min. before the time of elongation and backsight on the fixed point. (3) Measure the angle from this backsight to the foresight on Polaris. This is done by foresighting on Polaris before elongation and then keeping the vertical cross hair on the star until it reaches elongation, i.e., until it appears to

move neither east nor west. (4) Determine the azimuth of Polaris for the time of observation and for the latitude. (5) From the azimuth of Polaris determined in (4) and the angle measured in (3) calculate the required azimuth of the line of reference. (6) If it is desired to determine the meridian, backsight on the fixed point of reference, turn backward the azimuth that was calculated in (5), and thus obtain the line of zero azimuth.

516(*b*). SETS OF OBSERVATIONS. There are two objections to establishing a meridian or to determining the true azimuth of a line by only *one* observation on Polaris. (1) There is no check on the work. (2) If the transit is out of adjustment so that the line of sight does not revolve in a vertical plane, the resulting error is likely to be large since the vertical angle through which the telescope is depressed after sighting at Polaris is large. Both objections may be overcome by taking a double observation. As Polaris does not move east or west more than about 5″ of arc for a period of 10 min. before and after elongation, there is ample time for two observations, the first with the telescope normal and the second with it inverted. (See "To Project a High Point to the Ground" (Art. 254).) Such a set of two observations may be taken regardless of whether the first or the second method is used. In the first method two points will be projected from Polaris to the ground. If these two points coincide, the instrument is probably in adjustment; otherwise the error may be eliminated by taking a point halfway between. During the period when Polaris does not appear to move either east or west, a skillful transitman can take at least two sets of two observations each, thus establishing four points.

Likewise, in determining the azimuth of a line from an observation station to a fixed point of reference, errors in adjustment may be eliminated by measuring the azimuth angle twice, once with the telescope normal, the other with it inverted. A still greater degree of precision can be obtained by measuring the angle by the method of repetition; two or more repetitions should be made with the telescope normal and an equal number with the telescope inverted, as explained in Arts. 265, 266. Note that when two or more measurements of an angle are made, one half with the telescope normal, the other half with it inverted, the transit should *not* be releveled during the entire set of observations. When, however, the position of Polaris is projected to a point on the ground by a set of two observations, the transit is not turned on its vertical spindle after backsighting on Polaris. Hence, by the use of a striding level, the supporting axle of the telescope may be made level immediately before each pointing on the star.

If an observation is taken at any time within a period of 4 min. before or after elongation, the observed azimuth will differ from the azimuth at elongation by less than 1″ of arc; hence it is far more important to eliminate instrumental errors by adjustment of the instrument and by taking sets of observations than it is to observe Polaris at the exact time of its elongation.

516(*c*). ADVANTAGES OF THE ALTERNATIVE METHOD. (1) In the first method, the point projected from Polaris to the ground should be at least 300 to 400 ft. from the transit. At a given observation station, obstacles may make it difficult or impossible to establish such a point at a sufficient distance from the station. In the alternative method, however, the fixed point of reference may be any well-defined point in any convenient direction from the station. (2) A backsight on a fixed point and a foresight on Polaris can usually be taken in much less time than that consumed in backsighting on Polaris and then lining in a point. The time thus saved is important when it is desired to make two or more observations on Polaris within the period of time when the star appears to move neither east nor west. (3) In the alternative method, the method of measuring angles by repetition can be used. It may be added that, in general, whenever more than one pointing on Polaris is to be made, the alternative method is usually the better, regardless of whether observations are at elongation, at culmination, or at any other time.

516(*d*). LIMITS OF ERROR. Under ordinary conditions, with ordinary instruments, and with ordinary skill, the error should never exceed 1′ of arc, and usually it should be less than 30″, provided precautions are taken to eliminate instrumental errors by adjustment of the instrument and by proper methods of procedures in observation.

In order to determine a meridian with a high degree of precision, a fixed point of reference should be used (alternative method) and the horizontal angle between the backsight to this point and the foresight to Polaris should be measured by one of the methods used in triangulation, i.e., by the method of repetition (Art. 265) or by the direction method (Art. 267). Since only a limited number of pointings can be made during the period when Polaris appears to be motionless, it may be well to make observations on each of several nights.

By means of high-precision instruments and the precise measurement of angles (Chapter 27), by the use of accurate methods of observation similar to those used in first-order triangulation, and by determining as nearly as possible the exact value of the

azimuth at elongation, the error should not exceed 10″ of arc and, usually, a considerably higher degree of precision can be attained.

516(c). PRACTICAL SUGGESTIONS.

1. In sighting first at Polaris and then to a point on the ground, or vice versa, the telescope is moved through a relatively large vertical angle; hence it is important that the line of sight should be perpendicular to the supporting axle and that this axle should be truly horizontal. The transit should, therefore, be adjusted on the day of the observation, and it should be carefully leveled up immediately before the observation, or *between* sets of two or more observations each. For greater accuracy, the special method of setting up the transit (Art. 247) may be used. In work of the highest precision, a transit equipped with a strident level is desirable. Precautions should be taken to keep the transit stable during the period of observation.

2. The point over which the transit stands during the observations should be so chosen that the point established after sighting at Polaris will be at least 300 or 400 ft. distant. It is well to select some regular station of a survey, or else, before nightfall, to set a point for this purpose on one of the main traverse lines so that the bearings or the azimuths of all the traverse lines may be easily calculated. (See station *Y*, p. 641.) When, instead of staking out a meridian, it is desired to determine the true azimuth to a fixed point, one station of the survey may be used as an observation station and another station as the fixed point.

3. A quick way of bringing Polaris into the field of the telescope is to clamp the latter inclined at an angle equal to the latitude of the observer, and to turn the transit on its spindle until, looking along the top of the telescope, Polaris is in line.

4. The telescope must be focused for distant objects. As this is sometimes difficult to do at night it is well to mark the correct position, once for all, on the objective slide. When the telescope is properly focused, the star should appear to be a clear, bright point and not a hazy disk.

5. Some form of light, such as a flashlight, an electric lantern, or an ordinary lantern, will be needed at the observation station and at the point on the ground to which the foresight is directed. To illuminate the cross hairs, hold a light within 3 or 4 in. of the objective lens so that its rays will enter the telescope. If the transit is not provided with a reflector, tear a hole about an inch in diameter in a sheet of white paper and by means of a rubber band attach the paper to the telescope in such a position that the star can be seen through the hole while the rest of the paper reflects light into the telescope. The position of the light, found by trial, should be such that the cross hairs and the star are clearly visible at the same time. If not visible, try focusing the eyepiece while sighting directly at the light. Some transits are equipped with a device for illuminating the cross hairs.

6. When a point is to be set on the meridian, no angle will be measured; consequently, no attention is paid to the graduated circle on the limb of the transit, and the two plates may be kept clamped together throughout the observation. Bring the vertical hair on Polaris about 15 min. before elongation and keep it on the moving star by means of the lower tangent screw. At eastern elongation, Polaris apparently ceases to move east, but appears to climb up the vertical hair. It will soon move off the hair westward, showing that it has passed its elongation. In like manner, at the western elongation, the star appears to stay on the vertical hair for some minutes, but then moves downward, and finally eastward. Before depressing the telescope to establish the point on the ground, make sure that both plate clamps are tight. To set the point, bring a light into line by sighting across the top of the telescope and signaling with another light held at the transit. Set a stake in line a few inches in front of the light, driving it until the top is a little above the ground. Hold a pencil on the stake and a handkerchief back of the pencil but in front of the light. The handkerchief not only makes the pencil stand out with surprising distinctness, but it also renders the cross hairs plainly visible; the pencil is easily brought into line and a tack set in the stake. Sometimes it is more convenient to hold a piece of white paper against the side of a fence or building and then direct a strong light on a pencil held against the paper. A box, open on two sides, with a thin sheet of paper stretched across one of the openings and a light behind the paper, is sometimes used. A narrow, vertical slit in the side of a box or of a tin can in

which a light has been placed is another form of target. An illuminated ranging pole (Art. 720) also makes an excellent target. A good code of signals, carried out with lights, will facilitate the work of setting the point exactly on the meridian. It is desirable to repeat the procedure just outlined at least once but with the telescope inverted as explained in Art. 516(*b*).

7. When it is desired to determine the azimuth of a line between the observation station and a fixed point, the details of the procedure are much the same as in (6) except that a horizontal angle must be measured. Since so much depends upon the correct measurement of this angle, the special method of checking angle readings, explained in Art. 241, is recommended. Moreover, for reasons explained in Art. 516(*c*), the backsight should be on the fixed point instead of on Polaris, particularly if there is to be a series of observations on the star. For the precise measurement of angles, see Chapter 27.

8. Since the most convenient period for observation is during the early part of the night, it is well to note that during the evening hours of the late summer and early fall months Polaris may be observed at eastern elongation and that during the evening hours of the late winter and early spring months it may be observed at western elongation.

9. As a guide to estimating the time of elongation, it may be noted that the two bright stars Zeta Ursae Majoris (or Mizar) and Delta Cassiopeiae are on the horizontal wire of the telescope, as the telescope is swung from one to the other, about 8 min. before Polaris is at western elongation and about 30 min. before it is at eastern elongation, for the middle latitudes of the United States.

517. Observation at Culmination. (*a*). FIRST METHOD. (1) Before the night of observation, ascertain as nearly as practicable the exact time of culmination (Art. 512). (2) Set up the transit at least 10 or 15 min. before the time of culmination, and, keeping the vertical cross hair exactly on Polaris as it approaches culmination, follow the star until it reaches culmination, i.e., until the exact time of culmination. (3) Leaving the plates clamped, depress the telescope and set a point on the meridian.

517(*b*). SECOND METHOD. Instead of setting a point on the meridian, the azimuth of the line from the observation station to any fixed point may be determined by measuring the horizontal angle from the backsight on the fixed point to the foresight on Polaris at culmination. This procedure is similar to that of the alternative method at elongation (Art. 516(*a*)).

The objections to a single observation have been stated in Art. 516(*b*). A set of two observations may be taken, one at an interval of a few minutes before the time of culmination, with the telescope normal, and the other at exactly the same interval of time after culmination, with the telescope inverted. The true meridian should be halfway between the two points thus established. More than one set may be taken by using more than one time interval before and after culmination.

A better method is to determine the azimuth of a line from the observation station to a fixed point by a similar set of two observations, one with the telescope normal, the other with it inverted. In both observations the horizontal angle is measured from a

backsight on the fixed point to a foresight on Polaris, but the first observation is taken a few minutes before culmination and the second at exactly the same interval of time after. (Do not relevel the transit between observations.) The mean of the two angles thus measured should be the true azimuth of the line to the fixed point.

With reasonable care, the error of observation at culmination should not exceed 1′ of arc and can be made much less (Art. 516(*d*)).

517(*c*). PRACTICAL SUGGESTIONS. (1) The star, when near culmination, appears to be moving so rapidly that an error in time of 2 min. is equivalent to about 1′ of arc. Hence it is important that the exact time of culmination should be known as nearly as possible and that the timepiece used in the observation should be set accurately. It is often possible to set the timepiece shortly before the observation by radio signals. It is also important that the line of sight be clamped on Polaris at the instant of culmination, or at the exact interval before and after culmination. (2) Many of the practical suggestions, given for observing Polaris at elongation, hold good for observing it at culmination.

517(*d*). A SPECIAL METHOD OF TIMING that may be used in observing Polaris at culmination is based on the interval of time which elapses from the instant when Polaris and Mizar (see Fig. 509(*a*)) are in the same vertical line to the instant of culmination. This special time interval is given to seconds in *The American Emphemis and Nautical Almanac* for different latitudes and for different days throughout the year. The procedure for observation is as follows: (1) Ascertain the special time interval and the approximate time of culmination; from these, determine the approximate time when Polaris and Mizar will be in the same vertical line. Before this time of coincidence, set up the transit. (2) Polaris and Mizar cannot be seen simultaneously within the field of the telescope; hence sight first on one star and then on the other, and repeat at short intervals until both stars appear to be in the same vertical line. Note the watch time of coincidence. (3) From the time of coincidence noted in (2) and the special time interval ascertained in (1), determine accurately the watch time of culmination. Note that in this method of timing it does not matter if the time of day by the observer's watch is somewhat incorrect.

When the watch time of culmination has been determined, a single observation may be taken at culmination or a set of two observations may be taken at equal intervals before and after culmination, as already explained. When Mizar cannot be observed, as in the spring months, the star Delta Cassiopeiae may be used instead in determining the time of culmination.

518. Observation at Any Time. (1) Select an observation station and a fixed point of reference at least 300 to 400 ft. distant. (2) Set up the transit and backsight on the fixed point. (3) Measure the angle from the backsight to the foresight on Polaris. Note carefully the exact time at which the foresight is taken. (4) Determine the azimuth of Polaris for the year, for the exact local civil time of observation, and for the latitude (Art. 515). (5) From the angle measured in (3) and the azimuth angle of Polaris determined in (4), calculate the azimuth of the line from the observation station to the fixed point of reference. (6) If it is desired to establish a meridian, backsight on the fixed point, turn backward the azimuth of the line to that point, and thus obtain the line of zero azimuth.

A set of two observations may be taken, one with the telescope normal, the other with it inverted, provided the exact time of each observation is noted. Similarly, two or more sets of two observations each may be taken.

When only one observation on Polaris is made, the local civil time of that observation and the corresponding Greenwich mean time are used in determining the azimuth of Polaris at the time of observation. (See Art. 515.) When a set of two observations is made, the mean of the local civil times of the two observations is used in determining the azimuth of Polaris. From the azimuth thus determined and from the mean of the two angles that were

measured from the backsights on the fixed point of reference to the foresights on Polaris, the azimuth of the line from the observation station to the fixed point may be calculated.

Obviously, in observing Polaris at any hour angle the accuracy of the results depends to a large extent on (1) the care with which the time of each observation is determined and (2) the accuracy with which the corresponding azimuth of Polaris is computed. Since the star appears to be moving more slowly as it approaches elongation, the nearer the time of an observation is to the time of elongation, the more accurate the results are likely to be.

518(*a*). Practical suggestions. Many of the practical suggestions in Arts. 516(*e*) and 517(*c*) hold good also for the method of observing Polaris at any time.

519. Methods Compared. (*a*). At elongation. The proper time for sighting on the star is known without depending upon the watch. There is time to make two or more observations during the period in which there is no apparent motion of Polaris east or west. The calculation of the azimuth at elongation is simple. With the same care and under the same conditions, errors are likely to be smaller than those made in either of the other two methods. The principal disadvantage is that elongations may occur at inconvenient times for observations, very late at night or in the daytime. Were it not for this, there would be little reason for the use of either of the other two methods.

519(*b*). At culmination. In theory this is the simplest method for it involves no calculation of the azimuth of the star—merely the time of its culmination. However, the observation is rendered more difficult by the fact that the star appears to be moving so rapidly that the observation must be taken quickly and at the exact moment. This means that the time of culmination must be determined with care and that the timepiece of the observer must be accurate. It is somewhat difficult, moreover, to take sets of two observations each.

519(*c*). At any time. This method has most of the disadvantages of the preceding method, and in addition it involves considerably more calculation than is necessary in either of the other two methods. Nevertheless, it is used regularly by skilled observers, particularly in extensive triangulation. It has the advantage that if the star is not visible at one time in the night, it may be at another time, and, on a clear night, as many observations can be taken as may be desired.

520. Rough Determination of a Meridian. If a transit is not at hand, any of the preceding methods, especially the first two, may be carried out roughly as follows. Suspend a long plumb line, and on a level support place the rear sight of a compass or anything having a small hole in it to look through. Shift the sight to the right or left to keep it in line with the star and the plumb line until the time of elongation or culmination, as the case may be. The line of sight thus established may be marked by stakes, but of course it will be shorter and much less accurate than if established with a transit. With care, however, a meridian can be established in this way with

sufficient accuracy for the purposes of compass surveying. For example, if the plumb line is 50 ft. from the peep sight and an error of, say, 0.2 ft. is made in sighting and setting the stake, the corresponding angular error will be about 14′ of arc. (See Art. 283(*d*).)

521. To Determine the Magnetic Declination. (1) Determine a true meridian by one of the methods previously explained. (2) With vernier at zero, backsight along the meridian, and measure the angle between this line and a line found by turning the upper plate until the needle points to the zero of the compass box. This angle is the magnetic declination.

521(*a*). PRACTICAL SUGGESTIONS. (1) If there is a declination arc, be sure that it is set at zero. (2) The chief difficulty in determining magnetic declination lies in the lack of sensitiveness of the needle. It will be found that when the needle is almost at zero, the upper plate can often be turned several minutes by using the tangent screw, with little if any effect on the needle. One way to overcome this is to observe the reading of the vernier when the needle points to N, then unclamp the upper plate, turn it several degrees in any direction, the needle having been raised off its pivot, lower the needle, quickly set the vernier at its original reading, and see if the needle settles at N. If not, repeat the process, changing the setting of the vernier a little each time until the needle does settle at zero. Even then it is often difficult to determine the declination within 3 or 4′. As this, however, is closer than one can read the compass, this error is usually of little importance. (3) If the *azimuth* of Polaris has been determined, it is not necessary of course, actually to stake out a N and S line. The declination may be measured from the line fixed by the observation and allowance made for azimuth. For example, a line is established by observation at eastern elongation, and the azimuth of this line for the date of observation is 1° 11′. Backsighting along this line with the vernier at zero, and then turning the upper plate until the needle points exactly at zero, the vernier reading is, say, 12° 10′. If the declination is *west*, it must be 12° 10′ − 1° 11′ = 10° 59′. If the declination is *east*, it must be 12° 10′ + 1° 11′ = 13° 21′.

CHAPTER 35

DETERMINATION OF A MERIDIAN

By Observations on the Sun

In this chapter three methods of determining a meridian are explained: (1) method of *single altitude* by direct observation on the sun; (2) method involving the use of the *solar attachment;* (3) method of *equal altitudes* of the sun or of a star. In each case the method of procedure is given in detail, so that it may be easily followed by anyone who is accustomed to the use of surveying instruments and who has a knowledge of spherical trigonometry. All three methods, however, will be understood better if studied in connection with a course in field astronomy. In Art. 537, common methods of determining latitudes are outlined.

522. Astronomical Data. In order to make solar observations and the corresponding calculations, certain astronomical data are required. These data may be obtained from one or more of the following publications: *The American Ephemeris and Nautical Almanac*—a bound volume of about 800 pages; *The American Nautical Almanac*—a paper-covered volume of about 200 pages; *The Ephemeris*—a small pamphlet prepared for surveyors employed by the General Land Office. These publications may be obtained from the Superintendent of Documents, Government Printing Office, Washington, D. C. The principal use made of astronomical data by the surveyor is in the determination of a meridian, or of the true azimuth of a line. For that purpose, *The American Nautical Almanac,* generally called "Nautical Almanac," is recommended. For the convenience of their clients, makers of surveying instruments also publish small catalogs or manuals that give useful data such as declinations with tables of corrections for refraction in altitude.

523. The Astronomical Triangle. When the sun is the only heavenly body observed, the celestial sphere may be considered to have a radius equal in length to the distance from the earth to the sun. If the axis of the earth is produced indefinitely it will intersect the celestial sphere in a point called the *celestial pole,* represented by P in Fig. 523. If a line from the center of the earth through any point on its surface is produced indefinitely it will pierce the celestial sphere in the *zenith* for that point; hence observers at different points will have different zeniths, but the celestial pole is the same for all. If in Fig. 523, Z is the zenith for an

observer at O, the circle $S''ENW$ is the *horizon.* The arc $EBDW$ represents the *celestial equator,* cut from the celestial sphere by a plane passed through the equator of the earth. In the northern hemisphere the sun is always north of the equator (between it and the pole) from March 21 to September 21, and always south of it between September 21 and March 21. The angular distance from the equator to the sun at any given moment is the sun's *declination for that moment, positive if north, negative if south.* Thus if the sun is at S in Fig. 523, the declination north (or positive) is the

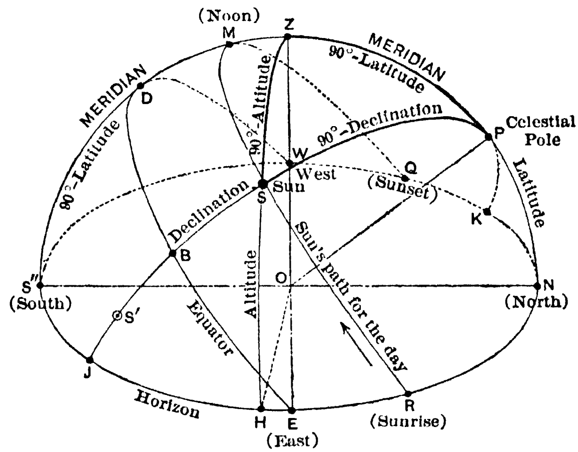


FIG. 523.

arc BS; if the sun is at S' the declination south (or negative) is BS'. The great circle $S''ZPN$, etc., is the meridian for the point on the earth's surface for which Z is the zenith. For any given day the sun appears to move in a circle which is approximately parallel to the equator. Thus for the particular day represented in the figure the sun rose at R, is now at S, at noon it will be at M, and will set at Q. (Suppose that the day is in May. Will the apparent path of the sun on the following day be nearer or farther from the celestial equator? Does the apparent path of the sun ever coincide approximately with the celestial equator?)

Remark. It is important to fix in mind the algebraic signs corresponding to north and south declinations: north +, south −.

The spherical triangle ZPS is called the *astronomical triangle.* For any given point on the earth's surface the side ZP is always the same, but the other two sides are always changing, ZS depend-

ing upon the time of day (or altitude of sun) and PS on the time of year and day (or declination of sun).

$$ZP = 90°\text{—latitude}$$

$$ZS = 90°\text{—altitude}$$

$$PS = 90°\text{—declination}$$

The altitude HS can be measured; the declination BS can be ascertained from *The American Nautical Almanac;* and if the latitude PN of the place of observation is known, the astronomical triangle can be solved for the angle SZP. This angle is the angular distance of the sun from the meridian at the time of the observation. If, therefore, a transit is at O and is pointed at the sun, the angular distance of the line of sight from the north can be found, for it is equal to NOH, equal to the computed angle SZP. When this angular distance has been determined it is a simple matter to convert it into bearing or azimuth. The line of sight to the sun may now be used as a reference line for calculating the bearings of other lines, or, if desired, a meridian may be established by turning off the bearing (or azimuth) right or left as the case may be.

It is evident that the astronomical triangle can be used to determine other quantities. For example, the three sides being known as above, the angle ZPS (called the *hour angle*) can be computed. This is the angular distance of the sun from the meridian and is easily converted into time of day. From this the longitude can be found by comparing with Greenwich time. Likewise if the time of observation (hour angle ZPS) and the sun's declination are known and the altitude is measured, the side ZP may be computed, and hence the latitude PN of the place of observation ascertained.

Question. What arc in Fig. 523 corresponds to the time of day?

523(*a*). DEFINITIONS. Some of the more important astronomical terms are defined in detail in the preceding chapter, including the various terms used in connection with time (Art. 511). The following summary of astronomical terms is taken from *The Ephemeris,* published by W. and L. E. Gurley.

Great Circle: Intersection of a diametral plane with the celestial sphere.
Zenith: Point directly overhead.
Meridian: Great circle passing through zenith and earth's poles.
Horizon: Great circle defining plane perpendicular to zenith and tangent to earth, as determined by level bubbles.
Polar Axis: Prolongation of earth's axis.
Celestial Equator: Great circle defining plane perpendicular to polar axis passing through earth's center.

Ecliptic: Sun's apparent path in the heavens. Making angle of about 23° 27′ with celestial equator.

Equinoxes: Two points at which ecliptic and celestial equator intersect.

Declination: Angular distance of sun above or below celestial equator.

Latitude: Angular distance of the zenith from the equator.

Longitude: Angular distance of meridian east or west of a starting meridian, usually meridian through Greenwich, England, measured along the equator.

Altitude: Sun's angular distance above the horizon.

Meridian Altitude: Altitude when passing the meridian.

Hour Angle: Angular distance of heavenly body east or west of the meridian.

Equation of Time: The difference between true sun time and mean clock time. The latter runs at an average rate throughout the year and is sometimes ahead of and sometimes behind the true sun.

524. The Sun's Declination is continually changing at rates varying from about 0″ per hr. on June 22 and December 22 to almost 1′ of arc per hr. on March 22 and September 22. In the methods discussed in this chapter the declination is required at the time of the observation. This value can be taken from one of the publications mentioned at the beginning of this chapter.

524(*a*). TO LOOK UP THE DECLINATION.

1. Change the standard or watch time of the observation to the corresponding Greenwich civil time of observation in order to determine the declination from the tables found in an ephemeris or in a nautical almanac. This is done by adding to the standard or watch time of observation, 4, 5, 6, 7, or 8 hr., if the observer is in the United States, according to the time zone in which the place of observation is located. If the observation was made in the afternoon, 12 hr. should, of course, be added to obtain the corresponding Greenwich civil time. (See Art. 511 for a fuller explanation of time.)

2. By means of an ephemeris or almanac, select the value of the declination for the day and hour of observation at the Greenwich meridian, interpolating if necessary. If the ephemeris gives the declination for each day at zero hours only, it will be necessary to correct it for any fractional part of a day by means of the hourly difference taken from the table. The hourly difference should be multiplied by the hours and decimal part of an hour of Greenwich civil time and the result added algebraically to the tabular declination for zero hours.

524(*b*). ILLUSTRATION. The time of an observation at New Haven, Connecticut, was, according to the watch, 9.30 A.M., eastern war time, on May 20, 1943; hence the corresponding Greenwich time was $13^h\ 30^m$ civil time on the same day. The sun's apparent declination, as given in the ephemeris, for Greenwich at 0 hr. on May 20 was $+19°\ 44'.7$ and the hourly change in declination was $+32''.2$; hence the declination at the time of the observation (13.5 hr. Greenwich time) was $(+32''.2 \times 13.5) + 19°\ 44'.7 = +19°\ 51'.9$.

524(*c*). ILLUSTRATION. The watch time of an observation at Denver, Colorado, was 3.40 P.M. war time on Nov. 10, 1943. The corresponding Greenwich civil time was $21^h\ 40^m$, i.e., $(3^h\ 40^m + 6^h\ 0^m + 12^h\ 0^m)$. The sun's apparent declination at 0^h at

Greenwich was −16° 50′ 10″, and the hourly variation was −42″.9. The declination at the time of observation (21.67 hr. Greenwich time) was (−42″.9 × 21.67) − 16° 50′.2 = −17° 05′.7.

524(*d*). NOTE. To obtain greater accuracy, some surveyors prefer to use the mean value of the hourly changes in declination. Thus, in the illustration just given, the mean of −42″.9 and −42″.2 (the hourly changes for November 10 and November 11) or −42″.55 would be used. The resulting declination at the time of observation would then be (−42″.5 × 21.67) − 16° 50′.2 = −17° 05′.6.

Since the time of observation was within 2.33 hr. of November 11 at Greenwich, the declination can also be computed by counting backward from 0 hr. on November 11. From the almanac the sun's declination at 0 hr., Greenwich time, November 11, was −17° 07′.2 and the hourly change −42″.2. The declination at the time of observation (2.33 hr. before November 11) was −42″.2 × 2.33 smaller than −17° 07′.2 = 17° 05′.6, which agrees with the declination already found.

525. Refraction. Because of refraction the sun appears to be higher than it really is. Hence all vertical angles found by sighting at the sun should be reduced by *subtracting* the correction for refraction. It will be seen from the table on p. 587 that the nearer the sun is to the horizon, the greater the correction for refraction. Below say 20 or 30° the correction for refraction is very uncertain, and therefore any observations depending principally upon a single altitude of the sun should be taken only after the sun is 20 to 30° above the horizon. The correction for refraction varies with the temperature, height of barometer, etc.

526. Parallax. Parallax (in altitude) is the angular decrease in the apparent elevation of the sun (or other heavenly body) owing to the observations being taken at the surface of the earth instead of at its center. The correction for parallax is always positive. The parallax of the sun is a small quantity, never exceeding about 9″ of arc for low altitudes and decreasing as vertical angles increase. Since the magnitude of the parallax correction is much smaller than the probable error in the observed altitude, its effect is of no significance and it is sometimes omitted. Owing to the relation between parallax and altitude both corrections are frequently combined as in the accompanying table taken from the solar ephemeris issued by C. L. Berger & Sons, Inc.

527. Methods of Viewing the Sun. The sun is too bright to be viewed directly through the telescope so that some device for sighting must be used. The more common devices will be described.

527(*a*). THE CARD METHOD. In the card method the images of the sun and the cross hairs are formed on a card (or in an emergency, on a notebook) held by the observer a few inches from the eyepiece of the telescope and in line with it. The method is fairly satisfactory except that it is difficult for the observer to hold the card steadily at the proper distance from the eyepiece with one hand while manipulating the horizontal and vertical tangent screws

CORRECTION TO OBSERVED ALTITUDE OF THE SUN FOR REFRACTION AND PARALLAX

Apparent Altitude	Temperature −5° C. +23° F.	0° C. +32° F.	+5° C. +41° F.	+10° C. +50° F.	+15° C. +59° F.	+20° C. +68° F.	+25° C. +77° F.	+30° C. +86° F.
	′	′	′	′	′	′	′	′
10	5.42	5.30	5.20	5.10	5.00	4.92	4.83	4.75
11	4.92	4.82	4.73	4.63	4.55	4.47	4.38	4.32
12	4.50	4.42	4.33	4.25	4.17	4.10	4.03	3.97
13	4.15	4.07	4.00	3.92	3.85	3.78	3.72	3.65
14	3.83	3.77	3.70	3.62	3.55	3.50	3.45	3.37
15	3.58	3.50	3.43	3.37	3.32	3.25	3.20	3.13
16	3.35	3.30	3.23	3.17	3.12	3.07	3.00	2.95
17	3.15	3.10	3.03	2.98	2.92	2.88	2.82	2.77
18	2.95	2.90	2.85	2.80	2.75	2.70	2.65	2.60
19	2.78	2.73	2.68	2.63	2.58	2.53	2.48	2.43
20	2.63	2.58	2.53	2.48	2.43	2.38	2.33	2.30
21	2.48	2.43	2.38	2.35	2.30	2.27	2.22	2.17
22	2.35	2.30	2.25	2.22	2.18	2.13	2.08	2.05
23	2.25	2.20	2.15	2.12	2.08	2.03	1.98	1.95
24	2.13	2.08	2.05	2.02	1.98	1.93	1.88	1.87
25	2.03	1.98	1.95	1.92	1.88	1.83	1.80	1.77
26	1.95	1.90	1.87	1.83	1.80	1.75	1.72	1.70
27	1.85	1.82	1.78	1.75	1.72	1.68	1.63	1.62
28	1.77	1.72	1.70	1.67	1.63	1.60	1.57	1.53
29	1.68	1.65	1.63	1.60	1.57	1.53	1.50	1.47
30	1.62	1.58	1.57	1.53	1.50	1.47	1.45	1.42
32	1.50	1.47	1.45	1.42	1.38	1.35	1.33	1.30
34	1.37	1.35	1.32	1.30	1.27	1.25	1.23	1.20
36	1.27	1.25	1.22	1.20	1.18	1.15	1.13	1.10
38	1.18	1.15	1.13	1.12	1.10	1.07	1.05	1.02
40	1.10	1.07	1.05	1.03	1.02	0.98	0.97	0.95
42	1.00	0.98	0.97	0.95	0.93	0.90	0.88	0.87
44	0.93	0.92	0.90	0.88	0.87	0.85	0.83	0.82
46	0.88	0.87	0.85	0.83	0.82	0.80	0.78	0.77
48	0.82	0.80	0.78	0.77	0.75	0.73	0.72	0.70
50	0.75	0.73	0.72	0.70	0.68	0.67	0.67	0.65
55	0.62	0.60	0.60	0.58	0.57	0.57	0.55	0.53
60	0.52	0.50	0.50	0.48	0.47	0.47	0.45	0.45
65	0.40	0.40	0.40	0.38	0.38	0.37	0.37	0.35
70	0.32	0.32	0.30	0.30	0.30	0.28	0.28	0.28
75	0.23	0.23	0.22	0.22	0.22	0.20	0.20	0.20
80	0.15	0.13	0.13	0.13	0.13	0.13	0.12	0.12
85	0.07	0.07	0.07	0.07	0.07	0.07	0.05	0.05
90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

with the other. It is well to have two operators when the card method is used, one to hold the card and manipulate one of the tangent screws while the other observer turns the other tangent screw and keeps the notes. If the sun is high its image may fall on the limb of the transit before reaching the card. In that case the

card method of viewing the sun may be used, provided a prismatic eyepiece is employed to reflect the image 90° to the card. In this, as in all other methods of solar observation, care must be taken to avoid the blunder of mistaking a stadia hair for the horizontal hair. When the telescope is reversed the surveyor must also make sure the sun's image is observed in the quadrant diagonally opposite from that used when the telescope is normal.

527(*b*). THE SOLAR SCREEN METHOD represents an improvement over the method in which the observer holds a card back of the eyepiece. The solar screen consists of a bracket, supporting a small screen, which can be attached to the eyepiece of the telescope. By careful focusing of both the eyepiece and the objective, the image of the sun and the cross hairs can be brought out sharply and held steadily upon the screen. Observations may be made either by centering the sun's image or by bringing the cross hairs tangent to the image as explained in Art. 528. As it is difficult to bisect accurately the sun's rather large image, the method of pointing at the sun's center is not used except when time is short and only approximate azimuths are desired. When using some transits the screen must be removed in order to transit the telescope when taking the second half of the set of pointings, which should always be done with the telescope inverted.

527(*c*). THE SIMPLEX SOLAR SHIELD.* A patented device, invented by Professor C. H. Wall, is an improvement on the usual

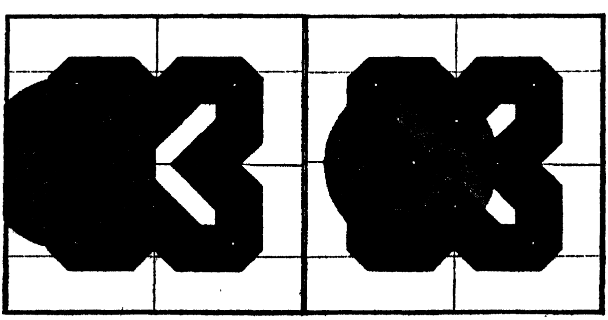


FIG. 527(*c*). The Wall Simplex Solar Shield with the sun in two positions. The sun's image travels ¼ diameter between pointings. Ten pointings—5 normal, 5 inverted—constitute a set.

type of plain solar screen. The simplex attachment is provided with a symmetrically designed shield placed between the eyepiece of the transit and a plate upon which the cross hairs and the sun's

* The use of this shield is explained in detail in a small pamphlet issued by Professor C. H. Wall, Columbus, Ohio, U. S. A.

image are focused. The advantages of this method are that it affords an accurate means for sighting on the sun's center and gives uniform time intervals between pointings. These intervals permit a valuable check on the altitude readings. The observer knows when to take the pointings, has ample time to record the necessary data, and is warned by the position of the image on the plate when to take the next pointing. Each observation consists of ten pointings, five direct and five with the telescope reversed.

527(*d*). THE PRISM METHOD. The prism is a very compact attachment to the eyepiece of the transit. It consists of a reflecting prism with a movable screen of colored glass through which the sun can be viewed directly. When taking a backsight on any terrestrial point, the screen is swung to one side. Before sighting at the sun, the colored glass should be put into position. The screen cuts down the intensity of the light so that the sun's disk can be centered by the cross hairs or brought tangent to them. Instrument makers can supply a reflecting prism and screen to fit any transit. Owing to lack of clearance between the reflecting prism and compass box of the transit, it may be necessary to remove the prismatic eyepiece in order to transit the telescope before taking the second half of the set of pointings. The reflecting prism is a most convenient and satisfactory device. When not in use, it should be kept on its base provided for the purpose in the instrument case.

528. Determination of the Meridian by Direct Observation on the Sun. It is difficult to set the intersection of the cross hairs on the exact center of the sun because of its size and rapid apparent motion. Better results will be obtained by using one of two methods of pointing: (1) the *quadrant-tangent method* or (2) the *center-tangent method.*

528(*a*). PROCEDURE BY THE QUADRANT-TANGENT METHOD OF POINTING

Step 1. Select a point *A* over which to set the transit and a second point *B*, at least 500 ft. distant, to serve as a permanent backsight. Thus the line *AB* becomes the line whose true azimuth will be determined by the observation. Since the purpose of the observation is generally to determine the true azimuths or bearings of the main lines of the survey, it is desirable to have either *A* or *B*, or both these points, on one of these main lines, preferably near the center of the survey.

Step 2. Backsight on *B* with the telescope normal and the A vernier set at zero or at a reading which is carefully recorded.

Step 3. Loosen the upper clamp and sight at the sun. When the observations are in the *forenoon* and the transit has an *erecting* telescope (image upright), proceed as follows. With the telescope normal, bring the sun's image into the upper left-hand quadrant formed by the cross hairs, with the lower edge of the image slightly below the horizontal hair and the right-hand edge tangent to the vertical hair, as shown by the dotted circle in Fig. 528(*a*), A.M.(1). Clamp the telescope and the horizontal plate in this position. As the altitude of the sun increases, keep the vertical hair tangent to the right-hand edge of the image by means of the

QUADRANT-TANGENT METHOD

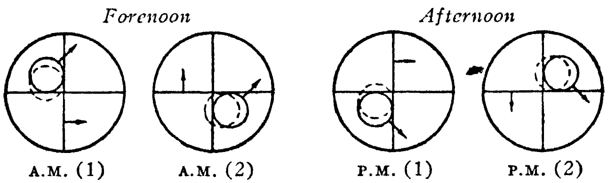


FIG. 528(*a*). These diagrams show the sun as viewed *directly* through an *erecting* telescope. For an inverting telescope, turn the page upside down. When using a solar screen, the diagrams should be viewed as if looking through the page from the back for an erecting telescope—upside down for an inverting telescope.

upper-plate tangent screw until the sun has risen sufficiently to bring the lower edge of its image tangent to the horizontal hair. The image will then be in the upper left-hand quadrant and tangent to *both* hairs, as shown by the full circle in Fig. 528(*a*), A.M.(1). Be careful during this procedure not to disturb the lower tangent screw or the clamp and tangent screw which control the telescope.

Step 4. Record the watch time of the observation, the reading for the horizontal angle, and the reading for the vertical angle. (See field notes, Art. 530(*a*).)

Step 5. Repeat the procedure just outlined three to five times, keeping the telescope normal. This completes one-half a *set*. A backsight may now be taken on point *B* as a check in detecting any disturbance of the transit.

Step 6. With the telescope *inverted* and the screen in position, complete the other half of the *set* by repeating steps 3 and 4 as many times as there were repetitions in step 5. In step 3, however, the sun's image is brought into the lower right-hand quadrant, diagonally opposite to that used for observations when the telescope was normal. Bring the image into this quadrant with its left-hand edge slightly to the left of the vertical hair and the upper

edge tangent to the horizontal hair, as shown by the dotted circle in Fig. 528(*a*), A.M.(2). As the image of the sun moves to the right, keep its upper edge tangent to the horizontal hair by means of the telescope tangent screw until the image is tangent to both hairs, as shown by the full circle in Fig. 528(*a*), A.M.(2). Be careful not to disturb either the upper or the lower clamp and tangent screws that control the horizontal plates.

When observations are made in the afternoon, the procedure given in steps 1 to 6 is slightly modified to conform to the methods of sighting indicated in Fig. 528(*a*), P.M.(1) and Fig. 528(*a*), P.M.(2).

The number of pointings in the first half of the set, when the sun's image was tangent to and above the horizontal hair, and the number of pointings in the second half of the set, when the sun's image was tangent to and below the horizontal hair, should be equal. The value of the vertical angle will then be the mean of all the vertical readings, and this value will be the same as if the pointings were made directly on the center of the sun, i.e., no corrections for semi-diameter are necessary. (See mean value of vertical angles in field notes, Art. 530(*a*).) Similarly the value of the horizontal angle will be the mean of all the horizontal angle readings, both those taken with the telescope normal and those taken with the telescope inverted, without any correction for semi-diameter. (See mean of horizontal angles in the field notes (Art. 530(*a*).) Because the quadrant-tangent method requires no correction for semi-diameter, it is preferred by many surveyors.

528(*b*). Procedure by the center-tangent method of pointing

For a forenoon observation with an erecting telescope (image erect) in the normal position, steps 1, 2, 4, and 5 are the same as in the procedure for the quadrant-tangent method of pointing. However, the method of pointing is different from that given in step 3. The procedure for steps 3, 4, 5, and 6 is as follows.

Step 3. Bisect the image of the sun with the vertical cross hair and move the telescope up or down until the lower edge of the sun's image is slightly below the horizontal hair, as shown by the dotted circle, Fig. 528(*b*), A.M.(1). Clamp the telescope in this position and let it remain clamped during the remainder of the pointing. As the image moves upward and to the right, keep it centered on the vertical hair by means of the upper-plate tangent screw until the image is tangent to and above the horizontal hair, as shown by the full circle in Fig. 528(*b*), A.M.(1).

Step 4. Record the watch time of the observation, the reading for the horizontal angle, and the reading for the vertical angle.

Step 5. Repeat the procedure just outlined in step 3 until three pointings have been made, keeping the telescope normal. This completes one-fourth a set.

Step 6. With the telescope still normal, bisect the image of the sun with the horizontal hair and move the telescope right or left until the left-hand edge of the image is slightly to the left of the vertical hair, as shown by the dotted circle in Fig. 528(*b*), A.M.(2). Clamp the upper plate in this position and let it remain clamped

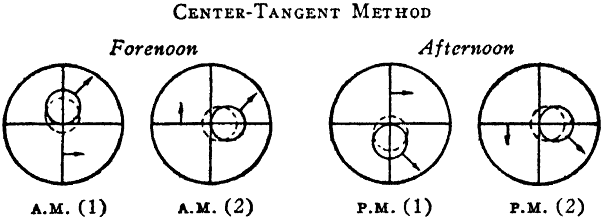


FIG. 528(*b*). These diagrams show the sun as viewed *directly* through an *erecting* telescope. For an inverting telescope, turn the page upside down. When using a solar screen, the diagrams should be viewed as if looking through the page from the back for an erecting telescope—upside down for an inverting telescope.

during the remainder of the pointing. As the image moves to the right and upward, keep it centered on the horizontal hair by means of the telescope tangent screw until the image is tangent to and to the right of the vertical cross hair, as shown by the full circle in Fig. 528(*b*), A.M.(2). Record the watch time of the observation, the horizontal angle, and the vertical angle. Repeat the procedure just outlined until three pointings have been made. This completes one-half a set consisting of six pointings, all with the telescope normal.

The six pointings in the second half of the set are made with the telescope inverted. The procedure is the same as that in the first half of the set. The methods of pointing are again those indicated in Figs. 528(*b*), A.M.(1) and 528(*b*), A.M.(2). The entire set consists of twelve pointings, six normal and six inverted.

When observations are made in the afternoon, the general method of procedure is exactly the same except that the methods of sighting must be modified as shown in Figs. 528(*b*), P.M.(1) and 528(*b*), P.M.(2).

Before solving the astronomical triangle, it will be necessary to correct those vertical angles which have been measured either to

the lower edge (forenoon observation, Fig. 528(*b*), A.M.(1)) or to the upper edge (afternoon observation, Fig. 528(*b*), P.M.(1)) of the sun's disk by the amount of the sun's semi-diameter. This can be taken from any ephemeris. The angle subtended by the sun's semi-diameter varies from about 15′ 45″ on July 1 to 16′ 18″ on January 1. Care must be taken to note the proper direction of the correction.

When observations are taken with the vertical hair tangent to the sun's edge as in step 6, Fig. 528(*b*), A.M.(2), a correction is necessary to bring the pointings to the sun's center. The *mean* of the values of the *horizontal* angles must be corrected by an amount equal to the sun's semi-diameter divided by the cosine of the mean of the corresponding observed altitudes of the sun (corrected for refraction as explained in Art. 525).

Some surveyors prefer this center-tangent method because only one cross hair is moved at each pointing and only one point of tangency is observed. The moving hair is kept centered on the sun while the other hair cuts from the sun a small and diminishing segment which can be bisected accurately.

528(*c*). PRELIMINARY TEST FOR THE ACCURACY OF A SET OF FIELD OBSERVATIONS. Although the sun's daily path shown in Fig. 528(*c*) is approximately a circle, a short portion of it, such as that

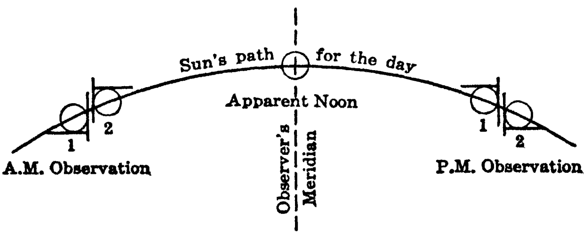


FIG. 528(*c*).

situated between positions 1 and 2, is practically a straight line. Therefore changes in successive values of horizontal angles from a fixed backsight to the sun as it apparently moves along its path should bear a very close relation to the changes in corresponding values of the successive vertical angles. The ratios of the differences between successive horizontal angles to the differences between corresponding pairs of vertical angles should be the same or approximately the same for all pointings. (See columns 5, 7, and 8, Art. 530(*a*).) If the ratios are uniform, the observations may be assumed free from serious errors and the subsequent solution

can be carried out with reasonable prospect of a dependable result. *If, however, the ratios differ by an appreciable amount, it will be a waste of time to carry out the solution of the astronomical triangle, and moreover the resulting azimuth will be incorrect.*

528(*d*). PRACTICAL SUGGESTIONS FOR OBSERVING.

1. The station (point *A*) selected for making the observations should afford unobstructed visibility toward station *B* and toward the sun at all times of the day. It should permit a firm set-up of the transit, and it should be easily accessible at all times. Both stations *A* and *B* should be carefully referenced.

2. The notebook should be prepared with proper headings in advance of the observations in order that all readings may be entered without confusion or delay. (See Art. 530(*a*) for a satisfactory form of notes.) A sketch showing the line *AB* and the horizontal projection of the line from station *A* to the sun should also be given in the notes.

3. After setting up the transit and before starting to level, it is well to have two diagonally opposite leveling screws in line with the sun's direction. This arrangement facilitates rapid releveling between sets of observations. However, releveling between the normal and inverted positions of the telescope during the progress of a set should *not* be attempted. The transit should be in good adjustment. It is particularly important that the horizontal axis be perpendicular to the vertical axis of the spindles and that in leveling the transit the vertical axis be made truly vertical. Unless the vertical axis is truly vertical, the horizontal axis will be inclined even though it is in good adjustment and the reversal of the telescope between sights will not eliminate the error caused by such inclination. In work of high precision, a sensitive striding level is used in leveling the horizontal axis prior to sighting. If the transit is not equipped with a striding level, the spindle axis should be made vertical as judged by the telescope level and not by the plate levels alone (Art. 247).

4. For an accurate method of measuring a vertical angle see Art. 271(*c*). For an accurate method of measuring a horizontal angle when one line of sight is to a high point (as to the sun) see Art. 264.

5. In sighting at the sun, guard against the common mistake of using a stadia hair instead of the middle horizontal hair. This can be done by raising and lowering the telescope slightly so that all horizontal hairs can be identified against the sun's disk.

6. If observations are taken too early in the morning (before the sun reaches an altitude of at least 20 to 30°), the correction for refraction will be large and uncertain; if they are taken too near noon, the altitude of the sun is changing too slowly and the astronomical triangle becomes poorly proportioned for definite results. For these reasons, the best period of the day for observations is 2 to 3 hr. before or after the time when the sun crosses the observer's *meridian*. In midwinter the sun is so far south of the celestial equator (declination minus) that its altitude, in northern latitudes, is low and the astronomical triangle is more poorly proportioned than at any other period of the year. For these reasons midwinter observations should be avoided. In case midwinter observations are urgently needed it will be found advisable to make them in both morning and afternoon and to take the mean of the resulting azimuths. For the effects of favorable and unfavorable conditions of observation due to the time of year, see the table of errors in suggestion 11.

7. The object in sighting at the sun twice, once with the telescope normal and once with it reversed, is to eliminate instrument errors. As a further precaution both the A and B verniers on the limb may be read to eliminate errors of eccentricity, as is done in triangulation. Likewise, if there are two verniers on the vertical circle the two readings may be noted. *The special method of checking angle readings, explained in Art. 241, is particularly useful for checking the readings on both the horizontal and the vertical circles.*

8. The time between the two observations should be as short as practicable; yet there is no need of undue haste. No appreciable error will occur if the pointings are not more than, say, 2 or 3 min. apart.

9. When using a transit with circles reading to single minutes the maximum error in determining an azimuth of the sun should not exceed 1 or 2′. With a precision instrument, the maximum error should be less than 1′

Mistakes in observing and in computing may be detected and the accuracy of the azimuth determination of an important line may be increased by taking several sets of observations or by combining the results of several morning observations with similar observations taken in the afternoon.

10. A slide rule will be found convenient when computing the ratios of angular differences in the last column of the field notes (Art. 530(*a*)). No fixed limits for variations in these ratios can be stated. Much depends upon the degree of precision desired in the survey. Probably a maximum variation of 20 per cent could be allowed in most cases without discarding the set of observations. It is a good plan to compute the ratios just as soon as each pair of angles has been recorded. Pointings can then be continued until readings with sufficiently close ratios have been obtained. The ratios do not constitute an infallible check, but they do provide a simple precaution against most of the blunders likely to occur in observing and recording.

11. Errors in finding the azimuth of the sun depend largely on: (1) errors in vertical angles, (2) errors in latitude, (3) errors in watch time, (4) the time of the year, i.e., the proportions of the astronomical triangle, (5) errors in horizontal angles. Since errors in time affect only the value of the sun's declination, they are generally unimportant except in months nearest the equinoxes when declinations are changing with maximum rapidity. Radio time signals are broadcast at convenient intervals throughout the day and it is advisable to check the observer's watch by this means either just before or soon after an observation. (See radio signals, Art. 511(*g*).) Errors in vertical angles and in latitude affect azimuth most when the sun's declination is south since the astronomical triangle then has its poorest proportions. This is shown in the accompanying table Errors in latitude can be avoided by examining good maps such as those issued by the United States Geological Survey. It will be helpful to keep in mind that, for short distances in a north-south direction, one mile is roughly equivalent to a little over one minute in latitude. In nearly every instance errors in horizontal angles will be detected by means of the ratios obtained as in 10. *Observations that are obviously erroneous should be discarded before beginning computations.*

To give an approximate idea of the changes in azimuth due to errors of 1′ in altitude or 1′ in latitude, the following table has been prepared for observations in latitude 41° 20′ N.

ERRORS IN FINDING AZIMUTH OF THE SUN AFFECTED BY CONDITIONS OF OBSERVATION

Conditions	Most Favorable Conditions Time about June 1 (Near Summer Solstice) Sun due east or west of observer	Average Conditions Time about May 1 (Approaching Summer Solstice) Sun 2½ hr. from Meridian	Most Unfavorable Conditions Time about December 15 (Winter Solstice) Sun 2 hr. from Meridian

Approximate Change in Computed Azimuth of Sun. Latitude 41° 20′ North

Change due to error of 1′ in altitude	50″	82″	124″
Change due to error of 1′ in latitude	30″	90″	114″

528(*e*). MODIFIED METHOD OF OBSERVATION

If the transit does not have a full vertical circle, or if for any other reason it is not feasible to take sights with the telescope inverted, proceed as in the center-tangent method with a group of three

pointings, as shown in Fig. 528(*b*), A.M.(1), followed by a second group of three pointings as shown in Fig. 528(*b*), A.M.(2), except that the telescope is kept normal for all pointings. Each altitude as observed in the first group of three pointings must be corrected for the sun's semi-diameter, i.e., add 16′ of arc if the horizontal hair was made tangent to the lower edge of the sun as indicated in Fig. 528(*b*), A.M.(1). The computed azimuth should be to the sun's center. Therefore, before the horizontal angles observed in the second group of three pointings (Fig. 528(*b*), A.M.(2)) can be used, each must be corrected by an amount equal to the sun's semi-diameter divided by the cosine of the corresponding observed altitude of the sun (corrected for refraction as explained in Art. 525). The mean of the six horizontal angles and the computed azimuth of the sun's center may then be used in calculating the azimuth of the line from the transit station to the point selected for a backsight.

529. Determination of a Meridian by Observation on a Star. Instead of sighting on the sun, observations to determine azimuths may be taken on a bright fixed star, preferably one near the prime vertical, i.e., a star nearly due east or west of the observer's meridian, and about 30 to 50° above the horizon. The method of observation is similar to that in Art. 528, except that the star is so small that there is no need of using either the quadrant-tangent or the center-tangent method of pointing.

For computations, the declination of the star must be taken from either *The American Ephemeris and Nautical Almanac* or *The American Nautical Almanac.* Since the declination of a fixed star changes very slightly, the time of observation does not affect the computations.

530. Method of Procedure for Computations.

1. Find the mean value of the observed altitudes (mean apparent altitude). Look up in the table on p. 587 the correction for refraction and parallax, which corresponds to the mean apparent altitude, and subtract it from the mean apparent altitude. Correct the result for the index error if there is any such error. This gives the corrected mean altitude used in the computation.

2. Find the sun's declination for the day and hour of the observation as explained in Art. 524.

3. Subtract the final corrected declination of the sun from 90°, thus getting the *polar distance p* of the sun at the time of the observation.

4. The latitude, the polar distance, and the corrected mean altitude being known, the angle Z of the spherical triangle can be found by spherical trigonometry, using any one of several formulas. Either of the following formulas is recommended (h = cor-

rected mean altitude; p = polar distance of the sun; L = latitude of the observer; and $S = \frac{1}{2}(h + L + p)$) :

$$\tan \tfrac{1}{2}Z = \sqrt{\frac{\sin (S - L) \sin (S - h)}{\cos S \cos (S - p)}} \qquad \text{(I)}$$

$$\cos \tfrac{1}{2}Z = \sqrt{\frac{\cos S \cos (S - p)}{\cos L \cos h}} \qquad \text{(II)}$$

5. The angle Z found from either of the above formulas is the angular distance of the sun east or west of the meridian, or it is equal to the *angular distance* of the line of sight to the center of the sun measured from the *north* point. For example, in Fig. 523, p. 583, the angle $Z = SZP$ is measured by the arc NH equal to the angle NOH, i.e., the *horizontal* angle that a line of sight from the place of observation O to the sun S makes with a north and south line ON. This angular distance $Z = NOH$ may easily be converted into azimuth or bearing as explained in Art. 530(b).

Remark. In astronomical work azimuths are measured from the south; and hence the left-hand members of formulas I and II are frequently given as cot $\frac{1}{2}Z$ and sin $\frac{1}{2}Z$, respectively. In that case Z is equal to the angular distance of the sun measured from the *south*.

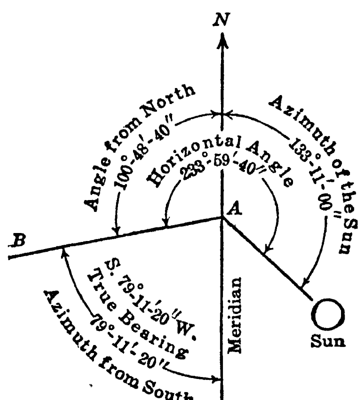


FIG. 530(b).

Questions. Assume azimuths to be measured clockwise from the north and the value of Z to be found from formulas I and II. (1) If the altitude of the sun is found from a morning observation, is the corresponding value of Z equal to the *azimuth* of the line of sight to the sun? (2) If the altitude is found from an afternoon observation, is Z equal to the azimuth? Assuming azimuths to be measured clockwise from the south and Z to be found from the cotangent or sine instead of from the tangent or cosine, answer the same questions 1 and 2.

530(*a*). An example illustrating a form of notes and computation for azimuth

Azimuth by Observing Altitude of Sun

At New Haven, Conn., Apr. 17, 1943 N Lat. 41° 18′ 40″

Pointing No.	Telescope and Sun	Watch Time	Hor. Angles	*H* Diff.	Vert. Angles	*V* Diff.	Ratio of *V* Differences *H*
1	Normal	9h 52m 50s	231° 42′		49° 10′	(Art.	528(c).)
				51		23	0.45
2	"	9h 55m 35s	232-33		49-33		
				33		15	0.45
3	"	9h 57m 20s	233-06		49-48		
4	Inverted	10h 00m 30s	234-59		50-45		
				34		15	0.44
5	"	10h 02m 20s	235-33		51-00		
				32		13	0.41
6	"	10h 03m 55s	236-05		51-13		

Sums		−349m 210s	−23° 58′	299° 149′	
Means		9h 58m 45s	233° 59′ 40″	50° 14′ 50″	
	Watch corr.	−10s	(Watch fast)		
Eastern War T	ime	9h 58m 35s			
Correction for	refraction and p	arallax		0° 00′ 40″	(From table on p. 587.)
	Altitude of sun	's center		50° 14′ 10″	
	Add	4h	(For Eastern	War Time)	
(1943)	April 17	13h 58m 35s	= Greenwich	M. Time of Ob	servation
Sun's apparent	declination	+10° 17′ 30″			
Sun's polar dis	tance	79° 42′ 30″	= 90° − (10°	17′ 30″)	

530(*b*). Computation for azimuth by formula I (Art. 530):
From field notes:

$h =$ 50° 14′ 10″
$L =$ 41 18 40
$p =$ 79 42 30

$2S =$ 171° 15′ 20″
$S =$ 85 37 40
$S - h =$ 35 23 30
$S - L =$ 44 19 00
$S - p =$ 5 55 10

Check $S =$ 85° 37′ 40″

L cos S = 8.882158
Log sin = 9.762800 + L cos $S - p$ = 9.997678
Log sin = 9.844243 ... 8.879836

9.607043
8.879836

2)0.727207

L tan ½Z = 10.363603
½Z = 66° 35′ 30″
Z = 133° 11′ 00″ = azimuth of sun from the north
233° 59′ 40″ = angle from *AB* to the sun
100° 48′ 40″ = angle from north to *AB*
79° 11′ 20″ = azimuth of *AB* from the south
259° 11′ 20″ = azimuth of *AB* from the north

The following information should be entered on the right-hand page of the field notebook: the date, place, watch error, names of observers, transit used with style of eyepiece, latitude and longitude of the station, and description of stations occupied and sighted. A sketch showing the observation station and the backsight with relation to a meridian line and the sun's position should always be drawn.

531. The Solar Attachment, frequently used in connection with an engineer's transit, is a device for determining a true meridian by an observation on the sun. Its use is based on the same general principles involved in the method of the preceding article, except that instead of solving the astronomical triangle by spherical trigonometry the instrument is made to solve it mechanically, so that at the end of the observation the line of sight through the transit telescope lies in the true meridian.

Three of the forms of solar attachment in common use are known as: (1) the Smith Solar Attachment, (2) the Saegmuller Solar Attachment, and (3) the Burt Solar Attachment. Although these three differ in appearance and in the method by which they are mounted on the transit, they are alike in principle. Each is provided with devices for setting off latitude (or colatitude) and declination. In addition, the Smith and the Burt attachments are provided with a circle for setting off the hour angle. In explaining use of these attachments it will be assumed that the observer is north of the equator. The general principles outlined apply, however, to observations in the southern hemisphere.

In the use of a solar attachment it is necessary to know only the latitude (or colatitude) of the place of observation and the apparent declination of the sun at the hour of observation corrected for refraction. When these have been set off on the attachment, the astronomical triangle can be solved solely by the manipulation of the instrument.

The chief advantages in the use of a solar attachment are (1) freedom from any except the simplest calculations and (2) the rapidity with which the transit can be oriented in the true meridian whenever the sun is shining.

A solar attachment should be kept in good adjustment. For methods of adjusting various forms of solar attachments, see Chapter 54, Arts. 750–752.

532. The Smith Solar Attachment (Fig. 532) consists of an auxiliary or solar telescope of low magnifying power mounted on a horizontal axis parallel to, but independent of, the supporting axle of the transit telescope. The solar telescope can be rotated in its supporting collars to permit the hour angle of observation to be set off by means of the graduated drum near the eyepiece at the upper end of the telescope. The index of the hour drum is attached to the upper collar. The eyepiece is provided with a screen of colored glass to cut down the brilliance of the sun so that it can be viewed through the telescope by means of a plane mirror mounted just below the objective lens.

The mirror is attached to the lower end of an arm which carries the index and vernier of the declination arc. The declination arc itself is attached to the shell of the solar telescope and rotates with it when the hour drum is turned. When the declination is set off on its arc and the transit is turned horizontally, the sun's image can be reflected into the telescope where it can be centered accurately by means of cross hairs which are spaced a distance

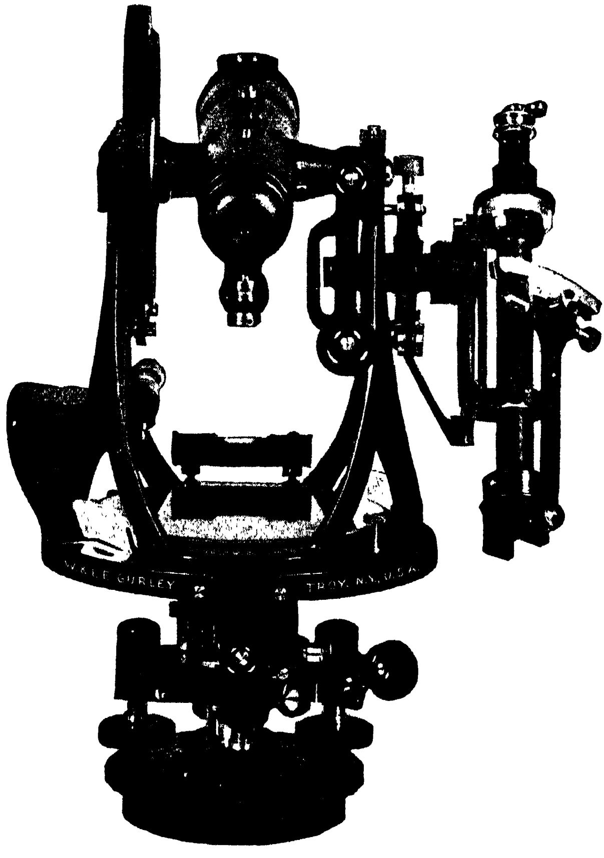


FIG. 532. The Smith Solar Attachment. (Courtesy of W. & L. E. Gurley, Troy, N. Y.)

apart equal to the sun's apparent diameter. The latitude arc is mounted close to the solar telescope in such a manner as to rotate freely about the outer end of the latitude axis. The collars supporting the solar telescope are so attached to the latitude arc that when the latitude is set off the axis of the solar telescope is made parallel to the earth's polar axis in only one position, namely, when the transit telescope is turned so as to lie in the meridian. At the same instant, the sun's image will be bisected in the solar telescope provided the declination is set off correctly and the hour drum is rotated to show local apparent time. Conversely, if the latitude of the station is set off and the declination and hour angle are turned off correctly for a given instant, the main telescope will lie in the meridian.

532(*a*). PROCEDURE FOR DETERMINING A MERIDIAN WITH THE SMITH SOLAR ATTACHMENT.

1. Look up the declination for the day and hour of the observation. (See Art. 524(*a*).)

2. Correct the declination for *refraction in declination.* See Table IX, p. 1112. Since the sun appears to be higher than it really is, the value of the declination found in the ephemeris must be corrected for *refraction in declination,* i.e., in the plane of the great circle through the sun and the celestial poles. Refraction in declination varies with the sun's declination, the latitude of the station, and the hour angle of the sun from the observer's meridian.

3. Set up over a convenient station *M,* where the sun can be viewed without obstruction. Level up approximately.

4. Set off on the declination arc the final value of the declination as obtained in (2). Since the declination arc is graduated in both directions from the zero mark in the center, the proper half must be selected. If the sun's declination is minus, the sun will be below the plane of the celestial equator. Hence the declination arm must be raised (setting made on the upper portion) in order to tilt the reflecting mirror downward toward the sun.

5. On the latitude arc set off the latitude of the station.

6. Rotate the solar telescope, thus tilting the reflecting mirror toward the sun, until the hour angle corresponding to the time of observation is set off on the drum. It will be noted that the graduations on the drum are rather coarse. This is because the declination does not change appreciably during an interval of 10 or 15 min.

7. Level up very carefully, using the telescope level in at least two positions at right angles for the final test (Art. 247).

8. Set the A vernier of the horizontal limb at zero, but leave the lower clamp loose so that both plates, clamped together, can turn freely.

9. With the screen in position turn the entire instrument until the image of the sun is centered by the cross hairs of the solar telescope. The final centering should be completed by means of the lower clamp and tangent screw. The line of sight through the main telescope will then be in the true meridian.

10. Loosen the upper-plate clamp and sight at *N* the other end of a reference line *MN*. The azimuth of the line *MN* is the angle turned off, i.e., the reading on the limb. If desired, a meridian *MO* may be established before the upper-plate clamp is loosened by lining in a point *O* on the ground.

Note. For practical suggestions, see Art. 535. For the adjustment of the Smith solar attachment, see Chapter 54, Art. 750.

Note. The Smith solar attachment designed in 1880 by Benjamin H. Smith, a United States surveyor, is the type of solar used by the General Land Office. An advantage of this attachment is the construction which permits the use of the transit telescope without disturbing the latitude and declination settings. On the other hand, the attachment constitutes a permanent addition to the transit. In some types it cannot be removed without disturbing the adjustments of the transit although attempts to reduce this disturbance are sometimes made through the introduction of a removable counterweight. The Smith solar attachment, unlike other types, can be added only to a new instrument.

533. The Saegmuller Solar Attachment is somewhat simpler than the Smith solar since it has no graduated arcs but instead is designed to make use of the vertical arc of the transit for the purpose of laying off both latitude and declination angles; moreover, it has no hour circle. The attachment is mounted on a polar axis fastened to a small base on the top of the transit telescope. The attachment consists essentially of a small solar telescope supported by short standards in such a manner as to allow the solar telescope to rotate about the polar axis. Rotation is controlled by means of a polar clamp and tangent. In place of the declination arc a spirit level is attached to the solar telescope to aid in setting off the declination as explained below. The solar telescope is provided with cross hairs spaced to include the sun's image. Instead of using a reflector, however, the Saegmuller attachment is designed so that the sun can be viewed directly through the solar telescope. Colored glass screens are provided to reduce the sun's brilliance. In some models the screen is placed outside the objective; in others the screen is attached to the eyepiece. A diagonal prism at the eyepiece is also used to allow sights to be taken without interfering with the main telescope. The method of procedure

is so similar to that just explained for the Smith solar that only a brief outline is necessary.

533(*a*). PROCEDURE FOR DETERMINING A TRUE MERIDIAN WITH THE SAEGMULLER SOLAR ATTACHMENT.

1. Find the final corrected value for the sun's declination at the time of observation. (See steps 2 and 3 for the Smith attachment above.)

2. Level up carefully over some station *M*, note whether the vertical circle has any index error. Level up the attachment (like a transit) by using its four base leveling screws upon the plane provided by the leveled transit telescope. Backsight on some station *N*. Point the solar telescope on the same backsight, thus bringing both telescopes into the same vertical plane.

3. Incline the transit telescope until the reading on the vertical circle is equal to the corrected value of the sun's declination found in (1). If this declination is north, depress the objective end of the telescope; if south, elevate it. Without disturbing the transit telescope, bring the solar telescope to a horizontal position by means of its level. The angle between the two telescopes is now equal to the sun's declination.

4. Without disturbing the relative positions of the two telescopes, unclamp the vertical circle and set it to the colatitude of station *M*, making the objective end of the telescope the higher.

5. Unclamp the upper horizontal plate, turn it (and the transit telescope) around the vertical axis, and at the same time turn the solar telescope around the polar axis until the image of the sun is brought into the field of the solar telescope. When the image is accurately centered, the transit telescope must be in the meridian, *with the transit telescope pointing south.* The azimuth of *MN* is then known from the reading on the horizontal limb, and if desired a true meridian may be staked out on the ground.

If desired, the plate verniers may be left clamped at zero until the sight has been taken at the sun and then the angle turned off to the point *N*, as in step 10, Art. 532(*a*).

Note. For practical suggestions, see Art. 535. For the adjustment of the Saegmuller attachment, see Chapter 54, Art. 751.

534. The Burt Solar Attachment shown in Fig. 534 is essentially the solar attachment invented by William J. Burt about 1835 for use with the surveyor's compass and since adapted to the engineers' transit. The principal parts are: (1) the *polar axis* mounted on an adjustable base attached to the top of the telescope directly above the center of its supporting axle so that the entire attachment, except the *hour drum* near the base, is free to rotate about the polar axis; (2) the declination arc marked in the figure; (3) the wide, movable *solar arm* about 5 in. in length, which has a small lens and small silver plate at each end. Each plate is ruled with fine lines which define a small square for the purpose of centering the sun's image formed by the solar lens. Each lens is provided with a shutter for use when the sun is very bright; otherwise it is difficult to tell when the image is exactly centered in the square. When the sun is dull the shutter is moved back, leaving the lens clear. It should be noted that the arm which carries the lenses and the declination arc is reversible about the polar axis. When the declination is south (sun below the equator), the arm is turned end for end from the position shown in the figure. This is the reason for two lenses and two ruled plates.

534(*a*). PROCEDURE FOR DETERMINING A MERIDIAN WITH THE BURT SOLAR ATTACHMENT. This procedure is so similar to the methods already described for the Smith and Saegmuller solar attachments that only a brief outline is necessary. Steps 1, 2, 3, and 4 are the same as for the Smith solar, except that after setting off the declination due care must be taken to turn the proper lens toward the sun. (5) Level up very carefully, using the telescope level in at least two positions at right angles for the final test. Note the amount and sign of the index error of the vertical circle. Then set off the colatitude of the station on the vertical circle. (6) On the hour drum set off the local apparent time of the observation. Other steps required for backsighting, turning the transit so as to bring the sun's image within the ruled square, and reading the azimuth of the line to the backsight are so nearly the same as with the other attachments that further description is unnecessary.

Note. For practical suggestions, see the next article. For the adjustment of the Burt solar attachment, see Chapter 54, Art. 752.

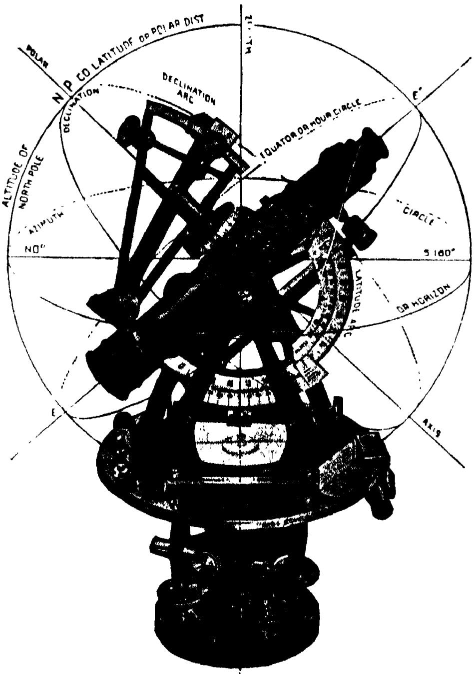


FIG. 534. The Burt Solar Attachment. (Courtesy of W. & L. E. Gurley, Troy, N. Y.)

535. Practical Suggestions for the Use of Solar Attachments. Many of the suggestions in Art. 528(*d*) apply equally well to the use of the solar attachment, especially that pertaining to the best time for observations.

1. In correcting the declination for refraction that portion of Table IX on p. 1112 should be used which most nearly corresponds to the latitude of the observer. Note the time of observation in hours before or after noon, and opposite this number in the table find the correction in the column corresponding most nearly to the declination. Notice that if the declination is north (+) the correction is to be added whereas if it is south (−) it is to be subtracted; or, if given the proper sign, the correction is always added algebraically (northern hemisphere).

2. In using the Burt attachment, when the image has been centered in the ruled square it will at once move out; but if no mistake has been made it may be kept approximately in the square by simply revolving the solar attachment about the polar axis. If it is necessary to disturb the main part of the transit, it shows that the polar axis is not pointing exactly at the pole or that some mistake was made in setting off the declination. Similar statements can be made with respect to corresponding motions of sun and instrument when either the Smith or the Saegmuller attachment is used.

3. When using the Burt solar the sun's image should be examined through a magnifying glass during the final stages of the observation. It is said that an error of one-quarter of a minute in azimuth or latitude can be detected by thus observing the sun's image; nevertheless, the instrument, under the best of conditions, cannot be relied upon to give as accurate results as may be obtained by the direct method of observation.

4. If at the end of an observation the plate verniers, originally set at zero, do not still read zero, it is likely that the wrong tangent screw has been turned. No harm has been done, however, as the reading can be noted and subtracted from the final reading to the backsight.

5. To save time in the field, some surveyors are accustomed to prepare, in advance, a table of declinations of the sun for each hour or half hour of the working day. Appropriate headings for such a table are: watch time, local apparent time, declination, refraction correction, declination setting.

6. Both the Smith and Saegmuller attachments can be used when the sun's disk is visible through thin clouds or haze. Under these conditions the colored screen over the solar eyepiece can be swung aside and the sun observed directly.

7. The accuracy obtained by use of the solar attachment is usually considered to be less than that obtained by observations of the sun's altitude and the solution of the astronomical triangle. For this reason the solar attachment is used less than formerly except in surveys where it is necessary to determine azimuth frequently.

536. Other Methods of Determining a Meridian. There are two other methods of determining a meridian: (1) Sight at the sun or a star when it is some distance from the meridian, carefully measuring its altitude, and then sight at it again when it has crossed the meridian and is at exactly the same altitude. In the case of a star the meridian is halfway between the two sights, and in the case of the sun it is nearly so, being found by applying a small correction. This method, called the method of equal altitudes, is explained in detail in Mr. Barber's article at the end of this chapter. (2) A second method is to sight at the sun at exactly local apparent noon. This method involves the determination of the true local time of the place of observation, i.e., the difference between the watch time and sun time. As the sun is moving rapidly, the observer's longitude and the time must be known very closely.

537. To Determine Latitude. (*a*). BY OBSERVATION ON POLARIS. (1) Observe Polaris at culmination and correct the vertical angle for refraction (Art. 525) and the index error if there is any (Art. 271(*b*)). The result is the true altitude. (2) If the star is at upper culmination, subtract the polar distance of Polaris for the given date; if at lower culmination, add the polar distance. The polar distance for any day in the year may be found from the apparent declination of Polaris as given for that day in *The American Nautical Almanac*. The result is the required latitude. (Why? See Art. 523.) Great care must be taken to eliminate instrumental errors. (See note, 537(*b*).)

537(*b*). BY OBSERVATION ON THE SUN AT NOON. (1) Sight at the lower or upper limb of the sun a little before local apparent noon, allowing for the difference between standard time and local mean time, and follow the sun until it reaches its highest point, i.e., when it crosses the meridian. (2) Correct the vertical angle for refraction and parallax (Art. 525) and for index error if there is any. The result is the true altitude of the sun's limb. If it is the lower limb add the sun's semi-diameter (about 16′, but given more

exactly in *The American Nautical Almanac*); if it is the upper limb subtract the semi-diameter. This corrected result is the true altitude of the sun's center. (3) From the true altitude of the sun's center found in (2) subtract *algebraically* the sun's declination. The result subtracted from 90° gives the required latitude. (Why? See Fig. 523, where $S''M$ represents the altitude at noon.)

Note. In both Arts. 537(*a*) and 537(*b*), only a single telescope position is possible; consequently instrumental errors affect the result seriously. Observations over two days, with normal and reversed telescope, may be averaged to improve results.

537(*c*). LATITUDE BY THE USE OF THE SOLAR ATTACHMENT. The latitude of a place may be determined very easily with a solar attachment by observing the sun at noon. (1) Set off the colatitude (as nearly as can be estimated) and the corrected declination of the sun in the usual manner. (2) Point the smaller telescope at the sun. (The two telescopes are in the same vertical plane but not parallel, the angle between them being equal to the corrected declination of the sun.) By means of the tangent screw which changes the inclination of the *larger* telescope, keep the *smaller* telescope pointing exactly at the sun until it ceases to move upward and begins to move downward. The reading of the vertical circle at this highest altitude is equal to the colatitude.

538. Determination of the Meridian in High Latitudes—Observations by Equal Altitudes.*

There is no operation in the field work of a land surveyor that is fundamentally more important than the establishment of a correct meridian line at the place of survey, from which to deflect the courses of boundaries.

The rules for surveying public lands prohibit a surveyor from assuming a probable "variation" of the magnetic compass to govern his work. They also prohibit him from depending on a variation of his needle throughout his work, though he may have obtained it by proper means at the outset, because all the successive courses may be influenced by unknown attraction.

A large number of surveys throughout the country are known to have been recorded with untrue courses, regardless of these requirements, causing many legal complications.

From a desire to aid surveyors to overcome the difficulties known to impede precision in this element of their work, especially in high latitudes like those of Alaska, and to determine exact meridians by astronomic means without very elaborate instruments, some suggestions are here offered as auxiliary to the means which are adapted to all places.

The use of equal altitudes has received slight notice in a few works on surveying, but has not been set forth in the detailed manner applied to the other processes. But since preparing this article and laying it aside for consideration, it has been learned that this principle was recommended to the deputies in one of the states by instructions issued some years ago by a surveyor general. It was given them "as being available in emergencies, calling their attention to the advantages in such cases of being able to take the forenoon observation, go on with the survey till the afternoon time approached,

* The remaining pages of this chapter are a reprint of a paper by Mr. A. W. Barber, published in *Michigan Engineer, The Proceedings of the Michigan Engineering Society*, for 1905. The paper contains many practical suggestions which are as useful today as in 1905.

and then complete the set. It is available also for P.M. followed by A.M. observation."

Thus the writer now finds, from correspondence, that one of the best authorities on these subjects then gave full indorsement to the equal-altitude method for using either sun or star, as being correct, practical, and of important utility in many contingencies.

The instructions now generally provided require that the meridian be ascertained by observation of either Polaris or the sun; and that the correctness of all solar work must be checked and confirmed by Polaris.

Various Obstacles

to compliance with these rules have been found in Alaska, and are evident:—

Polaris being 55 to 70° above the horizon, it is too high to be observed without a reflecting eyepiece.

Its position so near the zenith greatly increases any instrumental error, in bringing a vertical line to the ground.

It is obscured by daylight during several months in the summer season.

Fogs and mists are said to be often prevalent, even after clear days.

The sun in winter reaches only a small elevation, moving in a low arc, with high and uncertain refraction, both causes unfavorable to accurate results.

The ordinary solar compass or transit is not constructed for use in such latitudes, and special constructions are inconvenient.

Yielding to these difficulties, it appears to have been a common but unauthorized practice to disregard accuracy, assume a magnetic declination, and work by the needle only.

A Simple and Practical Method

is therefore very desirable, to enable surveys to be made with courses that will bear the test of a correct examiner. Such a method is now suggested, which may be used by any careful observer having a good common transit with vertical arc.

This process requires no reflecting eyepiece; no full vertical circle; no mathematical tables; no calculations of local mean time, standard time, sidereal time, or astronomical and civil day. It does not depend on the date or the hour of the day, or any reckoning of culmination, hour-angle, elongation, or polar azimuth. Neither does the sun's declination or the atmospheric refraction enter at all into the calculation.

Moreover, should there be unsuspected error in the graduation or setting of the vertical arc, or some defect of collimation in the telescope, it would equally affect both parts of the observer's work, and produce no effect in the resulting meridian.

The Method by Equal Altitudes

depends on the fact that the circumpolar stars describe invariable circular arcs below the true pole; and that when a peg is set on the ground to mark the course to a certain star when west of north at an altitude of say 35°, and another peg is afterward set to mark the position of that star when it has passed to the northeast and risen to the same precise elevation, the meridian of the transit will lie exactly midway between the pegs. Hence the surveyor may be advised to proceed as follows in any latitude or time of the year when the stars are sufficiently discernible:

1. Choose a suitable station, set transit firmly and level precisely, by the telescope level turned in all directions.
2. Set the index arm for vertical angles at zero, and keep it tightly clamped for reading elevation angles.
3. Provide suitable illumination for cross wires, and also for reading angles, horizontal and vertical.
4. Select a conspicuous star, perhaps 30 or 40° from the pole, which appears two hours more or less before its lower culmination—that is, which stands west of north and is rapidly descending. Identify this star beyond all chance of error, noting it on a diagram for certainty some hours later.

5. Direct the telescope to this point, fixing the star at the intersection of the cross wires, and clamp the axis so it will retain the altitude shown on the arc. Read and note down the angle of elevation, and read more than once.

6. Unclamp the axis, bring the telescope to the earth, and have an assistant drive a peg in line with the cross wire, three to five chains distant. A candle held there before a white surface will exhibit the wire and give the exact point for a tack.

7. Repeat the observation once or twice, at intervals of 10 or 15 min., for confirmation of results, marking successive pegs, *A*, *B*, etc., with degrees and minutes of elevation found.

8. Be ready to observe the upward path of the same star after it has passed east below the pole. Correct the leveling, set the vertical index successively at each altitude previously noted (beginning with the lowest) ; and when the star (diagonally ascending in the field of the glass) approaches the horizontal wire, bring the vertical wire also upon the star at the intersection, using the slow-motion screw of the horizontal plate.

9. Keep the plate at that point, bring the telescope down, and set peg in line as before. Repeat the process for each observation *A*, *B*, etc., taken before midnight, marking each peg *B*, *A*, etc., with the elevation in figures.

10. Measure the arc between pegs *AA*, denoting equal altitude, and take one half. Lay off this half from either peg, and set a peg and tack for the true meridian. As a test of correctness, the middle point between pegs *BB* and between *CC* should be found to coincide with the one first found. A single pair is sufficient, except for confirmation.

By Equal Altitudes of the Sun

In this operation a reflecting eyepiece with dark glass will be necessary. The sun's large image in the field cannot be centered as truly as a star. It is therefore found best to place the intersection of the wires at the lower limb (apparently the sun's upper edge, as reversed by the mirror) and at the precise point of tangency, when the sun is just leaving the horizontal wire, apparently descending, with the vertical wire bisecting its disk.

This is convenient for the forenoon observations; hence at the corresponding times after noon, with the proper altitude of telescope, one must be ready at the moment the sun (now apparently ascending) first reaches the level wire, having the vertical one bisect the sun by the point of tangency, as before.

The pair of pegs in this case will be set southeast and southwest of the station. The center or bisecting line of the included arc would be the meridian, were it not for the sun's change of declination in the intervening time. This slight change requires a calculation for correction which the star process avoids. The formula calls for a short arithmetical process, with a table of natural sines.

Working by a Reference Mark

Instead of using pegs and tacks for day work in this line, the writer has found it easier to use a reference point or mark.

On April 17, after careful leveling, I set the horizontal plates at zero with the telescope directed at a distant spire for my mark. Clamp the lower plate fast, and direct the telescope to the sun, observing it as above shown. Find by the horizontal angle that the sun's azimuth to the left or east of the spire is 30° 27′; and by the vertical arc I find his altitude 47° 09′. (The semi-diameter may be disregarded in each pair of observations if the same limb of the sun is used each time.)

For the corresponding afternoon observation, I have the index of altitude fixed at 47° 09′, and watch the sun rise (apparently) to the proper position. At the right moment clamp the plate, use the slow motion, and when the disk is in position find from the horizontal plate that the sun is 84° 49′ west of the spire or mark.

The whole arc is 30° 27′ + 84° 49′ = 115° 16′; and the bisecting meridian is 57° 38′ from either position of the sun. From this one half I subtract the first azimuth of the sun from the mark; 57° 38′ − 30° 27′ = 27° 11′ as the true bearing of the mark from the uncorrected meridian, and it apparently bears S 27° 11′ E from the transit.

The Correction for Declination

at or near the times of the solstices will be merely theoretical, as an hourly difference of declination less than 10″ will be quite negligible. But during the rest of the year it should be ascertained, for it would amount to as much as a change of 10′ were observations taken 6 hr. apart on September 25, in latitude 65°.

To calculate this correction: Take one-half the change in declination between observations at equal altitude; divide these minutes of change by the product of the cosine of the latitude by the sine of half the difference in time expressed in degrees (15° per hour); the quotient will be the minutes of arc for the correction. This is to be applied from south to west from June 21 to December 21 (declination decreasing) and from south to east the rest of the year.

Example of Above Operation

on April 17, in latitude 39°, using standard watch time; second and third pairs of observations, the first pair being already noted.

B.	10:01 A.M.	altitude 49° 36′	azimuth E	from mark 26° 07′
C.	10:16 A.M.	altitude 51° 48′	azimuth E	from mark 21° 38′
C.	1:59 P.M.	altitude 51° 48′	azimuth W	from mark 76° 00′
B.	2:14 P.M.	altitude 49° 36′	azimuth W	from mark 80° 29′

The sum of the measured arcs of the *BB* positions, 26° 07′ + 80° 29′ = 106° 36′. The middle point for meridian is at 53° 18′. As the reference mark is 26° 07′ from the forenoon sun, its arc from the south meridian point must be 53° 18′ − 26° 07′, and its course from the transit is again found S 27° 11′ E. (Uncorrected.)

The third pair, *CC*, gives 21° 38′ + 76° 00′ = 97° 38′. One half of this, or 48° 49′, less 21° 38′, gives the same resulting arc, 27° 11′.

Correction for Change of Declination

One-half the change on that day in 4¼ hr. was 1.9′. The cosine of the latitude 39° is 0.78; the sine of half the difference in time (2⅛ hr. = about 32°) is 0.53; their product is 0.78 × 0.53 = 0.41. Dividing 1.9′ by 0.41, the quotient is 4½′ of arc for correction of the south meridian point eastward. This gives the bearing of the spire or mark from the true meridian, S 27° 06′ 30″ E.

The Direct Transit Observation of the Sun

for meridian is an operation highly commended by many; and with a perfect instrument used by an unerring workman it should be wholly reliable. Yet the writer meets numerous capable engineers who object to its use and will not depend on it.

There are so many uncertainties attending its use by unknown persons with the possibly disordered instruments, and so many lines nominally established by it have been found out of limits, that the method has not been approved in the Manual of Instructions unless the results are verified by the independent check of Polaris. The latter being direct and simple and having no possible connection with the solar methods cannot be dispensed with as the court of last resort to prove the accuracy of the tedious and intricate "direct solar" problems, where the mean of repeated observations is likely to involve repetition of error.

To use the sun in that manner, one must sacrifice much time, in the best working hours, in observing and in solving his spherical triangles, usually under conditions of physical discomfort, while his party waits; or else he must proceed to work and secure his solution at another time and place.

Of all the methods of taking an observation for meridian, this appears most thickly beset with chances for error. The result is certainly affected by any index error in the vertical arc, or in the collimation of the telescope, unless one has a full circle and eliminates the error by reversing, repeating, and

averaging. Any arithmetical error in logarithmic readings or written work will vitiate the results.

The surveyor is obliged to pay due regard to latitude and colatitude, altitude and zenith distance, declination and codeclination. He must take care of his standard time, longitude, and azimuth from reference mark. The exact refraction should be used, with due regard to temperature, barometer, and humidity. Having safely determined all these, he is then ready to perform his calculation; yet at last may well suspect whether some error has crept in, unless he has some independent test.

Such a check, suited to the latitude of Alaska or elsewhere, is believed to be found in the method of equal altitudes. A surveyor may set a tack beneath his star at 10 in the evening, noting its altitude; and should fog and storm set in for a day or a week, it will only delay but not defeat his completion of the work. Whenever it clears off he can secure his counter observation about 2 in the morning, with no doubt of finding the correct meridian.

The foregoing pages will not appeal strongly to surveyors that are indifferent to certain precision, and averse to losing sleep; but they have been approved by several of a different class—men who have kept many a long vigil of the small hours to secure the most critical use of Polaris, and lay courses from which no competent inspector could materially differ.

PART II: OFFICE WORK

INTRODUCTION

OFFICE WORK may be divided into two parts: (1) *computing* and (2) *drawing.*

539. Computations are of two kinds.

1. Computations required for purposes of plotting, which include such work as *calculation of bearings and azimuths* (Chapter 37), *latitudes and departures* (Chapter 38), *omitted measurements* (Chapter 39), and other miscellaneous computations preparatory to plotting maps and profiles.

2. Computations made for the purpose of obtaining data to be used in laying out work in the field, or for ascertaining desired information concerning the tract surveyed. Under this head may be classed such work as *calculation of areas* (Chapter 40), *subdivision of land* (Chapter 41), *earthwork computations* (Chapter 42), and miscellaneous work in computing dimensions and elevations.

For the most part, office computations involve only the simplest principles of plane trigonometry and of geometry. Accuracy and facility in the use of logarithms are essential. Much of this work is now commonly done by the use of a calculating machine and tables of natural trigonometric functions.

540. Drawing consists in *plotting* and in *finishing maps and profiles,* and, occasionally, in the graphic solution of problems.

1. *Plotting* consists mainly in reversing on paper operations in the field, drawing instruments being used in place of field instruments, and measurements being laid off to a greatly reduced scale. The nature of this work is suggested by the following chapters: *methods of plotting angles* (Chapter 45), *methods of plotting traverses* (Chapter 46), *plotting details* (Chapter 48), and *plotting profiles* (Chapter 50). Many of the methods are analogous to field methods, the scale being used in place of the tape, and the protractor in place of the transit or compass.

2. *Finishing maps and profiles* (Chapter 49) consists in inking the pencil drawing, in lettering the map, in putting on conventional signs, and in such miscellaneous work as drawing border lines and meridian needles.

CHAPTER 36

OFFICE COMPUTATIONS

GENERAL DIRECTIONS AND SUGGESTIONS

In this chapter are given general directions and suggestions which apply to all kinds of office computations. The aim is to outline methods of avoiding mistakes and of economizing time.

541. Checking Computations is recognized in practice as a very essential part of office work. No important result is allowed to stand until it has been checked. Each student, therefore, should learn to check his own work as a matter of course; otherwise he will find himself severely handicapped when he enters upon actual practice. Once he understands a method, he should find his own mistakes, and make sure of his own results, before submitting work to an instructor.

Remark. Frequently students may work together with profit *when studying methods,* but the habit of comparing results at every opportunity is a real weakness which is likely to cling to one long after graduation. It is true that in practice two men often work on the same problem, but unless they work independently any agreement in results is of little value as a check.

There are several general methods of checking computations, among which are these. (1) Checking arithmetical work step by step, as, for example, checking multiplication by casting out nines (Art. 545(*h*)). (2) Checking results by known conditions, as, for example, checking the three computed values of the interior angles of a triangle by comparing their sum with 180°. (3) Checking a result by using a different method of computation, as, for example, finding the length of the hypotenuse of a right-angled triangle by trigonometry and then checking it by squaring the other two sides. (4) Checking a result by using different data, as, for example, checking the length of a diagonal of a quadrilateral by solving each of the two triangles of which it is a common side. Usually one or more of these methods of checking can be used, no matter what the nature of the computations may be.

541(*a*). Approximate checks are of great value in making sure that no mistakes have been made. Computed areas may be checked approximately by means of the planimeter. Latitudes and depar-

tures of a traverse which does not close may be checked roughly by plotting the traverse. Computed lengths may be checked by scaling the map. Accurate computations may be checked roughly by the use of the slide rule. Thus many approximate methods are constantly used to check results which have been obtained by more accurate methods.

542. Systematic Methods. The main object of systematic methods in office work is to economize time and to insure correct results. Checks are used to detect mistakes; systematic methods to avoid them. The remarks in Art. 20 are especially true of office computations.

Remark. It is sometimes difficult to convince students that systematic methods actually *shorten* work. In the long run it will take less time to avoid mistakes than to correct them. Why not take a little trouble to avoid much trouble?

The following suggestions are offered for systematizing office computations:

1. A *"separate-leaf" notebook* with pages about 8 by 10½ in. is recommended for class work. It will be easier to keep computations in columns and to make good sketches if the pages are divided into quarter-inch squares or slightly smaller. The computations for each exercise can be handed in, checked, returned, arranged in logical order in the notebook, and preserved for future reference. Avoid the use of loose scraps of paper for computations. In practice, the calculations in connection with a survey may be carried on for a considerable period of time, and it is important to be able at any moment to find easily and quickly any particular computation that has been completed. For this reason pages should be numbered, and the character of the computations should be indicated with running titles at the top of the pages. Each notebook should have an index, and all books should be numbered for filing purposes.

2. *Arrangement.* Adopt some economical and systematic arrangement. If the same kind of work is always arranged in the same way, blunders and omissions often become self-evident. It is usually advantageous to arrange work in columns. (See pp. 644 and 689.) The head of each column, each final result, and all other important quantities should be labeled so that every step in the computations may be followed by simply glancing through the work. At the head of the page the problems should be concisely stated, and the data given.

3. *Order of Procedure.* Having brought together all the necessary data, *check them* before beginning computations. Too frequently one who is inexperienced completes his calculations only to find that some mistake in the data renders them useless. It is particularly important if data have been copied from the field notes to compare them with the original notes in order to avoid mistakes. Adhere to the same order of procedure for similar problems, so that much of the work will become automatic, and the chances of error will be diminished.

543. Speed in computing may be gained in several ways: (1) by adopting economic methods of procedure; (2) by using short cuts in arithmetical work; (3) by eliminating unnecessary steps and worthless refinements; (4) by the use of tables; (5) by the use of graphic methods; (6) by the use of mechanical methods.

544. Economic Methods of Procedure consist largely in following a logical sequence of operations and in performing at one time all operations that are alike. Frequently the entire work in a problem can be blocked out on paper in some convenient form before any numerical work is begun so that afterward the attention may be concentrated upon the mechanical work of computation. (See Art. 573(*a*).) This is especially true of logarithmic computations and work to be done by the use of a calculating machine. The less the mind is compelled to turn from one thing to another, the more quickly and accurately it works.

545. Short Cuts in Arithmetical Work. There are a few simple methods of shortening arithmetical work in common use that are of great value in computing. If the student is not already familiar with those given in this article, he can easily learn them; and, once he becomes accustomed to their use, he will be surprised at the amount of time they save.

545(*a*). Mental multiplication. The product of composite numbers may be obtained mentally by resolving them into factors and finding the product of these factors.

(1) 32 × 28 = 16 × 8 × 7 = 896. (2) 32 cu. yd. × 27 = 16 × 6 × 9 = 864 cu. ft.
When the multiplier is a factor of 100, multiply by 100 and divide by the proper quantity:

(3) 68 × 25 = 6800 ÷ 4 = 1700. (4) 98 × 50 = 9800 ÷ 2 = 4900.
(5) 76 × 75 = 3800 + 1900 = 5700. (6) 41 × 150 = 4100 + 2050 = 6150.

Mental division is the reverse of the above.
(7) 864 cu. ft. ÷ 27 = 864 divided by 9 and again by 3 = 32 cu. yd.

545(*b*). Long multiplication can be shortened in many cases by a similar method. It is surprising how many times two adjacent figures in the multiplier form a number which is an even multiple of another figure in the multiplier.

(1)	(2)
2875	Circum. = 2468
273	0.3183
8625	7404
77625	44424
	7404
784875	
	Diam. = 785.5644

Illustration. In the first illustration 8625 is multiplied by 9, since 27 = 9 × 3. In the second illustration multiply first by the left-hand 3, and then 7404 by 6, since 18 = 6 × 3.

(3) Multiply 265894 × 497328, multiplying by only 4 digits instead of 6. Suggestion: 32 = 4 × 8 and 49 = 7 × 7.

545(*c*). LONG DIVISION. There is little gained in applying the corresponding process in division except when the divisor can be factored into two divisors, each of which does not exceed 12.

Illustration. 48926 ÷ 63. The work as shown requires 16 figures, whereas long division by the regular method would require at least 29.

```
7|48926.0
9|6989.4
  776.6
```

545(*d*). ABRIDGED MULTIPLICATION may be used when the product is required only to a given number of places of figures. The method of procedure is as follows. (1) Reverse the order of the figures of the multiplier, writing them directly below those of the multiplicand. (2) Multiply in the usual way, except that in multiplying by any figure of the multiplier begin with the one in the multiplicand directly above it, mentally adding anything which should be carried from the multiplication of the preceding figure of the multiplicand by this figure of the multiplier. The result is the first right-hand figure of the corresponding partial product. The partial products are written so that their right-hand figures are in the same vertical column. The decimal point is determined by inspection.

Illustration. Let it be required to multiply 140.66 by 52.13 to one decimal place. The long method of multiplication and the short method are given side by side.

```
      1 4 0.6 6          1 4 0.6 6
        5 2.1 3            3 1.2 5
      ---------          ---------
      4 2 1 9 8          7 0 3 3 0
    1 4 0 6 6              2 8 1 3
  2 8 1 3 2                  1 4 1
7 0 3 3 0                      4 2
---------------          ---------
7 3 3 2.6 0 5 8          7 3 3 2.6
```

Notice that in multiplying by 1 the result was put down as 141 instead of 140. This was because the amount to carry (0.6) was nearer 1 than 0.

545(*e*). NOTE. If a result is required to n places of figures, it is usually best to write the first or right-hand figure of the multiplier under the $n + 1$ place of the multiplicand, i.e., carry the computation one more place than required. This is illustrated on p. 616, where the result was required to four places of figures. (Or two decimal places.)

4 8.6 6 7 8	4 8.6 6 7 8	4 8.6 6 7 8
1.0 6 2 3	3 2 6 0.1	3 2 6 0.1
1 4 6 0 0 3 4	4 8 6 6 8	4 8 6 7
9 7 3 3 5 6	2 9 2 0	2 9 2
2 9 2 0 0 6 8	9 7	1 0
4 8 6 6 7 8	1 4	1
5 1 6 9 9 8 0 3 9 4	5 1.6 9 9	5 1.7 0

The right-hand multiplication is accurate enough for most purposes, but by carrying the work to five places, as in the middle example, the result is exact to four places. (In this case it happens to be exact to five places.)

545(*f*). Note. Always begin by multiplying that figure in the multiplicand which stands in the first place to the right of the multiplier. Before carrying over the result, change it to the single digit which most nearly expresses it in the next left-hand place. For example, if the product is less than 5 there is nothing to carry; if it is 5 or more but less than 15 there is one to carry; if it is 15 or more but less than 25 there is two to carry, and so on. Thus in the middle example shown above when multiplying by 6 the process is $6 \times 6 = 36$ or nearly 4 and $(6 \times 6) + 4 = 40$, giving a 0 for the first figure. In the right-hand example in multiplying by 2 the process is $2 \times 8 = 16$ or nearly 2 and $(2 \times 4) + 2 = 10$.

545(*g*). Abridged division consists in deleting a figure of the divisor in each partial division after the first, instead of bringing down the next figure in the dividend.

3.1416)41.689348(13.27	3.1416)41.689348(13.27
31416	31416
10273	102733
9425	94248
848	84854
628	62832
220	220228
220	219912
	316

Illustration. Notice that in multiplying the divisor after deleting, the figure deleted is multiplied mentally and the nearest digit carried over. Thus in multiplying by 3 begin with 6, the figure deleted, then $3 \times 6 = 18 = 2$ (nearly) to carry, and $(3 \times 1) + 2 = 5$, the first digit in 9425. Likewise, in multiplying by 7, $7 \times 4 = 28 = 3$ (nearly) and $(7 \times 1) + 3 = 0$, the first digit in 220.

Remark. If a quotient is desired to n places of figures, it is better in some cases to obtain $n \div 2$ or one-half the required figures before beginning to delete.

545(*h*). Rule for checking multiplication. Cast out nines from (1) the multiplicand, (2) the multiplier, and (3) the product. The result in (3) should equal the product of the results of (1) and (2).

Methods of Casting Out Nines. Casting out nines from any number is equivalent to subtracting from that number the nearest multiple of nine. This may be accomplished by subtracting from the *sum* of all the digits in the number the nearest multiple of nine, but it is somewhat easier to cross out first all 9's and all figures whose sum is 9 or a multiple of 9, and then to add the remaining figures. For example, let 4 867 428 be the given number; first cross out 4, 8, and 6, also 2 and 7, thus leaving 4 and 8; then $4 + 8 = 12$ and $12 - 9 = 3$ (or what amounts to the same thing, $1 + 2 = 3$). The same result is obtained by adding the digits, thus: $4 + 8 + 6 + 7 + 4 + 2 + 8 = 39$ and $39 - 36 = 3$ (or $3 + 9 = 12$, and $1 + 2 = 3$). Another method is to begin at one end to add the digits, but to keep casting out nines as the addition progresses, thus: $4 + 8 + 6 = 0$, and $0 + 7 + 4 + 2 = 13 = 1 + 3 = 4$, and $4 + 8 = 12 = 1 + 2 = 3$. Hav-

ing once become accustomed to casting out nines, the method affords a very simple and rapid check for multiplication, as illustrated below.

2875	= 4	2468	= 2
273	= 3	3183	= 6
784875 = 12	= 12	7855644 = 21	= 12
		3	= 3
14066	= 8	486678	= 3
5213	= 2	10623	= 3
73326058 = 16	= 16	5169980394 = 0	= 9

Illustration. Taking the multiplicands, multipliers, and products from Art. 545(*b*), the nines may be cast out as indicated above. Notice that 7848 = 27 = 0 in the first illustration. In the number 5 169 980 394, the three 9's were first crossed out, then 6 and 3, then 8 and 1, leaving 5 and 4 equal to 0. In this way, by crossing out from inspection two or more figures whose sum is 9 or 18 or 27, the work becomes very simple.

Remark. Of course this method is not an absolute check, for it will happen occasionally that two mistakes are made which offset each other. For example, one figure in the product may be 1 too large and another 1 too small; the sum would check though the product would be wrong.

If the remainder in either facor is 0, it is not necessary to cast out nines from the other factor, but simply to note whether the remainder of the product is 0. Why?

545(*i*). Rule for checking division. Cast out nines from (1) the dividend, (2) the divisor, (3) the quotient, and (4) the remainder if there is one. Multiply the result in (2) by the result in (3) and add the result in (4). This final result should equal the result in (1). This process is, of course, the reverse of the preceding.

Illustration. Take the dividend, divisor, quotient, and remainder from Art. 545(*g*):

Dividend	41.689348	= 7
Divisor	3.1416	= 6
Quotient	13.27	= 4
Remainder	316	= 1 + 24 = 7 (check)

545(*j*). To square a small number. (1) Add to or subtract from the given number a quantity such that the result will be a multiple of ten. (2) Subtract from or add to the number the same quantity. (3) Multiply the result in (1) by the result in (2), and add the square of the quantity subtracted or added in (1) and (2).

$84^2 = 80 \times 88 + 16 = 7056$ $\qquad 48^2 = 50 \times 46 + 4 = 2304$

$(8\tfrac{1}{2})^2 = 8 \times 9 + (\tfrac{1}{2})^2 = 72\tfrac{1}{4}$ $\qquad (8\tfrac{1}{4})^2 = 8 \times 8\tfrac{1}{2} + (\tfrac{1}{4})^2 = 68\tfrac{1}{16}$

The above rule is particularly easy of application in squaring multiples of 5.

$(25)^2 = 20 \times 30 + 25 = 625$ $\qquad (45)^2 = 40 \times 50 + 25 = 2025$

545(k). SQUARE ROOT OF SMALL NUMBERS. The following approximate method usually gives results which are accurate enough for most purposes. Divide the given number by that number whose square is nearest the given number and find the arithmetical mean of the quotient and the divisor. Expressed algebraically, let N = the given number, S = the nearest square, and $r = \sqrt{S}$.

$$\sqrt{N} = \left(\frac{N}{r} + r\right) \div 2$$

$$\sqrt{84} = \left(\frac{84}{9} + 9\right) \div 2 = (9.33 + 9) \div 2 = 9.17 \text{ (9.1652 correct)}$$

$$\sqrt{47} = \left(\frac{47}{7} + 7\right) \div 2 = (6.71 + 7) \div 2 = 6.86 \text{ (6.8557 correct)}$$

545(l). SQUARE ROOT OF LARGE NUMBERS may be found in the same way provided a table of squares is also used. The above formula may be written in a more convenient form when applied to large numbers.

$$\sqrt{N} = r + [(N - S) \div (2 \times r)]$$

$$\sqrt{32200} = 179 + [(32200 - 32041) \div (2 \times 179)]$$

$$= 179.444 \text{ (179.44358 correct)}$$

$$\sqrt{175440} = 419 + [(175440 - 175561) \div (2 \times 419)]$$

$$= 418.856 \text{ (418.85558 correct)}$$

The value of S, the nearest square, is taken from a table of squares. Not only is this method very much shorter than the ordinary process of extracting a square root, but results are usually correct to three and often to four decimal places. Deleting the divisor (see Art. 545(g)) shortens the process still more.

545(m). CHECKING BY SECOND DIFFERENCES. This method is useful in a variety of problems involving equations of the second degree. It is especially useful in checking ordinates, taken at *equal* intervals, to horizontal or vertical curves. In this case the method consists simply in finding the "first differences" between successive ordinates, and then finding the "second differences" between the "first differences"; the "second differences" should be equal. Due regard should be paid to algebraic signs.

Illustration. Let it be required to check the elevations given in Art. 479(e) for a vertical curve. The work on p. 619 shows that the "second differences" are equal, and hence the elevations are probably correct.

Elevations	*First Differences*	*Second Differences*
98.6000		
	+0.4125	
98.1875		+1750
	+0.2375	
97.9500		+1750
	+0.0625	
97.8875		+1750
	−0.1125	
98.0000		+1750
	−0.2875	
98.2875		+1750
	−0.4625	
98.7500		+1750
	−0.6375	
99.3875		+1750
	−0.8125	
100.2000		

545(*n*). ADDITIONAL ARITHMETICAL CHECKS. *Addition.* The usual check is to add both up and down the columns. Another check is to go over the work a second time, adding in the carried figures after the others, instead of before, or vice versa.

Subtraction. Add the subtrahend and the remainder, but cover up the minuend while doing so.

Multiplication. In place of the check afforded by casting out nines a similar check, more laborious but more reliable, is that by excess of elevens. Thus in the examples in Art. 545(*h*):

2875 ÷ 11 =	261 +	remainder of	**4**
273 ÷ 11 =	24 +	"	" **9**
784875 ÷ 11 =	71352 +	"	" **3**
9 × **4** = 36 and 36 ÷ 11 =	3 +	"	" **3** (check)
2468 ÷ 11 =	224 +	remainder of	**4**
3183 ÷ 11 =	289 +	"	" **4**
7855644 ÷ 11 =	714149 +	"	" **5**
4 × **4** = 16 and 16 ÷ 11 =	1 +	"	" **5** (check)

When a product is not carried to the full number of places, as in abridged multiplication, neither the check by excess of nines nor that by excess of elevens can be used. In such a case a good check is to reverse the position of the multiplier and multiplicand.

Division. Instead of casting out nines a check by excess of elevens may be used. When the work has not been carried to the full number of places, as in abridged division, about the only check is to multiply divisor and quotient together and add in the remainder.

546. Eliminating Worthless Refinements consists chiefly in avoiding the use of unnecessary figures. Data may be given to more places of figures than there is any need of, or time may be wasted in trying to attain a higher degree of precision than the

data will warrant. The aim in this article is to show that in all calculations it is desirable to make a critical inspection of the numbers which enter into every computation for the purpose of rejecting figures which cannot affect the accuracy of the final results. Not only is it a waste of time to use an unnecessary number of figures in computations, but also the results thus obtained are misleading, for they appear to be more accurate than they really are. Worthless figures may be avoided by an intelligent use of the following facts. (1) Numbers expressing measurements cannot be exact, for the true value of a single measurement is never known. (2) A result cannot be more accurate than the data from which it is obtained, no matter how many places of figures are used in each step of the calculation. (3) In dealing with inexact numbers *consistent* accuracy should be the aim instead of *absolute* accuracy, and consistent accuracy is not gained by carrying all numbers to the same number of decimal places regardless of how many figures each number contains.

546(*a*). ILLUSTRATION. The length of one line measured to the nearest *foot* is 3216 ft.- a second line measured to the nearest *tenth* is 321.6 ft.; a third line measured to the near. est *hundredth* is 32.16 ft.; a fourth line measured to the nearest *thousandth* is 3.216 ft; The last figure in each number is *uncertain*, whereas the first three are *certain*, and the *relative precision* of each measurement is $\frac{5}{32160} = \frac{1}{6432}$. The position of the decimal point has nothing to do with the *relative precision* of a measurement, and the mistaken idea that the number of decimal places to which a result is carried indicates the accuracy of that result is responsible for a great deal of stupid and unnecessary labor.

546(*b*). THE ACCURACY TO BE ATTAINED in any computation is determined by the *number* of certain figures in the data. In order to determine the number of certain figures in any measurement some approximate expression of its accuracy must be obtained. For example, 5600 ft. may contain one, two, or three certain figures according to whether the measurement was made to the nearest 100 ft., to the nearest 10 ft., or to the nearest foot.

546(*c*). NOTE. In office computations it frequently happens that one man works from field notes taken by others. In this case, unless it is evident that a measurement has been given to too many places, it may be assumed, as a rule, that each number contains but one doubtful figure. In the case of a decimal fraction the zeros used merely to indicate the position of the decimal point are not counted as certain figures. Thus, for example, 0.00043 contains only one certain figure; 0.000430 may contain two, since the cipher at the end indicates that the measurement was carried to one more place. A final zero used for this purpose should never be omitted, for it may be the only doubtful figure.

546(*d*). REMARK. The number of *significant figures* in any number representing a measurement is always one greater than the number of certain figures; the *last* significant figure is the first uncertain figure. It must not be assumed that because a figure is uncertain it has no significance. Its uncertainty may be very small, hence it has significance; but there can be no advantage gained in writing a number beyond the *second* doubtful figure.

546(*e*). THE EFFECT OF UNCERTAIN FIGURES in the data upon the results of computations may be illustrated as follows. Four measurements are: 6843.2 ft., 253 ft., 282.42 ft., and 1200 ft. Each may be assumed to contain only one uncertain figure, except the distance 1200 ft., which was measured to the nearest hundred feet and therefore contains three uncertain figures. It is required to add the four numbers, and also to find the product of the first two. Let the uncertain figures be indicated throughout the work by smaller type.

6843.2	6843.2	684	6843.2
253.	253	25	352
282.42		28	
1200	205296	120	1369
	342160		342
8578.62	136864	8570	20
	1731329.6		1731000

Since the uncertain figures in the data render so many figures worthless in the sum and product, the results 8570 and 1,731,000, obtained by abridged methods, are better than 8578.62 and 1,731,-329.6, since the latter appear to be much more accurate than they really are. It is seldom necessary to give a final result beyond the first doubtful figure except when it is desired to show that the uncertainty of that figure is small. Thus, ordinarily, the preceding results may be written 8600 ft. and 1,730,000 sq. ft.

Not only is it misleading to retain more than two uncertain figures in any result, but it is a reflection upon the intelligence of the computer.

The preceding examples illustrate two general propositions:

I. *In addition or subtraction the result cannot be accurate beyond the first doubtful place in any of the numbers added or subtracted.*

II. *In multiplication or division the relative accuracy of the product or quotient cannot exceed that of the factor whose relative accuracy is least.*

546(*f*). NOTE. [The first of these propositions is self-evident and the second is easily proved. Let a and b be the errors of two numbers A and B. Then a/A is the relative accuracy of A and b/B of B. $(A + a) \times (B + b) = AB + Ba + Ab + ab$. Dropping the very small quantity ab, the error of this product is $Ba + Ab$ and the relative accuracy of the product is $\frac{Ba + Ab}{AB} = \frac{a}{A} + \frac{b}{B}$. Hence the relative accuracy of a product is practically equal to the sum of the relative (or fractional) accuracies of the factors, and hence less than either.

It is unwise to adopt hard and fast rules in shortening computations, but by following intelligently the suggestions given below a

great deal of time may be saved without sacrificing accuracy in the results.

546(*g*). SUGGESTIONS FOR SHORTENING COMPUTATIONS. *Multiplication.* (1) Before beginning a multiplication ascertain by inspection which of the factors contains the smallest number of certain figures. Call this the *limiting* factor. (2) Use the abridged method. Reverse the limiting factor for a multiplier, count off from left to right one more place in the multiplicand than there are certain places in the multiplier, and begin under this place to write the multiplier backward from right to left. *The number of certain figures in the product can never exceed the number of certain figures in the limiting number by more than one; frequently the number of certain figures in both will be the same.*

Remark. This method gives the result exactly to one uncertain place, and the error in the second uncertain place, if there is any, will rarely exceed one unit. (See Art. 545(*e*).) By observing what figures are affected by the uncertain figure in the multiplier the number of uncertain figures in the product will become evident. (See abridged multiplication, Art. 546(*e*).)

Addition and Subtraction. (1) The limiting number will be the one whose first uncertain figure is farthest to the left. (2) In writing down the quantities omit all figures in each number beyond the second doubtful place of the limiting number. (See abridged addition, Art. 546(*e*).)

Division. Use the abridged method in Art. 545(*g*), and determine by inspection the figures rendered uncertain in the divisor.

General Suggestions. (1) Retain throughout the different steps of a calculation enough places of figures to correspond to two uncertain figures in the result in order that the first uncertain figure may be free from accumulated rejection errors. (2) In dropping figures, add 1 to the last figure retained when the first left-hand rejected figure is greater than 5; when it is less than 5 leave the last figure unchanged. If the rejected figure is 5, add 1 to the last figure retained if this will make it even; otherwise leave it unchanged.

546(*h*). APPLICATION OF RULES. We usually know before beginning a calculation what degree of precision is required in a result, or else we desire to compute a result as accurately as the data will permit. In either case we may be guided by the rules just given. In the following illustrations of the applications of the rules it is understood, unless otherwise stated, that the last digit in a number is the only uncertain figure.

Illustration I. The lengths of two sides of a rectangular lot are given as 626.8 ft. and 73.2 ft. What is its area? There cannot be more than two certain figures in the result, since there are only two in 73.2. Hence reverse

732, and writing the 7 under the *third* figure in 626.8 obtain the result by abridged multiplication = 45,880. Since the last three figures are uncertain, this result may be written 45,900 sq. ft. By direct multiplication 626.8 × 73.2 = 45,881.76, the last five figures of which are uncertain; hence the result is no more accurate than before.

Illustration II. The lengths in the above illustration are given as 626.834 ft. and 73.253 ft. The data will bear inspection. Was allowance made in chaining for temperature and other sources of error? (Art. 77). If not, there cannot be more than three certain figures in either measurement. Assuming this to be true, 626.8 × 73.25 = 45,912 (by abridged multiplication), or 45.910 sq. ft. Taking the measurements as given, by direct multiplication 626.834 × 73.253 = 45,917.471002. Only four figures of this result can be certain; hence it may be written 45,917 sq. ft., with the last figure uncertain. This is a good illustration of the folly of carrying computations to an excessive number of places.

Illustration III. The radius of a circle is given as 21.84. What is the circumference? There can be only three certain figures in the answer; hence use 2π to four places. By abridged multiplication 6.283 × 21.84 = 137.22 ft., with the last two figures uncertain. Using π to one more place, 2 × 3.1416 × 21.84 = 137.225088 (by direct multiplication), a result with only three certain figures and no more accurate than before.

Illustration IV. The base of a right-angled triangle is given in the field notes as 428.63 ft. and the adjacent angle is 44° 24′, measured to the nearest minute. What is the length of the perpendicular side? The natural tangent for 44° 24′ is 0.97927 and for 44° 25′ it is 0.97984. Since the maximum error in the angle may be as large as 30″ the natural tangent may vary nearly 3 in the fourth place, and there can be only three certain figures. Hence drop the last figure in 428.63 and by abridged multiplication 428.6 × 0.9793 = 419.72, or since there are but three certain figures the result is 419.7. If the length is taken as it stands and a five-place tangent is used, the result by direct multiplication is 428.63 × 0.97927 = 419.7445001 ft., only three figures of which are certain. Hence the result is 419.7, the same as before.

Illustration V. Suppose in the above illustration that the angle was measured to the nearest 10″. The variation in the tangent would be only ⅙ × 0.00057 or 0.00009; hence the first four figures are certain and five places may be retained in the length and in the tangent. Or, by abridged multiplication, 428.63 × 0.97927 = 419.744, four figures of which are certain; hence the result is 419.74.

Illustration VI. Suppose in the above illustration that the base, measured to the nearest foot, is 428 ft. There can be only two certain figures in the result; hence use only three figures in the tangent. By abridged multiplication 428 × 0.979 = 419.0 ft.

546(*i*). TRIGONOMETRIC TABLES. The following general rules may be given for simple computations such as ordinarily arise in office work. They hold good for logarithmic tables as well as for tables of natural functions.

For two certain figures in the result use three-place tables; for three certain figures use four-place tables; for four certain figures use five-place tables. In computations involving several steps it may be necessary to use tables to one more place in each case.

If angles are given to the nearest minute use four-place tables; if angles are known to be in error less than 30″ use five-place tables.

546(*j*). REMARK. Six-place tables are seldom needed in office computations except for measurements of the highest precision, such as those which are made in accurate triangulation. In direct logarithmic multiplication and division with four-, five-, and six-place tables the work is about as 2 : 3 : 4, i.e. the work is doubled by using six-place instead of four-place tables.

547. Logarithmic Computations. The following suggestions are intended to supplement the directions commonly given for the use of logarithmic tables.

1. A large proportion of the time required for the use of tables is spent in merely turning leaves, a fact not always appreciated. Make it a rule, therefore, when the tables are open at a given page to look up all that is to be taken from that page before turning to another. In order to do this it will be necessary, at the beginning of a calculation, to prepare a form or schedule in which a place is left for each logarithm, and for the result of each step throughout the computation. The undivided attention can then be given to looking up the logarithms, thus saving time and diminishing the chances of error.

Illustration. On p. 644 the bearing, the length, and the letters of each given line were entered in the first column, and this column was entirely completed before the logarithmic tables were used at all. (See also the form for triangulation computations, p. 658.)

2. Mistakes are less likely to occur if like operations are always carried out in the same way. For example, in *interpolation*, it is well to work always from the next smaller towards the next larger minute, regardless of whether this takes one up or down the page. Thus if the sine of 42° 20′ 20″ is desired, add one-third the difference in the table to the sine of 42° 20′ rather than subtract two-thirds the difference from the sine of 42° 21′, though the latter method would be correct. Likewise if the cosine of 64° 10′ 15″ is desired, subtract one-fourth the difference in the tables from the cosine of 64° 10′ rather than add three-fourths the difference to the cosine of 64° 11′. Of course one must always be careful to observe whether the tabular values of functions are increasing or decreasing, as one glances from the next lower to the next higher minute, for this will determine whether the proportional difference is to be added or subtracted.

Remark. Form the habit of checking roughly a quantity obtained by interpolation. Thus, for example, if one-fourth the tabular difference has been added to one tabular value, see if the result, as nearly as can be estimated, is three-fourths the tabular difference from the next tabular value. A quick inspection of this sort is likely to reveal any serious blunder that has been made in interpolating.

Some tables give proportional parts; but if they are not given, aim to perform interpolation *mentally* without setting down figures.

3. When an angle is more than 90° there is danger not only of ignoring the algebraic sign of a function but also of using the wrong heading or the wrong column of minutes. In some tables the degrees are indicated from 0° to 180° by giving four different

values on each page, one in each corner. In that case follow the simple rule *to use the line of headings and the column of minutes which are nearest to the degree number used.*

207° 27° TOP OF PAGE 332° 152°

0′	Log Sin	d.	Log Tan	c.d.	Log Cot	Log Cos	d.	60′
60′	Log Cos	d.	Log Cot	c.d.	Log Tan	Log Sin	d.	0′

117° 297° BOTTOM OF PAGE 62° 242°

It will pay to go through a new set of tables and to draw indexes or arrows on each page, similar to those shown above. This simple device will often save mistakes by reminding one of the proper minute column and line of headings to use. If many calculations are to be made with angles greater than 180°, additional numbers may be marked on each page of the tables, e.g., the numbers 207°, 332°, 242°, and 297°, shown above.

4. Another useful device is to letter a narrow strip of cardboard with headings at the top and bottom, as shown above, to fit the tables used, and to slide this strip up or down in taking quantities from the tables.

5. When, as is often the case in trigonometric computations, any one of several methods may be used, the following suggestions will help to decide which to choose.

547(*a*). *An angle is best determined from a function which changes rapidly, i.e., with large tabular differences.*

547(*b*). *A quantity to be found from a given angle is best determined by means of a function which changes slowly.*

547(*c*). *Tangents and cotangents vary more rapidly than sines and cosines; hence use the former to determine angles and the latter to determine lengths.*

Remark. The more poorly proportioned a triangle the more important these rules become.

Illustration. Let the three sides of a right-angled triangle be 20.00 ft., 140.00 ft., and 141.42 ft., respectively. Required the angles A and B. Each can be found in four ways:

$$\tan A = \cot B = \frac{20}{140} = 0.14286 = 8^\circ\ 7\frac{24'}{29} = 8^\circ\ 7'.82 +$$

$$\tan B = \cot A = \frac{140}{20} = 7.00000 = 81^\circ\ 52\frac{282'}{1456} = 81^\circ\ 52'.19 +$$

$$\sin A = \cos B = \frac{20}{141.42} = 0.14142 = 8^\circ\ 7\frac{23'}{29} = 8^\circ\ 7'.79 +$$

$$\sin B = \cos A = \frac{140}{141.42} = 0.98996 = 81^\circ\ 52\frac{2'}{4} = 81^\circ\ 52'.5$$

The denominator in each fraction of a minute is the tabular difference for the corresponding function. This is largest for tan B or cot A, i.e., the function is changing about 0.01456 per minute of arc, and hence this is the most accurate determination of A or B. The least accurate result, and one of comparatively little value, is that obtained from sin B or cos A, the tabular difference being only 4 for one minute of arc. Of course so poorly proportioned a triangle should be avoided in practice when great accuracy is desired.

6. The student will find it a great help to memorize a few logarithms, especially those of the whole numbers up to ten, and of those constants most frequently used. Thus, for example, he can learn, in a few minutes, the logarithms of 2, 3, 4, 5, 6, 7, 8, and 9. Then he will know the logarithms of their squares, and of simple multiples, and hence the logarithms of at least a third of the numbers from 1 to 100 will be at his command.

7. *Finally,* the student is urged to use logarithms in preference to arithmetical calculations whenever the work involves many figures. He must overcome the antipathy to the use of logarithms which seems prevalent among students, and recognize that in long multiplications, in long divisions, in squaring or cubing numbers, and in extracting square or cube roots a table of logarithms is a great labor-saving device.

Note. In multiplying or dividing two numbers of three digits or less, it is usually better to use arithmetical methods than to use logarithms. For a single computation involving numbers which do not exceed *four* significant figures, there may be some doubt as to whether or not logarithms can be used to advantage. When, however, there are a number of similar calculations, involving numbers of four significant figures, time will be saved and mistakes avoided by the use of logarithms. When a calculating machine is available, results may often be obtained by arithmetical methods more quickly and with fewer mistakes than by means of logarithms or any other method.

548. Useful Tables. Among the tables most useful in office computations in addition to the tables of natural functions and the logarithmic tables are the following: stadia reduction tables, tables of latitudes and departures (see Art. 567(*b*)), multiplication tables, such as those of Crelle, which give the products of three figures by three figures to 999 × 999, tables of squares and cubes, decimals of a foot for each 1/64 in., earthwork tables, table of chords, and tables giving squares of feet and inches to each 1/32 in. up to 100 ft.

Many simple tables may be prepared for special use. For instance, any arithmetical calculations which occur again and again may be performed once for all and the results tabulated. Thus, for example, such constants as 43,560 (sq. ft. per acre), 27 (cu. ft.

per cu. yd.), and $\pi = 3.1416$ may be multiplied by each digit from 2 to 9 once for all, thus saving time and avoiding errors in subsequent multiplications or divisions in which these quantities are involved.

549. Graphic Methods often give satisfactory results in less time than they could be obtained by computation. For example, horizontal curves may be plotted and scaled (see Art. 495), gradients and vertical curves may be determined in the same way; areas may be checked by scaling distances and by means of ruled squares; contour points may be interpolated graphically (Arts. 598 and 660(*b*)).

Diagrams may be constructed to be used in place of tables, such, for example, as those for reducing stadia readings (Art. 645(*c*)), for determining latitudes and departures (Art. 567(*c*)), and for estimating quantities in earthwork.

Remark. Excellent devices for shortening office work appear in the engineering papers from time to time, either in the form of diagrams or tables. When engaged in a special line of work, be on the alert to devise similar diagrams or tables. By exercising a little ingenuity a great deal of drudgery may be avoided.

550. Mechanical Methods. Among the most common mechanical devices for supplementing office computations are the slide rule, the planimeter (Art. 589), the pantograph (Art. 663), the protractor, and the calculating machine. There are several forms of slide rules designed for special purposes, such, for example, as the stadia slide rule for reducing stadia readings. There are also various forms of planimeters, pantographs, and calculating machines. The slide rule so greatly facilitates calculations which involve not more than three significant figures that it has become indispensable for office work. For computations which involve more than three significant figures, the calculating machine is recommended. Results can be obtained, by its use, more quickly than by any other method; and where there is a large amount of office work to be done, the machine will pay for itself in time saved. With the better types of machine the principal arithmetical operations are instantly proved without further checking and the operator is relieved of much of the mental fatigue which is likely to accompany other methods of calculation. The details of operation may be mastered easily and quickly. Because of these various advantages, the use of the calculating machine is replacing to a large extent other methods of computation.

CHAPTER 37

CALCULATION OF BEARINGS AND AZIMUTHS

IF the bearing or the azimuth of any line of a survey is known and the angle which that line makes with a second line is also known, the bearing or the azimuth of the second line can be calculated. The methods of calculation are given in this chapter.

551. Algebraic Signs of Bearings and Angles. For convenience in computation, all angles measured to the *right*, or *clockwise*, are plus; to the *left*, or *counterclockwise*, minus. This applies not only to angles measured in the field *but also to angles of bearing*. Thus NE and SW bearings are *plus* and NW and SE bearings are *minus*.

552. The Angular Distance. If through one end of any line a north-south line is established, four angles are formed, two of which are measured from N (one clockwise and one counterclockwise) and the other two from S (one clockwise and one counterclockwise). Each of these angles may be called the *angular distance* of the given line. In giving the angular distance of any line, it is necessary, therefore, to state whether it is *plus* or *minus* and whether it is measured from N or from S.

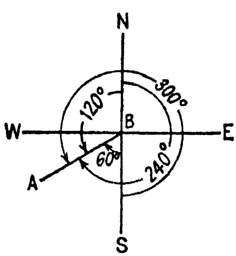


FIG. 552.

Illustration. The bearing of a line, *BA*, is S 60° W (Fig. 552). The four angular distances are: (1) 240° *plus* from N. (2) 120° *minus* from N. (3) 60° *plus* from S. (4) 300° *minus* from S.

Note. Notice that when *any* angular distance is 90° or *less*, it is identical with *bearing*; and when an angular distance is measured clockwise from the north point, it is identical with *azimuth* as that term is used when azimuths are measured from the north.

553. To Convert Angular Distance into Bearing. (1) Notice whether the angular distance is measured from N or from S and whether clockwise or counterclockwise. (2) Imagine the angle turned off from N or from S as the case may be, and from the number of quadrants passed over, determine the *quadrant* of bearing (i.e., whether it is NE, SE, SW, or NW), and then the *angle* of bearing.

553(*a*). ILLUSTRATION. The angular distance of a line from N is 160° *plus*. Since this is more than 90° clockwise from N, but less than 180°, the line must be in the SE quadrant, 20° from S, i.e., the bearing is S 20° E.

553(*b*). ILLUSTRATION. The angular distance of a line from S is 220° *minus*. Since this is more than 180° counterclockwise from S, but less than 270°, the line must be in the NW quadrant, 40° from N, i.e., the bearing is N 40° W.

553(*c*). ILLUSTRATION. The angular distance of a line from N is 280° *plus*. This is more than 270° (or three quadrants) clockwise from N; hence the line is in the NW quadrant, i.e., bearing is N 80° W.

Note. It is important that the student should be able to convert angular distance into bearing or vice versa without puzzling over it, and without relying upon rules. Form a clear mental picture of the four quadrants of bearing, the starting point N or S as the case may be, the direction plus or minus, the number of quadrants passed over, and the final location of the line. The mental process is a simple one; and once the idea is grasped, it ceases to be a matter of memory. Practice on the following illustration, checking the columns of bearings until sure of the method.

Angular Distance	*Bearing*	*Angular Distance*	*Bearing*
N 60°+	N 60° E	S 20°+	S 20° W
N 170°+	S 10° E	S 130°+	N 50° W
N 80°−	N 80° W	S 280°−	S 80° W
N 260°+	S 80° W	N 310°+	N 50° W
N 300°−	N 60° E	S 260°+	N 80° E

554. Calculation of Bearings. The bearing of one line is known; the angle which that line makes with a second line is also known. Required the bearing of the second line.

1. *Set down the given* BEARING *followed by the corresponding sign* (Art. 551).

Note. Every line has two bearings, forward and back (Art. 205). Be sure to use the one corresponding to that end of the line at which the given angle was measured. Change the given letters if necessary. See Art. 554(*d*).

2. *Set down the given* ANGLE *followed by the corresponding sign* (Art. 551).

3. *Add* (1) *and* (2) ALGEBRAICALLY. *Put the first letter of the given bearing in front of the result and the proper algebraic sign after it.*

Note. The result in (3) is an *angular distance.* The letter in front tells whether it is measured from N or from S, and the sign gives the direction—clockwise if plus, counterclockwise if minus.

4. *Convert the angular distance in* (3) *into bearing* (Art. 553).

554(*a*). ILLUSTRATION. The bearing of a line *BA* is N 60° E.; angle *ABC* = 80° clockwise. Required the bearing of *BC*.

BA = N 60° E +
ABC = 80° +
Angular distance of *BC* = N 140° +
Bearing of *BC* = S 40° E

The student should draw a figure showing the four compass points and the lines *AB* and *BC*.

554(*b*). ILLUSTRATION. The bearing of a line *BA* is N 60° W; angle *ABC* = 80° clockwise. Required the bearing of *BC*.

BA = N 60° W −
ABC = 80° +
Angular distance of *BC* = N 20° +
Bearing of *BC* = N 20° E

Since one sign is plus and the other minus, the algebraic sum is really one angle subtracted from the other. Show by a figure that this is correct.

In each of the two preceding illustrations the given angle was measured clockwise. The method holds good for angles measured counterclockwise, or to the left.

554(*c*). ILLUSTRATION. The bearing of a line BC = S 40° E; angle CBA = 80° counterclockwise. Required the bearing of AB.

BC =	S 40° E	−
CBA =	80°	−
Angular distance of BA =	S 120°	−
Bearing of BA =	N 60° E	

By the same figure used in connection with the first illustration the student can see that the process here indicated is correct.

554(*d*). AN IMPORTANT DISTINCTION. The bearing taken at one end of a line should never be used in connection with an angle taken at the other end. The numerical value of the bearing at one end is the same as that at the other end, but the *quadrant of bearing* at one end is different from that at the other. The importance of this distinction is shown in the following illustration. Assume that the bearing of a line AB, measured at A, is S 60° E; that the angle ABC is 105°; and that it is desired to calculate the bearing of BC. Since the angle was measured at B, the bearing used must also be that at B. This bearing is N 60° W not S 60° E; it is the back bearing of the line AB. In referring to the bearing of any line, the letter that represents the end of the line at which the bearing applies will be given first. For example, the bearing of a line AB at A may be expressed as follows: AB = S 60° E. The bearing at B of the same line will then be: BA = N 60° W (Art. 205).

Incorrect

AB =	S 60° E	−
ABC =	105°	+
Angular distance of BC =	S 45°	+
Bearing of BC =	S 45° W	
	Incorrect	

Correct

BA =	N 60° W	−
ABC =	105°	+
Angular distance of BC =	N 45°	+
Bearing of BC =	N 45° E	
	Correct	

It is seen that if the incorrect bearing SE is used to start with, the result makes the point C appear to be SW of B when it really is NE of B.

554(*e*). CHECK. When the notation just suggested is adopted, the letter which represents the station where the angle is measured and the first letter of the line whose bearing is used should be the same. For example, in the correct illustration above, B represents the station and it is also the first letter of the line BA; whereas, in the incorrect illustration, B represents the station but A is the first letter of the line AB. (See also the note, Art. 555(*a*).)

554(*f*). REMARK. The above method holds good no matter in which quadrant of bearing the given line may lie, or whether the given angle is measured to the left or to the right. The method is a very simple one, and the somewhat extended explanation on the preceding pages is justified only by the fact that for some reason many students find difficulty in calculating bearings, especially in the field. At first, in order to be sure of the correctness of the method in any given case, it may be necessary to make a sketch, drawing the

given line in its proper quadrant of bearing, and then roughly plotting the given angle. It is far better, however, to do all this mentally. Until the student can *see* in a mental picture the reason for each step of the method as he takes it, he cannot consider himself proficient in calculating bearings. To this end, practice on the examples given below, checking the bearings in the last column. The calculation is purposely omitted in some cases. Notice in each case whether the given bearing is *forward* bearing or *back* bearing.

EXAMPLES IN CALCULATING BEARINGS

R = Angle to the right; *L* = Angle to the left

Given Bearings	Angle	Bearing Used	Calculation	Angular Distance	Required Bearing
AB: N 30° E	*ABC:* 50° *R*	*BA:* S 30° W	+30 + 50	S 80°+	*BC:* S 80° W
AB: N 30° W	*ABC:* 50° *R*	*BA:* S 30° E	−30 + 50	S 20°+	*BC:* S 20° W
BC: S 50° E	*BCD:* 30° *R*	*CB:* N 50° W	−50 + 30	N 20°−	*CD:* N 20° W
BA: N 30° W	*ABC:* 50° *R*	*BA:* N 30° W	−30 + 50	N 20°+	*BC:* N 20° E
AB: N 30° E	*ABC:* 120° *R*	*BA:* S 30° W		S 150°+	*BC:* N 30° W
CD: N 30° E	*DCB:* 200° *R*			N 230°+	*CB:* S 50° W
CD: N 30° E	*CDB:* 200° *R*	*DC:* S 30° W		S 230°+	*DB:* N 50° E
BC: S 80° W	*BCD:* 300° *R*	*CB:* N 80° E		N 380°+	*CD:* N 20° E
BC: S 80° E	*BCD:* 300° *R*	*CB:* N 80° W		N 220°+	*CD:* S 40° W
AB: N 50° W	*ABC:* 320° *R*				*BC:* N 90° E
BC: N 80° W	*BCD:* 40° *R*				*CD:* S 40° E
BA: N 60° E	*ABC:* 150° *R*				*BC:* S 30° W
AB: N 30° E	*ABC:* 50° *L*	*BA:* S 30° W	+30 − 50	S 20°−	*BC:* S 20° E
AB: N 30° W	*ABC:* 50° *L*	*BA:* S 30° E	−30 − 50	S 80°−	*BC:* S 80° E
AB: N 60° E	*ABC:* 200° *L*	*BA:* S 60° W	+60 − 200	S 140°−	*BC:* N 40° E
AB: S 70° E	*ABC:* 300° *L*				*BC:* N 10° W
BC: S 60° W	*BCD:* 90° *L*				*CD:* N 30° W
BA: N 80° W	*ABC:* 200° *R*				*BC:* S 60° E
AB: S 70° W	*ABC:* 200° *L*				*BC:* S 50° W
AB: S 70° W	*ABC:* 200° *R*				*BC:* S 90° W

555. Application of the Method to a Traverse. In calculating the bearings of traverse lines, it is important to keep in mind the distinction between forward bearing and back bearing, explained in Art. 205 and illustrated in Art. 554(*d*). Unless one observes this distinction, one is likely to make the mistake of using a forward bearing, taken at one end of a traverse line, in connection with the traverse angle measured at the other end.

The correct procedure at any station is as follows: (1) Set down the back bearing of the traverse line from the preceding station. (2) Set down the traverse angle, measured at the station. (3) Add these two angles algebraically after giving to each its proper algebraic sign (Art. 551). (4) Convert the angular distance found in (3) into bearing. This will be the forward bearing of the line to the next station.

This procedure may be illustrated by the traverse shown in Fig. 555. Assume that at station *A* the forward bearing of *AB* is S 30° E. Then at station *B*, the *back* bearing of that line is *BA* = N 30° W. This bearing and the traverse angle of 90° are used in

calculating the forward bearing BC = N 60° E. At station C, the back bearing CB = S 60° W and the traverse angle 75° are used in calculating the forward bearing CD = N 45° W. Thus it can be continued around the traverse.

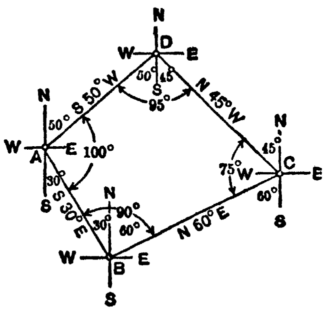


FIG. 555.

The calculations for the entire traverse are given below, first on the assumption that traverse angles are measured to the right (+), and then on the assumption that they are measured to the left (−). In the calculations for angles to the right, the numerical values of the bearing for BC, CD, and DA appear twice, once for the forward bearing and once for the back bearing. In the calculations for angles to the left, this repetition of numerical values is avoided by placing in parentheses, after the forward bearing, the letters which designate the quadrant of the back bearing, i.e., the bearing to which the next traverse angle is added algebraically.

It is seen from these two sets of computations that it is immaterial in which direction angles are taken, provided proper account is kept of algebraic signs.

Angle to Right

AB =	S 30° E	−	
BA =	N 30° W	−	
ABC =	90°	+	
BC =	N 60° E	+	
CB =	S 60° W	+	
BCD =	75°	+	
CD =	S 135°	+	
CD =	N 45° W	−	
DC =	S 45° E	−	
CDA =	95°	+	
DA =	S 50° W	+	
AD =	N 50° E	+	
DAB =	100°	+	
AB =	N 150°	+	
AB =	S 30° E		(check)

Angle to Left

AB =	S 30° E	−	
BAD =	100°	−	
AD =	S 130°	−	
AD =	N 50° E	+	(SW)
ADC =	95°	−	
DC =	S 45° E	−	(NW)
DCB =	75°	−	
CB =	N 120°	−	
CB =	S 60° W	+	(NE)
CBA =	90°	−	
BA =	N 30° W	−	(check)

555(*a*). NOTE. Apply, by inspection, the check of Art. 554(*e*). In the calculations for angles to the right, in every case the letter which represents the station at which the traverse angle was measured and the first letter of the line from the preceding station are unlike, which indicates that the forward bearing of the line is given and must be changed to the back bearing. This is also true of the calculations for angles to the left except for the first station A. The first letter of the line whose bearing is S 30° E is also A. Hence, this is a back bearing as it stands and is not changed.

555(*b*). Calculating bearings from deflection angles. Precisely the same method holds good, except that the *forward* bearing of the preceding line is used each time instead of the *back* bearing. (See Art. 305), for an illustration.) Prove that this method is correct.

555(*c*). To change the bearings of all lines of a traverse by the same amount. This is also a special case. Give the proper sign to the angle by which each bearing is to be changed, according to whether the lines are to be swung clockwise or counterclockwise. Add this angle algebraically to each bearing.

Example. (1) In Fig. 555 change the bearing of *AB* to S 50° E and find the corresponding bearings of the other three lines, changing each by the same amount. (2) Change *BC* to N 30° E and find the corresponding bearings of the other three lines.

556. To Calculate the Angle between Two Lines When Their Bearings Are Known. This is simply the reverse of the process already given.

Method. (1) Make sure that the two bearings are both from the same station, i.e., if necessary reverse the letters. (2) Choose either line as a backsight and convert the bearing of the other line into *angular distance,* measured from the same point (N or S) from which the bearing of the backsight is measured. (3) Give to the angle of bearing of the backsight and to the angular distance of the other line each its proper algebraic sign, i.e., to the right +, to the left −, and subtract *algebraically* the former from the latter, i.e., the bearing from the angular distance. (4) The result is the required angle measured clockwise or counterclockwise *from the assumed backsight,* according to whether the sign is plus or minus.

If the student thoroughly understands the method given on the preceding pages he will have no difficulty in proving the correctness of the method given above.

Illustration. Calculations of the angles in Fig. 555 from the given bearings. Given: *AD* = N 50° E and *AB* = S 30° E. To find *DAB.* This can be done four different ways—two assuming *AD* as a backsight and two assuming *AB* as a backsight.

AD assumed as a backsight.

AB =	N 150°	+	or	*AB* =	N 210°	−
AD =	N 50° E	+	or	*AD* =	N 50° E	+
DAB =	100°	+	or	*DAB* =	260°	−

AB assumed as a backsight.

AD =	S 130°	−	or	*AD* =	S 230°	+
AB =	S 30° E	−	or	*AB* =	S 30° E	−
BAD =	100°	−	or	*BAD* =	260°	+

In the preceding illustration the algebraic sign of each result indicates in which direction, clockwise or counterclockwise, the angle is measured from the assumed backsight. Find the angles at B, C, and D in a similar manner.

557. Calculation of Azimuths. The general method of calculating azimuths is similar to that of calculating bearings. The procedure at any station is as follows: (1) Set down the back azimuth of the line from the preceding station. (The back azimuth of a line is equal to the forward azimuth plus 180°.) (2) Set down the traverse angle with the proper algebraic sign. (3) Add these two quantities algebraically. The result will be the forward azimuth of the line to the next station. When this result exceeds 360°, it should be reduced to an azimuth less than 360°. When the result is *minus*, it means that the azimuth is measured counterclockwise from the north point, and it should be subtracted from 360° in order that it be measured clockwise.

The procedure may be applied to the traverse in Fig. 555. Assume that the forward azimuth of the line AB is 150°. The calculations first for traverse angles to the right and then for traverse angles to the left are given below.

Angle to the Right	*Angle to the Left*
$AB = 150°$	$AB = 150°$
$BA = 330°\ (150° + 180°)$	$BAD = -100°$
$ABC = 90°$	
	$AD = 50°$
$BC = 60°\ (420° - 360°)$	$DA = 230°$
$CB = 240°\ (60° + 180°)$	$ADC = -95°$
$BCD = 75°$	
	$DC = 135°$
$CD = 315°$	$CD = 315°$
$DC = 135°\ (495° - 360°)$	$DCB = -75°$
$CDA = 95°$	
	$CB = 240°$
$DA = 230°$	$BC = 60°\ (420° - 360°)$
$AD = 50°\ (230° + 180° - 360°)$	$CBA = -90°$
$DAB = 100°$	
	$BA = -30° = 330°$
$AB = 150°$ (check)	$AB = 150°$ (check)

Note that in the calculation of the azimuth for BA, a *negative* azimuth was obtained, namely, −30°. This can happen only when the transit angle is *negative* and greater than the back azimuth to which it is added algebraically.

The calculation of azimuths is somewhat simpler than the calculation of bearings and offers fewer opportunities for mistakes, particularly mistakes in algebraic signs. For this reason some engineers prefer to use this method even when traverses are run by the direct-angle method.

CHAPTER 38

LATITUDES AND DEPARTURES

THE theory of latitudes and departures explained in this chapter is as simple as it is important. Latitudes and departures are used in plotting maps (see Chapter 48); also in calculating omitted measurements (Chapter 39) and areas (Chapter 40).

558. Latitude and Departure are terms for two coordinate distances, one (*the latitude*) measured parallel to a *north and south* line, the other (*the departure*) measured parallel to an *east and west* line.

Illustration. In Fig. 558 a line AB is in the NE quadrant. $AC = DB$ is its *latitude* and $AD = CB$ is its *departure.* The angle CAB is the *bearing* of the line.

Latitude difference and *longitude difference.* Terms preferred by some writers to the old and established terms latitude and departure.

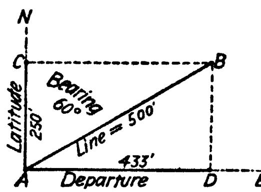


FIG. 558.

559. Latitude and Departure Found from the Bearing. It is evident from Fig. 558 that

$$\textit{Length of the line} \times \textit{Cosine bearing} = \textit{Latitude of the line}$$

$$\textit{Length of the line} \times \textit{Sine bearing} = \textit{Departure of the line}$$

Illustration. A line AB in Fig. 558 is 500 ft. long. Its bearing is N 60° E. The natural cosine of 60° = 0.500; the natural sine of 60° = 0.866.

Length × Cosine bearing = Latitude	Length × Sine bearing = Departure
500 ft. × 0.500 = 250 ft.	500 ft. × 0.866 = 433 ft.

Note. For other trigonometric relations, see Art. 574.

560. Algebraic Signs of Latitudes and Departures are determined by the quadrants of bearing. *Latitudes* are *plus* when the bearing is *north, minus* when the bearing is *south.* Departures are *plus* when the bearing is *east, minus* when the bearing is *west.* Thus the first letter of a bearing (N or S) determines the sign of the latitude, and the second letter of the bearing (E or W) determines the sign of the departure.

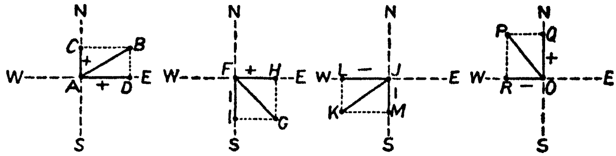


FIG. 560.

Illustration. In Fig. 560 a line has been assumed in each of the four quadrants of bearing, thus illustrating all possible cases. The algebraic signs are shown on the figure and indicated also in the following table:

		Latitude		*Departure*	
Line	*Bearing*	N(+)	S(−)	E(+)	W(−)
AB	NE	*AC*		*AD*	
FG	SE		*FI*	*FH*	
JK	SW		*JM*		*JL*
OP	NW	*OQ*			*OR*

Remark. These algebraic signs correspond to those in common use for ordinate and abscissa, i.e., *above* the axis of reference *plus; below—minus;* to the *right* of the axis *plus;* to the *left—minus.*

Illustration. The calculation of latitudes and departures according to the method of Art. 559 is indicated below.

Line × Cosine bearing = Latitude	N(+)	S(−)	Line × Sine bearing = Departure	E(+)	W(−)
AB × Cos *BA* N	*AC*		*AB* × Sin *BA* N	*AD*	
FG × Cos *GF* S		*FI*	*FG* × Sin *GF* S	*FH*	
JK × Cos *KJ* S		*JM*	*JK* × Sin *KJ* S		*JL*
OP × Cos *PO* N	*OQ*		*OP* × Sin *PO* N		*OR*

Remark. It is evident that the *departure* of any *north and south* line is zero, and that the *latitude* of any *east and west* line is zero.

561. The Latitudes and Departures of a System of Traverse Lines may be used for determining the position of any station with respect to any other station. The *algebraic* sum of the latitudes of all the lines between any two stations gives the distance that one of those stations is north or south of the other; and the *algebraic* sum of the departures gives the distance one station is east or west of the other. This is obvious from inspection of Fig. 561.

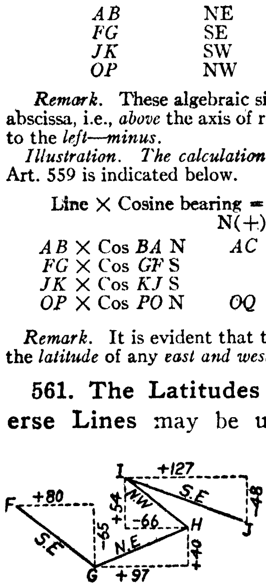


FIG. 561.

Illustration. In Fig. 561 let it be required to find the position of *J* with respect to *F*. Suppose that the latitudes and departures have been found by calculation to be the quantities in feet indicated on the figure. Tabulate the quantities thus:

		Latitude		*Departure*	
Line	*Bearing*	+(N)	−(S)	+(E)	−(W)
FG	SE		65	80	
GH	NE	40		97	
HI	NW	54			66
IJ	SE		48	127	
		+94	−113	+304	−66
			+ 94	− 66	
			− 19	+238	

It is seen that J is 19 ft. south of F and 238 ft. east of F. It is evident that the position of any point with respect to any other point can be determined in like manner. Thus:

I is $+97 - 66 = 31$ ft. east of G and $+40 + 54 = 94$ ft. north of G

Remark. This principle is made use of in plotting (Chapter 48) and in the calculation of omitted measurements and of areas (Chapters 39 and 40); also in subdividing land (Chapter 41).

562. Check for a Closed System of Traverse Lines. The algebraic sum of the latitudes should be zero and the algebraic sum of the departure should be zero.

Remark. This simply means that if a man walks around a closed system of traverse lines, returning to the starting point, he will have walked as far north as he walked south, and as far east as he walked west, i.e., *northings* and *southings* are equal and *eastings* and *westings* are equal.

563. The Error of Closure of a Survey (see Art. 307) may be ascertained by finding the difference between the north latitudes and the south latitudes and the difference between the east departures and the west departures.

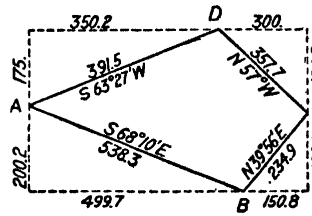


FIG. 563.

Illustration. In Fig. 563 is shown a closed system of traverse lines. Suppose the latitude and departure of each line have been found from calculation to be equal in feet to the numbers indicated in the figure. They may be tabulated as follows:

Line	*Length*	*Bearing*	N Lat.(+)	S Lat.(−)	E Dept. (+)	W Dept. (−)
AB	538.3	SE		−200′.2	499′.7	
BC	234.9	NE	180′.1		150′.8	
CD	357.7	NW	194′.9			−300′.0
DA	391.5	SW		−175′.0		−350′.2
	1522.4		+375′.0	−375′.2 +375′.0	+650′.5	−650′.2 +650′.5
			Error =	−0.2 S	and	+0.3 E

$$\text{Error of closure} = \sqrt{(0.2)^2 + (0.3)^2} = \sqrt{0.13} = 0.36 \text{ SE}$$

Since the total distance around the polygon is 1522.4, the error of closure is $\frac{0.36}{1522.4}$ or about 1/4230. This is the *apparent* error of the field work. The *real* error in field work may have been much greater, since the apparent error includes compensating errors. For example, a north latitude which is too large may be offset by a south latitude which is also too large, or a west departure which is too small may be offset by an east departure which is also too small.

The direction of the error is indicated by the algebraic signs. Thus in this example the error in latitude (−0.2) and the error in departure (+0.3) indicate error south and east, i.e., 0.36 southeast.

Permissible error of closure. (See Arts. 356 to 358; also Art. 360.)

564. Each Line Has Two Bearings. Which Is Used? In computing latitudes and departures one may go around the polygon

in either direction, but the bearings used must all be *forward* bearings *for the direction chosen.*

Illustration. If in Fig. 563 one goes around the polygon from A to B to C to D to A, the bearings used are those noted on the figure because they are *all* forward bearings for the direction chosen. But if one goes around in the opposite direction (A to D to C to B to A), the bearings should be changed so that they *all* will be *forward* bearings for that direction, i.e., AD = N 63° 27′ E, DC = S 57° E, CB = S 39° 56′ W, and BA = N 68° 10′ W. The principal thing to avoid, therefore, is the careless use of a back bearing in place of a forward bearing. For example, suppose that in Fig. 563 the bearing of CD had been put down S 57° E instead of N 57° W, the other bearings remaining as they are in the figure. In the tabulated form the latitude of CD would then be entered in the south column and its departure in the east column, and of course the survey would not close.

565. Latitudes and Departures Computed from Azimuths. It is not necessary to convert azimuths into bearings. If the azimuths are measured clockwise from the north point as zero, and the usual trigonometric rules for signs are observed, the signs of the trigonometric functions of any azimuth will agree with the signs of the corresponding latitudes and departures as given in Fig. 560. Thus the NE quadrant will correspond to the first quadrant in trigonometry, the SE quadrant the second, SW the third, and NW the fourth. This is the principal reason for reckoning azimuths from N instead of from S. (See Art. 209(*b*).)

Illustration. In Fig. 563 the azimuth of DA would be 243° 27′. The cosine and sine of 243° 27′ are both minus; hence the latitude and departure of DA are both minus, or south and west respectively. The student can easily satisfy himself in regard to the signs in the other quadrants in a similar manner.

566. Adjustment of the Angular Error. Before computing latitudes and departures, the traverse angles should be adjusted to satisfy geometric conditions. In a simple closed traverse the error in the sum of the angles (Art. 298) may be distributed equally to each angle of the traverse, provided it can be assumed that all angles have been measured with the same degree of precision. Such a distribution eliminates the effect of constant errors but not necessarily of accidental errors; hence the adjusted values of the angles may not be the true values though they may be the most probable that can be assigned. In a survey in which there are two or more closed traverses interconnected, additional geometric conditions are imposed. For example, in the traverse net on p. 641 the sum of the four angles at Y should equal 360°. In such a survey, the adjustment of angles becomes more complex and can be made only by taking into account such considerations as those outlined in Art. 571(*c*), particularly those in (3) and (4). (See also the last paragraph in Art. 571(*d*).) The adjustment of a traverse net is explained in Art. 572 at the end of this chapter.

567. Different Methods of Finding Latitudes and Departures. (*a*). BY COMPUTATION. The length of each line may be multiplied by the *natural* cosine and the *natural* sine of its bearing, but, unless a calculating machine is used, usually it is better to work with logarithmic tables. Systematic forms of computation facilitate the work and diminish chances of error. A complete example, showing a form for logarithmic work and a method of tabulating results, is given on pages 644 and 645.

567(*b*). TRAVERSE TABLES are simply tables of natural sines and cosines from 0° to 90° multiplied by 1, 2, 3, etc., to 10, and in some tables to 100. In the common forms, values are given for every quarter degree, but such tables are good only for rough compass surveys and work of a similar nature. Tables such as those of Boileau giving values for every minute of arc are exceedingly useful. In finding the latitude and departure of any length expressed by several figures, the latitude and departure for each digit are read from the table (using the given bearing), and each is brought to the correct value by moving the decimal point. The several quantities are then added.

Illustration. The following line of values is taken from a traverse table. Only half a line is shown; the other half is usually given on the opposite page.

	1		2		3		4		5	
	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.	Lat.	Dep.
30°	0.866	0.500	1.732	1.000	2.598	1.500	3.464	2.000	4.330	2.500

Suppose that a line is N 30° E and its length is 4 ft.; then its latitude is 3.464 and its departure is 2.000. If the line is 40 ft. instead of 4 ft., the two values are 34.64 and 20 ft.; if 400 ft. long, the values are 346.4 and 200.0 ft., and so on.

Let it be required to find the latitudes and departures of the two lines *AB* and *CD*, whose lengths and bearings are given below.

AB		*CD*	
Length = 542.3 ft.,	Bearing = N 30° W	Length = 503.1 ft.,	Bearing = S 30° W
Latitude	Departure	Latitude	Departure
433.0	250.0	433.0	250.0
34.64	20.0	00.0	00.0
1.732	1.0	2.598	1.5
.26	.15	.086	.05
469.63 N	271.15 W	435.68 S	251.55 W

Remark. When a zero occurs in a number, instead of inserting the zeros, as shown above, move the next number over two places. A natural mistake which might be made in the second illustration is that of putting down 0.8 instead of 0.08 and 0.5 instead of 0.05 for the latitudes and departures, respectively, of the digit 1 in the quantity 503.1.

567(*c*). TRIGONOMETER. A device for finding graphically the latitude and departure of any line. A diagram is constructed on regular cross-section paper, or squares may be ruled to any desired scale. From the lower left-hand corner of the diagram as an origin, distances are numbered along the horizontal and vertical

edges. A revolving arm, pivoted at this origin, has a straightedge, which if produced would pass through the origin. This straightedge is graduated to correspond to the scale of the cross-section paper. The upper and right-hand edges of the diagram are marked to correspond to degrees and fractions of degrees, with the origin as a center. By means of this angular graduation the revolving straightedge is set to the bearing of the given line, the length of the line is noted on the straightedge, and the corresponding latitudes and departures are read from the cross-section scale. If azimuths are used instead of bearings, the numbering of angles should be continued up to 360°. This instrument is useful in rough determinations, such for example as those needed in plotting to a small scale and for checking computed values, but of course the degree of precision which can be attained is far less than that obtained by computation.

567(*d*). MECHANICAL METHODS. For calculations which involve not more than three significant figures, the slide rule can generally be used to advantage.

The calculating machine for determining latitudes and departures from the natural trigonometric functions is recommended when such a machine is available. When results to a relatively large number of significant figures are required, tables of natural trigonometric functions carried to a corresponding number of places should be used. Seven-place tables, such as Tables XIX and XX in the back of this book, are often used. In multiplying by means of a calculating machine, time may be saved by the short method of multiplication explained in Art. 545(*d*).

568. A Complete Example in the Computation of Latitudes and Departures. The computations in Fig. 568(1) to Fig. 568(5) illustrate every step in finding the coordinates for plotting by the method of latitudes and departures.

1. A rough sketch of the traverse lines was drawn (Fig. 568(1)). Angles and distances taken from the field notes were copied on this sketch and checked by again comparing with the field notes.
2. Angles were adjusted so that the sum of the interior angles in each polygon would be correct. (See Art. 298.) The error was not distributed equally in this case, as there were certain checks on some of the angles, whereas on others there were no checks. Where adjusted values differed from the original values, the latter were crossed out and the adjusted values written in red ink.
3. The field notes gave the angle between Polaris at its eastern elongation and the line *YV* as 63° 25′ (Fig. 568(1)); hence the true bearing of *YV* was 63° 25′ + 1° 37′ = N 65° 02′ E.
4. Starting with *YV* = N 65° 02′ E the bearings of all the other lines were calculated *from the adjusted values of the angles.* The form shown in Fig.

568(2) is not quite so compact as that given in Art. 555. For example, the quantity 180° may be omitted in each case. There are other minor changes,

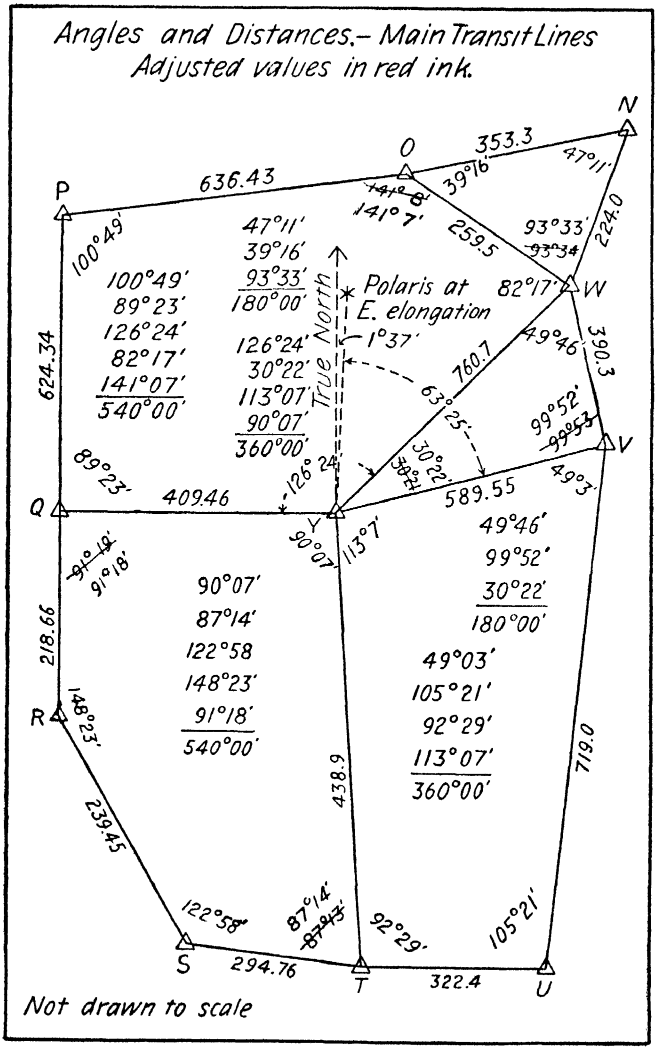


FIG. 568(1).

but the method is the same as that explained in the preceding chapter. The student is advised to follow the form in Art. 555.

Remark. In many surveys no observations are taken to determine the true meridian. In that case the magnetic bearing of any line which is thought to be correct may be assumed to start with, or this magnetic bearing may be changed to a true bearing if the magnetic declination is known. So far as plotting is concerned, it makes no difference what value is taken for the bear-

ing of the first line. An absolutely fictitious bearing may be used, provided the bearings of all other lines are computed from the traverse angles. The

Calculations for Bearings

Outside Lines

Elongation		1°37'
Polaris		63°25'
YV	=	N 65°02'E + (S.W.)
YVW	=	99°52' +
		164°54' +
		180° 0' −
VW	=	N 15°06'W − (S.E.)
VWN	=	225°36' +
		210°30' +
		180° 0' −
WN	=	N 30°30'E + (S.W.)
WNO	=	47°11' +
NO	=	S 77°41'W + (N.E.)
NOP	=	180°23' +
		258°04' +
		180° 0' −
OP	=	S 78°04'W + (N.E.)
OPQ	=	100°49' +
		178°53' +
		180° 0' −
PQ	=	S 1°07'E − (N.W.)
PQR	=	180°41' +
		179°34' +
		180° 0' −
QR	=	S 0°26'E − (N.W.)
QRS	=	148°23' +
		147°57' +
		180° 0' −
RS	=	S 32°03'E −

Outside Lines

SR	=	N 32°03'W −
RST	=	122°58' +
		90°55' +
		180° 0' −
ST	=	S 89°05'E − (N.W.)
STU	=	179°43' +
		90°38' +
		180° 0' −
TU	=	S 89°22'E − (N.W.)
TUV	=	105°21' +
UV	=	N 15°59'E + (S.W.)
UVW	=	148°55' +
		164°54' +
		180° 0' −
VW	=	N 15°06'W − (Check)

Inside Lines

TS	=	N 89°05'W −
STY	=	87°14' +
TY	=	N 1°51'W − (S.E.)
TYV	=	113°07' −
		114°58' −
		180° 0' +
YV	=	N 65°02'E +
VYW	=	30°22' −
YW	=	N 34°40'E +
WYQ	=	126°24' −
		91°44' −
		180° 0' +
		S 88°16'W +

Fig. 568(2).

relative positions of all points on the map will be correct, and this in many cases is all that is required.

5. Another rough sketch was made of the traverse lines (see Fig. 568(3)), and the length and the calculated bearing marked along each line.

6. Latitudes and departures were computed by logarithms (Fig. 568(4)). The data from the sketch in Fig. 568(3) were entered in the first column as shown, then all the logarithm sines and cosines were looked up, then all the

logarithms of the lengths, then the additions were performed and finally the latitudes and departures were ascertained from corresponding logarithms.

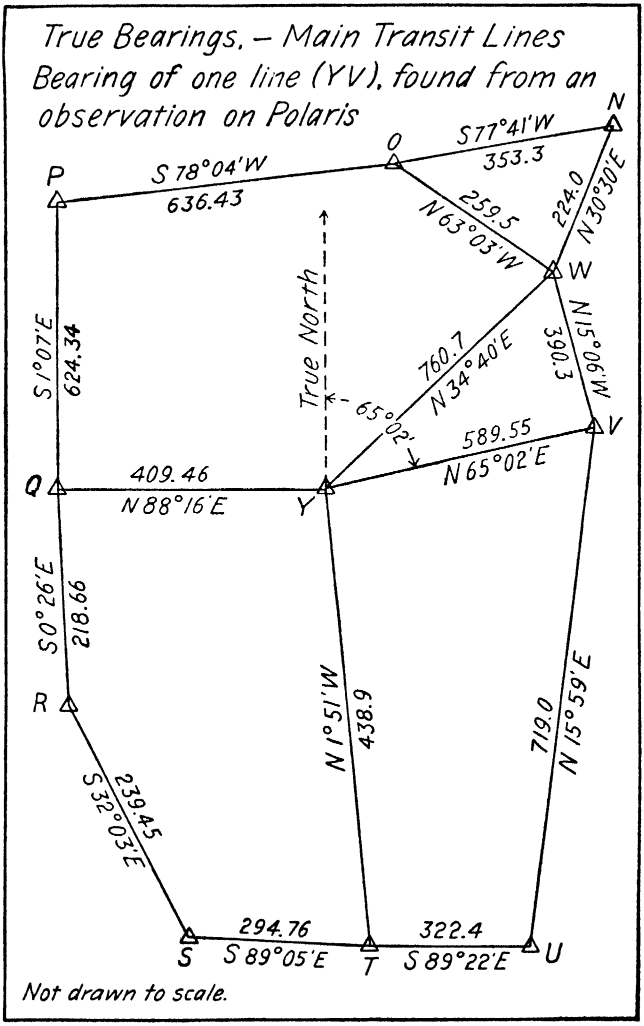


FIG. 568(3).

In this way like operations were grouped to save time. (See Art. 547 for other suggestions for the use of logarithms.)

7. The results obtained in (6) were tabulated in the first seven columns in Fig. 568(5). The errors are found to be 0.02 and 0.14 ft. for latitudes and departures respectively.

8. The last step was to compute and tabulate the last three columns (Fig. 568(5)) for purposes of plotting as explained in Art. 638(a).

Checks. If the errors in (7) are found to be very large, apply two checks: (1) Compare each latitude and each departure with the first and last letters, respectively, of the bearing of the corresponding line to see if each has been

Calculation of Latitude and Departures.

Quantity	Departure	Log. Sin	Log. Cos	Latitude
S 1°07′E		8.28977	9.99992	
624.34		2.79542	2.79542	
PQ =	12.17	1.08519	2.79534	624.23
S 0°26′E		7.87870	9.99999	
218.66		2.33977	2.33977	
QR =	1.65	0.21847	2.33976	218.65
S32°03′E		9.72482	9.92818	
239.45		2.37922	2.37922	
RS =	127.07	2.10404	2.30740	202.95
S89°05′E		9.99994	8.20407	
294.76		2.46946	2.46946	
ST =	294.72	2.46940	0.67353	4.72
S89°22′E		9.99997	8.04350	
322.4		2.50840	2.50840	
TU =	322.38	2.50837	0 55190	3.56
N15°59′E		9.43990	9.98288	
719.00		2.85673	2.85673	
UV =	197.98	2.29663	2.83961	691.22
N15°06′W		9.41582	9.98474	
390.3		2 59140	2.59140	
VW =	101.68	2.00722	2.57614	376.82
N30°30′E		9.70547	9.93532	
224.0		2.35025	2.35025	
WN =	113.69	2 05572	2.28557	193.00
S77°41′W		9.98989	9.32902	
353.3		2.54814	2.54814	
NO =	345.17	2.53803	1.87716	75.36
S 78°04′W		9.99051	9.31549	
636.43		2.80375	2 80375	
OP =	622.67	2.79426	2.11924	131.59

FIG. 568(4).

entered in the proper column, N or S and E or W. (2) See if a latitude and a departure have not been interchanged, the latitude being put in a departure column and vice versa. To do this, notice if the bearing of each course is greater or less than 45°. If it is greater, the departure should be greater than the latitude; otherwise it should be less. (*Continued on p. 646.*)

Tabulation of Latitudes and Departures for Plotting

Line	Bearing	Length	Latitude		Departure		Sta.	Total from P	
			North	South	East	West		Latitude	Departure
PQ	S 1°07′E	624.34		624.23	12.17		Q	− 624.2	12.2
QR	S 0°26′E	218.66		218.65	1.65		R	− 842.9	13.8
RS	S 32°03′E	239.45		202.95	127.07		S	−1045.8	140.9
ST	S 89°05′E	294.76		4.72	294.72		T	−1050.5	435.6
TU	S 89°22′E	322.4		3.56	322.38		U	−1054.1	758.0
UV	N 15°59′E	719.0	691.22		197.98		V	− 362.9	956.0
VW	N 15°06′W	390.3	376.82			101.68	W	+ 13.9	854.3
WN	N 30°30′E	224.0	193.00		113.69		N	+ 206.9	968.0
NO	S 77°41′W	353.3		75.36		345.17	O	+ 131.6	622.8
OP	S 78°04′W	636.43		131.59		622.67	P	0.0	0.1
			1261.04	1261.06	1069.66	1069.52			
			Error = 002		Error = 0.14				

PQ							Q	− 624.2	12.2
QY	N 88°16′E	409.46	12.39		409.3		Y	− 611.8	421.5

Note: Plotting co-ordinates are from P, (most westerly station) as the origin.

FIG. 568(5).

If no mistakes are discovered in (1) and (2), make sure that the data were copied correctly and go over the computations again. One can often tell from inspection which latitude and departure is most likely to be incorrect, and therefore which computation to check first.

569. Degree of Precision. The considerations which govern the degree of precision in the field work of a survey usually govern the degree of precision in the computation of latitudes and departures. For example, if the purpose of a survey is merely to secure data for plotting a map, the number of significant figures in the computed latitudes and departures need be only great enough to enable the draftsman to plot points correctly within the scale of the map. If, on the other hand, the purpose of the survey is such that a high degree of precision is required in the field work, a consistently high degree of precision should be attained in office calculations. There is a tendency to carry the computation of latitudes and departures to more places of figures than the field data warrant. (See Art. 546.)

570. Balancing a Survey (also known as the *adjustment of a survey*) means making such corrections that the sum of the north latitudes shall equal the sum of the south latitudes, and the sum of the east departures shall equal the sum of the west departures. This applies, of course, only when the survey forms a closed polygon. In such a case there is practically always some error of closure, but it does not follow that this error must always be corrected. Thus, for example, on page 645, the errors of 0.02 ft. in latitude and 0.14 ft. in departure may be entirely ignored so far as plotting is concerned. This is usually the case, for if the error of closure is large enough to be appreciable in plotting it usually exceeds the permissible error, and measurements have to be corrected in the field. Hence it is seldom necssary to balance a survey for purposes of plotting. When, on the other hand, the data are to be used for other purposes, as, for example, in describing boundaries of land, it may be necessary to distribute the errors so that the polygon described shall be a closed figure.

571. Two Methods of Balancing a Survey. It is assumed that before the latitudes and departures were calculated, the angular errors were distributed. (See Art. 566.) The survey may then be balanced by any one of several methods. The two most commonly used are: (1) When all measurements have been made with the same degree of precision, the errors may be distributed among the sides of the polygon in proportion to their respective lengths. (2) When some measurements have been made with a greater degree of precision than others, the less accurate measurements should be given the larger corrections.

571(*a*). First method. The following proportions are used:

$$\frac{E_l}{P} = \frac{\text{Total error in latitude}}{\text{Total length of perimeter}} = \frac{\text{Error in latitude of one side}}{\text{Length of that side}} = \frac{e_l}{s}$$

$$\frac{E_d}{P} = \frac{\text{Total error in departure}}{\text{Total length of perimeter}} = \frac{\text{Error in departure of one side}}{\text{Length of that side}} = \frac{e_d}{s}$$

$$e_l = \frac{E_l}{P} \times s \quad \text{and} \quad e_d = \frac{E_d}{P} \times s$$

It is not worth while to split hairs—the corrections can usually be made mentally. Thus approximate values are found for $\frac{E_l}{P}$ and $\frac{E_d}{P}$ once for all and these are multiplied by the lengths of the different sides in succession. Corrections should be subtracted from numbers in a column where the sum is too large and added to numbers in a column where the sum is too small.

This method of adjusting a traverse is based on the assumptions (1) that errors in the field work are accidental and that consequently the correction to each side of the traverse is proportional to its length; and (2) that the angular and the linear measurements are made with the same consistent degree of precision. For these assumptions the method is logical.

Illustration. Let it be required to distribute the errors of −0.2 in latitude and +0.3 in departure in Art. 563.

$$\frac{E_l}{P} = \frac{0.2}{1522.4} = \frac{1}{7500} \qquad \frac{E_d}{P} = \frac{0.3}{1522.4} = \frac{1}{5000}$$

AB	$538 \times \frac{1}{7500} = 0.07$	$538 \times \frac{1}{5000} = 0.11$
BC	234 × " = 0.03	234 × " = 0.04
CD	357 × " = 0.05	357 × " = 0.07
DA	391 × " = 0.05	391 × " = 0.08
	0.20	0.30

The sums of the corrections equal 0.2 and 0.3 respectively. In this example (p. 637) the sum of the south latitudes exceeds the sum of the north latitudes; hence subtract 0.07 and 0.05 from 200.2 and 175 respectively, but add 0.03 and 0.05 to 180.1 and 194.9 respectively. The east departures exceed the west departures; hence subtract 0.11 and 0.04 from 499.7 and 150.8 respectively, but add 0.07 and 0.08 to 300.0 and 350.2 respectively. The corrected latitudes and departures may be written in the table in red ink directly over the original corresponding values.

Note. Correcting the latitude and departure of a line is equivalent to changing the two short legs of a right triangle. Hence, to be strictly consistent, the line itself (or hypotenuse) should be changed to correspond, i.e., a new bearing and a new length should be found which correspond to the corrected latitude and departure. This is often a needless refinement, but when it is necessary it can be done easily by simply reversing the trigonometric method used in finding latitudes and departures. (See Art. 559.)

An alternative method of computation is this. Let S = the corrected length of any side, L = uncorrected latitude, D = uncorrected departure, and s, E_l, E_d, and P same as in Art. 571(*a*).

$$S = s - \frac{E_l}{P} L - \frac{E_d}{P} D \text{ (formula for correcting lengths)}$$

Let B = the uncorrected bearing of any side and C_b the correction in minutes to be applied to this bearing; then

$$C_b = 3400 \left(\frac{E_l}{P} \sin B - \frac{E_d}{P} \cos B \right) \text{ (formula for correcting bearings)}$$

The quantities $\frac{E_l}{P}$ and $\frac{E_d}{P}$ having already been worked out, the extra work involved is small. Use natural functions in the last formula.

A modification of the first method of adjusting a traverse which is sometimes used is based on the assumptions that (1) the errors in field work are accidental and (2) that the angular measurements are more accurate than the linear measurements. In this method the proportions are:

$$\frac{\text{Total error in latitude}}{\text{Arithmetical sum of all the latitudes}} = \frac{\text{Error in latitude of one side}}{\text{Latitude of that side}} \quad \text{(III)}$$

$$\frac{\text{Total error in departure}}{\text{Arithmetical sum of all the departures}} = \frac{\text{Error in departure of one side}}{\text{Departure of that side}} \quad \text{(IV)}$$

This rule-of-thumb method is logical for the assumptions made only to the extent that each side is parallel to one or the other of the coordinate axes.

571(*b*). ADJUSTMENT OF A TRAVERSE BETWEEN TWO FIXED POINTS. Assume that an open traverse is run between two fixed points, *A* and *B*, whose adjusted coordinates were determined in a previous survey. The fixed points, for example, may be two monuments which mark two control stations of a state rectangular coordinate system (Chapter 56). Assume that the coordinates are: *A* = northing 51,755.3 ft.; easting 17,822.7 ft. *B* = northing 55,928.9 ft. Easting 15,571.6 ft. From these coordinates the latitude and the departure of the line *AB* are: $L = 4173.6$ ft., $D = 2251.1$ ft.

Assume that the algebraic sum of the latitudes of all lines in the traverse between *A* and *B* is 4174.9 ft. and that the algebraic sum of all departures is 2251.9 ft. The total error in latitude for the traverse is $4174.9 - 4173.6 = 1.3$ ft. The total error in departure is $2251.9 - 2251.1 = 0.8$ ft. These two total errors may be distributed among the lines of the traverse by the first method of balancing a survey. However, in the equations (Art. 571(*a*)), the *total length of the perimeter* is replaced by the *total length of the traverse.*

Note. The open traverse between *A* and *B* may be conceived as forming a closed traverse with the straight line *AB* as one side. Since the coordinates of *A* and *B* are fixed, the latitude and departure of the line *AB* cannot be changed. Hence the total errors of closure in latitudes and departure must be distributed solely among the lines of the open traverse.

If it is desired to compute the northings and eastings of the intermediate stations of the traverse, this may be done by means

of the corrected latitudes and departures of the lines of the traverse. Beginning with the northing and easting of either *A* or *B*, the computations are carried through the traverse from station to station.

When some of the measurements in the traverse survey are considered more accurate than others, the second method of adjusting latitudes and departures may be used instead of the first. (See the next article.)

571(*c*). SECOND METHOD OF BALANCING A SURVEY. This consists in distributing the errors where it is most likely that errors in field work were made. On the whole this method is often preferable to the preceding method. In using it the following points should be kept in mind.

1. Measurements made under the most unfavorable conditions may be given the greatest corrections. Thus, for example, linear measurements on steep slopes, or over other ground unfavorable for chaining, are more likely to be too long than if made under favorable conditions.

2. All linear measurements are likely to be too long provided the length of the tape is not longer than it should be. Hence in making corrections it is better, as a rule, to decrease distances than to increase them, i.e., it is better to subtract from the latitudes and departures in the columns whose sums are the greater than to add to latitudes and departures in columns whose sums are the smaller.

3. A slight change in the bearing of a long line will affect the latitudes and departures more than the same change in a short line.

4. If a line is nearly north and south, a slight change in its bearing will change its departure without greatly affecting its latitude, whereas a slight change in its length will change its latitude without greatly affecting its departure. Likewise, if a line is nearly east and west a change in its bearing affects its latitude; a change in its length affects its departure.

571(*d*). SUGGESTIONS. In an article by Mr. E. Sherman Gould (*Engineering News,* Jan. 4, 1900) some excellent suggestions were given which are as useful today as they were in 1900. These suggestions follow:

The error of closure must be due either to a mistake in calculating the traverse or to an error in field work.

To detect it, the traverse must be first examined. We assume that the courses have been called back so that there is no error in copying, also that the taking out of sines and cosines and multiplications have been checked, so that there is no error there. We also assume that before the columns were footed up care was taken to see that all entries were made in the proper columns. In this connection it may also be stated that it will be found conducive to accuracy to head the columns of Latitudes "N" and "S," and those of Departures "E" and "W," rather than + and −. It is then very easy to

see if any mistakes have been made in putting the latitudes and departures in the wrong column. To see if they have not been interchanged, that is, latitude placed in the departure column, or vice versa, observe if the bearing of each course has more or less than 45° of departure. If it has more, the departure is greater than the latitude; if it has less, the latitude is the greater. Thus, suppose the bearing to be N 36° 24′ E and the distance 463 ft. The sine and cosine of this angle are, respectively, 0.59342 and 0.80489. Multiplying both by the distance gives 274.753 and 372.664. A glance shows that the latter must be the latitude and the former the departure.

If all this is correct, then without a shadow of a doubt the error, if a considerable one, is in the chaining. If numbered stakes have been driven every hundred feet, and leveled over as in the case of a railroad line, there will be no mistake in the full stations, because the leveling party would detect and report it. The chief source of error is in the pluses, and these errors are commonly made, if at all, by reading the chain backward, for instance, taking 64 ft. for 36. The distances used in the traverse must be then carefully examined to see if such a change, or any change in fact, in one or more distances would reduce the error to limits of normal inaccuracy. By diligent scrutiny such an error is sure to be detected by an expert eye, and then a couple of trustworthy chainmen are sent out to verify the suspected measurement, not being informed, of course, of what the error is suspected to be. In almost all cases a resurvey can be avoided by an intelligent guess at the point where the mistake was probably made. Undoubtedly a great deal of unnecessary work is frequently gone through with in such cases merely for the lack of a little reflection.

All actual blunders or "busts" being thus eliminated, there will still remain some discrepancies between the Northings and the Southings and Eastings and Westings, unless by pure chance they should happen to balance. . . . The practice of the old land surveyors, when given a survey to locate from a description, was to first work out a traverse for it, which as a matter of course never closed, and then retain all the courses and distances intact till the last one was reached. A closing course and distance were then calculated and the area of the survey determined accordingly. This, I believe, is the legal way of treating a survey which does not close, unless some metes and bounds are cited which conflict. It is, however, awkward when the last course is a course of an adjoining tract, particularly if of an earlier date.

I believe the best way to balance such a survey as we have been considering, that is, one in which there can be found no absolute mistake, and only a slight discrepancy remains to be eliminated, is to go very carefully over the traverse and see to what changes of distance it will respond most sensitively. For instance, supposing that in a large survey of several hundred acres the North latitudes exceeded the South by a few feet, and the East departures were less than the West by a somewhat smaller discrepancy. In the first place we would look to see if this difference was about in the right proportion; that is to say, if the total latitudes were greater than the total departures. Then the difference between the Northings and Southings should be greater than between Eastings and Westings, and the fact of their being so would give us additional confidence that the errors were simply due to ordinary small inaccuracies, although this evidence would not be conclusive either way. We would then see what courses, if any, ran nearly North and South. It is clear that by lengthening and shortening these by a small amount we would make a great impression on the latitudes. And the same in reference to the departures. On the other hand, the change of a single minute in the bearing of a long course running nearly North and South will make a great change in the departure. In a word, by an intelligent study of the traverse all minor discrepancies can be eliminated with a small number of changes when placed where they do the most good, and this method will prove the most satisfactory in practice.

572. Adjustment of a Traverse Net. In surveying of the highest precision, the discrepancies due to errors in a traverse net are generally adjusted by methods based on the theory of least squares. However, unless the net is quite extensive, an approximate method may be used. The method of successive distributions

devised by Professor George H. Dell * is easily understood, is mathematically correct, involves only simple arithmetical computa tions for its application, and will give results which agree very closely with those obtained by the method of least squares. (See the introductory comments in Art. 171 (*d*).) Two separate, independent adjustments are necessary, one for the errors in latitudes, the other for errors in departures. The procedure in *each* of these independent adjustments is, in general, practically identical with the procedure explained in detail in the adjustment of a level net (Art. 171 (*d*)). Consequently, only an outline of the procedure is given in the illustrative example which follows. (See Art. 171 (*d*) for definitions.)

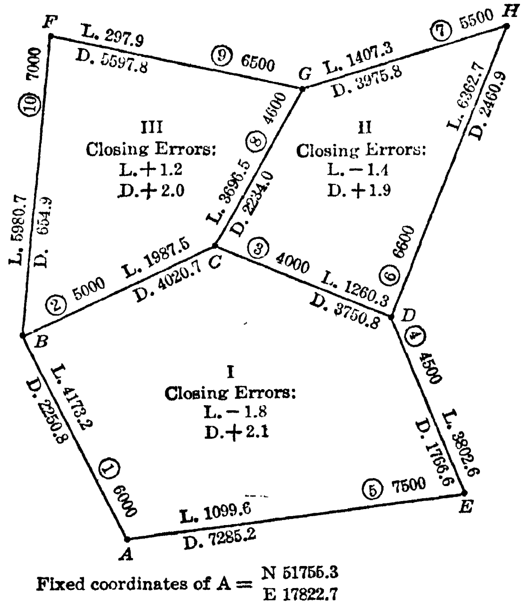


FIG. 572(1).

Illustrative Example. Given the traverse net shown in Fig. 572(1) in which the known rectangular coordinates of a fixed control station *A* are Northing 51,755.3 ft., Easting 17,822.7 ft. Required the adjusted values of the coordinates of each of the other stations. In the figure, the lines between stations are straight. Since none of these lines is less than 4000 ft. in length, the most general case is to assume that no station is visible from any other and that the line actually surveyed between any two stations was not straight but zigzagging from one intermediate station to the next like an open traverse. For example, between stations *A* and *B* the open traverse might be a zigzag line with a number of intermediate stations, depending upon the lengths of sight between successive stations. From the field notes

* "The Adjustment of a Level Net," by George H. Dell, published in the *Transactions of the American Society of Civil Engineers,* Volume 101.

of this traverse the latitude and the departure of the imaginary line AB are calculated. In like manner, the latitude and the departure of each of the other imaginary straight lines are computed. From here on, no attention is paid to the traverses as run in the field, and the net is adjusted precisely as if the straight lines between stations were traverse lines.

The general procedure is to adjust the values of all latitudes and then to adjust the values of all departures, or vice versa, as two independent problems. From the adjusted values thus obtained, the adjusted values of the two rectangular coordinates of each station are calculated by simple arithmetic.

Only the procedure for adjusting the values of latitudes will be explained here.

Procedure for the Adjustment of Latitudes

Step 1. Draw a figure similar to Fig. 572(1). It is helpful to plot the traverse net roughly to scale from the latitudes and departures, starting from station A. For this purpose cross-section paper is useful. Record on the figure the latitude and departure of each line and its *approximate* length. Number the closed traverses or circuits I, II, and III.

Draw a figure similar to Fig. 572(2) and place direction arrows on it. (See Art. 171(d).) Note the auxiliary direction arrow in circuit III. These

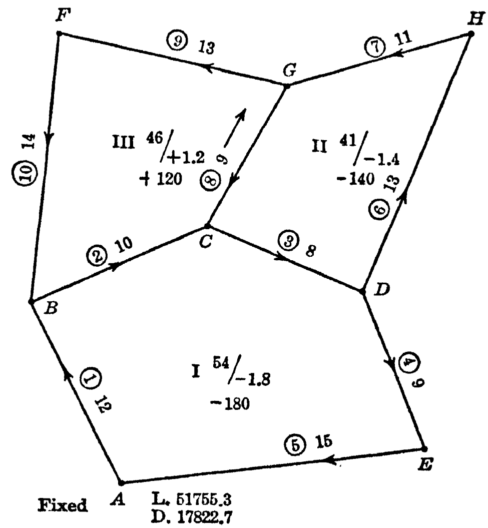


Fig. 572(2).

direction arrows indicate a convenient order of computation. They are not dependent in any way either on the order or on the direction in which traverse lines were run. In this problem each line between two stations is a *section*. Number the sections in the order indicated by the arrows, placing each number within a circle. Likewise number the sections in Fig. 572(1). Number the circuits in Fig. 572(2). The remaining data in Fig. 572(2) will be added later.

Note. A section can be assumed to extend over two or more sides of a circuit, as, for example, from D through E to A. (See also the level net, Fig. 171(d), 3.) Little would be gained, however, by lengthening the sections in the small traverse net of this example.

Step 2. Prepare Table I. In this table the approximate length of each section is divided by an arbitrary number (500 in this case) to obtain the relative length. The relative length is divided by the weight for the section

to obtain the effective length. (See Art. 171(*d*).) In this survey, all sections were surveyed with the same degree of precision and are therefore given the same weights. Consequently, the effective length of each section is the same as its relative length. Had one line been given, say, twice the weight of each of the other lines, its effective length would have been half as great as its relative length. Algebraic signs of the latitudes must be observed throughout the adjustment. The error of closure for latitudes in each of the three circuits is computed as indicated in the table. If suitable methods of eliminating systematic errors were used in the field work, these errors of closure indicate a degree of precision of about 1 to 10,000 or traverses of the second order.

TABLE I

Section	Between Stations	Approximate Lengths	Effective Lengths	Latitudes	Circuit Latitudes		
					Circuit I	Circuit II	Circuit III
1	*A* to *B*	6000	12	N 4173.2	+4173.2		
2	*B* to *C*	5000	10	N 1987.5	+1987.5		+1987.5
3	*C* to *D*	4000	8	S 1260.3	−1260.3	−1260.3	
4	*D* to *E*	4500	9	S 3802.6	−3802.6		
5	*E* to *A*	7500	15	S 1099.6	−1099.6		
6	*D* to *H*	6600	13	N 6362.7		+6362.7	
7	*H* to *G*	5500	11	S 1407.3		−1407.3	
8	*G* to *C*	4600	9	S 3696.5		−3696.5	*C* to *G* +3696.5
9	*G* to *F*	6500	13	N 297.9			+297.9
10	*F* to *B*	7000	14	S 5980.7			−5980.7
				Closing Errors	−1.8	−1.4	+1.2

Step 3. Enter on Fig. 572(2) the following quantities obtained from Table I. The *effective length* of each section placed near the section number, the *effective length* of each circuit placed near the number of the circuit, and, directly beneath that length, the *error of closure* for the latitudes of the circuit. It may be well also to show the error of closure multiplied by 100, which is the quantity used later to avoid decimals in computations.

Step 4. Distribute the errors in latitudes among the sections, as indicated in Table II. Three cycles of distribution, the number in this example, are generally sufficient to reduce the error in each section to a negligible quantity. The general method is identical with that used in the adjustment of a level net (Art. 171(*d*), step 7). In each of the three circuits, the ratio used in distributing the closing error among sections in that circuit is the effective length of a section divided by the effective length of the circuit. For example, $^{12}/_{54}$ is equal to the effective length of section 1 divided by the effective length of circuit I. The lengths may be taken from Fig. 572(2). Note that in carrying a value for section 8 from circuit II to circuit III, or vice versa, the algebraic sign is reversed. This is because the direction of computation for section 8 in circuit II is from *G* to *C* (southerly), whereas in circuit III the direction is from *C* to *G* (northerly), as indicated by the auxiliary arrow in Fig. 572(2). Section 8 is the only one for which there is a reversal of sign in the computations, and such a reversal is indicated by an asterisk wherever it occurs in Table II.

For convenience in computation, each of the closing errors in latitude recorded in Fig. 572(1), i.e., 1.8, 1.4, and 1.2 ft., were multiplied by 100 before entering them in Table II. Consequently all closing errors and all corresponding corrections in that table and in Fig. 572(3) are 100 times too great. The corrections are reduced to their true values in Table III by dividing the sum of the computed corrections for each section (Fig. 572(3)) by 100.

The following suggestions may be helpful in carrying through the computations in Table II. (1) In circuit I and the first cycle, the only closing

error used is the known closing error of the circuit, i.e., −180. (2) In circuit II and the first cycle, look back to see if there is a section correction in circuit I which affects the error of closure for circuit II. If there is, it must be added algebraically to the error of closure for circuit II. Thus the correction +26.7 for section 3 in circuit I is added algebraically to the closing error of −140 in circuit II. (3) In circuit III and the first cycle, look back similarly to both circuits I and II for corrections that affect the closing error of circuit III. Thus the correction +33.3 for section 2 in circuit I and the correction +24.9 for section 8 in circuit II are added algebraically to the closing error of +120 for circuit III. Note the change of sign from +24.9 to −24.9 for reasons already given. (4) In the second cycle, look back for corrections in the two circuits *immediately preceding* which affect the distribution of errors in the second cycle. Thus in circuit I and the second cycle, the correction −27.9 for section 2, is taken from circuit III, first cycle, and the correction +22.1 for section 3 is taken from circuit II, first cycle. In circuit II and the second cycle, the correction +0.8 for section 3 is taken from circuit I, second cycle, and the correction +25.1 for section 8 is taken (with reversed sign) from circuit III, first cycle. Corrections are carried in a similar way from circuit to circuit and cycle to cycle throughout the table. (5) The error to be distributed in any circuit and cycle is the *total* closing error in that circuit and that cycle.

Step 5. Record the partial corrections for sections obtained in Table II on a sketch of the network as shown in Fig. 572(3). For each cycle of

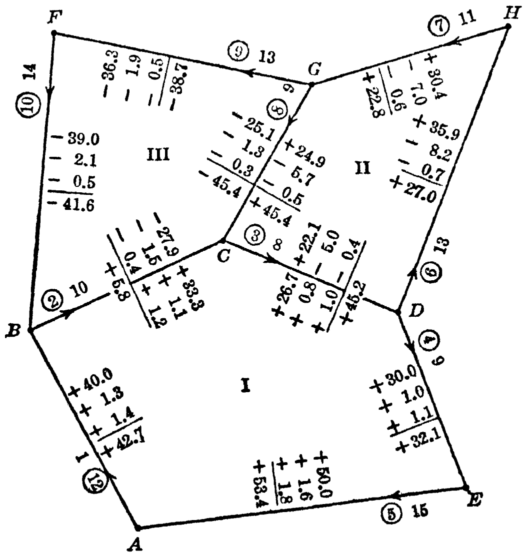


FIG. 572(3).

distribution there is a partial correction for each section in each circuit. There will be twice as many partial corrections for an interior section as for one in the perimeter of the net since each interior section is a portion of each of two circuits. It may be helpful at the end of each cycle of distribution in Table II to record on the sketch the partial corrections obtained in that cycle instead of waiting until all cycles have been completed. The algebraic sum of the partial corrections for each section is equal to the *total correction* for that section. For an interior section, the total correction is the algebraic sum of the *six* partial corrections.

Step 6. Verify the section corrections and circuit closures as shown in Table III. The correction for each section entered in this table is the sum

TABLE II

DISTRIBUTION OF ERRORS IN LATITUDE

Circuit	Section	First Cycle: Closing Errors	First Cycle: Distribution to Sections	Second Cycle: Closing Errors	Second Cycle: Distribution to Sections	Third Cycle: Closing Errors	Third Cycle: Distribution to Sections
I		−180	Corrections		Corrections		Corrections
	1		$\frac{12}{54} \times +180 = +40.0$		$\frac{12}{54} \times +5.8 = +1.3$		$\frac{12}{54} \times +6.5 = +1.4$
	2		$\frac{10}{54} \times +180 = +33.3$	−27.9	$\frac{10}{54} \times +5.8 = +1.1$	−1.5	$\frac{10}{54} \times +6.5 = +1.2$
	3		$\frac{8}{54} \times +180 = +26.7$	+22.1	$\frac{8}{54} \times +5.8 = +0.8$	−5.0	$\frac{8}{54} \times +6.5 = +1.0$
	4		$\frac{9}{54} \times +180 = +30.0$		$\frac{9}{54} \times +5.8 = +1.0$		$\frac{9}{54} \times +6.5 = +1.1$
	5		$\frac{15}{54} \times +180 = +50.0$		$\frac{15}{54} \times +5.8 = +1.6$		$\frac{15}{54} \times +6.5 = +1.8$
Closing Errors		−180	+180.0	−5.8	+5.8	−6.5	+6.5
II		−140	Corrections		Corrections		Corrections
	3	+26.7	$\frac{8}{41} \times +113.3 = +22.1$	+0.8	$\frac{8}{41} \times -25.9 = -5.0$	+1.0	$\frac{8}{41} \times -2.3 = -0.4$
	6		$\frac{13}{41} \times +113.3 = +35.9$		$\frac{13}{41} \times -25.9 = -8.2$		$\frac{13}{41} \times -2.3 = -0.7$
	7		$\frac{11}{41} \times +113.3 = +30.4$		$\frac{11}{41} \times -25.9 = -7.0$		$\frac{11}{41} \times -2.3 = -0.6$
	8		$\frac{9}{41} \times +113.3 = +24.9$ *	+25.1 *	$\frac{9}{41} \times -25.9 = -5.7$ *	+1.3 *	$\frac{9}{41} \times -2.3 = -0.5$ *
Closing Errors		−113.3	+113.3	+25.9	−25.9	+2.3	−2.2
III		+120	Corrections		Corrections		Corrections
	2	+33.3	$\frac{10}{46} \times -128.4 = -27.9$	+1.1	$\frac{10}{46} \times -6.8 = -1.5$	+1.2	$\frac{10}{46} \times -1.7 = -0.4$
	8	−24.9 *	$\frac{9}{46} \times -128.4 = -25.1$ *	+5.7 *	$\frac{9}{46} \times -6.8 = -1.3$ *	+0.5 *	$\frac{9}{49} \times -1.7 = -0.3$
	9		$\frac{13}{46} \times -128.4 = -36.3$		$\frac{13}{46} \times -6.8 = -1.9$		$\frac{13}{46} \times -1.7 = -0.5$
	10		$\frac{14}{46} \times -128.4 = -39.0$		$\frac{14}{46} \times -6.8 = -2.1$		$\frac{14}{46} \times -1.7 = -0.5$
Closing Errors		+128.4	−128.3	+6.8	−6.8	+1.7	−1.7

* Note the reversal of the algebraic sign when a closing error or a correction for section 8 is carried across from circuit II to circuit III or vice versa.

of the corrections for that section indicated in Fig. 572(3). Note, however, that each section correction is divided by 100 since in step 4 errors of closure were multiplied by 100 to avoid decimals. This method of verification is a means of checking the distribution of closing errors as carried out in Table II, but it is not a check on the work done in previous steps in computing the closing errors themselves.

TABLE III

VERIFICATION OF SECTION CORRECTIONS AND CIRCUIT CLOSURES

Section No.	Circuit I		Circuit II		Circuit III	
1	+42.7 ÷ 100 =	+0.4				
2	+05.8	+0.1			+5.8 ÷ 100 =	+0.1
3	+45.2	+0.4	+45.2 ÷ 100 =	+0.4		
4	+32.1	+0.3				
5	+53.4	+0.5				
6			+27.0	+0.3		
7			+22.8	+0.2		
8			+45.4	+0.5	−45.4	−0.5
9					−38.7	−0.4
10					−41.6	−0.4
Summation:	+179.2	+1.7	+140.4	+1.4	−119.9	−1.2
Should be:	+180.0	+1.8	+140.0	+1.4	−120.0	−1.2

Step 7. Apply the section corrections for latitudes to the original computed latitudes, and from the corrected latitudes compute the adjusted northings as indicated in Table IV.

Adjusted Eastings. The corrected departures are computed by a procedure similar to that outlined in the first six steps in the computation of corrected latitudes, and from the corrected departures the adjusted eastings are determined as indicated in Table IV.

Coordinates of Intermediate Stations. The coordinates of each transit station on the zigzag traverse between two control stations of the traverse net may be computed from the northings and eastings of the two control stations, as explained in Art. 571(*b*).

572(*a*). OTHER METHODS OF ADJUSTING TRAVERSES. The adjustment of traverses by the method of least squares is explained in detail in the manual, *Technical Procedure for City Surveys*, American Society of Civil Engineers. Among other publications in which the subject is treated are several manuals by the U.S. Coast and Geodetic Survey which may be obtained from the Government Printing Office, Washington. The use of the method of least squares is justifiable only when procedures in field work are such as to reduce greatly the systematic errors.

Another method of adjusting a traverse net, which gives results identical with those obtained by the method of least squares, is explained in a paper, "An Improved Method for Adjusting Level and Traverse Surveys," by Clarence Norris and Julius L. Speert, *Transactions Number*, 1940, American Society of Civil Engineers; also *Proceedings*, Vol. 65, October 1939.

TABLE IV

COMPUTATION OF COORDINATES

Circuit	Section	Computed Latitudes	Latitude Corrections	Corrected Latitudes	Adjusted Northings	Computed Departures	Departure Corrections	Corrected Departures	Adjusted Eastings
				A (Fixed)	= 51755.3			*A* (Fixed)	= 17822.7
I	1, *AB*	+4173.2	+0.4	+4173.6	*B* = 55928.9	−2250.8	−0.3	−2251.1	*B* = 15571.6
	2, *BC*	+1987.5	+0.1	+1987.6	*C* = 57916.5	+4020.7	−0.7	+4020.0	*C* = 19591.6
	3, *CD*	−1260.3	+0.4	−1259.9	*D* = 56656.6	+3750.8	−0.6	+3750.2	*D* = 23341.8
	4, *DE*	−3802.6	+0.3	−3802.3	*E* = 52854.3	+1766.6	−0.2	+1766.4	*E* = 25108.2
	5, *EA*	−1099.6	+0.5	−1099.1	*A* = 51755.2	−7285.2	−0.3	−7285.5	*A* = 17822.7
					Check −0.1				(Check)
II	6, *DH*	+6362.7	+0.3	+6363.0	*H* = 63019.6	+2460.9	−0.7	+2460.2	*H* = 25802.0
	7, *HG*	−1407.3	+0.2	−1407.1	*G* = 61612.5	−3975.8	−0.6	−3976.4	*G* = 21825.6
	8, *GC*	−3696.5	+0.5	−3696.0	*C* = 57916.5	−2234.0	0.0	−2234.0	*C* = 19591.6
					(Check)				(Check)
III	9, *GF*	+297.9	−0.4	+297.5	*F* = 61910.0	−5597.8	−0.6	−5598.4	*F* = 16227.2
	10, *FB*	−5980.7	−0.4	−5981.1	*B* = 55928.9	−654.9	−0.7	−655.6	*B* = 15571.6
					(Check)				(Check)

CHAPTER 39

CALCULATION OF OMITTED MEASUREMENTS

WHEN all the angles and sides of a polygon are known except two, those two can be found by calculation. In that case, however, it is necessary to assume the measurements actually taken in the field as correct. There can be no check on the field work, nor can the survey be balanced. Hence the importance of taking closing angles and distances notwithstanding the fact that they can be calculated if omitted. There are times, however, when because of some oversight or because of obstacles to field work it becomes necessary to compute omitted measurements. Methods for doing this are outlined in this chapter.

573. Omitted Parts of a Triangle. The most general case is that of the oblique triangle. If any three parts, including at least one side, are known, the remaining three may be found by trigonometric methods, which need not be explained here. The case, however, in which one side and the two adjacent angles are known arises so often in triangulation that suggestions are given below for the order and arrangement of the work.

573(*a*). COMPUTATIONS FOR TRIANGULATION. (1) Find the most probable value of the base line and adjust the angles in each triangle of the triangulation net so that the sum will be 180°. (2) Make out a form similar to one of the two given below, leaving spaces for logarithms and results. (3) When the form is completed, and not before, look up the logarithms. (4) Using the computed length of a side as a base, solve each of the triangles of the triangulation net in a similar manner.

573(*b*). ILLUSTRATION. Given $a = 416.32$ ft., $A = 64°\,18'$, $B = 58°\,40'$, and $C = 57°\,02'$.

To find b	To find c	Check
$b = \dfrac{a \sin B}{\sin A}$	$c = \dfrac{a \sin C}{\sin A}$	$b = \dfrac{c \sin B}{\sin C}$
$\log a =$	$\log a =$	$\log c =$
$\log \sin B =$	$\log \sin C =$	$\log \sin B =$
$\text{colog} \sin A =$	$\text{colog} \sin A =$	$\text{colog} \sin C =$
$\log b =$	$\log c =$	$\log b =$
$b =$	$c =$	

A check which is longer but more sure than that given above is to find b from the formula $\dfrac{b - a}{b + a} = \dfrac{\tan \frac{1}{2}(B - A)}{\tan \frac{1}{2}(B + A)}$. This check is quite certain to reveal mistakes if any have been made.

573(*c*). REMARK. In triangulation, one side of the triangle farthest from the base may be measured, and an agreement between its measured and its calculated length is a check on the calculated sides of all the triangles in between.

573(*d*). COMPACT FORM. The following form is more compact than that just given.

(1)	log sin C =		
(2)	log a =		
(3)	colog sin A =		
(4)	log sin B =		Add the logarithms in the order indicated to get results (5) and (7).
(5)	log c =	(1) + (2) + (3)	
(6)	colog sin C =		
(7)	log b =	(2) + (3) + (4) = (4) + (5) + (6) (check).	

573(*e*). REMARK. Various formulas for the solution of oblique triangles are given in the back of this book. While the use of these formulas usually shortens the work of computation, attention is called to the fact that any oblique triangle can be divided into two right-angled triangles by dropping a perpendicular from one vertex to the opposite side and the required parts found by solving these two right-angled triangles. Whatever solution is used, prepare a form before looking up the logarithms.

574. Trigonometric Relations of Bearing, Length, Latitude, and Departure. The *latitude* and *departure* of any line form two sides of a right-angled triangle in which the *line itself* is the hypotenuse. One angle in this triangle is the bearing of the line, i.e., the angle between the hypotenuse and the latitude. Any two parts of this triangle, including at least one side, being known, the other two can be found by trigonometry.

	Given	*Required*	*Formula*
(*a*)	*Bearing and Length*	*Latitude*	= *Length* × *Cosine Bearing*
(*b*)	*Bearing and Length*	*Departure*	= *Length* × *Sine Bearing*
(*c*)	*Latitude and Departure*	*Tangent Bearing*	= *Departure* ÷ *Latitude*
(*d*)	*Latitude and Bearing*	*Length*	= *Latitude* ÷ *Cosine Bearing*
(*e*)	*Departure and Bearing*	*Length*	= *Departure* ÷ *Sine Bearing*
(*f*)	*Latitude and Length*	*Cosine Bearing*	= *Latitude* ÷ *Length*
(*g*)	*Departure and Length*	*Sine Bearing*	= *Departure* ÷ *Length*
(*h*)	*Latitude and Departure*	*Length*	$= \sqrt{(Latitude)^2 + (Departure)^2}$

575. Omitted Measurements. There are no omitted measurements in any polygon when the bearing and length of each side is known. When the exterior or interior angles are given in place of the bearings, and the bearing of one side is known (or assumed for purposes of calculation), the bearings of all the other sides can easily be found by calculation (Chapter 37). When not more than two bearings, or two lengths, or one length and one bearing are omitted the missing parts may be calculated. This applies of course only to closed polygons. When there are three or more missing parts they cannot be calculated. Missing parts are understood to mean bearings or lengths or both.

There are four general cases:

I. (*a*) When the *bearing* of *one* side is omitted.

(*b*) When the *length* of *one* side is omitted.

(*c*) When the *bearing* and the *length* of *one* side are omitted.

II. When the *bearing* of *one* side and the *length* of *another* side are omitted.

III. When the *lengths* of *two* sides are omitted.

IV. When the *bearings* of *two* sides are omitted.

Remark. When two unknown sides are parallel, case III is indeterminate unless the area is known. Cases II and IV are also indeterminate unless it is known which of two angles that correspond to a given sine or cosine is the right one to use.

Notice that in case I only one side of the polygon is affected. This is the simplest and most common case, and the solution of each of the other three cases involves the general method used in case I. This method consists simply in computing and tabulating the latitudes and departures by the method of the preceding chapter just as if no side were missing. Of course, the survey will not balance, the error of closure being very large, i.e., equal to the required side. Hence the discrepancy between the sums of the north and south latitudes will be the latitude of the missing side, and the discrepancy between the sums of the east and west departures will be its departure. Two sides of a right triangle being known, its hypotenuse (the required side) is easily found.

576. Methods of Calculation for each of the four cases are outlined in succeeding articles.

Caution. Before beginning calculations make sure that the given bearings are either all forward bearings or all back bearings.

576(*a*). Case I: When the bearing of one side or the length of one side is omitted, or when both are omitted

1. Calculate and tabulate the latitudes and departures of the given sides (Chapter 38).

2. Find the difference of the sums of the north and south latitude, and the difference of the sums of the east and west departures. These two differences are the latitude and departure respectively of the omitted side.

3. From the latitude and departure thus found, calculate the omitted bearing or the omitted length, or both.

Remark. The choice of several trigonometric formulas may be had. Thus formulas *c* and *d* or *c* and *e* (Art. 574) may be used first, and then formula *h* furnishes a check.

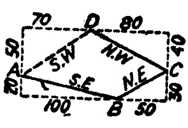


Fig. 576(*b*).

Question. What determines the *quadrant* of the omitted bearing thus found, i.e., NE, SE, SW, or NW?

576(*b*). Illustration. In Fig. 576(*b*) suppose that the bearing and the length of *AD* are omitted in the field notes and that the latitudes and departures of the other three sides have been calculated and tabulated as follows:

Line	*N Latitude*	*S Latitude*	*E Departure*	*W Departure*
AB		20	100	
BC	30		50	
CD	40			80
	—	—	—	—
	70	20	150	80

$$\text{Tan bearing of } AD = \frac{\text{Dep.}}{\text{Lat.}} = \frac{70}{50} = 1.4 = \tan\ 54°\ 30'; \text{ hence } DA = \text{S } 54°\ 30' \text{ W}$$

$$\text{Length of } AD = \frac{\text{Lat.}}{\cos B} = \frac{50}{0.5807} = 86.0 \text{ ft. or } \sqrt{50^2 + 70^2} = 86.0 \text{ ft. (check)}$$

576(*c*). ILLUSTRATION. In Fig. 576(*c*), let the broken lines represent boundary lines, the lengths and bearings of which were not taken in the field. The traverse lines *A–B–C–D* were measured, the interior angles were taken, and each corner of the boundary line was located by the angle and distance. No bearings were recorded. Required to find the lengths and bearings of the boundaries (broken lines). This problem simply involves four applications of the method of case I. Assume a bearing for any line as *AB* and calculate in the usual way the bearings of the eight lines shown by full lines. Then each boundary line is an omitted side of a four-sided polygon, and its length and bearing can be calculated. The bearings of the four boundary lines having been calculated, the angles *EFG, FGH, GHE,* and *HEF* are easily found (Chapter 37). If the lengths of the boundary lines had been measured in the field they would have furnished an important check on the calculation of the bearings of the boundary lines.

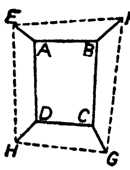


FIG. 576(*c*).

Remark. This problem could be solved by the method explained in Art. 577(*a*), and one method could be used to check the other.

576(*d*). GENERAL METHOD FOR CASES II, III, AND IV

When two sides are affected by omitted measurements. Let the sides *AB* and *BC* be affected by omitted measurements. Leaving these two sides out altogether, the polygon may be completed by a *closing line, AC,* the bearing and length of which can be calculated by the method of case I. This having been done, enough parts of the triangle *ABC* will be known to give the omitted parts by the usual trigonometric methods. Cases II, III, and IV can all be solved in this way. It is immaterial whether the deficient sides adjoin or not. This will become evident in any given case if in omitting the two deficient sides the other sides are shifted, each parallel to itself until they form a connected series of lines.

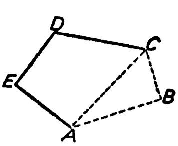


FIG. 576(*d*).

Illustration. Let the sides *C* and *E* in Fig. 576(*e*) be affected by omitted measurements. In Fig. 576(*f*) *D* has been moved parallel to itself to the end of *B*. Omitting *C* and *E* the closing line is *L*. The length and bearings

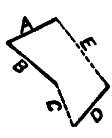


FIG. 576(*e*).

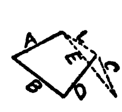


FIG. 576(*f*).

of the three sides *A*, *B*, and *D* remain unchanged, and hence the line *D* must form a triangle with *C* and *E* as indicated by broken lines in Fig. 576(*f*).

576(*e*). Case II: When the bearing of one side and the length of another side are omitted

1. Find the length and bearing of a closing line (case I).

2. Find the angle between the closing line and the side of omitted length but known bearing, using the bearing found in (1) and the known bearing (Art. 556).

3. Given two sides of a triangle and an angle opposite one of them to find a side and either of two angles. (Formula in the back of this book.)

4. From the angles found in (3) calculate the omitted bearing.

Remark. Two answers are possible in this case. (See Remark, Art. 575.) It is usually evident in any given case, however, which result corresponds to the line of the survey.

576(*f*). Another method for case II is to assume for purposes of calculation the side of unknown length to be due north and south, the bearing of the other sides being changed to correspond. This eliminates one unknown quantity, i.e., the departure of the side assumed as north and south becomes zero. The other unknown quantities can then be found by latitudes and departures. It is immaterial whether the deficient sides adjoin or not.

1. Assume the side of unknown length to be a north and south line, and calculate the bearings of the other side.

If the bearings of the sides are already known, the bearing of the side of unknown length can be changed temporarily by an amount (clockwise or counterclockwise) required to make it N and S. The bearings of the other sides should then be changed by the same amount and in the same direction (clockwise or counterclockwise). (See Art. 555(*c*).)

2. Compute and tabulate the latitudes and departures of all sides except the two affected by omitted measurements. Of these two, the side of unknown length has no departure since it was assumed N and S; hence the departure of the other side must be equal to the difference of the sums of the E and W departures of the known sides.

3. From the departure of the unknown side (found in (2)) and the length of that side, calculate its bearing (Art. 574(*g*)) and its latitude (Art. 574(*a*) or Art. 574(*h*)).

4. Insert in the table of (2) the latitude found in (3) and find a *new* difference between the sums of the N and S latitudes. The difference is the latitude of the side assumed as N and S; and,

since its departure is zero, the length and latitude of this side are identical.

576(*g*). Case III: When the lengths of two sides are omitted

1. Find the length and bearing of a closing line (case I).

2. Find the angle between the closing line and each of the deficient sides, using the bearing found in (1) and each of the two known bearings (Art. 556).

3. Given one side and two adjacent angles of a triangle. To find the other two sides and the remaining angle. (Formula in the back of this book.)

Remark. When the two unknown sides are parallel, this case is indeterminate unless the area is known.

576(*h*). Another method for case III is to assume either side of unknown length to be a north-south line and then to proceed as in the second method of case II (Art. 576(*f*)). The only difference is that in (3) the *length* is required and the *bearing* is known instead of vice versa.

576(*i*). Case IV: When the bearings of two sides are unknown

1. Find the length and bearing of a closing line (case I).

2. Given the lengths of three sides of a triangle. To find the three angles. (Formula in the back of this book.)

3. From the angles found in (2) and the bearing of the closing line find the two omitted bearings (Chapter 37).

Remark. As in case II, there will be two answers possible.

577. Simple Trigonometric Methods of computing omitted measurements are frequently used instead of those already explained. An example of this sort is shown in Fig. 577(*a*), and explained in detail in Art. 577(*a*).

577(*a*). Illustration. Given the measurements shown in the sketch. To find the lengths and bearings of *AB*, *BC*, and *CD*.

(1) $Aa \times \cos Aaa' = aa'$ and $Aa \times \sin Aaa' = Aa'$

$28.7 \times \cos 86° 55' = 1.54$ " $28.7 \times \sin 86° 55' = 28.66$

$a'a + ab = a'b$ " $Bb - Aa =$

$1.54 + 382 = 383.54$ $30.9 - 28.66 = 2.24$.

$(Bb - Aa) \div a'b$ = tan of angle which AB makes with $a'b$

$2.24 \div 383.54$ = tan of 0° 20′.

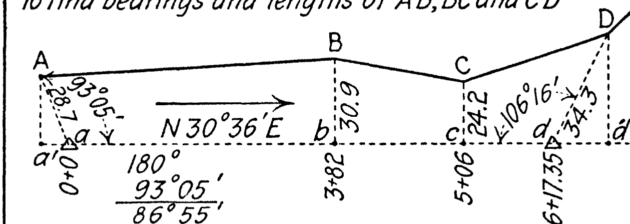
N 30° 36′ E − 0° 20′ = N 30° 16′ E = bearing of *AB*.

Length of *AB* is practically the same as length of *a′b* in this case.

Traverse tables give for *AB*: lat. = 331.26, dep. = 193.30.

The computations for *BC* and *CD* are similar except that the length of each is found by dividing the difference in the offsets at each end by the sine of the angle which the line makes with *a′d*; thus (see p. 665):

To find bearings and lengths of AB, BC and CD



28.7 = 1.457882   1.457882
Cos 86°55' = 8.730688   Sin 9.999371
+1.54 = 0.188570   28.66 = 1.457253
382.00   30.90 ··· Offset at A
AB = 383.54   2.24 = 0.350248
−0°20' ⟵⟶ 383.54 = 2.583810
N 30°36'E   −0°20' = 7.766438 (Tan)
AB = N 30°16'E
(AB = Lat. 331.26 Dep. = 193.30)

6.7 = 0.826075   = 0.826075
124.0 = 2.093422   = 8.732028 (Sin)
+3°05'34" = 8.732653   2.094047
N 30°36'00E (Tan)
N 33°41'34"E = BC = 124.18
(BC = Lat. 103.31 Dep. = 88.88)

34.3 = 1.535294   = 1.535294
Cos 73°44' = 9.447326   Sin = 9.982257
+9.60 = 0.982620   32.93 = 1.517551
111.35   24.20 ··· Offset at D
120.95   8.73
8.73 = 0.941014   = 0.941014
120.95 = 2.082606   = 8.857280 (Sin)
−4°08' = 8.858408   2.083734
N 30°36'E (Tan)
N 26°28'E = CD = 121.27
(CD = Lat. 108.54 Dep. 54.04)

FIG. 577(*a*).

$BC = (Bb - Cc) \div$ sine of angle which BC makes with $a'd'$
$124.18 = 6.7 \div \text{sine } 3° \, 05' \, 34''$

After the computations were completed the latitudes and departures of *Dd*, *da*, *aA* were looked up in a traverse tables as a check.

	N	S	E	W
AB	331.26		193.30	
BC	103.31		68.88	
CD	108.54		54.04	
Dd		25.03	23.45	
da		531.38		314.26
aA	13.26			25.45
	556.37	556.41	339.67	339.71
	+.04		+.04	

It is seen that the polygon closes within a reasonable limit of error.

Question. Assuming that angles are measured to minutes and distances to the nearest tenth, does the use of six-place tables in this example give better results than could be obtained by five-place tables? Can the bearing of *BC* be obtained to the nearest second as implied in the answer N 33° 41′ 34″ E? (See Art. 546(*i*).)

CHAPTER 40

CALCULATION OF AREAS

In this chapter are explained the methods of obtaining areas from field notes and from maps, including the geometric and trigonometric methods, the method of latitudes and departures, the method of coordinates, and the mechanical (planimeter) method.

578. The Area of Land in plane surveying means the area as projected on a *horizontal* plane.

579. Units of Area. Areas in surveying are usually given in square feet or in acres: 1 acre = 43,560 sq. ft. = 10 sq. chains = 10×66^2 sq. ft. (See Art. 8(*g*).)

Square Miles	*Acres*	*Square Rods*	*Square Yards*	*Square Feet*
1	640	102,400	3,097,600	27,878,400
	1	160	4,840	43,560
		1	30.25	272.25
			1	9

1 sq ft.	= 0.0929 sq. meter	1 sq. meter	= 10.7639 sq. ft.
1 sq. yd.	= 0.8361 sq. meter	1 sq meter	= 1.1960 sq. yds.
1 acre	= 0.4047 hectare	1 hectare	= 2.4710 acres
1 sq. mile	= 2.5899 sq. km.	1 sq. km.	= 0.3861 sq. mile

In Spanish measure 1 acre = 5650 sq. varas, more or less. (See Art. 8(*i*).)

580. Geometric and Trigonometric Principles. The following formulas are for areas:

GEOMETRIC

1. *Triangle* = base × half perpendicular height.
2. *Parallelogram* = base × perpendicular height.
3. *Trapezoid* = half the sum of the parallel sides × perpendicular height.
4. *Trapezium*, found by dividing into two triangles.
5. *Regular polygon* = length of perimeter × ½ perpendicular distance from center to sides or $A = \frac{n}{4} L^2 \cot \frac{180°}{n} = nr^2 \tan \frac{180°}{n} = \frac{n}{2} R^2 \sin \frac{360°}{n}$, in which A = area; L = length of one side; n = number of sides; r = radius of inscribed circle; R = radius of circumscribed circle.
6. *Circle* = diameter squared × 0.7854 = diameter squared × $\frac{1}{4}\pi$.
7. *Sector of circle* = $A = \frac{1}{2} LR = \frac{\pi R^2 I}{360°}$, in which L = length of the arc of the sector; R = radius of the curve; I = angle between the two sides or radii.
8. *Segment of circle* = area of sector minus area of triangle, formed by the chord and the two radii. (See also 14.)
9. *Ellipse* = major axis × minor axis × 0.7854.
10. *Parabola* = base × ⅔ height or ⅔ area of circumscribed rectangle.

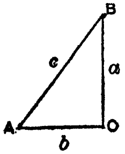


FIG. 580.

TRIGONOMETRIC

11. *Right triangles.*

Given	*Area*
Hypotenuse and acute angle	$(c^2 \sin 2B) \div 4$
Angle and opposite side	$(b^2 \cot B) \div 2$
Angle and adjacent side	$(b^2 \tan A) \div 2$
Hypotenuse and another side	$[a\sqrt{(c+a)(c-a)}] \div 2$
Two sides about right angle	$ab \div 2$

12. *Oblique triangle* (three sides given). Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where a, b, and c are the sides and $s = (a + b + c) \div 2$.

13. *Oblique triangle* (two sides and included angle given). Area $= \frac{1}{2}ab \sin C$.

14. *Area of segment of a circle* $= \frac{R}{2}(L - c \times \cos \frac{1}{2}I)$, where R = radius of curve, c = length of chord, I = intersection angle, i.e., angle between the two radii (see Art. 483 (*c*)), and L = length of arc.

Approximate formulas for area of segments: $A = \frac{2}{3}hc = \frac{c^3}{12R}$, where h = middle ordinate $= \frac{c^2}{8R}$ (approximately).

Geometric method of finding the area of a segment is: Area of the sector minus the area of the triangle formed by the chord and the two radii $A = \frac{\pi R^2 I}{360°} - \frac{1}{2}c(R - h) = \frac{\pi R^2 I}{360°} - R^2 \sin \frac{1}{2}I \cos \frac{1}{2}I$, in which c = length of chord; h = height of segment between the chord and the arc, i.e., middle ordinate; and the other symbols the same as for the sector of a circle.

15. *Area between tangents and curve* $= R\left(T - \frac{L}{2}\right)$, where R = radius; L = length of arc; and T = length of tangents. (See Art. 482.)

580(*d*). To REDUCE SQUARE FEET TO ACRES, divide by 43,560, or multiply by 0.0000229568. Using the abridged method of multiplication it is fully as easy to multiply by the decimal quantity as to divide by 43,560. A table in which each of these two quantities is multiplied by every digit from 2 to 9 is useful. (See Art. 548.)

581. General Methods of Determining Areas.

1. *By computations based directly on field measurements:* (*a*) from triangles into which the area is subdivided; (*b*) from offsets to a base line; (*c*) from latitudes and departures; (*d*) from co-ordinates.

2. *By computations based on measurements scaled from a map.*

3. *By a mechanical method* applied to a map, usually by means of an instrument called a planimeter (Art. 589).

582. Areas Computed by Subdivision into Triangles. An area can often be subdivided into triangles, the area of each triangle computed, and the total area found by adding the computed areas of the triangles.

582(*a*). ILLUSTRATION. In the chain survey shown in Fig. 582(*a*) and described in Art. 94, the lengths of the four fence lines and of the four broken lines were measured. Hence the three sides of each triangle are known, and the area of each can be found from formula 12, Art. 580.

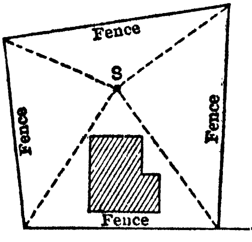


FIG. 582(*a*).

582(*b*). ILLUSTRATION. Suppose that in the above survey a transit was set up at *S*, the angles between the broken lines taken, and the length of each broken line measured. Then in each triangle two sides and the included angle would be known and the area could be found from formula 13, Art. 580.

582(*c*). ILLUSTRATION. Suppose that in the above survey the lengths of the fence lines and the angle at each corner had been measured but not the broken lines, the method of calculation would be: (1) Divide the polygon into *two* triangles by a diagonal. (2) Find the length of this diagonal from the two sides and included angle of one triangle and check it by solving the other triangle. (3) The three sides of each triangle are

now known and the area of each can be found from formula 12, Art. 580.

Check. Divide the polygon into two different triangles by the other diagonal and find the area of each of these triangles as before.

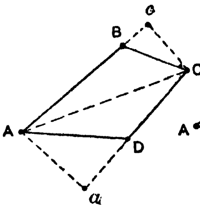


FIG. 582(*d*).

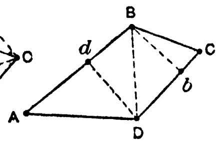


FIG. 582(*e*).

Remark. Usually the traverse lines do not coincide with the fence lines. This case will be discussed in Art. 587.

582(*d*). ANOTHER METHOD. Instead of solving the oblique triangles the polygon may be divided into right triangles. Draw one diagonal as AC, Fig. 582(*d*), and from each end drop a perpendicular to one side. These perpendiculars, Aa and Cc, are easily calculated from the right triangles, and each is the altitude of one of the two triangles into which the quadrilateral is divided. Thus:

$$AD \sin ADa = Aa \quad \text{and} \quad \tfrac{1}{2}Aa \times DC = \text{area of } ADC$$
$$CB \sin CBc = Cc \quad \text{and} \quad \tfrac{1}{2}Cc \times AB = \text{area of } ABC$$

The work may be checked by drawing the other diagonal BD, Fig. 582(*e*), dropping the perpendiculars Bb and Dd, and proceeding as before.

582(*e*). CURVED BOUNDARY. Suppose that in Fig. 582(*a*) the lower left-hand corner were rounded by an arc of a circle and the points of tangency were known. A chord drawn between these points of tangency would form with the other sides an irregular polygon, the area of which could be found as in the preceding illustration. To this area must be added the area of the segment between the chord and the arc. If the length of the chord and the middle ordinate are known, the approximate area is $\tfrac{2}{3}hc$. (See Art. 580(14).) If the intersection angle, etc., is known, use formula 14, Art. 580. If the length of the arc and its radius are known, use the geometric method, Art. 580.

583. Areas from Offsets at Regular Intervals. It is assumed that offsets are measured from a straight base line. There are several formulas designed to shorten the work of calculating each

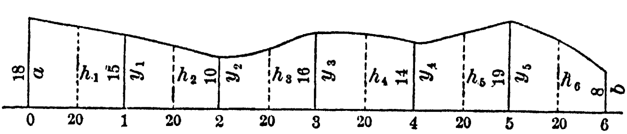


FIG. 583.

trapezoid separately. In Fig. 583 the first offset is numbered 0 and the other offsets 1, 2, 3, etc., . . . 6. Let a and b be the end offsets at 0 and 6 respectively, and $y_1, y_2, y_3, \ldots y_5$ the intermediate offsets, d the common distance between offsets, and S the sum of the intermediate offsets y_1, y_2, y_3, etc., . . . y_5.

$$\textit{Trapezoidal rule: } \text{Area} = \frac{d}{2}(a + 2S + b) \qquad \text{I}$$

$$1740 \text{ sq. ft.} = \frac{20}{2}(18 + 148 + 8)$$

This rule obviously is based on the assumption that the figures are trapezoids. It holds good for any number of offsets, and gives results accurate enough for ordinary purposes.

Another form of the trapezoidal rule is

$$\text{Area} = d\Sigma h \qquad \text{II}$$

in which Σh is the sum of the middle ordinates h_1, h_2, h_3, etc., . . . h_n.

$$1740 \text{ sq. ft.} = 20 \times (16.5 + 12.5 + 13 + 15 + 16.5 + 13.5)$$

Formulas I and II are used not only in computing areas from field measurements, but also in obtaining areas directly from the map, the offsets or the middle ordinates, whichever are used, being scaled at regular intervals.

583(*a*). FORMULAS FOR CURVED BOUNDARIES. When the boundary line between two offsets is curved the assumption that the figure is a trapezoid may give an area too large or too small, according to whether the boundary line curves away from or towards the base line. If the curve departs very much from a straight line more accurate results may be obtained by either of the following formulas than by the trapezoidal rule.

Poncelet's rule. Let h_n = the last middle ordinate.

$$\text{Area} = d\Sigma h + \frac{d}{12}(a - h_1 + b - h_n) \qquad \text{III}$$

$$1733 \text{ sq. ft.} = 20 \times 87 + {}^{20}\!/_{12}(18 - 16.5 + 8 - 13.5)$$

Francke's Rule:

$$\text{Area} = d\Sigma h + \frac{d}{72}(8a + h_2 - 9h_1 + 8b + h_{n-1} - 9h_n) \qquad \text{IV}$$

$$1730.8 \text{ sq. ft.} = 20 \times 87 + {}^{20}\!/_{72}(8 \times 18 + 12.5 - 9 \times 16.5 + 8 \times 8 + 16.5 - 9 \times 13.5)$$

Remark. Formula IV is more accurate than III. Either is more convenient to use and generally gives better results than Simpson's rule, which is sometimes used for the same purpose.

583(*b*). SIMPSON'S ONE-THIRD RULE may be stated as follows: *The area is equal to the sum of the two end ordinates plus four times the sum of the even intermediate ordinates plus twice the sum of the odd intermediate ordinates, the whole multiplied by one-third the common interval between offsets.* The end ordinate is considered to be the first ordinate, the second ordinate is considered as even, the third as odd, the fourth as even, and so on. Using Fig. 583 as an illustration,

$$A = \tfrac{1}{3}d(a + 4\Sigma h + 2\Sigma y + b)$$

The rule is based on the assumption that the boundary line between two successive odd ordinates is a parabolic arc. Consequently, unless the boundary line departs considerably from a straight line, there is little gained by the use of this rule; nothing is gained, for example, in applying it to Fig. 583. The rule is applicable only when the number of offsets is odd.

584. Areas from Offsets at Irregular Intervals. (*a*) FIRST METHOD. The most obvious method is to find the area of each trapezoid separately by multiplying its base by its middle ordinate. Thus in Fig. 584(*a*) the total area is:

$$8530 \text{ sq. ft.} = 50 \times \tfrac{1}{2}(40 + 30) + 40 \times \tfrac{1}{2}(30 + 36)$$
$$+ 60 \times \tfrac{1}{2}(36 + 44) + 30 \times \tfrac{1}{2}(44 + 28) + 60 \times \tfrac{1}{2}(28 + 38)$$

Another form of this equation is:

$$8530 \text{ sq. ft.} = \tfrac{1}{2}[40 \times 50 + 38 \times 60 + 30(50 + 40) + 36(40 + 60) + 44(60 + 30) + 28(30 + 60)] \text{ .}$$

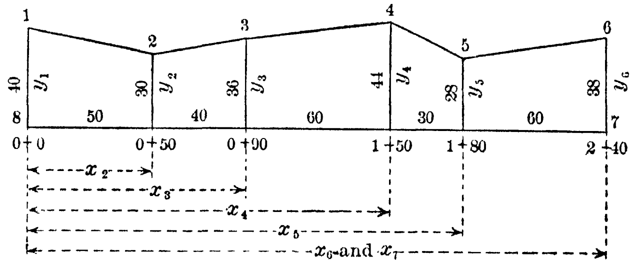


FIG. 584(*a*).

The procedure corresponding to this equation is: Multiply each end offset by the base of the trapezoid of which it is a part, and each intermediate offset by the sum of the bases of two trapezoids, one on either side. Divide the sum of all these products by two.

584(*b*). SECOND METHOD. Divide each trapezoid by a diagonal into two triangles. Consider each end offset as the base of one of these triangles and multiply it by the altitude of the triangle, i.e., the horizontal distance from the base to the nearest offset. Consider each intermediate offset as the common base of *two* triangles, one on either side, and multiply it by the sum of the altitudes of these two triangles, i.e., the sum of the distances from the common base to the two nearest offsets, one on either side. Divide the sum of all these products by two to obtain the required area. Obviously this method is merely a modification of the first method, and the corresponding numerical expression is identical with that in the second equation of the first method. This second method is often used in calculating the areas of cross sections of highways and railways. (See Art. 601(*d*) and (*g*) for an example.)

584(*c*). THIRD METHOD. The method of calculating areas by coordinates, explained in Art. 586, may be used. The procedure for the application of the first formula or rule in that article is as follows: Point 8 in Fig. 584(*a*) is assumed as the origin. Beginning with point 1, the abscissa and the ordinate of each of the points 1, 2, 3, 4, 5, 6, and 7 are recorded as a fraction with the abscissa of the point as the numerator and the ordinate (offset) as the denominator. These fractions are then arranged for convenience in computation as shown on p. 671. Note that x_1, x_8, y_7, and y_8 are all zero.

$$\frac{1}{2}\left(\frac{x_1}{y_1}\times\frac{x_2}{y_2}\times\frac{x_3}{y_3}\times\frac{x_4}{y_4}\times\frac{x_5}{y_5}\times\frac{x_6}{y_6}\times\frac{x_7}{y_7}\times\frac{x_8}{y_8}\times\frac{x_1}{y_1}\right)$$

$$\frac{1}{2}\left(\frac{0}{40}\times\frac{50}{30}\times\frac{90}{36}\times\frac{150}{44}\times\frac{180}{28}\times\frac{240}{38}\times\frac{240}{0}\right)$$

To find the area: (1) *Multiply the numerator of each fraction by the algebraic sum of the denominators of the adjacent fractions, one on either side, regarding the denominator to which a full diagonal line extends as positive and the denominator to which a broken diagonal line extends as negative. The numerator of the first and last fraction is multiplied by the denominator of the only adjacent fraction.* (2) *Divide the algebraic sum of all the products by two.* The rule just given corresponds to rule I, Art. 586. Omitting products that are zero, the area may be indicated as follows:

$$\tfrac{1}{2}[50(40 - 36) + 90(30 - 44) + 150(36 - 28) + 180(44 - 38) + 240 \times 28 + 240 \times 38] = 8530 \text{ sq. ft.}$$

The same result may be obtained when the rule is applied to fractions in which the ordinates are numerators and the abscissas are denominators, as shown below. A different system of indicating positive and negative values is employed merely to illustrate its use. The algebraic sign that indicates in which direction from the origin an abscissa is measured is placed in *front* of the corresponding denominator, and the opposite sign is then placed *after* the denominator. In obtaining the algebraic sum of two denominators, the algebraic sign used for each denominator is the one that is on the side toward the other denominator.

$$\frac{1}{2}\left(\frac{y_1}{+x_1-}\;\frac{y_2}{+x_2-}\;\frac{y_3}{+x_3-}\;\frac{y_4}{+x_4-}\;\frac{y_5}{+x_5-}\;\frac{y_6}{+x_6-}\;\frac{y_7}{+x_7-}\;\frac{y_8}{+x_8-}\;\frac{y_1}{+x_1-}\right)$$

$$\frac{1}{2}\left(\frac{40}{0}\;\frac{30}{+50-}\;\frac{36}{+90-}\;\frac{44}{+150-}\;\frac{28}{+180-}\;\frac{38}{+240-}\;\frac{0}{+240-}\right)$$

Applying the rule, the area is:

$$\tfrac{1}{2}[+40 \times 50 + 30(0 + 90) + 36(-50 + 150) + 44(-90 + 180) + 28(-150 + 240) + 38(-180 + 240)] = 8530 \text{ sq. ft.}$$

If at any point the line located by offsets turns back, as for example, if x_5 were less than x_4, the method still holds good.

A good example of the use of this third method in calculating the area of a cross section of a highway will be found in Art 601(*h*).

584(*d*). ANOTHER CONVENIENT FORM is to place the coordinates of the corners in determinant array. Rule: *Multiply diagonally from left to right, and consider a downward diagonal plus, upward minus, or vice versa. Divide the algebraic sum of all the products by two.*

Illustration. Taking the example in Fig. 485(*j*) for illustration, and assuming *D* as an origin, the coordinates of the corners and the computations are:

Station	*y*	*x*	*Downward* (+)	*Upward* (−)
D	0	0		
E	+10	+14	0 × 14 = 0	10 × 0 = 0
A	−14	+24	10 × 24 = +240	−(−14 × 14) = +196
B	+16	+36	−14 × 36 = −504	−(16 × 24) = −384
C	+22	+16	16 × 16 = +256	−(22 × 36) = −792
D	0	0	22 × 0 = 0	0 × 16 = 0
			−8	−980

[−8 + (−980)] ÷ 2 = −494 sq. ft. *Answer.*

Remark. The minus sign in the above answer signifies that the area was obtained by taking the corners around the field clockwise. If taken counterclockwise the area would be the same, but with a plus sign.

When total latitudes and departures have been found for plotting, this method is very convenient. It is equally applicable, of course, to an example like that of Fig. 584(*a*).

585. Computation of Areas from Latitudes and Double Meridian Distances. This method is the one most often used for computing the area of a closed traverse. When the latitudes and departures of the traverse lines have been computed for purposes of plotting, very little additional labor is required in finding the area. The method is a simple one, though considerable space is required to explain it clearly in detail. Several definitions will be given first.

585(*a*). REFERENCE MERIDIAN. The north-south line through the most westerly station of a survey.

Remark. This is the sense in which the term is usually understood in the calculation of areas. It is not essential, however, to assume the reference meridian through the *most westerly* station. It may pass through any station provided proper account is taken of algebraic signs. It is usually assumed through the most westerly station in order to simplify matters and avoid mistakes.

585(*b*). TO DETERMINE WHICH IS THE MOST WESTERLY STATION. This can usually be done by inspection of the bearings. Start with any line and follow around the survey until the station is reached where the bearing changes from W to E. Thus if *AB* is NE, *BC* is SE, *CD* is SW, *DE* is SW, *EF* is NW, and *FA* is NE, without consulting the figure at all it is seen from the bearings that *F* is the most westerly station, because at *F* the lines stop running west and begin to run east. When, however, there are *two* such points where lines change from west to east, as, for example, if *F* in Fig. 585(*b*) were at *F'*, it is sometimes necessary to add up the departures each way, starting from some easterly station. Thus, if the sum of the departures of *CB* and *BA* is greater than the sum of the departures of *CD* and *DE*, then *A* is the most westerly station; otherwise *E* is farther west than *A*. A still better method, especially when

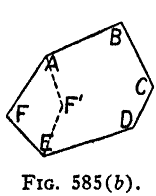


FIG. 585(*b*).

there are a large number of sides to the polygon, is to start with any station as an origin, and find by a preliminary tabulation the *total* departure of each station from the assumed origin. The station having the largest *minus* departure is the most westerly station.

585(*c*). THE FIRST AND LAST COURSES of every polygon are the two courses which meet at the most westerly station. The *first* course is that one for which the given bearing is *easterly*, the last course is that for which the bearing is *westerly*, it being understood that the given bearings *are all forward* bearings. (See Art. 205.)

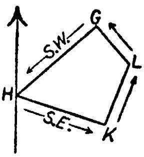


FIG. 585(*c*).

Illustration. In Fig. 585(*c*), let *H* be the most westerly station. Then *HK* is the *first* course and *GH* the *last* course, provided the bearings indicated on the figure were those used in calculating the latitudes and departures. If, however, the *opposite* bearings were used, *KH* = NW and *HG* = NE; then *HG* would be the first course and *KH* the last course. *Note that the two courses can never both be easterly, or both westerly.*

585(*d*). THE MERIDIAN DISTANCE of any point in a traverse is the distance of that point to the reference meridian, measured at right angles to the meridian. The meridian distance of any course or side of the traverse is the meridian distance of the point halfway between the two ends of that course. Meridian distance will be represented by the symbol M.D. Many surveyors use the term *longitude* (represented by *L*) instead of *meridian distance.*

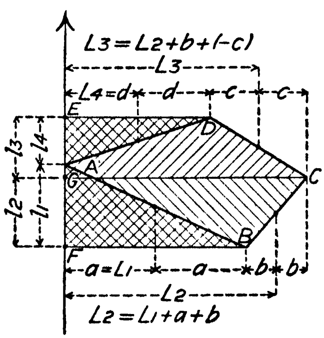


FIG. 585(*e*).

Illustration. In Fig. 585(*e*) L_1 is the meridian distance of *AB*, L_2 of *BC*, L_3 of *CD*, and L_4 of *DA*.

The meridian distance of any course is equal to the meridian distance of the preceding course plus half the departure of the preceding course plus half the departure of the course itself.

For the *first* course there is no preceding course, so that the first two terms are zero and *the meridian distance of the first course is therefore half its departure.* In applying the rule, east departures must be given *plus* signs and west departures *minus* signs.

Illustration. The correctness of the rule just given will become evident from inspection of Fig. 585(*e*).

M.D. of Course	=	*M.D. of Preced.*	+	*½ Dep. of Preced.*	+	*½ Dep. of Course Itself*
L_1 of *AB*	=	0 of 0	+	0 of 0	+	*a* of *AB*
L_2 of *BC*	=	L_1 of *AB*	+	*a* of *AB*	+	*b* of *BC*
L_3 of *CD*	=	L_2 of *BC*	+	*b* of *BC*	+	$-c$ of *CD*
L_4 of *DE*	=	L_3 of *CD*	+	$-c$ of *CD*	+	$-d$ of *DE*

Notice that *c* and *d* both have minus signs.

585(*e*). General method of calculating areas by latitudes and meridian distances. If in any survey an east-west line is drawn from each station to the reference meridian, triangles and trapezoids will be formed. *One side* of each triangle (or trapezoid) will be some *course* of the survey, the *base* of the triangle (or trapezoid) will be the *latitude* of that course, and the mean *length* or *altitude* of the triangle (or trapezoid) will be the *meridian distance* of that line. The *area* of each triangle (or trapezoid), therefore, will be the *latitude of some course multiplied by the meridian distance of the same course.* (Thus in Fig. 585(*e*) the area of $FBCG$ is $l_2 \times L_2$ = Latitude of BC × M.D. of BC.) The areas of all the trapezoids and triangles thus found in any survey are *not* to be added together to find the total area, for each trapezoid, as $FBCG$, includes area which is *not* within the boundaries of the survey, while the area of each triangle may be wholly outside of those boundaries. If, however, all areas found from *north* latitudes (called north products) are added and all areas found from south latitudes (south products) are added, the difference of these two sums will be the required area.

Illustration. The area of $ABCD$ in Fig. 585(*e*) may be found as follows:

North Product	*South Product*
Area = Latitude × M.D.	Area = Latitude × M.D.
$GFBC = l_2$ of $BC \times L_2$ of BC	$FAB = l_1$ of $AB \times L_1$ of AB
$GCDE = l_3$ of $CD \times L_3$ of CD	$EDA = l_4$ of $DA \times L_4$ of DA

It is evident that the area of $ABCD$ = (area of $GFBC$ + area of $GCDE$) − (area of FAB + area of EDA), or equals the difference of the sums of the north and south products.

585(*f*). Double meridian distances. If the meridian distance of any course is found from the meridian distance of the preceding course by the method of Art. 585(*e*), it is necessary to use *half* departures. As *whole* departures and not *half* departures are usually given in the tabulated form for latitudes and departures, it is more convenient to multiply each latitude by twice the meridian distance (double meridian distance) and then to divide the final result by two. The double meridian distance (D.M.D.) of any course is numerically equal to the sum of the meridian distances of the two ends of that course.

When the term *longitude* is used in place of *meridian distance, double longitude* is generally represented by D.L.

To find a double meridian distance of any course C from the double meridian distance of any preceding course P.

Double meridian distance of C = Double meridian distance of P + Departure of P + Departure of C. In the case of the first

course there is no preceding course; the first two terms disappear and the double meridian distance of the *first* course equals its departure. The double meridian distance of the *last* course also equals its departure, but this fact is used simply as a check.

585(*g*). RULES FOR COMPUTING AREAS BY METHOD OF LATITUDES AND DOUBLE MERIDIAN DISTANCES (SUMMARY).

1. Compute and tabulate the latitudes and departures in the usual manner.

2. Add three columns to the table, headed respectively D.M.D., North Products, South Products. (See p. 689.)

3. Ascertain which station is *farthest west,* and which of the two lines from this station is the *first line.* (See Arts. 585 (*b*),(*c*).)

4. Complete the D.M.D. of each course by adding to the D.M.D. of the preceding course the departure of the preceding course plus the departure of the course itself. West departures are *minus.* The D.M.D. of the *first* course equals the departure of the first course.

5. Check the double meridian distances by the fact that the D.M.D. of the *last* course is equal to the departure of the last course.

6. Multiply the latitude of each course by the corresponding D.M.D. If the latitude is *north* put the result in the column headed North Products; but if the latitude is *south* put the result in the column headed South Products.

7. Add up the north products, add up the south products, subtract the smaller sum from the larger, and *divide the result by two.*

If the latitudes and departures are in feet, as is usually the case, the result will be in square feet. To reduce to acres divide by 43,560, or multiply by 0.0000229568. If the latitudes and departures are in chains, the result will be in square chains. To reduce to acres, divide by ten.

8. Having computed an area by the method of latitudes and double meridian distances, always check the result roughly either by calculations based on measurements scaled from the map, or by means of the planimeter, or by some other method. If a more accurate check is desired, repeat the work with the most easterly station as an origin, thus using a new set of coordinates.

Remark. In steps 6 and 7 reject all figures beyond the second doubtful place. (See Art. 546.) In step 6 use abridged multiplication or logarithms, or a calculating machine.

585(*h*). THE METHOD OF DOUBLE PARALLEL DISTANCES and the method of double meridian distances are identical in principle. A parallel distance of any course of a traverse is the perpendicular distance from the middle point of that course to a reference line at right angles to the meridian—positive if north of the reference line, negative if south. It simplifies computations

to assume the reference line through the most southerly point of the traverse. The rules just given for computing areas may then be changed by substituting "D.P.D." (double parallel distance) for "D.M.D." and "latitude" for "departure." The first line will start from the most southerly station. The principal use of the D.P.D. method is for checking areas obtained by the D.M.D. method.

585(*i*). COMMON MISTAKES. Starting with the wrong line for a *first line;* adding departures instead of subtracting when they are *west;* entering north products in the column for south products or vice versa; *forgetting to divide the final product by two.*

585(*j*). ILLUSTRATIVE EXAMPLE. Let it be required to find the area of the polygon *A–B–C–D–E–A* in Fig. 585(*j*). Assume that the latitudes and departures have been calculated and tabulated in the usual manner. The next thing to do is to add a columr of double meridian distances. It is found that *D* is the most westerly station (Art 585(*b*)), and that therefore *DE* is the first line (Art. 585(*c*)).

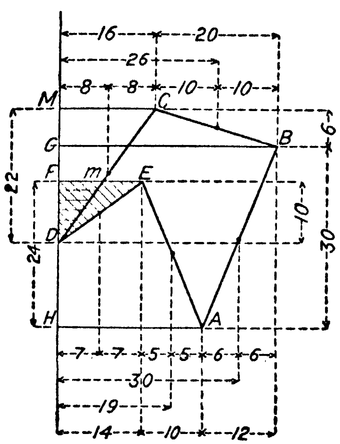


FIG. 585(*j*).

It is not necessary to tabulate the work done in finding double meridian distances It is tabulated below for sake of clearness.

D.M.D.	of	Preceding Course	+	Departure	of	Preceding Course	+	Departure	of	Course Itself	=	D.M.D.	of	Required Course
0	of	0	+	0	of	0	+	14	of	*DE*	=	14	of	*DE*
14	of	*DE*	+	14	of	*DE*	+	10	of	*EA*	=	38	of	*EA*
38	of	*EA*	+	10	of	*EA*	+	12	of	*AB*	=	60	of	*AB*
60	of	*AB*	+	12	of	*AB*	+	−20	of	*BC*	=	52	of	*BC*
52	of	*BC*	+	−20	of	*BC*	+	−16	of	*CD*	=	16	of	*CD*

Notice that the D.M.D. of *CD* comes out equal to its departure—a check on the work. In surveys which do not exactly close, the D.M.D. of the last line will differ from the departure of that line by twice the *total* error in departures (unless errors are adjusted). Inserting the D.M.D. found above and multiplying each latitude by its corresponding D.M.D., the tabulated form becomes that given on p. 677:

Line	*Bearing*	*Length*	*Latitude* N	*Latitude* S	*Departure* E	*Departure* W	*D.M.D.*	*Products* N	*Products* S
AB	NE		30		12		60	1800	
BC	NW		6			20	52	312	
CD	SW			22		16	16		352
DE	NE		10		14		14	140	
EA	SE			24	10		38		912
			46	46	36	36		2252	1264
								1264	
								2)988	

Area = 494 sq. ft.

Question. In the illustration the work practically amounts to subtracting the sum of the areas *MCD* and *FEAH* from the sum of the areas *MCBG*, *GBAH*, and *FED*. Why, from inspection of the figure, should *FED* be included in the latter sum?

Note. For more practical examples in areas, see pp. 689 and 692.

586. To Compute Areas from Coordinates. This is simply a more general case of the method of computing areas from latitudes and double meridian distances. In Fig. 586 let 1–2–3–4 be any irregular polygon, the coordinates of the corners with respect to two coordinate axes being known and represented by x_1, x_2, x_3, x_4, and y_1, y_2, y_3, and y_4.

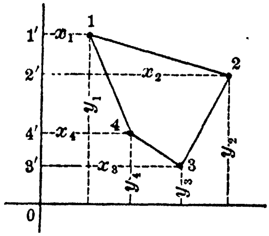


FIG. 586.

The area of the polygon is equal to the sum of the areas of two trapezoids minus the sum of the areas of two others, i.e.,

$$A = (\text{Area } 1' - 1 - 2 - 2') + (\text{Area } 2' - 2 - 3 - 3') - (\text{Area } 3' - 3 - 4 - 4') - (\text{Area } 4' - 4 - 1 - 1')$$

$$A = \tfrac{1}{2}[(y_1 - y_2)(x_1 + x_2) + (y_2 - y_3)(x_2 + x_3) - (y_4 - y_3)(x_4 + x_3) - (y_1 - y_4)(x_1 + x_4)]$$

$$A = \tfrac{1}{2}[y_1(x_2 - x_4) + y_2(x_3 - x_1) + y_3(x_4 - x_2) + y_4(x_1 - x_3)] \qquad \text{(I)}$$

$$A = \tfrac{1}{2}[y_1x_2 + y_2x_3 + y_3x_4 + y_4x_1 - (y_1x_4 + y_2x_1 + y_3x_2 + y_4x_3)] \qquad \text{(II)}$$

These two equations may be expressed as follows:

Rule I. *Multiply the ordinate of each corner by the difference between the abscissas of the two adjacent corners, always subtracting the following abscissa from the preceding abscissa. The area is one-half the sum of the products.*

Rule II. *Multiply the ordinate of each corner by the abscissa of the next corner and sum the products. Multiply the ordinate of each corner by the abscissa of the preceding corner and sum the products. One-half the difference of the two sums is the area.*

Remark. To avoid mistakes in algebraic signs it is better to choose the axes so that the whole polygon lies in the first quadrant. In the above rules it is assumed that the corners are numbered *consecutively* in either direction around the polygon. Of course the rules hold good if the words abscissa and ordinate are interchanged wherever they occur.

Rule I generally involves less work than rule II. When rule II is used it is convenient to arrange the ordinates and abscissas as shown in Art. 584(*d*).

Note. The D.M.D. method of obtaining areas (Art. 585(*f*)) has been very popular, but where a calculating machine is available, the easy cross multiplication of coordinates by machine has made the coordinate method possibly more popular and surely more rapid.

587. Summary of the Different Methods of Computing Irregular Areas. The most general case likely to arise in practice may be stated as follows. Boundary lines are irregular. Traverse lines do not coincide with boundary lines. Boundaries located from the traverse lines by offsets taken at every point of abrupt change and near enough together so that the boundary between two successive offsets may be considered straight.

The lines bounding the area to be computed form a polygon. In the perimeter are included not only any straight boundary lines which may exist, but also the series of straight lines which lie between offsets and which follow approximately irregular boundaries such as winding roads and streams. To distinguish this polygon from that formed by traverse lines, it will be referred to as the boundary polygon.

587(*a*). First method. (1) Compute the length and bearing of every line in the boundary polygon. (See practical illustration, Art. 577(*a*).) (2) Compute and tabulate the latitudes and departures for all the lines in the boundary polygon (Art. 568). If the error of closure is small, the work in (1) is probably correct. (3) Find the area within the boundary polygon by the method of latitudes and double meridian distances.

Remark. The length and bearing of each side in step 1 may be found in two ways: (1) If the boundary lines are irregular the method explained on p. 664 is preferable. (2) If, however, the boundary line is straight and corresponds to the omitted side of a polygon the method of case I, Art. 576, may be used although the first applies equally well.

587(*b*). Second method. (1) Compute the area within the polygon formed by the traverse lines by latitudes and double meridian distances (Art. 585(*g*)). (2) Compute the areas of the triangles, trapezoids, etc., which lie between the traverse lines and the boundary lines. (3) Add to the area found in (1) any area found in (2) which lies within the boundary polygon and subtract any area found in (2) which lies outside.

587(*c*). Remark. In step 2 the area between any traverse line and an *irregular* boundary may often be computed by the trapezoid rule (Art. 583)

or by one of the rules in Art. 583(*a*). In the case, however, of the area between a traverse line and a ***straight*** boundary line the method explained in the following illustration is frequently used.

587(*d*). ILLUSTRATION. Let it be required to compute the area of the piece of land shown in illustration III, p. 365. (1) The area *ABCD* is found by the method of latitudes and double longitudes. (2) The area of the polygon 9–*C*–*D*–11 is equal to the area of the trapezoid 9–*x*–*y*–11 minus the areas of the two shaded triangles. Since the distances *C*–9 and *D*–11 were measured in the field, and angles taken to 9 and 11, the sides 9–*x*, *Cx*, *Dy*, and 11–*y* are easily calculated. (3) The areas of the polygons *D*–*A*–7–11 and *A*–*B*–8–7 are found in a similar manner. (4) To find the area between *CB* and the brook, drop perpendiculars to *CB* at *C* and *B* and find these approximate lengths. (It would have been well to have measured them in the field.) The area between these two perpendiculars from *B* to *C* may be found by the method in Art. 584, and to this area the areas of the small triangle at each end must be added.

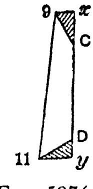


FIG. 587(*d*).

587(*e*). THE TWO METHODS COMPARED. The chief advantage of the first method is the excellent check in step 2. There is no such check in the second method. The first method would seem to involve more labor, but, in most cases, the work in step 1 of calculating lengths and bearings of boundary lines has to be done anyway, so that the extra labor involved is not so much as would at first appear. Whichever method is used, it is well to check the computed area roughly either by calculating the area from map measurements or by determining it mechanically by means of the planimeter as explained in the next two articles.

588. Areas Computed from Map Measurements. (*a*). BY SUBDIVISION INTO GEOMETRIC FIGURES. The area is subdivided into geometric figures, such as triangles, rectangles, squares, trapezoids, and the quantities necessary for computing the areas of these different figures, such as lengths of bases and altitudes, are scaled from the map. Geometric and trigonometric principles used in calculation are given in Art. 580. Areas from offsets scaled on the map may be calculated by the methods explained in Arts. 583, 584.

588(*b*). BY SUBDIVISION INTO SQUARES. Two sets of lines drawn at right angles to each other across the map divide it into squares of any desired size, say 100 ft. squares to the scale of the map. The number of squares which fall completely within the boundaries gives part of the area at once. The fractional areas inside the boundaries, formed by the boundary lines cutting through squares, are usually triangles and trapezoids, the areas of which are found as in Art. 588(*a*) and added to the area found by counting the number of whole squares. (See trapezoidal rule, Art. 583.)

Remark. It is usually more convenient to keep pieces of tracing cloth upon which squares have been drawn to the scales commonly used in plotting, and to lay the cloth over the map instead of drawing parallel lines. Another

method is to plot the boundaries on cross-section paper. A large square or a rectangle may be drawn in place of any number of the small squares, and its area calculated from the scaled lengths of its sides.

Methods 588(*a*) and 588(*b*) are used extensively as rough checks on computed areas.

589. To Determine an Area by Means of a Planimeter. A planimeter, a mechanical integrator, is used by engineers for measuring the area of any figure which has been plotted to scale. There are several types of this instrument, the most common of which is the *polar* planimeter.

When any area has been measured on a map, the corresponding land area can be calculated from the measured map area and the scale of the map. If, for example, the scale of the map is 1 in. = 20 ft., 1 sq. in. of the map represents $20 \times 20 = 400$ sq. ft. of land. In a more general form, if the scale of a map is N feet to the inch, 1 sq. in. on the map represents N^2 feet of land.

The following explanations of the polar planimeter and its use may be modified to apply as well to other forms of planimeter.

589(*a*). PRINCIPAL PARTS OF A POLAR PLANIMETER. A *tracing arm* which moves parallel to the surface of the map and carries near one end a *tracing point*. An *anchor point* near the other end of the tracing arm which remains at a fixed point (*pole*) on the map when the tracing arm and tracing point are moved. A *roller* which rolls on the surface of the map when the tracing arm is moved. When in use, the planimeter touches the surface of the map at only three points, namely, at the anchor point, the tracing point, and the circumference of the roller. The circumference of the drum of the roller is graduated into one hundred parts, and beside the drum is a vernier by means of which revolutions of the roller may be read to thousandths. Connected with the roller by a worm drive is a disk which is graduated by radial lines into ten equal sectors. One revolution of the roller turns this disk through one sector, so that for one complete revolution of the disk there are ten complete revolutions of the roller.

In a common form of small planimeter, the tracing arm is of *fixed* length, and this length is such that if the tracing point is moved completely around the perimeter of any closed figure whose area is 10 sq. in., the roller will make *one* complete revolution. If it is moved around the perimeter of any figure whose area is 5 sq. in., the roller will make one-half a revolution; around an area of 1 sq. in. the roller will revolve through an arc equal to one-tenth the graduated circumference of its drum. These statements are true only when the anchor point remains fixed at a point (any convenient point) *outside* the perimeter of the figure.

In the larger planimeters the tracing arm slides through a sleeve in the frame of the planimeter and, by means of a clamp and a tangent screw, it can be lengthened or shortened. By means of this adjustment the tracing arm may be set so that some simple relation exists between area and the number of revolutions, as, for example, 10 sq. in. to one revolution of the roller. A planimeter with an adjustable arm has two advantages. (1) It may be adjusted to the unit area of the map, such as a square inch or a square centimeter. (2) It may be tested for any unit area and, if found to be incorrect, the error can be eliminated by adjustment.

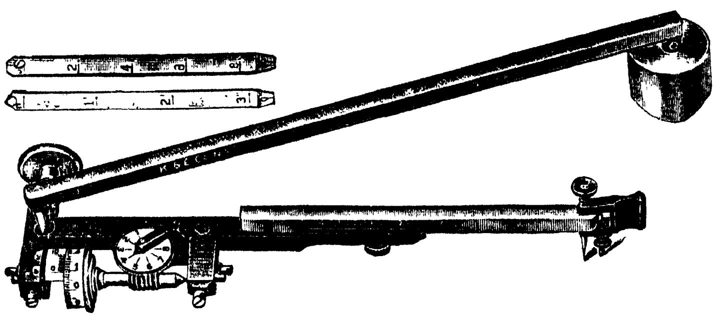


FIG. 589(*a*). Polar Planimeter. (Courtesy of Keuffel and Esser.)

589(*b*). TO TEST A PLANIMETER. Assume that the relation of 10 sq. in. to one revolution is desired. (1) Draw accurately the perimeter of any figure that contains an area of 10 sq. in., as, for example, a rectangle 2 by 5 in. (2) Set the anchor point *outside* the perimeter and in such a position that the tracing point can be placed on the most remote point of the perimeter. (3) Beginning at any point of the perimeter, move the tracing point *clockwise* completely around the perimeter. The roller should make exactly one revolution. Intermediate graduations may be tested by tracing the perimeters of smaller areas. If there has been any appreciable shrinkage or expansion of the paper since the map was plotted, the trial rectangle should be drawn to agree with the slightly changed scale of the map.

An accessory furnished with some planimeters consists of a metal disk, on the surface of which there is a groove that forms the circumference of a circle of known area, say 4 sq. in. The anchor point is set *outside* the disk, and the tracing point is then moved clockwise in the groove around the entire circumference. The roller should make exactly four-tenths of a revolution as read from the graduations on its drum.

589(*c*). USE OF THE PLANIMETER. It is required to determine the area of a plot of land from the corresponding map area by means of a planimeter. It is assumed that the instrument has an adjustable tracing arm.

1. Test the planimeter if there is any uncertainty of its correct adjustment.

2. Stretch the map sheet until it is flat and smooth.

3. Assume a starting point in the perimeter of the area and mark this point clearly and definitely, either by pricking it lightly with the tracing point or by some other means. Draw a small arrow toward the point so that it may be found easily.

4. Set the anchor point *outside* the perimeter and in such a position with reference to the area to be measured that the tracing point can reach the whole boundary without difficulty.

5. Set the tracing point exactly at the starting point and, using the vernier, observe the reading on the graduated drum of the roller. Set down this initial reading.

6. Guide the tracing point carefully around the area in a *clockwise* direction, using one hand to keep the point exactly on the perimeter and the other hand to keep the anchor point from moving. Stop exactly at the starting point.

7. Observe the reading on the graduated drum and set down this final reading over the initial reading.

8. Subtract the initial reading from the final reading and set down the result in square inches of map area.

9. Convert the result found in (8) into square feet of land area by multiplying by N^2, i.e., by the square of the number of feet represented by one inch on the map.

589(*d*). PRACTICAL SUGGESTIONS.

1. For accurate work, large planimeters with adjustable arms should be used.

2. The reason for setting the anchor point outside the perimeter is given in Art. 589(*h*).

3. Some planimeters have a *metal block* resting on the paper, into the top of which the fixed point is set. The block is held by its own weight, and no needle mark is made in the paper. Before starting to measure the area it is well to make a rough preliminary excursion of the tracing point over the area to see that all points of the boundary can be reached.

4. The roller drum may be set to a zero reading when the tracing point is exactly at the starting point in the perimeter so that the final reading alone gives the number of square inches of map area covered; but little if anything is gained, and accuracy in reading may be lost by doing this.

5. It is important to begin and end exactly at the starting point, to keep the tracing point accurately on the perimeter, and to prevent any movement of the anchor point. It is important also to keep the roller on the same smooth surface throughout the circuit. The passing over a joint or an edge of the paper may invalidate the result.

6. If the tracing arm is set so that one revolution of the wheel measures 10 sq. in. and the initial and final readings are both *less* than 10, i.e., the roller has completed less than one revolution, the numerical difference of

the two readings multiplied by 10 is equal to the area in square inches. For example, if the initial and final readings are respectively 0.242 and 0.732, the area is $(0.732 - 0.242) \times 10 = 4.900$ sq. in. When, however, the roller completes more than one revolution, the digit representing the number of revolutions should be placed before the decimal point of the final reading before subtracting the initial reading. For example, if the two readings are, as before, 0.242 and 0.732, but the drum has made four complete revolutions, the area is: $(4.732 - 0.242) \times 10 = 44.90$ sq. in. The graduated disk is for the purpose of keeping count of the number of revolutions. Since an area can generally be estimated by eye closer than 10 sq. in., the number of revolutions should habitually be checked by inspection of the area.

7. When an area is too large to be measured from a single position of the anchor point, it may be subdivided and the area of each portion determined by means of the planimeter in the usual manner. The anchor point should be outside the perimeter of each subdivision when it is being measured. Frequently, however, time may be saved by subdividing an area in such a way that a large portion of it may be calculated from data obtained by scaling the map as explained in Art. 588. For example, assume that one side of a plot of land is bounded by a river and that the remaining sides are straight lines. It may be possible to draw on the map a straight line near the line representing the bank of the river, to determine by means of the planimeter the area between this line and the river, and then to calculate the remaining area from map measurements. Even when an area is bounded by straight lines, it is often advantageous to draw one or possibly two squares or rectangles as large as practicable within the boundary lines, to calculate the corresponding areas, and then to determine the remaining area within the boundary lines by means of the planimeter.

589(e). To DETERMINE THE MULTIPLICATION CONSTANT. The constant by which the revolutions of the roller must be multiplied in order to obtain the area was assumed in the procedure previously outlined to be 10. The multiplication constant for any setting of the tracing arm may be determined as follows. (1) Determine the planimeter reading for a known area, say 10 sq. in. (2) Divide the known area by the planimeter reading. The result is the required constant of multiplication.

Illustration. The planimeter reading for a known area of 10 sq. in. is 0.861. The constant of multiplication is $10 \div 0.861 = 11.61$. For this setting of the tracing arm, every planimeter reading must be multiplied by 11.61 sq. in. in order to obtain the corresponding area.

589(f). ANCHOR POINT INSIDE THE PERIMETER. When the direction of the movement of the tracing point is parallel to the axis of the roller, the roller will slide instead of revolve. Hence, for any setting of the tracing arm, there is a circle whose radius is such that the tracing point can be moved completely around the circumference without causing the roller to revolve. This circle, called the *zero* circle (zero revolutions), differs for different settings of the tracing arm. If the circumference of a circle *larger* than the zero circle is traced clockwise with the anchor point as a center, the area obtained from the planimeter reading will not be the area of this larger circle but merely the area between the circumference traced and the circumference of the zero circle. Hence, in order to obtain the area of the larger circle, the area of the zero circle must be added to the area obtained from the planimeter reading. If the circumference of a circle *smaller* than the zero circle is traced clockwise with the anchor point as a center, the planimeter reading will be *negative*, and the area of the smaller circle will be equal to the algebraic sum of the area of the zero circle and the minus area obtained from the negative reading. For convenience in reading the planimeter, it is well to trace a circle smaller than the zero circle in a counterclockwise direction. The statements just made hold true for obtaining the area of a closed figure of any shape when the anchor point is within the perimeter of the figure.

For any planimeter, the area of the zero circle is constant for any setting of the tracing arm. Constants for various settings, furnished by the instrument maker, are usually given on the tracing arm or inside the planimeter box. When not furnished, any constant may be determined by tracing any closed figure, first with the anchor point outside the perimeter and then with it inside. The difference between the two areas thus obtained should be equal to the area of the zero circle. It is well to repeat the test one or more times on figures of different areas.

589(*g*). ILLUSTRATION. TO FIND THE AREA OF THE ZERO CIRCLE. The area of a 15-in. square, measured with the anchor point of a planimeter set outside, is found to be 224.94. The same area measured with the anchor point set inside is found to be 61.04. The area of the zero circle for this particular setting of the planimeter is, therefore, 224.94 − 61.04 = 163.90 sq. in. From a similar test applied to a 10-in. square, as a check, the area of the zero circle is found to be 164.10 sq. in.

589(*h*). ILLUSTRATIONS. TO MEASURE AREAS WITH THE ANCHOR POINT INSIDE THE PERIMETER.

1. The area of the zero circle of a planimeter is found to be 164.00 sq. in. The anchor point is set *inside* the perimeter of an area *greater* than the area of the zero circle and the perimeter is traced *clockwise*. The numerical difference between the final and initial readings is:

$$6.425 - 0.120 = 6.305$$

The area of the figure is: $6.305 \times 10 + 164.00 = 227.05$ sq. in.

2. Conditions are the same except that the area to be measured is *less* than that of the zero circle, and the perimeter is traced *counterclockwise*. For the sake of comparison, assume that the initial and final readings are the same as before, except that both must be *negative*. The numerical difference is:

$$-6.425 - (-0.120) = -6.305.$$

The area of the figure is $-6.305 \times 10 + 164.00 = 100.95$ sq. in.

Unless there is some good reason for placing the anchor point inside the perimeter, it is better to place it outside and thus avoid the correction for the zero circle.

589(*i*). DEGREE OF PRECISION IN PLANIMETER MEASUREMENTS. For the polar planimeter the absolute error is nearly constant for all areas, but the relative error is less as the area measured is larger. In measurements made with an ordinary polar planimeter, the degree of precision may be expected to vary from about 1/100 for an area of 2 or 3 sq. in. to 1/1200 for an area of 30 or more sq. in. With precision planimeters, such, for example, as a precision rolling planimeter, degrees of precision can be obtained varying from 1/1000 for an area of 2 or 3 sq. in. to 1/10,000 for an area of 40 or more sq. in.

CHAPTER 41

SUBDIVISION OF LAND

THE mathematical problems which arise in the subdivision of land are so numerous and varied that it is impracticable to discuss each one; hence only the more important typical cases are explained in this chapter.

590. Land Subdivision Design. Subdivision of land involves much more than the mere application of mathematical methods such as those given in this chapter. First of all, the subdivision must be planned to satisfy the requirements of some definite purpose. Frequently the purpose is to create building sites and streets. Thus subdivision of land becomes an important part of city and town planning. Even the subdivision of a small area into home sites may be a complex problem. The requirements to be considered in the subdivision of land are too varied and complex to be treated in this book. Land subdivision with particular reference to these requirements was for several years the subject of study by a committee of the American Society of Civil Engineers. The results of that study have been published by the Society in Manual 16, entitled *Land Subdivision.*

591. Simple Problems. Many of the mathematical problems in land subdivision involve only simple applications of geometric or trigonometric principles. An example of such a problem is the following:

Given an area ABCD. Required to subdivide it into two parts by a line FE perpendicular to AD and so located that the polygon ABFE shall contain a specified area (Fig. 591).

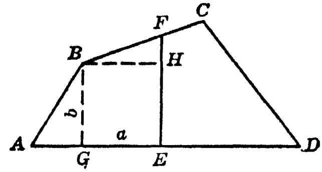


FIG. 591.

Let BG and BH be parallel respectively to FE and AE. To solve the problem, it is necessary merely to find the unknown distance $GE = a$.

$AG = AB \cos BAD$ and $BG = b = AB \sin BAD$

Area of $ABG = \frac{1}{2} AG \times BG$

Area of $BFEG$ = area specified for $ABFE$ minus area of ABG

Area of $BFEG = ba + \frac{1}{2}a \times a \tan FBH$

$$2 \times \text{Area of } BFEG = 2ba + a^2 \tan FBH$$

$$\frac{2 \times \text{Area of } BFEG}{\tan FBH} = \frac{2ba}{\tan FBH} + a^2$$

$$a^2 + \frac{2ab}{\tan FBH} + \left(\frac{b}{\tan FBH}\right)^2 = \frac{2 \times \text{Area } BFEG}{\tan FBH} + \left(\frac{b}{\tan FBH}\right)^2$$

$$a = -\frac{b}{\tan FBH} \pm \sqrt{\frac{2 \times \text{Area } BFEG}{\tan FBH} + \frac{b^2}{\tan^2 FBH}}$$

The problem may also be solved by a simple application of calculus. Assume BG to be the Y axis and GD the X axis. Then the equation of the line BC becomes $y = kx + b$, where the constant k is the tangent of the angle FBH which BC makes with the X axis and b is the intercept BG on the Y axis. Let a represent the distance GE.

$$\text{The specified area for } BFEG = \int_0^a y\,dx = \int_0^a (kx + b)dx = \frac{ka^2}{2} + ba$$

$$2 \times \text{Area } BFEG = a^2 \tan FBH + 2ba$$

From this equation the unknown distance a may be found. (See preceding solution.) Computations of this type can usually be made quite simple by properly choosing the origin and orientation of the axes.

592. General Problem. The following general problem frequently occurs as a step in the solution of a more complex problem in land subdivision.

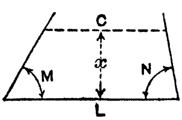


Fig. 592.

Given a line L intersected by two lines making angles with L of M and N degrees respectively. Required the distance x from L of a line C parallel to L, which shall form with L and the two inclined lines a trapezoid of given area A. Also required the length of C and of the non-parallel sides of the trapezoid.

$$C = L - x(\cot M + \cot N)$$

$$A = (L + C)\frac{x}{2} = Lx - \frac{x^2}{2}(\cot M + \cot N)$$

$$-2A = x^2(\cot M + \cot N) - 2Lx$$

$$x^2 - \frac{1}{\cot M + \cot N} \cdot 2Lx = -\frac{1}{\cot M + \cot N} \cdot 2A$$

$$\text{Let } t = \frac{1}{\cot M + \cot N}$$

$$x^2 - 2Ltx = -2tA$$

$$x = Lt \pm \sqrt{L^2t^2 - 2tA} \qquad \text{(I)}$$

From this value of x and the angles M and N, the other unknown distances may be found by trigonometry, for x is the value of the perpendicular side of a right triangle one angle of which is M, and also of another right triangle one angle of which is N. Compute the two bases of these triangles and subtract their sum from L to get C. The lengths of the non-parallel sides of the trapezoid are equal respectively to the hypotenuses of the triangles.

593. To Part Off a Required Area from a Given Irregular Polygon. The two general cases which arise most frequently in practice are: (1) When the *direction* of the dividing line is known, and it is required to find the two points where this line meets the perimeter of the polygon. (2) When a *starting point* on the perimeter is known, and it is required to find the direction of the dividing line. The different steps in the calculation will be outlined and then illustrated by a practical example for each case.

593(*a*). CASE I. *To divide an irregular polygon into two parts, one of which shall have a specified area, by a line having a given direction.*

If no bearings are given, the dividing line may be assumed as north and south, and the corresponding bearings of the sides of the polygon calculated by the methods of Chapter 37, beginning with the side which makes a known angle with the dividing line. If the bearings are given, it is better to use them as they stand.

1. Compute the latitude and departure of each side of the given polygon to see if it is a closed figure. (If it does not close within a reasonable limit of error, nothing further can be done with the problem.) If it does close, compute its area by the method of double meridian distances (Art. 585).

2. Draw the polygon accurately to as large a scale as can be conveniently used, and check the area obtained in (1) by some approximate method. (See Art. 581.)

3. Find by trial the approximate position of the dividing line. Usually this may be done very roughly by a method similar to that used in (2), since the object is simply to find which vertex of the polygon is nearest to the dividing line.

4. Through the nearest vertex found in (3), assume an auxiliary line having the same direction as the dividing line. A trapezoid

is thus formed having for its two parallel bases this auxiliary line and the dividing line (Fig. 593(*b*)). If the altitude of this trapezoid can be found, the position of the dividing line is determined.

5. Compute the area on one side of the auxiliary line by the method of latitudes and double meridian distances. To do this it will be necessary first to calculate the missing parts of the polygon formed by the assumed line and that part of the original perimeter bounding the area to be computed; one of these missing parts is the length of the auxiliary line (Chapter 39).

6. Find the difference between the area found in (5) and the area to be parted off on the corresponding side of the dividing line. This difference is the area of the trapezoid referred to in (4).

7. Substitute this difference for A in formula I, Art. 592, and solve for x, the altitude of the trapezoid. Find also by trigonometry the length of the non-parallel sides of the trapezoid, thus giving the two points where the dividing line meets the perimeter of the polygon. Since the dividing line is a base of the trapezoid, its length may easily be found.

8. As a final check, compute the area on each side of the dividing line by latitudes and double meridian distances, to see if the area which was to be parted off has been obtained, and also if the sum of the two areas is equal to the total area of the original polygon as found in (1). (This is a double check, but see Caution, Art. 593(*d*).) It is well to check the area of the trapezoid from the computed lengths of the two parallel bases and its altitude x, before applying the double check just given.

593(*b*). ILLUSTRATION. Let it be required to subdivide the tract of land shown in Fig. 593(*b*) by a line *de* whose direction shall be N 55° 10′ E, in such a way that the area *dEABcd* shall be 8 acres. The solution of this problem is shown on p. 689. The different steps were as follows:

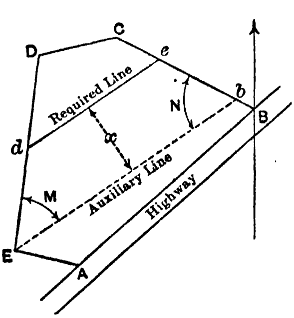


FIG. 593(*b*).

1. Calculated the latitudes and departures and found that the figure closed. Found the area by method of double meridian distances (Art. 585) to be 478,810 sq. ft. or 10.992 acres.

2. Drew the polygon to scale, and using one of the approximate methods in Art. 581 checked the area found in (1).

3. Found approximately where *de* would have to be in order to make the area between it and the highway equal to 8 acres. This was done *very roughly* by the approximate method used in (2), the sole object being to find which vertex in the perimeter is nearest *de*. This vertex is evidently *E*.

4. Assumed a line through *E* parallel to *de* (N 55° 10′ E) intersecting the opposite side of the polygon at *b*. In the polygon *ABbEA* there are two missing parts, the lengths *bE* and *Bb*. By the method given in Art. 576(*g*)

	Line	Bearing	Length	N	S	E	W	D.M.D.	N.Prod.	S.Prod.
Area of ABCDEA	AB	N.47-48 E.	1000.6	672.2		741.2		1072.2	720 730	
	BC	N.62-38 W.	573.3	263.5			509.1	1304.3	343 680	
	CD	S. 76-40 W.	330.3		76.2		321.4	473.8		36 100
	DE	S. 5-26 W.	805.0		801.4		76.2	76.2		61 070
	EA	S. 70-39 E.	175.4		58.1	165.5		165.5		9 620
				935.7	935.7	906.7	906.7		1064410	106 790
	Total Area of ABCDEA = (1064410 − 106 790) ÷ 2 = 478 810 Sq.ft. = 10.992 Acres.									
Area of ABbEA	AB			672.2		741.2		1072.2	720 730	
	Bb	N.62-38 W.	(15.6)	(7.2)			(13.9)	1799.5	12 960	
	bE	S.55-10 W.	(1087.7)		(621.3)		(892.8)	892.8		554 700
	EA				58.1	165.5		165.5		9 610
	Missing Parts in parenthesis			679.4	679.4	906.7	906.7		733690	564 310
	Area of ABbE = (733690 − 564310) ÷ 2 = 84690 Sq.ft. = 1.944 Acres									
To find x, eb, etc.	8 Acres = 348 480	M = 49°44′ cot = 0.84706; t = 0.7277						$x = Lt \pm \sqrt{L^2t^2 - 2tA}$		
	Area of ABbE = 84 690	N = 62°12′ cot = 0.52724; Lt = 791.5						$x = 791.5 - \sqrt{626472 - 383920}$		
	" " dEbe = 263 790	cot M + cot N = 1.37430; $L^2t^2 = 626472$						x = 299.0		
	ANSWERS: x = 299.0 eb = 338.0 eB = 338.0 + 15.6 = 353.6 Ed = 391.8 de = 1087.7 − 410.9 = 676.8									
Check Areas	Area dEbe = ½(de+Eb)x = ½(676.8+1087.7)·299 = 263 790 = (check)									
	dE	S. 5-26 W	391.8		390.0		37.1	37.1		14470
	EA	S.70-39 E	175.4		58.1	165.5		165.5		9620
	AB	N.47-48 E	1000.6	672.2		741.2		1072.2	720 730	
	Be	N.62-38 W	353.6	162.5			314.1	1499.3	243 640	
	ed	S.55-10 W	676.8		386.6		555.5	629.7		243440
				834.7	834.7	906.7	906.7		964 370	267530
	Area of dEABed = (964370 − 267530) ÷ 2 = 348420 Sq.ft. = (Check within 60 Sq.ft.)									

these lengths were found to be $bE = 1087.7$ ft. and $Bb = 15.6$ ft. (The computations made in getting these results are not shown in the work on p. 689.)

5. Found the area of *ABbEA* by latitudes and double meridian distances to be 84,690 sq. ft.

6. Found the area of the trapezoid *dEbe* by subtracting 84,690 from 348,480 sq. ft. (or 8 acres) to be 263,790 sq. ft. Substituting in formula I, Art. 592, found $x = 299$ ft. Found by trigonometry $eb = 338.0$ ft. and $Ed = 391.8$ (work not shown).

7. Checked the area of the trapezoid as indicated. Also checked the area between *de* and the highway by double meridian distances. Found it to be correct within 60 sq. ft. If worth while, this error could be corrected by moving the line *de*, which is 676.8 ft. long, an amount equal to $60 \div 676.8$ As this is less than 0.1 ft. it was allowed to stand in this case. As a final check the area *DdeCD* was calculated by double meridian distances (work not shown), and found equal to 2.993 acres. Adding the areas on either side of the dividing line *de*, $7.999 + 2.993 = 10.992 =$ the area of the entire tract (check).

593(*c*). Suggestions. (1) It is well to keep all areas in square feet until the problem is finished. (2) This method can be used for subdividing a polygon into any number of parallel strips, each of any required area. (3) In the case of very irregular boundary lines it may be impossible to find a vertex from which an auxiliary line can be drawn and a trapezoid formed. In that case the auxiliary line should be assumed as near to the position of the required line as can be estimated and the area computed as in (5). This area will probably be incorrect, but it will indicate about how much to move the auxiliary line for another trial. Two or more trials may be necessary. This trial method makes no use of formula I, and is frequently used in place of the trapezoid method. (4) It is not essential even in the trapezoid method to have the auxiliary line pass through a vertex of the polygon. It may start at any point on the perimeter, but it usually saves work to have it start from a vertex.

593(*d*). Caution. It often happens in subdivisions of areas that several of the same double meridian distances and N or S latitudes are used in two or more area computations (as in the total area and in one or more of the component areas). In such cases an error in the N or S products, if they are not multiplied independently each time, might extend through the whole computation, yet would not be revealed by the checks in step (8). (See preceding illustration.) In such cases it is often best to use the most easterly meridian as a starting point in computing the last set of double meridian distances.

593(*e*). Case II. *To divide an irregular polygon into two parts, one of which shall have a specified area, by a line starting from a given point on the perimeter.*

If no bearings are given, assume some line of the polygon as north and south, and calculate the corresponding bearings of the remaining sides; otherwise use the bearings given. Steps 1, 2, 3, 5, and 8 are practically identical with the corresponding steps of the preceding case.

1 and 2. Compute the latitudes and departures, find the area and draw the polygon to scale, as directed in steps 1 and 2 of the preceding case.

3. Find by trial the approximate position of the dividing line, in order to ascertain which vertex of the polygon is nearest to the *unknown* end of the dividing line. (See step 3 of the preceding case.)

4. From the vertex found in (3) draw an auxiliary line to the *known* end of the dividing line, and find its length and bearing (Art. 576(*a*)).

5. This auxiliary line is the base of a triangle having two unknown sides, one the required dividing line, the other a portion of one side of the polygon. (See *dBe*, Fig. 593(*f*).) The solution of the problem depends upon finding the area of this triangle.

6. Calculate the area on one side of the auxiliary line by the method of latitudes and double meridian distance.

7. Find the difference between the area found in (6) and the area to be parted off on the corresponding side of the dividing line. This difference is the area of the triangle referred to in (5).

8. The base of the triangle being known from (4) and its area from (7), its altitude follows at once. Then the lengths of the two unknown sides may be computed by trigonometry.

9. As a final check, calculate the area on each side of the dividing line as explained in step 8 of the preceding case.

593(*f*). ILLUSTRATION. Required to subdivide the tract of land shown in Fig. 593(*f*) by a line of unknown direction, starting from a given point *d* (391.8 ft. from *E*) and parting off an area *deBAEd* = 7.999 acres. The solution of this problem is shown on p. 692.

1 and 2. Found the area of the given polygon to be 478,810 sq. ft. (See Art. 593(*b*).) Checked it roughly by drawing the polygon to scale.

3. Found approximately the direction of *de* such as to make the area between it and the highway equal to 7.999 acres. Found *B* to be the vertex nearest the unknown point *e*.

4. Drew the auxiliary line *Bd* and found its bearing and length by the method for missing measurements to be, respectively, S 75° 33′ W and 898.0 ft. (See p. 692.)

5. Found the area of *dEABd* to be 242,565 sq. ft.

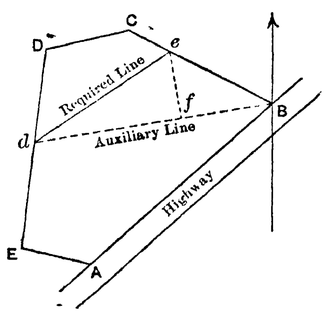


FIG. 593(*f*).

6. Found the area of the triangle *deB* by subtracting 242,565 from 348,436 sq. ft. (or 7.999 acres) to be 105,871 sq. ft.

7. Found the altitude *fe* of the triangle to be 235.8 ft. The two unknown sides by trigonometry are: *Be* = 353.6 ft. and *ed* = 676.8. The angle *edB* = 20° 23′, from which the bearing of *ed* = S 55° 10′ W.

8. Checked the position of *ed* by calculating the area on each side as in step (7) of the preceding illustration. (Work for area *dEABed* shown on p. 689.) (See Caution, Art. 593(*d*).)

Suggestions. See suggestions at the end of the preceding case.

593A. Subdivisions Involving Curved Boundary Lines.* The surveyor in developed areas is frequently confronted with the problem of computing lengths of lines which intersect curves. This is particularly true where subdivisions are being planned with

* This article was prepared for this book by Professor Walter L. Willig

Area of dEABd

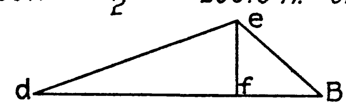
Line	Bearing	Length	N	S	E	W	D.M.D.	N.Prod.	S.Prod.
dE	S. 5-26 W	391.8		390.0		37.1	37.1		14 470
EA	S.70-39 E	175.4		58.1	165.5		165.5		9 620
AB	N 47-48 E	1000.6	672.2		741.2		1072.2	720 730	
(Bd)	(S 75-33 W)	(898.0)		(224.1)		(869.6)	943.8		211 510
			672.2	672.2	906.7	906.7		720 730	235 600

Area of dEABd in sq. ft. = 720 730 − 235 600 = 485 130 ÷ 2 = 242 565

Length and Bearing of Bd. Altitude ef

log. 869.6 = 2.93932
" 224.1 = 2.35044
" tan. Bear. = 0.58888
Bearing Bd = S. 75° 33′ W
log 869.6 = 2.93932
" sin 75-33 = 9.98604
" Bd = 2.95328
Length Bd = 898.0

7.999 Acres = 348436 sq. ft.
dEABd = 242565 " "
Area of Δ deB = 105871 " "
Base of Δ = Bd = 898.0 ft.
Altitude of Δ deB = 105871 ÷ $\frac{898.0}{2}$ = 235.8 ft. = ef



Length and Bearing of de (dividing line)

Bearing Bd = S 75-33 W = N 104-27 −
" Be = Bearing BC = N 62 38 −
" Angle dBe = 41° 49′

log 235.8 = 2.37254 — 2.37254
" sin 41-49 = 9.82396 — tan 9.95164
" Be = 2.54858 — fB 2.42090
Be = 353.6 — fB = 263.6
Bd = 898.0
fB = 263.6
df = 634.4

2.37254
log. 634.4 = 2.80236
" tan. edB = 9.57018
edB = 20° 23′
Bearing Bd = S 75-33 W
edB = 20-23
Bearing ed = S 55-10 W

2.37254
Sin. = 9.54208
log. ed = 283046
ed = 676.8

ANSWERS

curved streets, or where surveys are being made in such existing subdivisions. A few typical examples may be given of such calculations.

Most of the work consists of the calculation of lengths of lot lines and cross street lines which intersect the curved street lines, rather than of progressive calculations along the route of the curve. Therefore, stationing of the curve is not necessary and it is customary to use the radius of curve instead of the value of the degree of curve. Because the desired lengths are boundary dimensions and will be used in deed descriptions, the calculations are usually precise and call for arc distances along the curves. Of course, for field work the corresponding chord distances would be obtained when and where needed.

To present the problem clearly, a brief outline will be given of the procedure by which curved subdivisions may be established. The road system is first sketched upon a contour map, in order to establish road grades that are practicable and block dimensions which in turn will permit of economical lot layouts. Next, curve radii and centers are selected arbitrarily to fit the sketch. By means of a coordinate grid superimposed upon the topographical map, and tied into the ground survey, coordinates of curve centers are scaled and noted. Block dimensions are then scaled or calculated from previously fixed data, consistent with the geometrical requirements in each case.

Following the complete fixing of the block perimeter, a straight "backbone" for the block is usually selected on the layout in order to divide the block longitudinally. Its end coordinates are scaled and marked down. The backbone may consist of a series of straight lines, but would seldom be curved, since there would be many disadvantages without compensating advantages in such a case. The foregoing outline of the paper work of establishing a curved layout indicates that the computer from this point forward is supplied normally with the complete block dimensions, which would include curve radii and the coordinates of such control points as P.C., P.T., P.C.C., and the curve centers.

Geometrical Considerations. It must be borne in mind by the computer that the scaling of coordinates of two points on a straight line fully fixes the line. From these end coordinates the length and bearing of the line may be calculated. We shall refer to this calculation of the length and bearing of a line from its end coordinates as the ***inverse computation***. Once the line is fixed, no more points along that line may be selected by scaling their coordinates. Such points may be selected, however, by scaling dis-

tances from one end of the line. These distances, together with the already determined bearing, suffice to determine the coordinates of the new points. Similarly on a curve, the scaling of coordinates of any three points fully fixes the curve. More often the coordinates of the two end points on the curve and the center are scaled; or the center coordinates, the radius, and the directions of the two radii to the P.C. and P.T. may be arbitrarily fixed. After once fixing a curve, no more points along it may be selected arbitrarily by their coordinates. However, an arc or chord distance along the curve from a fixed point may be used to fix another point; or a direction of radius may be selected to the point. Also, when one side of a street is fully fixed, selection of the width of the street fully determines the other side, if the two sides are to remain parallel, as is normally true.

General Solution. In general, the length and/or bearing of any straight line intersecting a curve may be calculated by solving a triangle such as AOE (Fig. 593*A*, 1). One side of the triangle, *AE,* is the desired line; the second side, *EO,* is the curve radius going to one end of the line; and the third side, *AO,* connects the other end of the desired line with the curve center. The coordinates of *A* (which may be any point on the desired line) must be known or determinable. As has been indicated, the radius and the coordinates of the curve center are normally available. The coordinates of *A* may be scaled directly, or may be obtained by calculating along a known line, such as *GA* in the figure, a scaled or a fixed distance. The length and bearing of *AO* may be calculated inversely from the coordinates of its ends. The variations of the problem may be classified into three specific cases which will be treated separately. A fourth case will consider the problem of the intersection of two curves without a common tangent.

Case I. Direction of the desired line fixed arbitrarily (Fig. 593*A*, 1). Where the bearing of the line *AE* is fixed arbitrarily as desired (*E* being its intersection with the curve), we have in the triangle *AEO* two sides (*OE*, the radius, and *AO,* found inversely) and an angle *EAO* (from the directions of *AE* and *AO*). Two applications of the law of sines will provide first the other angles of the triangle and then the desired distance *AE.*

Any arc, such as from *E* to P.C., is usually found by the formula $L = R\beta$, where β is the angle, expressed in radians, included between radii to the two ends of the arc. β is readily converted from degrees, minutes, and seconds by the use of tables available in most books of logarithms or natural trigonometric functions. Such

tables are variously titled *Lengths of Circular Arcs for Radius 1.00* or *Conversion of Degrees, Minutes, and Seconds to Radians.*

Case II. Curved lengths (arcs) fixed arbitrarily (Fig. 593*A*, 1). A new line such as *BH* is to be fixed by arbitrary selection of the arc length *FH* from a previously determined point, *F*, and the selection of the rear distance *AB* along the known line *AG*. Coordinates of *B* may be calculated from the selected distance *AB*

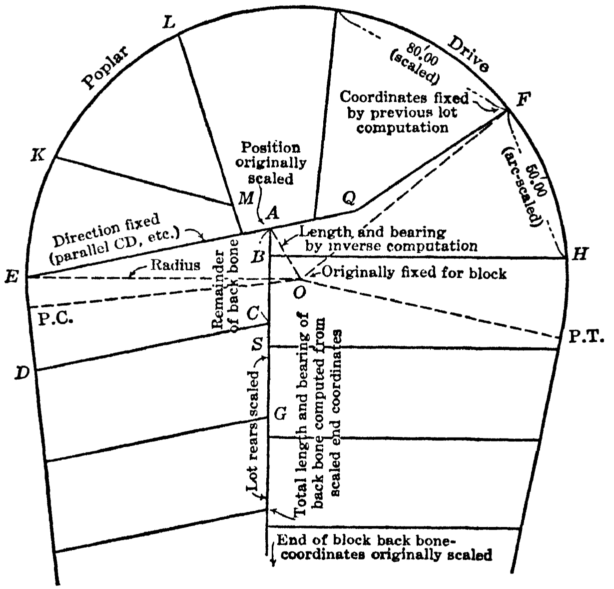


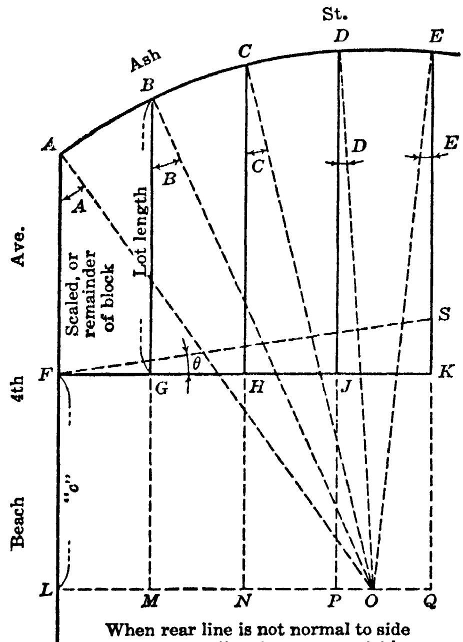
FIG. 593*A*,1.

and the known bearing of *AG*. By means of the formula $L = R\beta$, in which *L* is the newly selected arc and *R* is the radius, the central angle, β, between radial lines to *F* and *H* is determined by reversing the process outlined at the end of case I. Angle β, applied to the bearing of the known line *OF*, provides the bearing of the new line *OH*, whose length is *R*. Now the coordinates of *H* may be calculated from those of the curve center, and the length and bearing of the desired line, *BH*, may be obtained by the inverse calculation from the coordinates of *B* and *H*.

Case III. A series of parallel lines intersecting a curve (Fig. 593*A*, 2). In this very common situation we have a series of parallel lot lines, *AF*, *BG*, *CH*, etc., which intersect street curve, *ACE*, in the front, and are cut off by a common rear lot line which

is normal to the side lines. The problem may be reduced to the computation of several right triangles.

An auxiliary base, *LQ*, for all the triangles, is constructed through the center of the curve perpendicular to the side lot lines, and the parallel side lines are then extended to *M, N, P,* etc., where they intersect the base. Radii, *AO, BO, CO,* etc., are drawn



When rear line is not normal to side lines as *FS*, "c" becomes variable, changing by *FG* tan *θ*, etc.

FIG. 593*A*,2.

to the intersections of the several parallel lot lines with the curve. We now have a series of overlapping right triangles, *ALO, BMO,* etc. The hypotenuse of each is the radius of the curve. All the bases, *LO, MO, NO,* etc., consist of parts of the auxiliary base, *LQ*, and differ only by the lot widths, *FG, GH, HJ,* etc. The other legs of the triangles, *AL, BM,* etc., are the respective side lines of the lots, *AF, BG,* etc., plus or minus a common correction, C, which is the perpendicular distance between the auxiliary base and the common rear line of the lots.

Assuming as before that the block data and curve centers are known, bearing *AL* and length *AO* are known. If not already available, bearing *AO* may be found inversely from the coordinates

of *A* and *O*. Right triangle *ALO* may now be solved from side $AO = R$ and angle *A*, giving us *AL* and *LO*. From *AL* and the selected lot dimension *AF*, the correction *C* is determined. From *LO*, by subtraction of lot widths, the bases of the remaining triangles are obtained. In the second triangle, angle *B* is found from $\sin B = \frac{MO}{R}$ and then $BM = R \cos B$. Applying correction *C* to *BM*, the desired lot line *BG* is found. In similar manner all other lines are obtained.

When the rear line of the lots is not normal to the side lines, but is like *FS* in the figure, the above procedure still holds. However, in this case *C* becomes a variable, differing for the various lot lines by amounts equal to the normal lot widths multiplied by the tangent of the angle between the rear line, *FS*, and the normal line, *FK*. Where many lines are involved, considerable time may be saved in these computations by doing like steps for all the triangles at one time, instead of solving one triangle completely at a time.

Case IV. Curves intersecting without a common tangent (Fig. 593*A*, 3). Occasionally there is the problem of two curves inter-

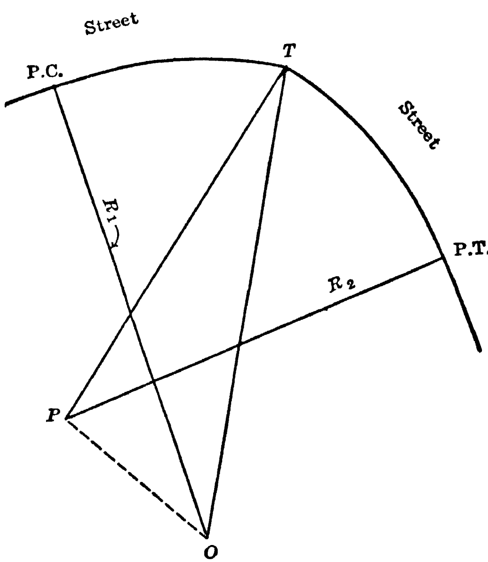


FIG. 593*A*,3.

secting one another without a common tangent. The solution is as follows, assuming again that both the radii and the coordinates of the centers of the curves are known. It is desired in this case

to find the two arc lengths from T to the P.C. and P.T., and the coordinates of point T.

By the inverse computation, obtain the length and bearing of the line OP joining the centers of the curves. A triangle, OPT, will then be formed by this line and the two radii to the point, T, where the curves intersect. Solve this triangle by a three-sides formula, obtaining all angles of the triangles. It is recommended that the formula yielding the tangent of one-half the desired angle be used, since the tangent function varies rapidly for all ranges of angle and thus provides precise results. It is also suggested that all three angles be found separately and their sum be compared with 180° as a check. As many of the factors are the same, the additional labor of finding extra angles is not excessive.

The bearings of the radii to the P.C. and P.T. would be known. These bearings, together with the angles TOP and TPO just found, and the bearing of OP previously found, permit the calculation of the values of the angles P.C.–O–T and P.T.–P–T and the bearings of PT and OT. Coordinates of point T may now be found from those of P and checked from those of O. Arcs P.C.–T and P.T.–T may be found from the respective central angles and radii by the formula $L = R\beta$.

Checking. Computation in all cases described above may be checked most thoroughly by calculating coordinates around the solved triangles as closed traverses. A perfect return to the starting coordinates of any triangle checks all computations subsequent to the original assumptions made for that triangle.

CHAPTER 42

EARTHWORK COMPUTATIONS

IN this chapter are given simple methods of computing earthwork, such, for example, as those most used in estimating cut and fill for ordinary grading. For a more comprehensive treatment of earthwork computations the student is referred to standard books on railroad engineering and on highway engineering.

594. Grade Elevations. The *grade* or *gradient* between two points of fixed elevations is equal to the difference in elevation divided by the distance between the points. This grade is generally given in *per cent*, as 3 per cent, 5 per cent, etc. (See Art. 136(a).) The per cent being known, the elevations of intermediate points are easily obtained. The usual method is to find the elevation of any point by adding to the elevation of the preceding point, if lower, a quantity equal to the distance from the preceding point multiplied by the grade. If the intermediate points are equal distances apart, the quantity added each time is of course a constant.

594(a). ILLUSTRATION. In Fig. 594(a) let the elevation of two points 200 ft. apart be respectively 52 and 60 ft., and let the finished grade be a straight line between the two points. Let the existing surface be represented by the full line. The grade is $\frac{60 - 52}{200} = 0.04$ or 4 per cent. For 50-ft. stations the constant change of grade is $0.04 \times 50 = 2$ ft. The elevation of A is $52 + 2 = 54$; of B, $54 + 2 = 56$; of C, $56 + 2 = 58$. As a check $58 + 2 = 60$, the original elevation. This check should never be omitted.

The grade elevation of any other point between 50-ft. stations may be found easily. Thus, for example, at a point 24 ft. from B toward C the grade elevation is $56 + (24 \times 0.04) = 56.96$.

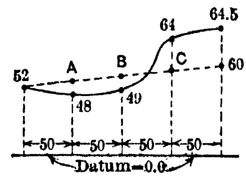


FIG. 594(a).

Note. For a more practical illustration, see Art. 188(b); also computations for sewer-grade points, Art. 191(e).

594(b). CUT AND FILL. The cut and fill at any point is the difference between the grade elevation at that point and the elevation of the existing surface. If the former is always subtracted from the latter *algebraically*, a *minus* result indicates *fill* (earth to be added), a *plus* result *cut* (earth to be taken away). Thus in Fig. 594(a) the fill at A is $48 - 54 = -6$ ft., and at B, $49 - 56 = -7$ ft. The cut at C is $64 - 58 = +6$ ft., and at the last point. $64.5 - 60 = +4.5$ ft.

595. Geometric Definitions and Principles. A *polyhedron* is a geometric solid bounded by planes. A *prism* is a polyhedron two

of whose faces (bases) are equal polygons lying in parallel planes and having their homologous sides parallel, the other faces being parallelograms formed by planes passed through the homologous sides of the equal polygons. A *truncated prism* is one in which the two bases are not parallel. A *right* truncated prism is one in which one of the bases is perpendicular to the lateral edges.

The volume of any truncated prism is equal to the area of a *right section* multiplied by the length of a line joining the centers of mass of the two bases. The volume of any *truncated triangular* prism is equal to the product of its right section by one-third the sum of its lateral edges. Any truncated prism whose base is a symmetrical polygon can be divided into equal truncated triangular prisms, and hence the volume of such a prism is equal to the area of a right section multiplied by a length equal to the mean of all the lateral edges.

596. Calculating Earthwork by the Method of Unit Areas. (*a*). WHEN UNIT AREAS ARE EQUAL SQUARES OR RECTANGLES. In Fig. 596(*a*) let *abcd* be a rectangle which has been staked out on the ground, and assume that the four corners are at different elevations, but lie in the same plane. The surface of the ground within the rectangle is, therefore, an *inclined* plane. Assume that it is desired to grade down to a level surface a certain distance below the lowest corner. The earth to be moved is a *right* truncated prism, with vertical edges at *a, b, c,* and *d;* the rectangle *abcd* represents the horizontal projection of the upper inclined base of the prism and also the lower horizontal base. Assume, on the other hand, that the finished grade is also an inclined plane (not necessarily parallel to the original surface). Then the earth to be moved is a truncated prism whose *right section* is the rectangle *abcd.* In either case the volume is equal to the area of the rectangle *abcd* multiplied by one-fourth the sum of the vertical edges through *a, b, c,* and *d,* or

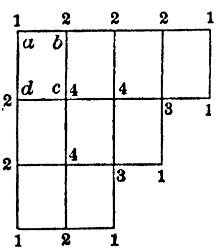


FIG. 596(*a*).

$$V\text{ (in cu. ft.)} = A\frac{a+b+c+d}{4} \quad\text{or}\quad V\text{ (in cu. yd.)} = \frac{A(a+b+c+d)}{4\times 27} \tag{I}$$

This formula is, of course, equally applicable to fills, i.e., when the grading is carried up to a plane wholly above the original surface.

596(*b*). WHEN UNIT AREAS ARE UNEQUAL. When any surface has been divided into a number of *unequal* squares or rectangles and the surface within each rectangle is a plane, there will be as many prisms as there are rectangles, and the total volume will be the sum of the volumes of all the prisms. By making the rectangles equal, however, the work of computation is simplified. Thus in Fig. 596(*a*) each corner height will be used as many times as there are rectangles joining at the corner (indicated on the figure by numbers). The total volume is therefore

$$V \text{ (in cu. yd.)} = \frac{A(\Sigma h_1 + 2\Sigma h_2 + 3\Sigma h_3 + 4\Sigma h_4)}{4 \times 27} \qquad \text{(II)}$$

A = area of one rectangle in square feet; Σ = sum of; h = corner height; subscripts indicate number of times a corner height is used. This formula is applicable (1) when one base of each prism is inclined and the other base is horizontal (right truncated prism); (2) when both bases, upper and lower, are inclined. In either case, a unit rectangle represents a right section of the corresponding prism.

596(*c*). REMARK. Both formulas I and II are based on the assumption that the original surface of the ground within a unit rectangle is a *plane*. In practice the unit area is assumed small enough so that this assumption is approximately true, but it is practically never exactly true, and the volume obtained from either formula will be slightly different from the true volume.

596(*d*). WHEN THE FINISHED SURFACE IS INCLINED it may slope in one direction (case II, Art. 194(*b*)), or it may slope in two directions (case III, Art. 194(*c*)). In either case the elevations of the finished surface at the different corners must be obtained either by calculation or graphically before the corner heights can be found. The method of ascertaining these grade elevations will depend upon how the unit areas were laid out in the field. The directions for setting grade stakes (Arts. 194(*b*), 194(*c*)) will suggest ways in which the method of Art. 594 may be used to find grade elevations at points in any given row.

596(*e*). METHOD OF PROCEDURE. (1) Draw a diagram of the rectangles as laid out in the field, not necessarily to scale but large enough so that the shortest side of a rectangle is at least an inch. (2) At each corner mark the elevation of the original surface as obtained from the field notes. (3) Calculate the elevation of the graded surface at each corner and put the result on the diagram directly under the original elevation. (4) Subtract on the diagram the original elevations from the grade elevations (or vice versa if it is fill instead of cut), thus getting the corner heights. (5) Sum

up all the corner heights in the perimeter used *once*. Sum up all those used *twice* and multiply by 2. Sum up all those used *three* times and multiply by 3. This will take into account all points in the perimeter. Sum up all *interior* corner heights and multiply by 4. (6) Sum up all the results obtained in (5) and multiply by $A/108$, where A = area in square feet of unit rectangle. The result is the total volume in cubic yards.

When the outline of the entire area is not rectangular this method of procedure must be modified. (See Fig. 596(g).)

596(f). SUGGESTIONS. Any systematic method of procedure may be followed in place of that given, provided that corner heights are correctly calculated and that there is no confusion as to the number of times each corner height is used. There is always danger of omitting a corner height altogether; hence in step (5) arrange corner heights in four columns according to whether they are used once, twice, thrice, or four times, taking interior corners in order, vertically or horizontally. Before adding, see if each of the quantities in the four columns agrees with the corresponding quantity on the diagram. If it does, put a check mark near the latter, and finally inspect the diagram to see if there are any corner heights without check marks.

596(g). WHEN THE OUTLINE OF AN AREA IS NOT RECTANGULAR, as in Fig. 596(g), it cannot be subdivided into squares or rectangles. As much as possible of the calculation is done by the method of the preceding article, however, and the rest by finding the volumes of the irregular solids around the edges. Thus in Fig. 596(g) the unshaded portion was divided into 50-ft. squares and the corresponding volume computed by the preceding method. Extending the lines to the boundaries, a number of irregular areas are formed (indicated by cross lines). From each area and the corresponding corner heights the volume of the corresponding prism may be found, and the sum of these volumes added to the volume of the unshaded portion. In the majority of cases the irregular areas will be trapezoids, i.e., each is a plane four-sided figure with two sides parallel. The middle height of each trapezoid (computed or scaled) multiplied by the base gives the area, and this area multiplied by the mean of the corresponding corner heights gives the volume. In Fig. 596(g) all the shaded areas are trapezoids except the two in the upper corners (double-cross-lined), and each of these may be divided into a trapezoid and a triangle.

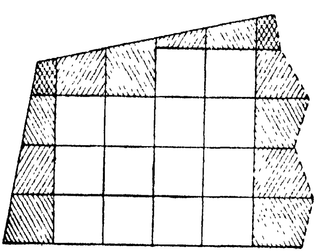


FIG. 596(g).

When a cut or fill "runs out to zero" the boundary line is usually irregular, as shown by the dotted line in Fig. 597. In a perfectly

general case of this kind all four sides of the figure would be bounded by irregular lines, but the method would still hold good.

596(*h*). *Suggestions.* Good judgment in choosing the size of the unit square or rectangle, and in planning the general arrangement, will save work both in the field and in the office. The surface within a unit area should be approximately a plane, and the aim should be to have the smallest number of corners consistent with this rule. (See Remark, Art. 596(*c*). Frequently large unit areas can be used for the greater portion of the work where the ground is even, smaller unit areas being adopted for places where the ground is uneven.

In very uneven ground the assumption that the four corners of a unit area lie in the same plane is seldom true, but any three points will lie in a plane. Hence better results may be obtained by subdividing each unit square or rectangle into two triangles by drawing that one of the two diagonals which more nearly fits the ground. Each corner height will then be used as many times as there are triangles meeting at that corner. (Why cannot this number exceed 8?) Formula II, Art. 596 (*b*), then becomes

$$V \text{ (in cu. yd.)} = \frac{A(\Sigma h_1 + 2\Sigma h_2 + 3\Sigma h_3 + \text{etc.}, \ . \ . \ . \ 8\Sigma h_8)}{6 \times 27} \qquad \text{(III)}$$

in which A is the area of a *unit rectangle* or *square*, and the subscripts denote the number of triangles meeting at a corner.

597. Estimates of Cut and Fill. In Figs. 596(*a*) and 596(*g*) it was assumed that the graded or finished surface would be entirely below or above the original surface, i.e., that the grading would be all cut or all fill. More often in practice the problem is one in both cut and fill. Assume that in Fig. 597 the elevations of the ground at all corners of the squares have been ascertained, and it is desired to grade down to an elevation of 20 ft. above datum. Let the outside broken line represent the boundary of the area to be graded. Assume that the elevations of all the corners within the shaded area are *above* 20 ft. and those in the unshaded area are *below* 20 ft. Then there must be a line somewhere between (represented by the inside broken line) which passes through points of exactly 20 ft. elevation, i.e., a line at grade corresponding to a 20-ft. contour. Points through which this line passes may be determined by interpolation as explained below. The volume of cut (shaded area) and the volume of fill (unshaded area) can each be calculated by the method of the preceding article, and the difference between the two is the amount of earth which must be "borrowed" or taken away, according to whether the fill or the cut is in excess, assuming that no allowance is made for shrinkage or expansion.

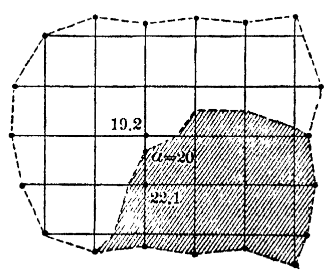


FIG. 597.

Remark. In the above illustration the finished surface is level; hence the grade elevation at any point, as, for example, at any corner of any square

or at any point between corners, is 20 ft. The interpolation for any point, as *a*, is very simple, since its elevation must be 20 ft. (See Fig. 598 for method.) When the finished surface is inclined the grade elevations at the corners of the squares will differ, but once the cut or fill at each corner is known the work of interpolation is no more difficult than for a level grade.

Sometimes portions of the outside boundary line are determined by the points where the grade "runs out," i.e., where there is neither cut nor fill. In such a case these points may also be found by interpolation.

598. Interpolation. (*a*). WHEN THE GRADE IS LEVEL. Let the elevations of the surface of the ground at *A* and *B*, two corners of a square, be 19.2 and 22.1 respectively. Suppose that the graded or finished surface is level and its elevation is 20 ft. It is required to find a point on the line *AB* where the existing surface is intersected by the graded surface, assuming the former to slope uniformly from *A* to *B*. There are several methods, arithmetical, graphic, and mechanical, based on the properties of similar triangles. One of the simplest graphic methods is shown in the figure. At *A* 19.2 − 20 = −0.8 (fill), and at *B* 22.1 − 20 = +2.1 (cut). Hence lay off to any scale 8 units in one direction and 21 units in the opposite direction, lay down a straightedge between the points thus obtained (indicated by the broken line), and mark the required point *C*. Since the 8 units and the 21 units may be laid off on any two parallel lines through *A* and *B*, it saves time to use the sides of the square, and points can be interpolated in a case like that of Fig. 598 without drawing any lines whatever. This method is equivalent to plotting a vertical section through *A* and *B* in which *A′B′* is a level line of 20-ft. elevation, *a* a point 0.8 ft. below this line, and *b* a point 2.1 ft. above it.

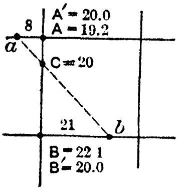


FIG. 598.

In other words the fill (−0.8) is laid off in one direction and the cut (+2.1) is laid off in the other. It is well, therefore, before beginning to interpolate to go over the plot and mark the amount of cut or fill at each corner.

598(*b*). WHEN THE GRADE IS NOT LEVEL. (1) Find the grade elevation at each corner. (2) Find the cut or fill at each corner. (3) Proceed as above, i.e., lay off the fill in one direction, the cut in the other on parallel lines, and lay a straightedge between the points thus obtained; it will cross the side of the square in the required point. (Draw a figure and prove this.)

Note. For other methods of interpolation see Art. 660.

599. Prismoids. *Definition.* A prismoid is a solid that has parallel plane bases or ends, and is bounded on the sides either by planes or by warped surfaces. It may be conceived as generated

by a right-line generatrix moving continuously around the perimeters of the two bases as directrices. It may be composed of any combination of prisms, pyramids, wedges, cylinders, cones, or frustums whose bases and apices lie in the bases of the prismoid. The common altitude of all solids in the combination is the perpendicular distance between the parallel planes of the bases. Since cylinders and cones are special forms of prisms and pyramids, it may be said that any prismoid may be resolved into prisms, pyramids, and wedges, and any formula for volume that is equally applicable to all three of these solids will also be applicable to any combination of them.

599(*a*). VOLUME OF A PRISMOID. Many excavations and embankments, especially those which occur in the construction of highways and railroads, are more nearly prismoids than prisms. There are several methods of computing the volume of a prismoid. Two of the most common are indicated by the following formulas, in which A_1 and A_2 represent the areas in square feet of the two parallel end faces, M the area of the middle section parallel to the end planes, and l the length or perpendicular distance (in feet) between the end planes:

$$\textit{End-Area Formula: } V \text{ (in cu. ft.)} = \frac{l}{2}(A_1 + A_2) \qquad \text{(IV)}$$

$$V \text{ (in cu. yd.)} = \frac{l}{54}(A_1 + A_2)$$

$$\textit{Prismoidal Formula: } V \text{ (in cu. ft.)} = \frac{l}{6}(A_1 + 4M + A_2) \qquad \text{(V)}$$

The end-area formula is not exact except when A_1 and A_2 are equal. The error in its use is due chiefly to the fact that when any of the solids into which the prismoid may be resolved is a pyramid, the formula gives the volume of that pyramid as *one-half* the product of its base area and its altitude (l), whereas the correct volume is *one-third* that product. The greater the difference between the two end areas of a prismoid, the greater the error in the volume obtained by the end-area formula. The maximum error of 16⅔ per cent will occur when one of the end areas is zero, as when cut changes to fill on a side hill. In such a case, however, the prismoid becomes a pyramid and its volume may be calculated as such.

The prismoidal formula is of wide application, since it gives the correct volume of any prismoid. It would be used more extensively than it is were it not for the extra labor involved in cal-

culating the areas of middle sections. The area M is *not* the mean of A_1 and A_2; it must be determined separately from its linear dimensions, each of which is the average of the two corresponding linear dimensions of the two end sections. The middle section is sometimes measured on the ground. (Prove that the volume of a pyramid or of a wedge obtained by the use of the prismoidal formula is correct.)

In a series of regular cross sections which are equally spaced and which are nearly equal in area, each alternate cross section may be taken as a *middle* section whose area is M; if l is the distance between sections, the length of the prismoid will be $2l$. For a continuous line of earthwork, the volume in cubic feet obtained by the prismoidal formula then becomes

$$V = \tfrac{1}{3}(A_1 + 4A_2 + 2A_3 + 4A_4 + 2A_5 + 4A_6 \ldots + A_n) \quad \text{(VI)}$$

The error in volume due to the use of the end-area formula is generally small, often less than 2 per cent, and often considerably less than the error due to irregularities in the surface of the ground which cannot well be taken into account in calculating volumes. Moreover, the end-area formula gives volumes that are generally somewhat too great and, therefore, the contractor suffers no injustice from its use. It involves less computation than is required by the prismoidal formula, particularly when the linear measurements for middle sections have to be calculated. For these reasons the end-area formula is commonly used for ordinary earthwork computations.

The use of the prismoidal formula is justified only when there is a large difference between the two end areas of a prismoid, when the intervals between cross sections are short, and when irregularities in the ground surface have been measured in the field. Obviously, these are conditions which imply that the unit cost per cubic yard of excavation is sufficiently great to require more than ordinary accuracy in the determination of volumes.

599(*b*). Prismoidal correction. When most of the cross sections are of the three-level type and prismoidal volumes are desired, results can be obtained more easily if, instead of using the prismoidal formula, volumes are computed by means of the end-area formula and prismoidal corrections are then applied. For any prismoid the correction C is the difference between the volume V_e obtained by the use of the end-area formula and the volume V_p obtained by the use of the prismoidal formula. For a three-level sections its value in cubic feet is:

$$V_e - V_p = C = \frac{1}{12}(H_0 - H_1)(D_0 - D_1) \text{ (cu. ft.)}$$

H is the cut or fill at the center line, D_0, the horizontal distance between slope stakes at one end section, and H_1 and D_1 are the corresponding distances at the other end section. As a general rule, the correction will be positive and is subtracted from the volume obtained from the end-area formula. When the algebraic sign is negative, the correction should be added.

600. Estimating Cut and Fill from a Contour Map. There are several methods in which approximate estimates for grading can be made from contour maps.

600(*a*). FIRST METHOD. In Fig. 600(*a*) the contours of the existing surface are represented by full lines. The figures around the outside of the bounding lines give the grade elevations at those points; hence the straight lines represent contours of the finished or graded surface at one-foot intervals. At every point where a straight line intersects a contour of the existing surface the cut or fill can be found by simply subtracting the elevation of the straight line from the elevation of the contour. By joining points of equal cut or fill, a set of lines is obtained (represented by broken lines in Fig. 600(*a*)) which are the horizontal projections of lines cut from the existing surface by planes parallel to the finished surface. In this case these planes are one foot apart, measured vertically. The layer of earth between any two of these parallel planes is a solid having for its two parallel bases the irregular areas enclosed by the two corresponding broken lines, and for its altitude the vertical distance of one foot. The volume of this solid may be obtained by the end-area method or by the prismoidal formula, the irregular areas being found from the map by scaling and computing or by the use of the planimeter. The sum of the volumes of all the layers is the total volume required.

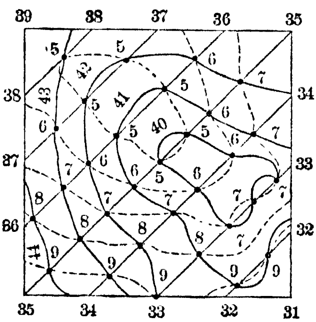


FIG. 600(*a*).

600(*b*). ILLUSTRATION. The line passing through the points marked 5 in Fig. 600(*a*) is the horizontal projection of a line on the existing surface which encloses an imaginary irregular surface that is parallel to and 5 ft. above the finished surface. Likewise the line through points marked 6 represents the boundary of an imaginary surface parallel to and 6 ft. above the finished surface. The area enclosed by the 5-ft. line and the area between the 6-ft. line and the top and left bounding line are ascertained, and the volume between these two irregular surfaces is computed as explained above. The volumes of the other layers may be found in a similar manner.

600(*c*). SECOND METHOD. (1) Divide the map into unit squares or unit rectangles by ruled lines. (2) Find by interpolation the elevation of the existing surface at the corners of each unit square. (This is done by a method the reverse of that explained in Art. 598.) (3) Ascertain the grade elevation at each corner and then the corner height. (4) Calculate the volume by the method of Art. 596.

600(*d*). REMARK. The two preceding methods are used most in estimates where the finished or graded surface is a plane—level or inclined. For warped or undulating surfaces, or for structures like dams or reservoirs and terraces having regular outlines, another method (given below) is often used.

600(*e*). THIRD METHOD. (1) Make an accurate map on which the existing surface of the ground is shown by contours drawn in black ink. (2) On this map lay out contours in red ink which *correspond in elevation* to those in black ink, but which represent the finished or graded surface. If this final surface is to be curving or undulating, the red-ink contours will be curving lines. If, however, it is a surface composed of *plane* faces, like the inside slopes of a rectangular reservoir, the red-ink contours will be a series of straight lines. (3) The area enclosed by a red-ink contour and a black-ink contour of the same elevation is the horizontal area of the cut or fill at that elevation. (4) The layer of earth between two of these successive areas is approximately a prismoid, the altitude of which is the contour interval. The volumes of successive layers may be obtained by the end-area method or by the prismoidal formula, but if the latter is used it is better to consider every other area as a middle area, and the height of the prismoid as *twice* the contour interval. Areas are usually found from the map by means of the planimeter.

601. Calculation of Earthwork from Cross Sections. In route surveying, the volumes of earthwork are determined from cross sections taken in the field by methods explained in Art. 195. The end-area method is generally followed except when more than ordinary accuracy is required, in which case either the prismoidal formula is used or prismoidal corrections are applied to volumes obtained by the end-area method. In any case, a large part of the work consists in determining the areas of the cross sections.

The area of a cross section may be determined either by calculation or by means of the planimeter. In order to use the planimeter a cross section must be plotted, and, for this reason, the planimeter method is justified only when sections are plotted for some other purpose, as, for example, when it is desirable to make

detailed studies of certain cross sections, a condition that occurs frequently in highway work.

The area of a cross section may be calculated by subdividing the area into simple geometric figures such as triangles, rectangles, and trapezoids. The area of each of these figures is calculated and the sum of all these areas is the required area.

601(*a*). THE CROSS SECTION OF A TRENCH may usually be taken as a trapezoid and its area as equal to one-half the sum of the top and bottom widths multiplied by the depth.

601(*b*). A LEVEL SECTION may be subdivided into a rectangle and two equal triangles by two vertical lines, one at either edge of the roadbed.

601(*c*). A THREE-LEVEL SECTION may be subdivided into four triangles by three lines drawn from the center of the roadbed as indicated by the broken lines in Fig. 601(*c*). The distance $w/2$

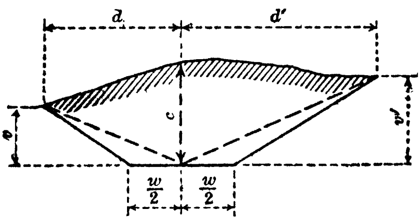


FIG. 601(*c*).

is the base of each of two triangles whose altitudes are v and v'. The cut c at the center is the common base of two triangles whose altitudes are the "distances out," d and d'.

$$\text{Area} = \tfrac{1}{2}\left[\frac{w}{2}(v + v') + c(d + d')\right] \qquad \text{(I)}$$

601(*d*). A FIVE-LEVEL SECTION may be subdivided into six triangles as indicated by the broken lines in Fig. 601(*d*). Each of

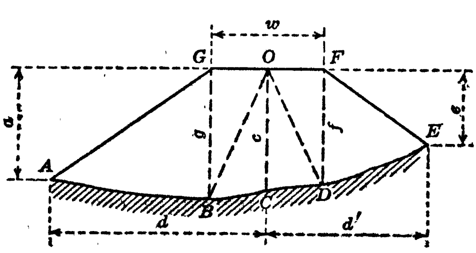


FIG. 601(*d*).

the vertical broken lines may be regarded as the common base of two triangles; hence the combined area of each pair of triangles is equal to one-half the product of the common base and the sum of the two altitudes. The total area is:

$$\text{Area} = \tfrac{1}{2}(g \times d + c \times w + f \times d') \qquad \text{(II)}$$

601(*e*). THE COORDINATE METHOD. A second method, which can be applied to any cross section, is the *coordinate method* explained in Art. 584(*c*). The section may be regarded as a polygon *ABCDEFGA*. If the point *O*, the center of the roadbed, is taken as the origin, the coordinates of each of the points on the ground surface *A*, *B*, *C*, *D*, and *E* are its "distance out" and its vertical distance from the plane of the subgrade. Both these quantities are given in the cross-section notes. Each pair of coordinates, including those for *G* and *F*, may be written in the form of a fraction thus:

$$\frac{0}{-\frac{1}{2}w+} \quad \frac{a}{-d+} \quad \frac{g}{-\frac{1}{2}w+} \quad \frac{c}{0} \quad \frac{f}{+\frac{1}{2}w-} \quad \frac{e}{+d'-} \quad \frac{0}{+\frac{1}{2}w-}$$

Since the ordinates in this case are all below the horizontal axis, their algebraic signs are negative, but the numerical result will not be affected if they are all taken as positive. The abscissas to the left of the center line are negative and those to the right are positive, as shown by the algebraic sign in *front* of each denominator. If, now, the algebraic sign opposite to the one in front is placed *after* each denominator, the following rule, explained in Art. 584(*c*), may be followed:

Multiply the numerator of each fraction by the algebraic sum of the denominators of the two adjacent fractions, one on either side, using for each denominator the algebraic sign that is toward the other denominator.

$$\text{Area} = \tfrac{1}{2}[a(\tfrac{1}{2}w - \tfrac{1}{2}w) + g(d + 0) + c(\tfrac{1}{2}w + \tfrac{1}{2}w) + f(0 + d') + e(-\tfrac{1}{2}w + \tfrac{1}{2}w)]$$

This reduces to $\frac{1}{2}(g \times d + c \times w + f \times d')$, which is the same as formula II. Obviously little if anything is gained by the use of the coordinate method in this case; it is more useful when applied to irregular cross sections having a considerable number of surface elevations. For a numerical application of this criss-cross method to such a section, see the next article.

601(*f*). THE AREA OF ANY IRREGULAR CROSS SECTION may be computed by any of the three methods explained in Art. 584. Probably for the more simple sections, the method based on the

subdivision of the section into simple geometric figures is the best; for the more complex sections, the crisscross application of the coordinate method (Art. 584(*c*)) is favored by many engineers. The use of both methods in determining the area of the irregular cross section in Fig. 601(*f*) will afford a comparison between the two.

601(*g*). CALCULATION OF THE AREA OF A CROSS SECTION BY SUBDIVIDING IT INTO TRIANGLES. In the field cross-section notes shown in Fig. 601(*f*) distances out from the center line of the

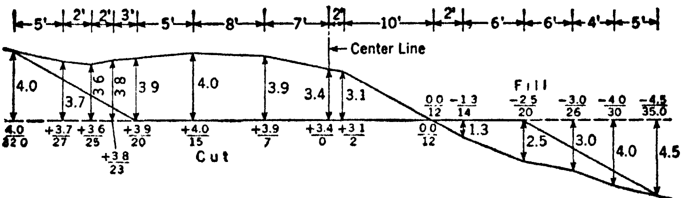


FIG. 601(*f*).

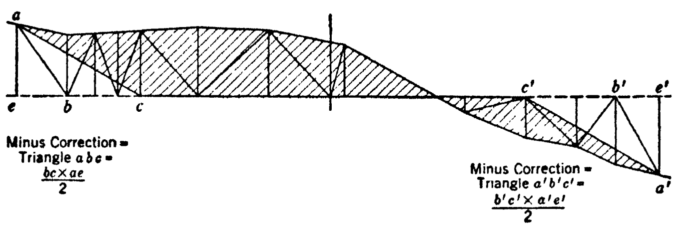


FIG. 601(*g*).

These figures appeared in an article, "Reducing Cross-Section Notes on Highway Construction," by Clifford B. Potts, Volume 10, Number 3, *Civil Engineering*, published by the American Society of Civil Engineers. The use of the mass diagram and a method of computing overbreak and underbreak are also illustrated in that article.

highway are given in the denominators of the fractions, and corresponding distances above or below grade are given in the numerators.

The point on the ground surface (0.0 elevation) at which cut changes to fill is 12.0 ft. to the right of the center line of the roadway. The cross-section area in cut to the left of this point and the cross-section area in fill to the right are section-lined in Fig. 601(*g*). These two areas will be computed separately.

The area in cut *plus the area of the triangle abc* will be considered as one figure. The double area of this figure minus the double area of the triangle *abc* will equal double the area in cut. Each vertical line in Fig. 601(*g*), except *ae*, may be considered to be the common base of two triangles. Twice the combined area of

each pair of triangles may be found by multiplying their common base by the sum of the two horizontal distances to the two adjacent vertical lines, one on either side of the common base. The necessary dimensions for such multiplications are given on the vertical lines and on the top horizontal line of dimensions in Fig. 601 (*f*). Taking the triangles in pairs from left to right and subtracting (4.0×7) the double area of the triangle *abc*, the double area in cut is:

$$(3.7 \times 7) + (3.6 \times 4) + (3.8 \times 5) + (3.9 \times 8) + (4.0 \times 13) + (3.9 \times 15) + (3.4 \times 9) + (3.1 \times 12) - (4.0 \times 7) = 240.80 \text{ sq. ft.}$$

Area in cut $= 240.80 \div 2 = 120.40$ sq. ft.

For fill:

$$(1.3 \times 8) + (2.5 \times 12) + (3.0 \times 10) + (4.0 \times 9) - (4.5 \times 10) = 61.40 \text{ sq. ft.}$$

Area in fill $= 61.40 \div 2 = 30.7$.

601 (*h*). Calculation of the area of a cross section by the coordinate method. The cross-section notes are given in Fig. 601 (*f*). To calculate the area in cut by the coordinate method, the point on the subgrade at the center line is taken as the origin, and the coordinates are arranged as they appear in the field notes. The rule in Art. 601 (*c*) can then be followed. Abscissas to the left of the origin are given minus algebraic signs. (See also Art. 584 (*c*) for an explanation and an illustration of the coordinate method.)

$$\frac{0}{-20+} \quad \frac{4}{-32+} \quad \frac{3.7}{-27+} \quad \frac{3.6}{-25+} \quad \frac{3.8}{-23+} \quad \frac{3.9}{-20+} \quad \frac{4.0}{-15+}$$

$$\frac{3.9}{-7+} \quad \frac{3.4}{0.0} \quad \frac{3.1}{+2-} \quad \frac{0.0}{+12-}$$

$$4(20 - 27) + 3.7(32 - 25) + 3.6(27 - 23) + 3.8(25 - 20) + 3.9(23 - 15) + 4.0(20 - 7) + 3.9 \times 15 + 3.4(7 + 2) + 3.1 \times 12 = 240.8 \text{ sq. ft.}$$

Area $= 120.4$ sq. ft.

When the quantities within the parentheses are replaced by a single quantity, the expression becomes identical with that obtained by the method of calculating the areas of triangles in pairs in Art. 601 (*g*). For the area in fill:

$$\frac{0.0}{+12-} \quad \frac{-1.3}{+14-} \quad \frac{-2.5}{+20-} \quad \frac{-3.0}{+26-} \quad \frac{-4.0}{+30-} \quad \frac{-4.5}{+35-} \quad \frac{0.0}{+20-}$$

$$-1.3(-12 + 20) - 2.5(-14 + 26) - 3.0(-20 + 30) - 4.0(-26 + 35) - 4.5(-30 + 20) = -61.4 \text{ sq. ft.}$$

Area in fill $= -61.4 \div 2 = -30.7$ sq. ft.

A common mistake in the crisscross multiplications is to use, every now and then, the wrong denominator. In order to avoid this mistake, many engineers draw diagonal lines from each numerator to the two adjacent denominators as shown in Art. 584(*c*).

Instead of being arranged in the form of fractions, the coordinates may be placed in two columns and the diagonal method of multiplication applied as explained in Art. 584(*d*).

601(*i*). FORM OF COMPUTATION SHEET FOR THE COORDINATE METHOD. It is highly desirable in computing areas of cross sections by the coordinate method to arrange the computations in such a form that, in checking results later, errors can be discovered and corrected without a complete recomputation. Professor John S. Worley recommends the following form:*

20.0	32.0	27.0	25.0	23.0	20.0	15.0	7.0	0.0	2.0	12.0
0.0	(4.0)	3.7	3.6	3.8	3.9	4.0	3.9	3.4	3.1	0.0
	(−7.0)	7.0	4.0	5.0	8.0	13.0	15.0	9.0	12.0	
	−28.00	25.90	14.40	19.00	31.20	52.00	19.50	30.60	37.20	
							39		30.60	
							58.50		58.50	
									52.00	
									31.20	
									19.00	
									14.40	
									25.90	
									268.80	
									28.00	
									2)240.80	
									120.40 sq. ft. excavation	

12.0	14.0	20.0	26.0	30.0	35.0	20.0
0.0	1.3	2.5	3.0	4.0	4.5	0.0
	8.0	12.0	10.0	9.0	(−10.0)	
	10.40	30.00	30.00	36.00	(−45.00)	
	30.00					
	30.00					
	36.00					
	106.40					
	45					
	2)61.40					
	30.70 sq. ft. embankment					

In this form, the fractions shown in the cross-section notes in Fig. 601(*f*) have been inverted so that the distances out are given in the numerators and the corresponding elevations above or below grade are given in the denominators. The procedure for computation, similar to that given in the rule in Art. 601(*e*), is as

* *Civil Engineering*, Vol. 10, No. 9, September 1940.

follows. Begin with the fraction which has 0.0 for its numerator. Multiply the denominator of this fraction (which is the elevation at the center line) by the sum of the two adjacent numerators. Multiply each remaining denominator by the algebraic difference of the two adjacent numerators, subtracting the inner (toward the center line) from the outer. One-half the algebraic sum of these computations equals the area of the cross section.

The computation sheet is preserved for checking. When there are many such sheets of computation, the checking is best done by computing machines. The computation sheet in this case would appear as follows.

20.0	32.0	27.0	25.0	23.0	20.0	15.0	7.0	0.0	2.0	12.0
0.0	4.0	3.7	3.6	3.8	3.9	4.0	3.9	3.4	3.1	0.0
	(−7.0)	7.0	4.0	5.0	8.0	13.0	15.0	9.0	12.0	
				2)240.80						
				120.40 sq. ft. excavation						

12.0	14.0	20.0	26.0	30.0	35.0	20.0
0.0	1.3	2.5	3.0	4.0	4.5	0.0
	8.0	12.0	10.0	9.0	(−10.0)	
				2)61.40		
				30.70 sq. ft. embankment		

601(*j*). Overbreak and underbreak. When excavation has been completed, the slope lines of any cross section may not be the straight lines that they were assumed to be in determining the area of the section. If the actual surface of a slope varies from the assumed slope line in such a way as to cause a decrease in the area of a cross section, this decrease in area is called *overbreak;* if the variation is such as to cause an increase in area, this increase in area is called *underbreak.* The variation is often great enough to justify a remeasurement of a slope to determine the amount of overbreak or underbreak.

601(*k*). The volume of earthwork on a curve. In ordinary work the error in volume due to curvature is generally negligible provided sections are taken, say, 50 ft. apart or less. When, however, the cost of earthwork is sufficiently great to justify the use of the prismoidal formula, as, for example, in rock excavation, curvature correction should be applied.*

601(*l*). Tables and diagrams. Most handbooks on railroad and highway work contain tables and diagrams, the use of which greatly facilitates the calculation of earthwork volumes. Special earthwork computation tables are also published.

601(*m*). Consistent accuracy. In the computation of areas and volumes intermediate results may be carried to one decimal place, but final results should not be recorded beyond the nearest square foot for areas, or beyond the nearest cubic foot or cubic yard for volumes.

602. Estimating Volumes of Earthwork is quite different from merely calculating volumes. In estimating the actual earth to be moved, questions of design are involved such as the economical distribution of earth materials and the physical charac-

* For curvature correction, consult books on route surveying, railroad engineering, or highway engineering.

teristics or properties of such materials. It is not within the scope of this book to discuss these questions of design, but some of the terms used in estimating earthwork will be defined.

602(*a*). SHRINKAGE AND SWELL. When the volume of a completed embankment is less than the volume of the excavation from which the material for fill was obtained, the difference is the *shrinkage;* when the volume of the embankment is greater, the difference is the *swell.* The shrinkage of ordinary soil varies from 0 to 30 per cent. A quite general allowance is 15 per cent. The swell for such excavated materials as ledge rock and shale varies from 10 to 50 per cent.

602(*b*). SETTLEMENT is the decrease in the height of an embankment which occurs subsequent to completion. When the fill is deposited with the usual earth-handling equipment the settlement is generally about 10 per cent. Rock fills, sand fills, and fills composed of mixtures of sand and a preponderance of rocky material settle very little. When special methods are used to compact the fill, there may be no appreciable settlement.

602(*c*). SUBSIDENCE of an embankment will occur when it is placed on a soil of insufficient supporting strength, such as that found in or near swamps.

602(*d*). OVERHAUL.* The estimates of the cost of excavation are usually based on a specified length of haul called the *free haul,* and the average excess of haul is known as the *overhaul.* In highway work, the free haul is commonly specified as 500 ft., and, if the average haul on any part of the work is 800 ft., the overhaul is 300 ft.

602(*e*). MASS DIAGRAM.* The economical distribution of earth materials depends largely upon how nearly cut and fill can be balanced so that excavated material will be used in embankments. Since overhaul is an added cost, it should also be considered. The problem can best be solved graphically by means of a distribution diagram known as a *mass diagram.* The principal feature of this diagram is a *mass curve,* similar to a profile, whose abscissas are distances and whose ordinates are volumes of earthwork.

* For overhaul and the use of mass diagrams see books on route surveying, railroad engineering, or highway engineering.

CHAPTER 43

THE USE OF THE DRAWING INSTRUMENTS *

In this chapter are given such directions and suggestions for the use of drawing instruments as are of special importance in mapping.

603. Drawing Instruments and Equipment. It is difficult even for an experienced draftsman to do good work with poor instruments. Much mapping may be done with ordinary equipment, but a first-class equipment is well worth the extra cost. The surface of the drawing board should be a true plane and smooth; a perfectly straight steel strip fastened to the left-hand edge is highly desirable. T-squares, straightedges, and triangles made of stainless steel are preferable to those made of wood or celluloid. The best scales obtainable are essential for accurate work.

603(*a*). THE DRAFTING MACHINE is so designed that the drafting tools just mentioned are embodied in principle in one unit, thus obviating the necessity for picking up and laying down a required tool. A good machine should expedite the manual work and, at the same time, make it less difficult to attain accuracy. Such a machine, however, is frequently not available; consequently explanations in this chapter will be confined to the use of ordinary drawing instruments. A draftsman skillful in working with these can readily use a drafting machine with equal skill.

604. Use of the Pencil.

1. For ordinary map drawing, use a 4-H or a 6-H pencil, depending upon the kind of drawing paper, but for penciling directly on tracing cloth a 3-H pencil (or softer) is better. Sharpen as follows. Cut the wood back a little more than an inch at *each* end of the pencil, *leaving at least* ⅜ *in. of lead exposed.* Sharpen the lead at one end to a *long thin wedge* by rubbing the flat sides on fine sandpaper alternately as in sharpening a knife, until they taper gradually from the wood to a knife edge. Sharpen the lead at the other end to a *long* cone-shaped point by rolling the lead as it is rubbed.

Remark. The above directions are made explicit because so many students neglect to sharpen their pencils properly. It is practically impossible to do

* This chapter is taken largely from a similar chapter in the author's *An Introductory Course in Mechanical Drawing,* American Book Co.

accurate work in plotting with a lead that is dull or soft—clear hairlines are absolutely essential.

2. Draw lines so close to the ruling edge that they are scarcely visible until the edge is moved away. To insure this, it may be necessary to incline the top of the pencil slightly outward, thus bringing the flat side of the lead in contact with the *lower* corner of the ruling edge. The pencil should be held nearly vertical, and it should be *drawn*, not *pushed*. Do not use the cone-shaped lead for ruling lines—it will wear off too fast; keep it sharp for marking points.

605. Use of the T-square.

1. Make sure that the upper edge of the blade is straight, and do not use the lower edge unless it is known to be truly parallel to the upper edge. To test an edge for straightness draw a fine hairline in exact contact with the edge, and then, turning the T-square end for end, see if the same edge coincides with the line throughout its length.

2. Use the T-square against the left-hand edge of the board for horizontal lines, and acquire the habit of *feeling* the head in perfect contact with the edge before starting to draw a line.

3. Do not use the T-square with its head against the right-hand edge of the board unless you draw with your left hand.

4. Draw horizontal lines from left to right. When near the end of the blade make sure that the pressure of the pencil has not sprung the edge of the blade away from the horizontal.

5. Do not draw vertical lines by placing the head of the T-square against the top or bottom of the board except in rough work, for adjacent edges of the ordinary drawing board are seldom exactly at right angles to each other.

6. In drawing long oblique lines it is sometimes necessary to turn the T-square over in order that it may lie flat. In that case use the edge of the blade which is known to be straight and take extra pains to keep the blade from slipping. There are so many long oblique lines drawn in mapping that a straightedge instead of a T-square can be used to advantage in much of the work.

606. Use of the Triangles.*

1. In drawing a line through a point the corner of the triangle should never be at the point. For example, do not erect a per-

* A new triangle should always be tested. The right angle may be tested by placing one edge against the T-square and drawing a vertical line along the other edge. Turn the triangle over, placing the same edge against the T-square, and if the triangle is not right-angled the vertical edge will not coincide with the vertical line. A 60° angle may be tested by constructing an equilateral triangle wholly with the triangle and T-square. Similar tests will suggest themselves for 30° or 45°.

pendicular to an oblique line by fitting one edge of a triangle against the line and drawing the perpendicular along another edge without moving the triangle, but fit one edge to the line, bring the edge of a second triangle or the edge of a T-square into contact, and slide the first triangle until the perpendicular passes through the point. Another method is to fit the hypotenuse of a triangle to the line, and bringing a straightedge into contact with a second side, turn the triangle until its third side is in contact with the stationary edge, thus bringing the hypotenuse perpendicular to the given line. In either method, exercise great care to keep the stationary straightedge from slipping.

2. By placing the 45° triangle against the horizontal edge of a T-square and the 30° or 60° triangle against the edge of the 45° triangle, 15° and 75° lines can be drawn.

3. In drawing a line parallel to a given line, slide the triangle along the edge of the T-square if the lines are too far apart to use the usual combination of two triangles.

4. It is difficult to make the edge of a triangle pass through a point or coincide with a line if that edge casts a shadow. Avoid shadows by turning the drawing board or by working from different sides of the board.

607. Use of the Scale. The scale commonly used in mapping is the *decimal* scale. The form recommended is the triangular

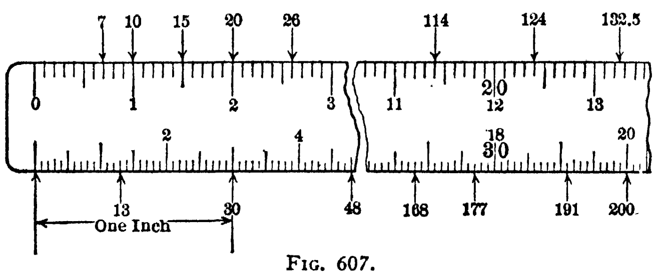


FIG. 607.

scale in which the smallest divisions on the six sides are respectively 1/10, 1/20, 1/30, 1/40, 1/50, and 1/60 in. For some kinds of mapping an architect's scale will also be needed. In using the scale for plotting do *not* reduce measurements by arithmetic but read distances directly from the scale itself *by changing in the mind the value of each division on the scale.*

Illustration. The smallest division on the upper edge of the scale, Fig. 607, is 1/20 in. When plotting 20 ft. to the inch consider each of these divisions as representing one foot. The portion to the right of the break is at the center of a 12-in. scale, and the "20" marked in large numerals indicates

the number of divisions to the inch. On the lower scale the "30" indicates that each inch is divided into thirty parts. The other four edges of the scale not shown in the figure are marked at the center 10, 40, 50, and 60 respectively.

In reading the scale simply add a cipher to the nearest number preceding the point. The distance in *feet* that each arrow in Fig. 607 is from the zero point is indicated by the corresponding number just above the arrow.

To plot 100 ft. to the inch use the 50 side and double the value of each division, i.e., 1/50 in. = 2 ft. To plot 200 ft. to the inch, let each 1/50 in. = 4 ft.; 300 ft. to the inch, 1/60 in. = 5 ft., or 1/30 in. = 10 ft. and so on.

For additional suggestions for using the scale see Art. 618.

608. Use of the Ruling Pen.

1. Never begin to ink a drawing until the ruling pen can be relied upon to produce a clear clean-cut line. If good ink is used and the pen is properly held, a ragged line indicates that the nibs are not exactly of the same length or that they are not sharp. A nib which is too short or too dull will leave *its* side of the line more or less broken. The best of pens become dull with use, and the draftsman should keep his pen sharp, but not sharp enough to cut the paper.

2. Hold the pen almost perpendicularly, *thumb screw out,* both nibs pressing evenly on the paper, and with the top inclined *slightly* in the direction in which it is moved. (The tendency is to incline it too much.) Acquire a certain "touch" which brings the pen *lightly* but *firmly* in contact with the ruling edge. Endeavor to get into the easiest position in which to ink a line, even though it is necessary to walk around the drawing. In this way the ruling edge may be kept between the line and the body, so that in moving the pen the tendency is to draw it *against* the ruling edge rather than away from it. Thus the pen will cling to the edge without that excessive pressure which pushes the nibs together and causes the width of line to vary. Steady the hand by sliding on the end of the little finger. The pen should be moved from left to right and should be *drawn,* not *pushed.* See that there are no small particles of dust or lint in its path.

3. Adjust the nibs of the pen to the width of line most often used in map drawing, and make a fine scratch across the milled head of the adjusting screw, so that by noting the position of this scratch the nibs can be set quickly to this width of line or to one either twice as wide or half as wide. When the pen is once set at the proper width avoid changing its nibs until all lines of that width are inked.

4. Tack a cloth to the drawing desk and clean the pen frequently when in use by forcing the cloth out through the nibs without disturbing the adjusting screw. Keep the pen bright and clean

when not in use, as ink left to dry in the pen will almost surely corrode the points.

608(*a*). To SHARPEN A PEN use a thin oilstone. Screw the nibs of the pen close together, and draw the pen across the stone a few times precisely as if drawing lines, but changing the inclination of the pen from one side of the vertical to the other, so as to keep a rounded point. This will make the nibs of equal length but *dull*. Unscrew the nibs and sharpen each of them separately by rubbing its *outer* side on the stone with a rotary motion, as if sharpening a knife, taking care to hold the pen at a small angle with the horizontal—almost flat. The burr on the inside of the nibs may be removed by a stroke or two on the stone, or with a flat file.

609. Use of Dividers. Dividers are used chiefly for dividing a line into equal parts and for transferring distances from one map to another, or from one part of a map to another part of the same map. *They should not be used for transferring distances from scale to paper*—a method both slow and inaccurate. Whenever possible use the scale in preference to the dividers. Dividers should by all means have a hairspring adjustment.

609(*a*). PROPORTIONAL DIVIDERS are used to transfer measurements to an enlarged or reduced scale. Thus if the measurements are to be reduced one-third, the dividers are adjusted so that any distance included between one pair of points is always one-third the distance included between the pair of points at the other end of the dividers.

Proportional dividers may be obtained which not only enable the draftsman to reproduce lines or a drawing to any desired ratio, but likewise to reproduce the *area* of any given plane surface to any desired proportion of the original.

Proportional dividers of the better type have graduations or scales on the legs to facilitate adjustments, and, in some cases, a vernier to aid in a very fine setting of the pivot on which the legs turn.

610. Use of the Compasses.

1. The leads that come with compasses are often too soft for accurate work. Cut off two inches or so of a 6-H pencil, take out the lead, sharpen it to a fine cone-shaped point, and keep it sharp by rubbing it frequently on a file or sandpaper.

2. Adjust the pivot needle so that it will be right for the pen and do not disturb it thereafter, but adjust the lead to the pivot needle. The point of the needle should project about 1/32 in. beyond the point of the pen. The shoulder of the needle point will then be at the end of the pen point or the pencil point when the compass legs are together, and the pencil alone will need readjustment from time to time as it wears off.

3. Bend the legs of the compasses until each is perpendicular to the paper when the arc is drawn, especially in inking, for in no other way can the nibs of the pen be made to bear evenly on the paper.

4. Guide the pivot needle to place with a finger of the left hand. Be sure it is set before starting the arc, lean the compasses a little in the direction of revolution, let the handle roll between the thumb and forefinger, ink the circle with one continuous motion, and stop the pen at the end of one revolution lest the line be made uneven by further inking. The circle is generally drawn in a clockwise direction.

5. Use bow compasses for very small circles, the lengthening bar for large circles, and beam compasses for circles too large to be drawn with ordinary compasses.

6. Note that the directions for filling, cleaning, and sharpening the ruling pen apply to the compass pen also.

611. Use of the Curve Ruler. It is assumed that points have been plotted close enough together to determine the curve completely. First draw the curve in pencil, using a hairline and noting which portions of the curve ruler are used. Ink in small portions of the curve, keeping the blades of the ruling pen tangent to the curve by turning the handle, or, better still, by using a swivel curve-line ruling pen. As the curve ruler will coincide with the pencil outline for a short distance only, great care is necessary to make the different sections join well. To insure a smooth curve it is well to let the edge of the ruler coincide with a small portion of the curve beyond the points where any particular section is begun or stopped.

In inking narrow ellipses, parabolas, and similar curves with a sharp turn, the common difficulty is to avoid a slight angular break in the curve at the turn. The part of the ruler used for one side of the bend should be such that if a line were continued it would nearly coincide with the curve for a short distance the other side of the bend, so that when the ruler is reversed for the curve on the other half of the turn the angular break referred to should not occur. A *small* portion of a narrow turn can usually be drawn better with the bow compasses than with the curve ruler.

Flexible curves which can be adjusted to any curved line, and hard-rubber splines held in place by weights, are useful for inking curves, especially irregular lines and those of gentle curvature. These curves are sold by dealers in drawing instruments.

612. The Use of the Pantograph. The pantograph is an instrument for copying drawings to the same, a larger, or a smaller

scale. The theory is based on the following proposition: If any parallelogram is jointed at the corners, as *A, C, D,* and *E* (Fig. 612), and the four sides (produced if necessary) are cut by a straight line, the four points thus obtained will continue to lie in a straight line no matter how much the angles of the parallelogram may be changed. Moreover, the distances between these points will remain in a constant ratio to each other.

In the pantograph, Fig. 612, two of the sides are cut by the straight line at their intersection *A*, and the other two are cut at *F* and *B* respectively. Hence if *F* is fixed and *A* is moved to any point as *A'*, the point *B* will move to a point *B'* in a straight line

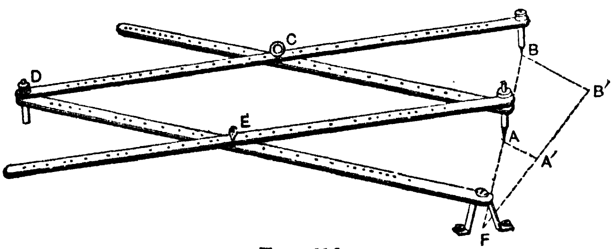


FIG. 612.

with *F* and *A'*. Moreover $FA : FB :: FA' : FB'$. Hence $AA' : BB' :: FA : FB$.

612(*a*). ILLUSTRATION. Suppose the parallelogram is adjusted at the points *C* and *E* so that $FB = 2FA$. Then if a point is moved along any line *AA'* the pencil at *B* will move parallel to it along a line *BB'* twice the length of *AA'*. It matters not whether the line *AA* is straight, curved, or irregular, and hence if the point *A* is moved over any outline a pencil at *B* will reproduce a similar outline twice as large. Conversely, if a point at *B* is moved over any outline a pencil at *A* will reproduce a similar outline half as large. If *A* is made the fixed point and either *F* or *B* is moved over any outline the remaining point will reproduce a similar outline of the same size, but in an inverted position.

612(*b*). TO SET THE PANTOGRAPH. (1) If the original drawing is to be enlarged, put the pencil at *B* and the tracing point at *A;* and, vice versa, if the drawing is to be reduced, put the pencil at *A* and the tracing point at *B*. (2) Whatever the setting, remember that the portion *ACDE* must form a parallelogram and that *F, A,* and *B* must lie in the same straight line. (3) Various pantographs have different systems of marking the arms for purposes of setting. The best test, however, is to measure the distances *FA* and *FB*. Thus to enlarge three times, *FB* must equal $3FA$; to enlarge 1½ times, $FB = 1\frac{1}{2}FA$; to reduce 1¾ times, $FB = 1\frac{3}{4}FA$, and so on. (4) To reproduce exact size, make *A* the fixed point, and $FA = AB$. The pencil may then be put at either *F* or *B*.

612(*c*). ANOTHER FORM OF PANTOGRAPH is shown in Fig. 612(*c*). Two sides of the parallelogram *CDFE* are cut by the straight line at *A* and *B*; the other two sides are cut at *F*. These three points, therefore, are always in the same straight line and may be used exactly as the three corresponding points of the pre-

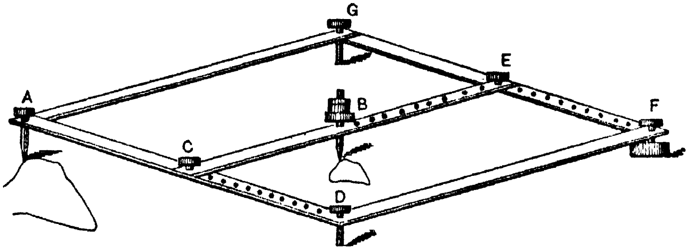


FIG. 612(*c*).

ceding form are used. The portions *EG* and *GA* serve simply to stiffen the arm *CA*.

612(*d*). REMARK. In both the forms shown one of the three working points is at the intersection of two sides of the parallelogram. This is by no means essential so long as the three working points lie in a straight line at the intersection of at least three sides of the jointed parallelogram. It is evident therefore that there may be an indefinite number of different forms of the pantograph. The two shown are inexpensive and are not intended for accurate work. To attain any considerable degree of accuracy one of the more expensive metal pantographs must be used. Such pantographs may be obtained in forms which vary from a simple type to the more complicated and accurate "precision pantograph."

CHAPTER 44

DRAFTING-ROOM METHODS *

In this chapter general directions and suggestions are given for penciling and inking, for mixing inks, for laying off measurements to scale, for section lining, for tinting, for tracing, and for making blueprints.

613. To Fasten the Paper to the Board. Insert a thumbtack in one corner, make the paper square with the board by means of the T-square, stretch it *diagonally* across to the other corner and insert a tack; stretch the paper diagonally in the other direction and insert tacks. It is not essential that the paper should be square with the board, but take pains to stretch it flat and smooth. In large sheets it will probably be necessary to put additional tacks along the edges. Press the tacks down until the heads are flush with the paper, so that the T-square can slide over them. If a tack should interfere with the work at any time, it may be temporarily removed. The paper, if much smaller than the board, may be placed at the lower left-hand corner several inches from the bottom.

Drafting tape may be used instead of thumbtacks; it will adhere to the paper and to the drawing board yet can be easily stripped from each without leaving traces of its use. Short pieces at the corners of the sheet, and along the edges if necessary, will hold the paper firmly to the board. The advantages are that no holes are made in the board and that the T-square and triangles slide over the tape more easily than over thumbtacks.

613(*a*). Note. In certain kinds of work the best results can be obtained only by stretching the paper while it is damp. Moisten the whole sheet with the exception of a half-inch border. Secure this dry border to the board with mucilage, beginning at the centers of opposite edges and leaving the corners until last. Use a sponge in smoothing the paper, working from the center out, and see that the mucilage is set before the body of the paper dries, so that the latter may be stretched uniformly by its own contraction. Cut the paper from the board when the map is finished, following the edges around successively in order.

If possible keep the paper in flat sheets. If kept in rolls it is more difficult to make it lie smoothly on the board.

614. Precautions to Insure Neatness. (1) Wipe the triangles and T-square free from dust before beginning work—occasionally

* This chapter is taken largely from a similar chapter in the author's *An Introductory Course in Mechanical Drawing,* American Book Co.

the triangles should be washed. (2) Keep the paper clean by frequently applying a brush or a cloth. After the eraser has been used never proceed with the work until the leavings have been brushed from the surface of the paper. (3) After sharpening a pencil on sandpaper wipe the lead clean with a cloth. Keep the sandpaper away from the drawing. (4) Never use a pencil soft enough to rub or blur.

615. Penciling. (1) *Use a hard pencil and draw hairlines.* (2) Draw accurately in pencil, for accuracy is seldom, if ever, increased in the inking in. (3) Do not try to stop pencil lines at exactly the right place, for just as much care is required to make the ink lines stop where they should, whether the pencil lines do or not. It saves time and usually makes intersections and corners more definite to let the pencil lines overrun *slightly,* the projecting ends being erased *after the whole drawing has been inked.* Likewise draw a pencil line a little longer than the distance which is to be marked off on it. (4) It is often better not to draw a construction line throughout its entire length. For example, to find where two lines intersect, draw an inch or so of one line near the estimated point of intersection, and simply cross it with the other. In drawing lines to plot corners of houses or other details the approximate positions of points may be estimated and only so much of each line drawn as is really necessary. Many similar cases occur where time would be wasted and unnecessary erasing caused by drawing entire lines.

616. Inking.

1. As a rule, do not begin to ink until the map is practically completed in pencil.

2. Make the lines wide enough to be clear and distinct. The tendency with beginners is to set the nibs of the pen too close together, making the lines too fine and often ragged. Keep a *separate* piece of paper or tracing cloth at hand for testing the pen to see if lines are of the right width.

3. Hold the pen nearly vertical, both nibs bearing evenly on the paper. (See Art. 608.)

4. Keep the pen sharp and in good condition, so that it will begin to ink a line the instant it touches the paper; no pressure should be required.

5. Do not ink too close to the edge of the T-square or triangle, but place the ruling edge so that it does not quite coincide with the pencil line. When the line is inked, remove the T-square or triangle by drawing it *away* from the line to avoid blotting or blurring.

6. Get into an easy position for inking each line, keeping the ruling edge between the line and the body.

7. Never push a pen backward over a line. It is difficult to patch up a ragged line, and it should not be necessary if the pen is in proper condition. Occasionally it may be necessary to go over a line a second time, but the pen should be moved *in the same direction as before,* great care being taken not to increase the width of line.

8. Ink in circles and arcs of circles first, as it is easier to draw straight lines tangent to circles than to draw circles tangent to straight lines.

9. Ink in *top* or *left-hand* lines first and work from the top down or from left to right to avoid waiting for lines to dry. In inking in small details, place one triangle so that the lines to be inked lie within the open space in the center. A second triangle can then be laid across the first and used as a ruler in any direction without blurring wet lines.

10. When several lines meet in a point, if possible ink *from* and not *towards* the point, otherwise too much ink will gather at the intersection of the lines. Allow one line to dry before inking another. The same precautions are necessary in inking acute angles.

11. In inking wide lines, as, for example, a heavy border, draw several parallel lines of medium width about their own thickness apart and fill in between with another set of parallel lines after the first are dry. This is better than drawing two outside lines and filling in between all at one time—a method that often leaves a large quantity of ink which warps the paper and takes a long time to dry.

12. Finally, while inking keep the surface of the paper clean, watch for particles of dust or lint in the path of the pen, and clean the pen frequently.

616(*a*). To mix India ink. Some draftsmen prefer to mix their own ink from Chinese stick ink when very fine and sharp lines are required. This is done as follows. Place as little water in the saucer as will suffice for the amount of ink required. Rub the end of the stick on the bottom of the saucer with a rotary motion until the mixture is of the desired density. The liquid will appear black before it really is. To get it just right test it in the inking pen—it should flow readily and dry quickly. The resulting line should be jet-black, and not easily erased or blurred when dry. If the line fades as it dries add more ink to the mixture from the stick. If it lacks the other desired properties try another make of ink. Chinese inks vary in quality, but it does not follow that the expensive kinds are the best. Try different sticks until one is found which is satisfactory. Wipe this dry after using to prevent it from crumbling, and it will last for a long time. Observe the followng precautions. Clean the saucer before using; avoid dust or undissolved particles of ink in the mixture—they clog the pen; keep the mixture covered when not in use.

617. Erasing. (1) Avoid erasing pencil lines until the map is inked. If the eraser must be used, as, for example, to correct the pencil drawing, brush the surface of the paper *clean* before doing anything else. (2) If the eraser does not of itself wear off fast enough to keep clean, rub it occasionally on the drawing board. (3) Avoid erasing *across* ink lines, rub in *between*. Much erasing with the pencil rubber will deaden any ink line, destroying the sharp black effect. (4) In erasing ink lines use an ink eraser, but do not press too heavily on the paper, as it injures the surface for further inking. For the same reason a knife should be used sparingly, though it is useful for taking out short portions of a line to make it "dashed" when it has been inked in full by mistake, or for removing the surface of a blot before applying the eraser. (5) If the surface of the paper has been injured in erasing, rub briskly with a clean hard substance until the paper becomes "polished" and ready for the ink. On some kinds of paper it may be well to apply soapstone to the surface before polishing. (6) To erase part of a drawing and leave adjacent portions untouched, cover the part not to be erased with a piece of stiff paper or a triangle. Better still, use an erasing shield made of sheet metal through which numerous narrow slots of various shapes have been cut. In lieu of such a shield, a flat piece of celluloid or cardboard with a narrow slit in it makes an excellent erasing shield for portions of lines, the rubber being applied through the slit. (7) A sponge rubber or crumbs of wheat bread can be used to clean the drawing when it is finished. (8) For erasing and cleaning tracings, see (12), Art. 621.

618. Laying Off Measurements to Scale.

1. Use a needle for pricking off distances. A handle can easily be made of wood or sealing wax. Make prick marks small, and draw freehand circles around them, if necessary, to keep track of the points.

2. Get the eye opposite the mark on the scale before pricking the point.

3. Do not lay off distances with the compasses or dividers when the scale can be used.

4. Avoid arithmetic work; note what the smallest division on the scale represents, and read directly from the scale. (See Art. 607.)

5. In laying off several successive distances along a line, set the scale once for all instead of moving the zero point up each time to the end of a space in order to lay off the next space. In the latter method an error in a single space affects all succeeding spaces, and thus errors will accumulate. For this reason it is better to adopt the system of plus distances in field work. (See note, Art. 322(*c*).)

6. Ordinarily it is not necessary to plot a point nearer than 1/100 in. Hence if the scale is 100 ft. to the inch, plot to the nearest half foot; 200 ft. to the inch, plot to the nearest foot, and so on. If the scale is 20 or 30 ft. to the inch, by using the "40" or "60" side the quarter of a foot may be more easily estimated, and thus distances plotted to the nearest eighth of a foot.

7. It is more accurate to mark off the radius of a circle on a straight line each side of a point, and open the compasses until the circumference of the circle passes through the extreme points, than to take the radius directly from the scale.

8. Form the habit of frequently checking measurements by the eye. Large errors may be detected usually by simple inspection. For small errors apply the scale a second time where there is the least doubt of accuracy. Other ways of applying checks should suggest themselves to one familiar with similar checks used in field work. (See Art. 658(*a*).)

9. A common mistake is to use one side of a triangular scale when intending to use another. A "scale-guard," costing but a few cents, will keep the scale right side up and help to avoid this common error.

618(*a*). Method of laying off plus stations. No subtraction is necessary. Suppose a station already plotted is 87 + 20 and it is desired to lay off the distance to station 90 + 40, using a scale of 1 in. = 100 ft. Let the zero of the scale represent station 87, and set the mark on the scale which represents 20 ft. from zero at station 87 + 20 on the paper. Then station 88 is at the first inch, station 89 at the second inch, station 90 at the third inch, and 40 ft. beyond the third inch is station 90 + 40.

A similar method may be used for any scale, although the 100-ft. stations will not fall on the inch marks to any other scale. Thus if 1 in. = 200 ft., the 100-ft. stations fall at the half-inch marks. If 1 in. = 30 ft., station 88 in the above illustration would fall at the point on the 30 scale marked 10 (see the figure in Art. 607), station 89 at 20, and station 90 at 30.

619. Section Lining. (1) As a rule draw the section lines at 45° and 1/16 in. to 1/8 in. apart—the larger the area covered the farther apart the lines. (2) Make the section lines finer than ordinary lines, usually about as fine as the pen will draw. Ink slightly thinned best serves the purpose. The common tendency is to make section lines too heavy and too close together. (3) Do not draw the lines in pencil. A draftsman should be able to draw the lines in ink, making the distances apart equal by the eye alone, but the

use of one of the standard section liners lessens the strain on the eyes and is a surer method of securing a uniform appearance.

620. Tinting.

1. Water colors are sometimes used in tinting maps, although not so much as formerly. To mix the tint, place as much water in the saucer as will suffice for the amount of tint required. Transfer tint from the cake to the water with a brush, or rub the cake on the finger end held in the water until the mixture is of the required density.

Note. Use a camel's-hair or sable brush similar to Keuffel and Esser's No. 3121–8N to 12N. A nest of saucers for mixing different tints will prove convenient.

2. The surface to be tinted must be clean. Avoid as far as possible all pencil lines within the area to be tinted. If drawn and not erased they can be seen through the tint; if they are erased the surface of the paper is often injured and a smooth, even wash cannot be laid on. It is well to tint before inking. If the drawing is inked first, waterproof ink must be used. The paper, of course, should be stretched flat and tight on the board; if necessary the edges may be glued. (See Art. 613.)

3. Raise the rear of the board slightly so that the surface will be sloping. If the top of the drawing table cannot be inclined, block up the board. Fill the brush from the *surface* of the tint to avoid specks in the bottom of the saucer. Small particles of coloring matter transferred to the paper will dissolve and leave small blots. Apply the brush moderately full of tint to the upper left-hand corner of the area and draw towards the right. This should leave a "pool" of tint extending across the area. This pool should contain enough tint so that it will almost but not quite run down the paper of itself. The secret of success is to *coax* this pool down the paper, keeping it moving constantly by applying the brush back and forth from side to side, not allowing the pool to stand in one place longer than in another. The pool must be kept full by adding tint from time to time with the brush. The tint *must not be painted on;* it is not even necessary to touch the brush to the paper. The whole operation must be done rapidly; it is better to let the brush overrun a little *outside* the outline than to spend so much time on the edges that tint on the interior soaks in or dries in streaks. The tint outside the boundary line can be removed *when thoroughly dry* by covering the portion not to be erased with a shield and using a pencil eraser. In laying on the wash do not go over the same place twice to "touch it up"; touching up will usu-

ally make it worse. When the pool reaches the bottom of the area, squeeze the brush comparatively dry, and use it as one would a sponge for taking up the surplus tint, working rapidly and drawing the brush frequently between the finger and thumb to discharge the tint. The whole surface should now have a uniform flat tint, free from streaks.

4. Better results may often be obtained by first going over the surface with clean water exactly as in tinting. When the surface, held towards the light, ceases to glisten, begin to lay on the tint. The work can be done a little more leisurely and carefully than if the surface were dry, since the tint is not absorbed so quickly by the paper and therefore dries more slowly and evenly. If, on the other hand, the paper is too wet the tint will run.

5. If a deep or very dark tint is desired, it is better to put on several successive layers of a lighter shade, waiting each time for the preceding one to dry, than to attempt a single dark tint at first. The tendency is to make the colors on maps too deep and striking.

6. When different colors are used for different portions of a map, they should harmonize.

7. To correct mistakes in coloring, wash the surface with clear water, take up with a clean blotter, and repeat as many times as may be necessary, working back and forth evenly each time over the surface with a brush. It is difficult to eradicate color completely, but with patience and persistence surprisingly good results may be attained.

620(*a*). Colored pencils can be used for tinting. Sharpen the pencil to a flat chisel edge, and rub the surface back and forth with uniform pressure to produce an even shade without streaks. If a more uniform coloring is desired, the penciled surface may be rubbed with a blotting-paper stump dipped sparingly in gasoline. If this is done well, a clear tint can be obtained.

620(*b*). To tint a tracing. When a map has been traced on the smooth side of the cloth, it may be tinted by rubbing colored pencils on the reverse or rough side, using the conventional tints given in Art. 676(*a*). Hold the pencil flat to spread the tint quickly and evenly, then use an old dried-up eraser to rub the color down until it is free from streaks and as uniform as a wash tint. Too soft an eraser will take the tint off instead of rubbing it in. Finish by rubbing with a cloth. The tracing cloth will take only about so much color. Lighter tints may be obtained by rubbing less color on in the first place. For streams and bodies of water the deepest blue is near the shore lines and this is gradually worked out to almost no color in the center. Sometimes excellent

effects may be obtained by section lining in color instead of spreading colors evenly.

621. Tracing.

1. Draftsmen differ as to whether the tracing should be made on the rough or smooth side of the cloth. If much penciling is to be done directly on the cloth, the rough side is preferable. It is easier to draw clean-cut lines on the smooth side and it is also easier to erase them if necessary, though erasures are more noticeable than on the dull side. Tracings will not curl or roll up to such an extent if ink lines are on the dull side, but blueprints will be slightly clearer if the lines are on the smooth side. When different portions of the map are to be colored with crayon pencils, the ink lines should be on the glazed side and the crayon rubbed on the back or unglazed side.

2. Tear off a strip about half an inch wide from each of the two finished edges of the tracing cloth and then fasten it down, stretched tight and smooth as possible.

3. Sprinkle the surface with powdered chalk, and rub it off with a cloth. This is necessary to make the tracing cloth take the ink, especially if working on the glazed side. Specially prepared chalk in a box with perforated cover may be had from dealers, but dust scraped from ordinary school crayon answers the purpose.

4. Make sure that the ruling pen is *sharp*. It is more important than ever that the pen should be in good condition. (See Art. 608.)

5. Lines, letters, and figures should be made slightly heavier on tracing cloth than on drawing paper, in order that the blueprint may be clear. It is not unusual to find a beginner carefully putting on almost microscopic figures with a crow-quill pen. Such fine work practically disappears on the blueprint. *Figures above everything else should be bold and clear,* since they cannot be guessed at.

6. *Erasures* should be made very carefully, using a hard-rubber ink eraser to start with and finishing with a soft-rubber pencil eraser; too much erasing wears a hole through the cloth. After erasing, the smooth surface should be restored before inking by rubbing with soapstone; otherwise the ink is likely to spread or blot. Many expert draftsmen prefer pumice powder for making erasures on the glazed side of the cloth. Dip the finger into the powder and rub over parts of the drawing which are to be erased, replacing the discolored powder at frequent intervals until a perfectly clean surface is obtained. Of course a rubber may be used in place of the finger. The advantages of this method are that it

is quick, erasures are less noticeable, and the surface is left in good condition for inking. Many draftsmen claim that pumice powder if vigorously applied is better than prepared chalk for preparing the surface of tracing cloth to take sharp unbroken lines, and that in addition the gloss is so effectually removed that pencil lines for alterations may be readily drawn on the glazed side.

7. If a considerable area of a tracing needs to be altered, it is possible to cut out a portion of the cloth and insert a new piece in its place, gluing the corners or edges. This can be done very neatly.

8. Tracing cloth is very susceptible to moisture, and if left on the board for more than a day is likely to become ruffled and uneven from expansion. All important lines, therefore, should be traced the same day, if possible.

9. If tracing cloth is of poor quality, the ink is likely to strike through and spread; such cloth is useless for tracing.

10. Red ink and colored inks in general do not give pure white lines on blueprints, but they are used for dimension lines and other portions of a drawing which it is desirable to render less prominent. Never use anything but black ink for dimensions themselves or for other important data on a map.

11. Tracing paper is often used, and is very satisfactory though less durable than cloth. Waterproof ink is best for drawing on either.

12. *To clean a tracing,* rub the surface with a cloth saturated with naphtha or gasoline (benzine). This will remove pencil marks and dirt without injuring the cloth or affecting ink lines if the ink is waterproof. It is an excellent method also of freshening the color of old and much-handled tracings.

Remark. Do not allow water to touch the tracing. A single drop will make a spot which will show in the blueprint. Ink lines may be entirely removed from spoiled or useless tracings, however, by soaking or boiling the cloth in water, leaving a clean white linen which is useful for many purposes in the office.

622. Methods of Reproducing Drawings. There are many forms in which drawings may be reproduced, among which the following are the most common: (1) blueprints, (2) brownprints, (3) ozalid prints, (4) blackprints, (5) photostats, and (6) reproductions by certain offset processes. Only the first four will be described here. (For photo reproduction, see Art. 684.)

622(*a*). BLUEPRINTS from tracings are easy to make and are used more than any other form of reproduction. Where blueprinting is carried on continually, an electric blueprinting machine is generally used. When only a relatively few prints are needed, it is

generally cheaper for the engineer to have the work done by a commercial company than to make the prints himself. Under certain conditions, however, it may be necessary or better for the engineer to do the printing, and, for this reason, the process is explained here.

The process is this. (1) Trim as many sheets of blueprint paper as there are prints required, each sheet slightly larger than the tracing, but not too large to go inside the frame. Keep the sheets away from the light. (2) Turning the frame until the glass side is down, remove the back boards and the pads. (3) Put the tracing in the frame *with the ink lines next to the glass.* (4) Place the blueprint paper *with the yellow or sensitized side next to the tracing.* (5) Insert the pad, taking pains to make the tracing lie flat, and insert the back boards. (6) Turn the frame over; make sure that the tracing is free from wrinkles, and that the ink lines are next to the glass. If they are not, words and numbers will read backward. (7) Expose squarely to the sun, making sure that no shadows fall across the glass. (8) After exposure, remove the paper and wash thoroughly in water until the greenish solution entirely disappears. (9) Allow the paper to soak for at least 10 min.—20 are better—and then hang it up to dry.

Practical Suggestions. (1) The length of time required for exposure varies for different kinds of blueprint paper. In a bright sun with quick paper 20 to 30 sec. will suffice; for medium paper 1 to 3 min.; for slow paper 3 to 6 min. The safest way is to test a new roll by exposing small strips until the correct time is ascertained. The same test should be made when it is cloudy, or when the paper is not exposed to the direct sunlight, for of course the weaker the light the longer the exposure. In an underexposed print the blue will be too light; in an overexposed print the blue will be too dark and the lines will appear bluish instead of pure white. (2) To get a good print it is important to keep hairs and specks from being reproduced. The stray hairs from the felt should be carefully dusted from the glass. The paper and the tracing should be in perfect contact, pressed smooth and flat against the glass. If wrinkles appear when the frame is turned over, take the back boards out and fix the tracing cloth. Sometimes the trouble is caused by too weak or too short springs on the back boards or by too thin a pad between the back boards and the paper. (3) Blueprint paper may be exposed to the light in an ordinary room for several minutes without being affected, but it should not be subjected to strong light through a window. It is safer, especially with very sensitive papers, to pull the window shade down and work in subdued light while placing the paper in the frame. (4) In washing the print make sure that is completely immersed, and move it back and forth or splash water over it so that the chemicals will be removed from all parts of the paper as quickly as possible. (5) Keep the tracing where no water can spatter on it, and do not handle it with moist fingers. A drop of moisture on the cloth will make a spot which cannot be taken out, and which will show in every subsequent print. (6) Keep blueprint paper in a dark dry place until wanted for use. It must be fresh to get the best results; with old paper the lines will be gray instead of white. (7) Various white-ink mixtures and bleaching solutions are used for making corrections on blueprints. A weak solution of quicklime or caustic soda and water, well shaken, is fairly satisfactory. Alterations may also be made or white lines may be drawn by using potassium acid oxalate (a poison) as ink. White lines may also be drawn by using Chinese white water color, and red

lines by using scarlet vermilion ink (Art. 669). To obliterate white lines or figures go over them with a blue pencil.

Note. Print frames can be purchased from dealers in engineers' supplies. Spring clips are useful for hanging up prints.

To prepare blueprint paper. Prepared paper for blueprints can be purchased in rolls of different widths at such a low price and of such excellent quality that it hardly pays to prepare it for oneself, unless a large quantity is required. If it is desirable, however, to make it, the chemicals may be used in the following proportions: 1 oz. (weight) prussiate of potash in 5 oz. water; 1 oz. (weight) citrate of iron and ammonia in 5 oz. water.

These two solutions may be prepared separately in the light, but they should be mixed (in equal parts) in subdued light or in a darkened room. Apply an even coating to one side of hard white paper with a soft sponge or a broad thin brush and hang the paper up to dry. The proportions of the chemicals in the blueprint mixture may be varied considerably without any effect except to change the time required for exposure. The larger the proportion of citrate of iron and ammonia the quicker the paper.

Blueprint cloth may be prepared in the same way as blueprint paper.

622(*b*). BROWNPRINT PAPER is a process paper for making positive prints, i.e., dark lines on a white background. The process is similar to blueprinting, except that the tracing is used to print a negative on the solar paper, and this negative is used to make positive prints on either solar or blueprint paper.

To make the negative place the tracing in the frame with the ink lines *against* the process paper instead of against the glass. Expose until the paper which projects from underneath the tracing is a dark tan color. This will take 2 to 5 min. in bright sunlight in summer, nearly twice as long in winter. Wash for at least 5 min. in clear water and then immerse in a fixing bath composed of 2 oz. of hyposulphite of soda to 1 gal. of water. The negative should be left in the bath just long enough for the lines to become white and the background a dark brown. If left too long it may become mottled. Some prefer to apply the fixing solution with a brush, especially if only a few prints are desired. After the negative is fixed it must be washed thoroughly to avoid "hypo" stains. Let the print soak in clean water for at least 20 min.—running water is desirable though by no means essential. Hang the negative up, and when it is practically dry, roll it tightly on a small round stick (such as comes in a roll of tracing cloth), so that the paper negative will be smooth for printing purposes. Any number of prints can now be made from this negative, while the tracing can be filed away and kept in good condition, one advantage of this process. If a large number of positive prints are required more than one negative can be made so that several positives can be printed at the same time, another advantage.

To make positive prints place the negative in the frame with the brown side next to the sensitized side of the printing paper, not against the glass. To make blueprints proceed exactly as outlined in Art. 622(*a*), except that the negative is used in place of the tracing, the result being blue lines on a white background. By using solar paper instead of blueprint paper, brown lines on a white background may be obtained, but these prints must be treated with a fixing solution as explained above. In either case corrections in pencil or ink can be made on the white background, another advantage as compared with ordinary blue-process paper. Corrections on the negative itself may be made by blocking out unnecessary lines with black ink or by inserting white lines by means of a bleaching solution furnished by the maker.

622(*c*). OZALID PAPER has a surface that is chemically sensitive to light, but after exposure it is developed or fixed in ammonia fumes. The result is a positive print with dark brown lines on a light cream background. Since it is developed in a gas, the paper does not shrink, and the print can, therefore, be scaled. Moreover, since it is a positive print it is much easier to read than a blueprint.

622(*d*). BLACKPRINT PAPER. This is a direct process paper which yields prints that have black lines on a white background. These prints are produced as positives and, therefore, no negative is required.

CHAPTER 45

METHODS OF PLOTTING ANGLES

FIVE methods of plotting angles are given in this chapter. The use of these methods in plotting maps is explained more fully in the next chapter.

623. Five Methods of Plotting Angles. (1) By the *protractor.* (2) By the *tangent* of the angle. (3) By the *cosine and sine* of the angle. (4) By the *chord* of the angle. (5) By *latitudes* and *departures* or *coordinates.*

624. The Protractor Method. There are various forms of protractors in common use. With the ordinary form, angles are laid off as follows. (1) Place the center of the protractor exactly at the point which is to be the apex of the required angle, and at the same time make the zero point fall on the given line so that the line from the center of the protractor to the zero point will coincide with one side of the required angle. (2) To get the other side, follow around the circular edge of the protractor the required number of degrees and minutes from zero, and make a fine dot or prick point. (3) Draw a line through this dot to the apex. If there are several angles to be laid off from the same line, make all the dots, if possible, before removing the protractor to draw the lines.

624(*a*). PRACTICAL SUGGESTIONS.

1. Make sure of the *exact center* of the protractor once for all by means of the following test. With a triangle and T-square or with hair dividers construct on paper two intersecting lines exactly perpendicular to each other. Place the protractor so that the 0° and the 180° are on one line and the 90° on the other. Observe carefully what point of the protractor coincides with the intersection of the two lines. This test should always be made when using any protractor for the first time. In some of the semicircular protractors the center is marked by a nick, the bottom of which is often mistaken for the center.

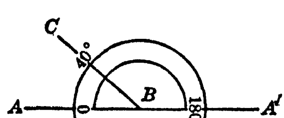


FIG. 624(*a*).

2. Accuracy depends largely on the correct centering of the protractor. Thus if in Fig. 624(*a*) an angle is to be laid off at *B* from a line *AB,* produce *AB* so that the 180° mark as well as the zero mark can be used in centering the protractor.
3. On most protractors there are two rows of numbers, the second increasing from 0° to 180° counterclockwise for use in laying off angles to the left.
4. There are three methods of laying off an angle greater than 180°: (*a*) Subtract 180° from the angle, lay off the remainder in the usual manner, and produce the line thus obtained through the center. For example, to plot 220°, mark off 40° clockwise, as shown in Fig. 624(*a*), and produce the line thus

obtained through *B*. (*b*) Subtract 180° from the angle, place the protractor on the opposite side of the line, and, considering 180° of the angle to have been plotted by simply producing the given line, plot the remainder with the protractor. (*c*) Invert the protractor as in the preceding method, but plot 360° minus the angle backward. This last method involves subtracting degrees *and minutes*, and hence mistakes are more likely to occur than if either of the first two methods are used.

624(*b*). REMARK. When a protractor is used which has a full circle graduated from 0° to 360° in both directions, an angle greater than 180° can be plotted directly instead of resorting to one of the three indirect methods just explained.

The ordinary protractor is seldom divided more finely than quarter degrees; hence anything less than 15′ must be estimated.

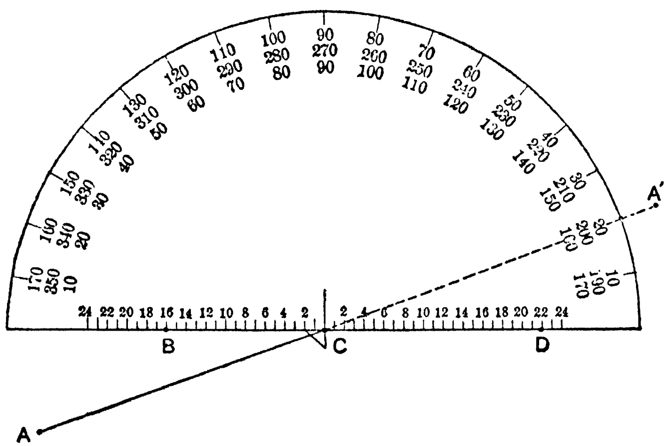


FIG. 624(*c*).

624(*c*). ANOTHER FORM OF PROTRACTOR is shown, much reduced in size, in Fig. 624(*c*). (On an actual protractor the circle is subdivided to degrees and quarter degrees, but, because of the small size of the cut, these subdivisions are not shown in the figure.) There are similar protractors on the market, but some draftsmen prefer to make their own as follows. (1) Purchase a *full circle* Bristol board protractor about 14 in. in diameter. Rule lines between 0° and 180°, and between 90° and 270°, thus getting the exact center. (2) Along the first of these lines construct a scale 1 in. = 50 ft., or 1 in. = 40 ft., or whatever scale is to be used in plotting. The zero of the scale is at the center of the protractor, the numbers increasing outward in either direction. (3) Number the 10° marks as shown in the figure. (4) Cut the circular protractor in two parts along the line from 0° to 180°, leaving, however, a small V-shaped point about ¼ in. deep as shown in the figure. Better results may be obtained by cutting clear across with a machine such as all bookbinders have. Then cut a small depression in the Bristol board, under the center, halfway through the back, and gum a thin piece of celluloid into it to form the V. The celluloid may be notched slightly to receive the needle about which the protractor revolves.

624(*d*). TO USE THE PAPER PROTRACTOR. Let is be required to draw a line *CB*, 160 ft. long, and making an angle of 20° clockwise with *CA*. Produce *AC* to *A′*, stick a very *fine* needle at the point *C*, bring the corner of the protractor (center) against the needle, turn the protractor until the 20° mark (outermost row) is on the line *CA′*, and mark *B* opposite 160.

For an angle over 180° use the middle row. Thus the same setting as above gives *ACD* 200° *clockwise*, and a point *D* any distance, say 220 ft., from *C*.

If angles *counterclockwise* up to 180° are desired, use the innermost row. Thus the same setting gives *ACD counterclockwise* = 160°.

624(*e*). REMARK. This form of protractor is exceedingly useful in plotting details where a large number of angles and distances are laid off from the same point, for it serves both as protractor and scale. It is necessary, however, to construct a separate protractor for each different scale. This, however, is not a serious drawback, since in ordinary work less than half a dozen scales are commonly used.

Notice that the required side of the angle is always found along the straight edge of the protractor. The fact that this straight edge is used as a scale offsets the disadvantage of resetting the protractor for each angle. A modified form of this protractor extends around nearly the entire circumference of the circle.

624(*f*). PRACTICAL SUGGESTIONS. (1) Fine hairlines should be ruled for the scale. Do not cut the Bristol board in two until these lines have been penciled and inked, if it is desired to ink them. Make the lengths of marks correspond to those on a regular scale. (See Art. 607.) (2) If only clockwise angles are to be plotted, it is better to omit the innermost row of numbers; otherwise print them in red ink to avoid confusion with the other two rows which may be printed in black. If counterclockwise angles larger than 180° are to be plotted, a fouth row of numbers may be added. (3) Instead of sticking a needle into the paper it is better to construct a gooseneck device like that shown in Fig. 624(*f*), the space between the needle point and the weight being large enough to permit the protractor to revolve. The needle point just touches the paper, and can be moved to any desired position by simply sliding the gooseneck. The device is easily made of brass tubing, filled with lead just over the center to increase its stability, and with a wooden plug to hold the needle.

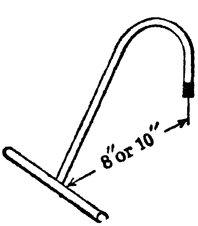


FIG. 624(*f*).

624(*g*). PROTRACTORS WITH VERNIERS. For work more accurate than can be done with an ordinary protractor, a metal protractor with a vernier may be used. The usual form is a circle or a half circle with a movable arm which projects beyond the circumference of the circle. This arm carries a vernier, the zero point of which coincides with one straight edge of the arm. By means of the vernier this straight edge can be set at any desired angle. Vernier protractors often read to minutes, but the difficulty in plotting to such a degree of precision is in properly centering the protractor. Moreover, unless the circular arc on the protractor is large, errors are likely to occur which render it nearly impossible to plot to minutes with any certainty.

625. The Tangent Method is a trigonometric method based upon the fact that in any right-angle triangle the *base* × *tangent* = *perpendicular*. The steps are: (1) Mark off on the given side of the angle, beginning where the apex is to be, a base of any convenient length, say 8 in. or 10 in. or 12 in., and at the end of the base erect a perpendicular extending toward the required side of the angle. (2) In a table of natural tangents look up the tangent of the required angle and multiply this tangent by the length of the base, 8 in. or 10 in. or 12 in. as the case may be. (When there is room the base is usually made 10 in. so that the multiplication can be performed by changing the decimal point one place to the right.) (3) Mark off the number of inches obtained from (2) on the perpendicular erected in (1) and join the point thus found with the apex.

625(*a*). When an angle is much over 45° the tangent becomes too long and the complement of the angle is plotted instead of the angle itself. If the angle is a little over 90°, plot 90° of it by erecting a perpendicular and the remainder by the tangent method, *using the perpendicular as a base.* In a similar manner any large angle can be plotted by laying off a portion of it either by erecting a perpendicular to the given line or by producing that line, so that the part of the angle remaining to be plotted by the tangent method is less than 45°.

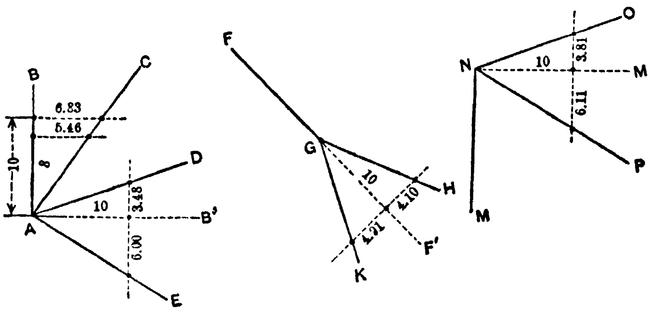


FIG. 625.

Required Angle	*Angle Plotted*	*Tangent*
BAC = 34° 20′	*BAC* = 34° 20′	0.683
BAD = 70° 49′	*B′AD* = 90° − 70° 49′ = 19° 11′	0.348
BAE = 120° 58′	*B′AE* = 120° 58′ − 90° = 30° 58′	0.600
FGH = 157° 42′	*F′GH* = 180° − 157° 42′ = 22° 18′	0.410
FGK = 206° 09′	*F′GK* = 206° 09′ − 180° = 26° 09′	0.491
MNO = 249° 08′	*M′NO* = 270° − 249° 08′ = 20° 52′	0.381
MNP = 301° 26′	*M′NP* = 301° 26′ − 270° = 31° 26′	0.611

The above examples indicate how the method is used. Notice that in every case except the first the angle actually plotted was not the required angle but the *smallest* angle that could be found with the required side either by producing the given side or by erecting a line at right angles to it.

625(*b*). PRACTICAL SUGGESTIONS. (1) The accuracy of this method depends very largely upon the correct laying off of perpendiculars. If a triangle is used it should be tested for its right angle. (See Art. 606.) In using the triangle to erect a perpendicular it is better to resort to one of the sliding methods (Art. 606) than to depend on fitting two sides of one triangle to a 90° angle. A steel square, however, may be used in that way, and is preferable to a triangle, but it also should be tested. (2) The inches laid off for the base and perpendiculars are of course *real* inches and not to any scale. As scales are not usually divided to hundredths of an inch, use the side with fiftieths. Thus in the first illustration 83 hundredths equals 41½ fiftieths. (3) If there is not room for a 10-in. base or its corresponding tangent, use a shorter base, say 5 or 8 in., *but be sure to use the corresponding length* for the perpendicular. Within reasonable limits, however, the longer the base the more accurate will be the angle, hence a 20-in. base is sometimes used. (4) After plotting an angle by the tangent method always check it roughly with the protractor. (5) If an angle is known to be wrong and no mistake can be found in the work, plot the angle over again, using a *new* length of base and its corresponding length of perpendicular. In addition it may be well to plot from some other direction, i.e., plot the complement or supplement of the angle if possible, thus using an entirely different base line. This is usually equivalent to checking by measuring the cotangent

(6) This same method can be used in the field for laying off angles with the tape. Measurements are then in feet instead of in inches.

625(*c*). TANGENT PROTRACTOR. A useful type of scale by means of which angles can be plotted by the tangent method without determining the numerical values of the tangents. The scale is subdivided into tangent distances corresponding to angles of multiples of 10′ between 0° and 45°. The subdivisions are numbered, however, to correspond to values of angles. Most tangent protractors are made only for a 10-in. base, and cannot conveniently be used for any other length of base.

626. The Cosine and Sine Method is practically a modification of the tangent method. The cosine of an angle is laid off as a base, and the sine as the perpendicular. The hypothenuse is then unity—a useful check.

Illustration. To plot 34° 20′. Cosine = 0.826, sine = 0.564. Lay off 8.26 in. as a base and 5.64 in. as a perpendicular. Check by seeing if the hypotenuse is exactly 10 in.

Note. When an angle is much larger than 45° its complement or its supplement may be plotted as in the tangent method. Most of the practical suggestions given for the tangent method apply to this method also.

627. The Chord Method. In this method the angle is constructed by two intersecting arcs. The first arc is of any convenient radius, say 5 in. or 8 in. or 10 in., and is described with the *vertex* of the required angle as a center. The length of the radius of the second arc must be equal to that chord of the given angle which corresponds to the radius of the first arc.

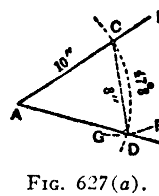


FIG. 627(*a*).

627(*a*). ILLUSTRATION. *At A to plot an angle of 47° 8′ clockwise from AB.* With *A* as a center describe an arc *CD* of any convenient radius, say 10 in. The chord for 47° 8′ for a unit radius is 0.7996 (found by one of the methods given below). For 10 in. the chord is 7.996 in., or for practical purposes 8 in. long. With *C* as a center and with a radius of 8 in. describe a second arc *FG* intersecting the first at *D,* a point which determines the angle.

627(*b*). TO FIND THE CHORD LENGTH. Take from a table of chords the chord for the required angle for a *unit length* and multiply by the length of the radius of the first arc. If a table of chords is not at hand look up in a table of natural sines the sine of *half* the required angle and multiply by *twice* the length of the radius of the first arc. (For the chord of any arc is equal to the radius of that arc multipled by twice the sine of half the subtended angle, or *chord* = *radius* × (2 × *sine* ½ *angle*) = 2 × *radius* × *sine* ½ *angle*. Draw a figure showing this to be true.)

Illustration. In Fig. 627(*a*) one-half the required angle is 47° 8′ ÷ 2 = 23° 34′ natural sine of 23° 34′ = 0.39982, and 20 × 0.39982 = 7.9964 = length of chord. (8 in. is near enough.)

627(*c*). PRACTICAL SUGGESTIONS.

1. The accuracy of the work depends upon the precision with which the arcs can be drawn and also upon how definite the point is in which the arcs

intersect. An arc of 10 in. radius can be drawn with ordinary compasses if the lengthening bar is used, but it takes time to set the compasses at *exactly* 10 in., and it is sometimes difficult to keep them there until the whole arc is completed. A better way is to keep beam compasses set at *exactly* 10 in., and to use them only for drawing the *first* arc of each angle. Then, instead of describing a second arc, find the chord by trial with a scale. Place the zero of the scale where one end of the chord must be, and knowing the length of the chord, fit the scale in place and thus find the point on the first arc where the other end of chord must be. If beam compasses are not at hand, use a straight stick with a needle at one end for a pivot, and a *very hard* lead at the other, sharpened to a *fine* point. Be sure that the distance between the needle and the point of the lead (which should be *exactly* 10 in.) cannot change while describing an arc. By exercising a little ingenuity beam compasses of this kind can be made with a device for adjusting the lead slightly to keep it exactly 10 in. from the needle point. All these precautions for drawing the arcs accurately are useless, however, if the arcs intersect so obliquely that the point of intersection is not definite.

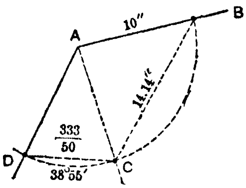


FIG. 627(*c*).

2. If a radius of 5 in. is used the natural sine of half the angle is multiplied by 2 × 5 = 10, which simply changes the decimal point. The objection to this, however, is that for accurate work the first arc should have a radius greater than 5 in. If 8 in. or 9 in. or 12 in. is used it is more convenient to multiply the sine by two and then by the radius 8 in. or 9 in. or 12 in., as the case may be. When there is room, however, use a radius of 10 in., multiply the sine by 2, and change the decimal point one place. If the fiftieths on the scale are used it is not necessary to multiply the sine of half the angle by two. Simply change the decimal point *three* places, call the result fiftieths, and read directly from the scale, adding of course a cipher to the number printed on the scale.

Illustration. At A to plot an angle of 128° 55′ *clockwise from AB.* From *A* as a center describe an arc of 10-in. radius. Plot 90° of the required angle by erecting the perpendicular *AC*. Half the remaining angle is (128° 55′ − 90°) ÷ 2 = 19° 27′.5. The sine of 19° 27′.5 is 0.33312. Hence with the zero of the scale at *C* find the point *D* where the 333 mark (fiftieth) comes, i.e., the required chord is 0.333 × 2 × 10 = 666/100 = 333/50. (This is really 6.66 in.)

627(*d*). LARGE ANGLES. The largest angle that can be laid off, using the ordinary scale, is 73° 46′, for the sine of half that angle is: sine of 36° 53′ = 0.60019. Hence the chord equals 600 fiftieths or the limit of a 12-in. scale. For any angle between 73° 46′ and 90°, erect a perpendicular and plot the complement of the required angle. In a similar manner any angle, no matter how large, can be plotted by either erecting a perpendicular if the angle is nearest 90° or 270°, or by producing the original line if the angle is nearest 180°. See Fig. 625 for parallel cases.

627(*e*). ERECTING PERPENDICULARS. If two sides of a right triangle are each 10 in. long, the hypotenuse is 14.142 in. long, i.e., the chord of 90° is 1.4142. Hence, if desired, perpendiculars can be erected by the chord method, as indicated in Fig. 627(*c*). Keep a straightedge with two fine scratches 14.14 in. apart especially for that purpose. It is convenient to have these marks on the

beam compass or on the straight stick used in place of the beam compass.

628. The Latitude and Departure Method consists in plotting points by coordinate distances corresponding to latitude and departure. While it is seldom used for laying off a single angle, it is of great value in plotting a series of connected lines, especially if they form a polygon, as, for example, the traverse lines of a survey which closes. This method will be explained in detail in the next chapter.

629. Methods Compared.

1. *The ordinary protractor* should not be used for plotting lines much longer than the radius of the protractor, if more than medium accuracy is required. With the expensive vernier protractors angles can be laid off to the nearest minute, but ordinarily 15′ is as close as the protractor can be read. The protractor is used for plotting short lines to relatively unimportant points, as, for example, in plotting the angles at a transit station to the corners of buildings, fences, and other points a short distance away. For the traverse lines some other method is usually used unless a vernier protractor is at hand.

2. *The tangent method, the cosine and sine method,* and *the chord method* are all good methods, and each has its advantages over the others. In the first two methods the difficulty lies in quickly and accurately erecting perpendiculars; in the chord method it lies in drawing arcs correctly and making their intersection perfectly definite. All three methods require the use of trigonometric tables, with the attending danger of blunders in looking up the values of functions.

3. *Latitude and departure method.* See Art. 639 for advantages and disadvantages.

630. The Three-Point Problem. Given three points, *A, B,* and *C,* whose relative positions are determined by the two distances *n* and *o* and the *interior* angle *M*. Let *N* and *O* represent the values of the angles at *P*, a point to be plotted.

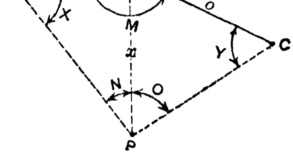


Fig. 630.

630(*a*). Graphic method. (1) Plot the two angles *N* and *O* on tracing cloth, making the lengths of the lines *PA, PB,* and *PC* indefinite. (2) Shift the tracing cloth over the drawing until each of the three lines passes through the plotted position of the corresponding point. (3) Prick through the point *P*.

A three-arm protractor should be used in place of the tracing cloth if there are many points to be located. Each arm is set to correspond to a line radiating from P.

630(*b*). PLANE-TABLE METHODS. (See Art. 447.)

630(*c*). ALGEBRAIC METHOD. Let $R = X + Y = 360° - (M + N + O)$. Then

$$x = \frac{n \sin X}{\sin N} = \frac{o \sin Y}{\sin O}$$

$$Y = R - X$$

$$\sin Y = \sin R \cos X - \cos R \sin X$$

$$\cot X = \cot R \left(\frac{n \sin O}{o \sin N \cos R} + 1 \right)$$

The angle Y may be found in a similar manner. Plotting these two angles, the lines AP and CP intersecting determine P.

630(*d*). NOTE. The problem is indeterminate when A, B, C, and P are on the circumference of a circle, i.e., when $M + N + O = 180°$. When $N + O = M$ or nearly, the solution is impossible.

CHAPTER 46

METHODS OF PLOTTING TRAVERSES

FOR every method of running traverses explained in Chapter 21 there is a corresponding method of plotting. Distances between stations are laid off to scale and angles or bearings are plotted by one of the methods given in the preceding chapter. The different steps for each method are outlined in this chapter, without entering upon the practical details of mapping. Such questions as the choice of scale, the general arrangement of the map, the accuracy required, and the best methods of plotting details are discussed in the next chapter.

631. Different Methods of Plotting Traverses, corresponding to the methods of running transit lines given in Art. 300, are:

I. { 1. *Plotting each traverse line by a* DIRECT ANGLE,
  2. *Plotting each traverse line by a* DEFLECTION ANGLE,

II. { 3. *Plotting each traverse line by its* AZIMUTH,
  4. *Plotting each traverse line by its* BEARING.

In addition there is a fifth method:

5. *Plotting traverses by* RECTANGULAR COORDINATES OR LATITUDES AND DEPARTURES.

In each of the first four methods the angle at each station may be plotted by means of the *protractor,* or by the *tangent method,* or by the *chord method,* or by the *cosine* and *sine method.*

632. Importance of Traverse Lines. Since the traverse lines form the basis of the whole map, they must be plotted with the greatest care, using fine hairlines. On a small scale map, say 1 in. = 200 ft., a line drawn with a soft or poorly sharpened pencil may easily be 1 to 2 ft. wide measured to scale; hence use a 6-H pencil sharpened to a *fine* wedge, and a needle to prick the points.

Where a high degree of precision is required *an ordinary protractor should not be used for plotting traverse lines.* It is inconsistent to measure angles in the field to the nearest minute, and then to plot them with a protractor which reads only to half degrees, especially if the lines between stations on the map are more than, say, 5 in. long. Of course there are many cases where the protractor will give results consistent with the accuracy required, but its chief use is in plotting details, and in roughly checking angles plotted by some other method. Even with vernier protrac-

tors it is difficult to attain a high degree of precision, though angles can be plotted very rapidly with it, and much more accurately than with an ordinary protractor.

The degree of precision attained in the field is usually such that when field measurements have been adjusted to conform to known conditions, the difference between the field measurements and their adjusted values is too small to affect the plotting. However, when the adjustment of field measurements is necessary, the work is usually completed before plotting is begun, in which case it is well to plot the adjusted values instead of the corresponding field measurements (Arts. 566, 570 to 572).

633. The Use of the Protractor in plotting direct angles, deflection angles, bearings, and azimuths requires no explanation in addition to that given in Art. 624. The line from which the angle is plotted should always be extended beyond the station to aid in centering the protractor, and occasionally a line at right angles is drawn in order that the 90° mark on the protractor may be used for centering in addition to the 0° and 180° mark.

Remark. In studying the remainder of the chapter keep in mind the fact that wherever the tangent method or the chord method is used for plotting an angle, the protractor can be used also, provided a high degree of precision is not required. Likewise the cosine and sine method can be used if desired in place of the tangent method, but the two are so nearly alike that the former is not explained in this chapter.

634. Plotting Each Traverse Line from the Preceding Traverse Line. This method consists simply in plotting the angle at each station by the protractor method, by the tangent method, by the chord method, or by the cosine and sine method, the distance between stations being laid off with a scale. When the angle is a large one it may be necessary to plot its complement or supplement as explained in Art. 625(*a*).

In order to start the map some traverse line must be drawn from an assumed point, and its length laid off to scale. The direction in which it is drawn and the position of the assumed point depend upon the general arrangement of the entire map. (See Art. 655 for directions for starting the map.)

634(*a*). TANGENT METHOD

TABLE I

Angle	*Value*	*Line*	*Length*	*Angle Plotted*	*Tangent*
FAB	56° 44′	*AB*	404	90° − 56° 44′ = 33° 16′	0.6560
ABC	258° 30′	*BC*	312	270° − 258° 30′ = 11° 30′	0.2034
BCD	31° 56′	*CD*	550	31° 56′	0.6232
CDE	150° 36′	*DE*	386	180° − 150° 36′ = 29° 24′	0.5635
DEF	67° 48′	*EF*	407.8	90° − 67° 48′ = 22° 12′	0.4081
EFA	154° 26′	*FA*	454.2		

Illustration (Fig. 634(*a*)). Before beginning to plot, Table I was prepared. The first four columns were taken from the field notes; the angles are interior angles measured clockwise. The fifth column gives the angles actually to be plotted, and the last column gives the corresponding *natural* tangents. Thus the draftsman has before him all the data necessary to plot the lines. In preparing the fifth column the fact was kept in mind that the tangent method is best adapted to plotting angles of 45° or less (Art. 625)(*a*). Hence at *A* and *E complements* of the angles taken in the field were used, at *D* the *supplement*, at *B* 270 *minus the angle*, but at *C* the original angle was used, since it was less than 45°. In the original drawing from which the figure was reduced the work was as follows. Since no bearings were given, the line *AF* was assumed in any convenient direction and its length laid off to scale. A perpendicular at *A* was erected and the complement of *FAB* = 90° − 56° 44′ = 33° 16′ was plotted. Ten *real* inches were used as a base and from Table I 6.56″ was the length of the tangent. Thus a point on *AB* was found, *AB* drawn, and its length laid off to scale. The angle at *B* is nearly 270°; hence a perpendicular was erected at *B*. 270° − 258° 30′ = 11° 30′, and since 258° 30′ is *less* than 270° the angle 11° 30′ must be plotted backwards or counterclockwise from the perpendicular. The dotted lines indicate how the tangent method was used for plotting the remaining angles. The plot closes if the last line passes through *F*, *and the distance EF scales correctly*. If another check is desired the angle at *F* may be tested by producing *EF* and measuring the perpendicular at a point 10 in. from *F*. In this case the distance *EF* came out one foot too long to a scale of 1 in. = 50 ft. The tangent measured .478 in., which corresponds to 25° 32′, to 25° 33′, and to 25° 34′. The correct angle is 180° − 154° 26′ = 25° 34′.

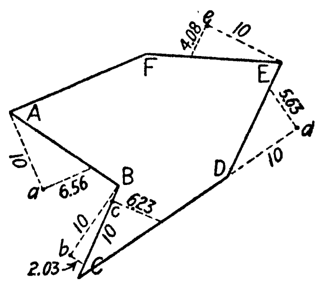


FIG. 634(*a*).

Remark. As pointed out in Art. 625, although any base may be used, 10 in. is the most convenient length. For such a base, a *tangent protractor* (Art. 625(*c*)) is a useful labor-saving device.

634(*b*). CHORD METHOD

TABLE II

Angle	*Value*	*Line*	*Length*	*Angle Plotted*	*Angle* ÷ 2	*Sine*	*Chord*
FAB	56° 44′	*AB*	404	56° 44′	28° 22′	0.4751	9.50
ABC	258° 30′	*BC*	312	11° 30′	5° 45′	0.1001	2.00
BCD	31° 56′	*CD*	550	31° 56′	15° 58′	0.2750	5.50
CDE	150° 36′	*DE*	386	29° 24′	14° 42′	0.2538	5.08
DEF	67° 48′	*EF*	407.8	67° 48′	33° 54′	0.5578	11.16
EFA	154° 26′	*FA*	454.2	25° 34′	12° 47′	0.2213	4.43

Illustration. Before beginning to plot, Table II was prepared. The first four columns were taken from the field notes and the last four were added for purposes of plotting. In preparing the fifth column the fact was kept in mind that in the chord method it is more convenient to use angles less than 73° (Art. 627(*d*)). Thus at *D* the *supplement* of the angle taken in the field was used; at *B* 270° *minus the angle* was the most convenient angle to plot. At *A*, *C*, and *E* the original angles are less than 73°. If a table of chords is at hand the sine column is unnecessary.

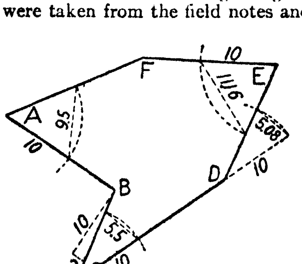


FIG. 634(*b*).

The line *AF* was assumed to start with. An arc of 10-in. radius described from *A* as a center, and the corresponding chord 9.5 in. (from Table II), gave a point on *AB*. The distance to *B* was laid off to scale. a perpendicular at *B* was erected, the angle 11° 30′ (270° − 258° 30′, see Table II) was plotted, and the direction of *BC* was thus obtained. The dotted lines in the figure indicate how the arcs and chords were used in a similar way to plot the remaining angles. If the last line passes through *F* and *the distance EF* is correct, the plot closes. The angle at *F* can be measured if an additional check is desired. The distance *EF* came out a little less than 407 instead of 407.8 to a scale of 1 in. = 50 ft. The chord measured 4.43, showing the angle to be correct.

Remark. Any radius may be used in the chord method, but 10 in. is the most convenient length, especially when the fiftieths on the scale are used as suggested in Art. 627(*c*).

634(*c*). CHECKS. Never lay off an angle by the tangent or chord method without checking it roughly with the protractor. When a traverse is plotted by direct angles an excellent check is to compute the bearings of all the lines, and at every fourth or fifth station measure the angle of bearing of the line that starts from that station. If the survey is a closed traverse, it is necessary only to check this angle roughly with the protractor; otherwise check it carefully by the tangent or chord method. (See Art. 658 for additional directions for checking traverses.)

Illustration. Let the bearing of AB = S 55° 0′ E. The calculated bearing of DE would be N 26° 02′ E. Draw a meridian at A making 55° 0′ with AB. Through D draw a parallel meridian, and see if the angle which it makes with DE is 26° 02′.

Remark. A check like the above is more necessary in a survey having a large number of stations.

635. Plotting by Deflection Angles. This method is a modification of the preceding method. It corresponds to the method of running traverse lines given in Art. 302, and is used for lines which progress more or less in one direction, as in railway work. The deflection angle at each station is plotted by the protractor method, by the tangent method, or by the chord method, preferably the latter. A mistake to be avoided is plotting the angle on the wrong side of the transit line produced. As some angles are measured to the right and some to the left, this mistake is likely to occur.

635(*a*). CHORD METHOD

TABLE III

Back-sight	*Sta.*	*Deflection Angle Right*	*Deflection Angle Left*	*Line*	*Length*	*Ang. Plot.*	*Ang. ÷ 2*	*Sine*	*Chord*
				AB	210.1				
A	*B*	34° 18′		*BC*	255.6	34° 18′	17° 09′	0.2948	5.90
B	*C*		64° 20′	*CD*	321.1	64° 20′	32° 10′	0.5324	10.65
C	*D*	76° 48′		*DE*	342.8	13° 12′	6° 36′	0.1149	2.30

Illustration. The first six columns in Table III are the field notes. The last four were added for purposes of plotting. Figure 635(*a*) studied in connection with these last four columns will illustrate the method of plotting when chords are used.

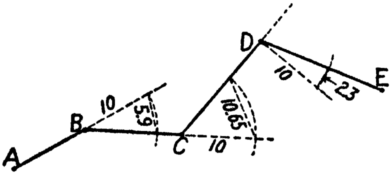


FIG. 635(*a*).

635(*b*). CHECKS are the same as those given for the preceding method. As this method is used most often for surveys that do not close, lines should be checked very carefully by calculated bearings at, say, every fourth station.

636. Plotting by Bearings. This method consists in plotting each line by laying off its angle of bearing from a line assumed as a north-south meridian. The protractor, the chord method, the tangent method, or the cosine and sine method may be used for laying off the angle of bearing.

Note. Except in compass surveying ***the bearings laid off are always calculated bearings,*** found by the method explained in Chapter 37. Sometimes these bearings are calculated from the *true* bearing of some traverse line, sometimes from the *magnetic* bearing. When no bearings are given the bearing of some traverse line is assumed, and the corresponding bearings of the others calculated. These bearings, though fictitious, will give the true *relative* positions of the different stations.

Any adjustments of angles which may be necessary (see Art. 642(*b*)) should be made before the bearings are calculated, and the adjusted values used in the calculations.

There are two general methods of procedure. (1) In the first, all bearings are plotted from the same line and afterwards transferred. (2) In the second, lines parallel to the assumed meridian are drawn through all the stations, and the bearing of each traverse line is plotted from the meridian at the station from which the traverse line starts. In either method it is convenient to assume the meridian at right angles to the T-square, i.e., a vertical line, with the north point at the top of the map. Sometimes, however, it is desired to assume the meridian in some other position, thus making the east-west line oblique to the T-square. In such cases it is convenient to use a T-square with an adjustable head so that the edge of the blade may be made to coincide with an east-west line when the head is in contact with the left-hand edge of the board. In accurate work, it may be best not to rely upon the T-square. (See *Caution*, Art. 638(*b*).)

The first method (plotting lines from the same meridian) will be explained in detail, both for the tangent and the chord method of plotting bearings.

636(*a*). Tangent method of plotting bearings

Construct near the center of the paper a 20-in. square and subdivide it into four 10-in. squares. This must be done with accuracy, and the large and small squares tested by measuring the diagonals. There is no use in proceeding further until the diagonals of the 20-in. square are equal, the sides each 20 in. long, and the diagonals of all the 10-in. squares equal to each other. Assuming the vertical line through the center of the 20-in. square to be a north-south line, each of the 10-in. squares will correspond to one of the four quadrants of bearing, NE, SE, SW, and NW.

Plot the bearing of each line of the survey by measuring ten times the natural tangent of the bearing (or if the bearing is over 45°, the complement of the bearing) along a side of that 10-in. square which corresponds to the quadrant of bearing. The *direction* of each line of the survey thus obtained may be transferred to its proper place in the plot by any of the accurate methods of drawing one line parallel to another.

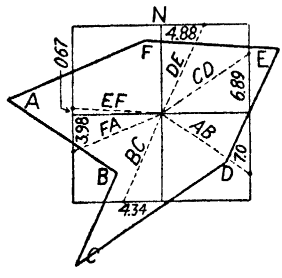


Fig. 636(*a*).

Remark. Some other length than 20 in. could be used for the side of the large square, but 10 in. is the most convenient base by which to multiply the natural tangent. For such a base, a *tangent scale* (Art. 625(*c*)) is a useful labor-saving device.

Illustration. In Fig. 636(*a*) the traverse lines in the figure in Art. 634(*a*) are plotted from bearings using the tangent method. The bearing of *AB* if not known may be assumed as any convenient bearing, and the bearings of the other lines calculated from the transit angles (Chapter 37). But in that case while the outline will be correct, none of the traverse lines will be in its really correct position with respect to a north-south line. In the illustration, however, the bearing of *AB* was known to be S 55° E. The following table was prepared for plotting.

Table IV

Line	*Bearing*	*Length*	*Angle Plotted*	*Tangent*
AB	S 55° 0′ E	404	E 35° 0′ S	0.7002
BC	S 23° 30′ W	312	S 23° 30′ W	0.4348
CD	N 55° 26′ E	550	E 34° 34′ N	0.6890
DE	N 26° 02′ E	386	N 26° 02′ E	0.4884
EF	N 86° 10′ W	407.8	W 3° 50′ N	0.0670
FA	S 68° 16′ W	454.2	W 21° 44′ S	0.3986

The letters in the fourth column tell how to lay off the tangent. Thus E 34° 34′ N means to measure the corresponding tangent (0.689) from E towards N.

Each dotted line in Fig. 636(*a*) was determined by the natural tangent of the corresponding angle in the fourth column of the table. Thus *AB* is E 35° S; hence measure down 7 in. along the right-hand side of the 20-in. square, beginning at the middle point, which represents the east. *BC* is S 23° 30′ W; hence from the middle of the lower side measure 4.34 (or better 4.35) westward. In this manner the *directions* of all the lines in the survey are ascertained.

Each line in the perimeter of the polygon may now be drawn parallel to the corresponding dotted line. A starting point *A* having been assumed (see Art. 655(*b*)), *AB* was drawn parallel to the corresponding line in the SE quadrant, and 404 ft. to scale gave *B*. Through *B*, *BC* was drawn parallel to the corresponding line in the SW quadrant, and 312 ft. to scale gave *C*. The final position of each traverse line was determined in a similar manner.

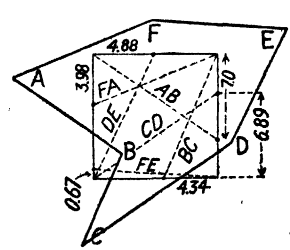


Fig. 636(*b*).

636(*b*). A modified method of plotting bearings by natural tangents is to draw near the center of the paper a 10-in. square instead of a 20-in. square, using the right-hand side for a north-

south line when a bearing is NW or SW and the left-hand side of the square for a north-south line when a bearing is NE or SE. Thus one square is made to answer for all four quadrants of bearing.

636(*c*). Chord method of plotting bearings

Proceed as follows. (1) Near the center of the paper draw a circle exactly 20 in. in diameter. (2) Through its center draw a north-south line and an east-west line. (3) Make sure that these two lines are mutually perpendicular. The four chords should be exactly equal to each other and each equal to 14.14 in. (4) Prepare a table similar to that below. (5) Plot each line in its proper quadrant by using the chord value of its bearing. (6) Transfer each line in order to its proper position in the plot by any of the accurate methods of drawing one line parallel to another.

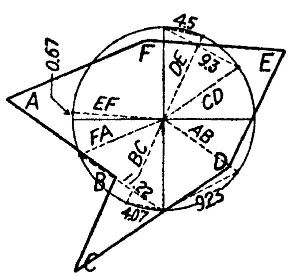


Fig. 636(*c*).

Remark. A circle of some other diameter could be used, but 10 in. is the most convenient radius, especially if the fiftieths on the scale are used.

Illustration. In Fig. 636(*c*) the same survey as plotted in the preceding examples is shown. If no bearings were taken in the field, one may be assumed for one of the lines and the bearings of the other lines calculated. In that case none of the lines will be in its correct position, but the outline of the figure will be correct, which in many cases is all that is required. In the illustration, however, *AB* was known to be S 55° E. The following table was prepared for plotting.

Table V

Line	*Bearing*	*Length*	*Angle Plotted*	*Angle* ÷ 2	*Sine*	*Chord*
AB	S 55° 0′ E	404	S 55° 0′ E	27° 30′	0.4617	9.23
BC	S 23° 30′ W	312	S 23° 30′ W	11° 45′	0.2036	4.07
CD	N 55° 26′ E	550	N 55° 26′ E	27° 43′	0.4651	9.30
DE	N 26° 02′ E	386	N 26° 02′ E	13° 01′	0.2252	4.50
EF	N 86° 10′ W	407.8	W 3° 50′ N	1° 55′	0.0334	0.67
FA	S 68° 16′ W	454.2	S 68° 16′ W	34° 08′	0.5611	11.22

The dotted lines radiating from the center of the circle in Fig. 636(*c*) were plotted by laying off the chords given in the last column of the table. Thus *AB* is S 55° 0′ E.; hence from the lower end of the vertical diameter lay off a chord easterly of 9.23 in. *BC* is S 23° 30′ W; hence lay off a chord from the same point westerly of 4.07 in. In this manner the directions of all the lines in the survey were determined.

Each line in the perimeter of the polygon was then drawn parallel to the corresponding dotted line. A starting point *A* having been assumed (see Art. 655(*b*)), *AB* was drawn parallel to the corresponding line in the SE quadrant, and 404 ft. to scale gave *B*. Through *B*, *BC* was drawn parallel to the corresponding line in the SW quadrant, and 312 ft. to scale gave *C*. The final position of each traverse line was obtained in a similar manner.

636(*d*). A modified method of plotting bearings by chords is shown in Fig. 636(*d*). The four quadrants occupy the same

space; otherwise the method is unchanged. This method may be used when a 20-in. circle will not come within the limits of the paper and it is not desirable to use a radius of less than 10 in. The dotted lines in the figure correspond to the traverse lines in Fig. 636(*c*).

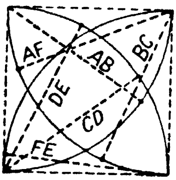


FIG. 636(*d*).

636(*c*). A SECOND METHOD OF PLOTTING BEARINGS BY TANGENTS OR CHORDS. In this method instead of plotting the bearings of all the traverse lines from *one* north-south line, the bearing of each line is plotted from a north-south line drawn through the station at the beginning of the line. Otherwise the method is the same as the preceding one. In this method errors are less likely to occur in drawing parallel lines, but more work is involved in the plotting.

Another method sometimes used for large maps is to draw a rectangle which includes as much as possible of the map and at the same time is divisible into 10-in. squares. The bearing of a line starting from any station may then be laid off in the square nearest the station and afterward transferred to the station. If the outside rectangle is carefully checked and the squares are laid off entirely by measurement, the errors due to the use of T-square and triangles are avoided. (See *Caution,* Art. 638(*b*).)

636(*f*). CHECKS. The angle at each station should be checked roughly with the protractor, or if the survey is not a closed traverse, more accurately by the tangent or by the chord method. When lines have been plotted by bearings there is no check which, if applied at any station, will check all the previous courses. This is one objection to this method of plotting, since if each angle must be checked by some accurate method, that method might as well be used to start with. The method of plotting by bearings is best suited, therefore, for closed traverses where it is sufficiently accurate to check angles with the protractor. (See Art. 658 for additional directions for checking traverses.)

637. Plotting by Azimuths. The method is practically the same as that given for plotting bearings. The only difference is that the angle plotted at each station is the *azimuth* of the next line instead of the bearing. The checks are similar to those given above.

638. Plotting Traverse Lines by Rectangular Coordinates or Latitudes and Departures. This method is by far the most practical one for plotting extensive systems of horizontal control, including triangulation systems as well as traverses. It consists in selecting one of the stations as a reference-point (origin) and

plotting each of the other stations by its total distance *east* or *west*, and its total distance *north* or *south* of the reference point (origin). These two coordinate distances by which each station is plotted are obtained from the latitudes and departures. For purposes of plotting, it makes no difference which station is chosen for a reference point. It is well, however, to select the *most westerly* station, especially as this is done when areas are computed from the latitudes and departures (Arts. 585(*a*) and (*b*)).

Note. Except in compass surveys the bearings used in computing latitudes and departures are always *calculated* bearings, based upon a magnetic, a true, or an assumed meridian, and upon the adjusted or final values of traverse angles. (See Note, Art. 636.)

Latitudes and departures may also be calculated directly from azimuths (see Art. 565), and the above holds true for azimuths as well as for bearings.

638(*a*). GENERAL METHOD OF PLOTTING BY LATITUDES AND DEPARTURES.

1. Compute and tabulate the latitudes and departures of all transit lines. (See Art. 568.)

2. Add to the table three columns for plotting, headed respectively *Station, Total Latitude, Total Departure* (p. 645).

3. Choose the most westerly station as a point of reference or origin. Its total latitude and its total departure (its two coordinates) are *both* zero.

4. With this station as an origin, compute the two coordinate distances for each of the other stations in turn as follows. Add *algebraically* the latitudes of all traverse lines passed over in going from the origin to the required point; this will give one coordinate, the *total latitude.* Add algebraically the departures of these same lines; this will give the other coordinate, the *total departure.* Enter the two results for each station in the last two columns (p. 645).

5. Find the size of the rectangle * which just encloses the survey. That is, the most westerly station will be on the left-hand side of the rectangle, the most easterly station on the right-hand side, the mostly northerly on the upper side, and the most southerly on the lower side. The largest total *north* latitude in the plotting column, plus the largest total *south* latitude, will be one side of the rectangle. Since there is no total *west* departure in the plotting column the largest total departure *east* will be the other side of the rectangle.

6. Draw the rectangle to the scale chosen for plotting the map. It is convenient to assume the rectangle with its east-west sides horizontal, but it is often best not to do so. (See suggestion 2,

* The author prefers to use this rectangle for measuring coordinates in place of two coordinate axes through the most westerly station.

Art. 638(*c*).) Check the rectangle by measuring each side, and by seeing that the diagonals are equal. The accuracy of the whole map depends upon the accuracy of this rectangle,—it is worth while to get it right to start with. (See suggestion 3, Art. 638(*c*).)

7. Locate each transit station by its two coordinate distances. *Measure from the two nearest sides of the rectangle*, subtracting if necessary values in the table from the lengths of the corresponding sides of the rectangle.

8. Connect the transit stations by fine lines and *measure the length of each traverse line* to see if it checks with the length used in calculating the latitudes and departures. (See also additional check, Art. 638(*f*).)

638(*b*). Caution. The object in measuring from the two *nearest* sides in (7) is to reduce the errors likely to arise when the T-square is used for projecting a point any considerable distance across the drawing board.

In very accurate work it is better not to rely upon the T-square and triangles but, having made sure that the rectangle is correct, to proceed as follows. Find by measurement a point on the east side of the rectangle exactly opposite the station used as the origin, and draw an east-west line from the origin to this point for use as a reference line in measuring latitudes. To plot any station, lay off its total departure twice, once along the north side of the rectangle and once along the south side. On the line between the two points thus obtained, lay off the total latitude of the station, north or south as the case may be, measured from the east and west reference line.

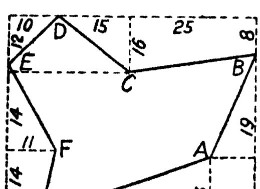


Fig. 638(*c*).

638(*c*). Illustration. For the sake of simplicity whole numbers are used for latitudes and departures in the following illustration. Since the values of the bearings and the lengths of the lines do not enter directly into the plotting, they are purposely omitted.

Line	*Bearing*	*Length*	*Latitude* *North* +	*Latitude* *South* −	*Departure* *East* +	*Departure* *West* −	*Station*	*Total Lat.*	*Total Dep.*
AB	NE		19		9		*B*	+4	50
BC	SW			8		25	*C*	−4	25
CD	NW		16			15	*D*	+12	10
DE	SW			12		10	*E*	0	0
EF	SE			14	11		*F*	−14	11
FG	SW			14		5	*G*	−28	6
GA	NE		13		35		*A*	−15	41
			48	48	55	55			

1. The latitudes and departures are first computed and tabulated.
2. The headings of the last three columns are entered as shown.
3. Running the eye down the second column, it is seen that either *E* or *G* must be the most westerly station (Art. 585(*b*)). Adding departures from *B* to *E* (25 + 15 + 10) = 50; from *B* to *G* = 9 + 35 = 44. Hence *E* is 6 ft. farther west than *G*. The coordinates of *E* are 0 and 0
4. Coordinates of the other stations from *E* as the origin are found as follows. Latitudes: $F = -14$. $G = -14 + (-14) = -28$. $A = -28 + 13 = -15$. $B = -15 + 19 = +4$. $C = +4 + (-8) = -4$. $D = -4 + 16 = +12$. $E = +12 + (-12) = 0$ (check). The departure coordinates are $F = 11$. $G = 11 + (-5) = 6$. $A = 6 + 35 = 41$. $B = 41 + 9 = 50$. $C = 50 + (-25) = 25$. $D = 25 + (-15) = 10$. $E = 10 + (-10) = 0$ (check).

5. In the column of total latitudes the largest north latitude is 12; the largest south latitude is 28. Hence one side of the rectangle is 12 + 28 = 40 ft. The other side of the rectangle is equal to the largest total departure of 50 ft. Size of rectangle is 40 ft. by 50 ft.

6. Draw the rectangle to scale, and check it by measuring its sides and diagonals.

7. Locate the origin *E*. It is on the western side of the rectangle, *a distance from the top equal to the largest* TOTAL LATITUDE NORTH = 12 ft. In this case this distance happens to be the latitude of *ED* also. Check *E* by measuring its distance from the lower side of the rectangle. It should be equal to the largest *total latitude south* = 28 ft. Locate each of the other stations by its coordinates, measuring as far as possible along the sides of the rectangle. Thus to locate *C* make a point 4 ft. below *E* on the west side of the rectangle, and another point on the north side 25 ft. from the corner, and draw projecting lines from these two points as indicated. Check by scaling the distance from *C* to the upper side of the rectangle to see if it is exactly 16 ft. For any station remote from *E* as *A*, it is usually more convenient *and more accurate* to measure from the two nearest sides.

8. *Test the length of every traverse line by scaling it.* Test also the angle *DEF*. (Why?) See Art. 638(*f*).

638(*d*). ILLUSTRATION. In Fig. 638(*d*) the traverse lines already plotted by the other methods are plotted by latitudes and departures. The actual measurements used are given on the figure. Station *D* was located by the departure of *DE* = 169.4 and by the latitude of *CD* = 312.0, because these two distances were more convenient to use than the two coordinates given in the table. Similar modifications of the method will become apparent from inspection.

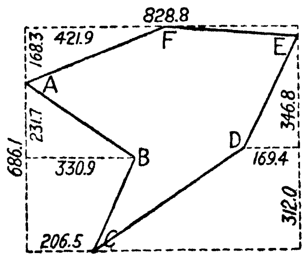


FIG. 638(*d*).

Line	Bearing	Length	Latitude North	Latitude South	Departure East	Departure West	Station	Total Lat.	Total Dep.
AB	S 55° 0′ E	404.0		231.7	330.9		*B*	−231.7	330.9
BC	S 23° 30′ W	312.0		286.1		124.4	*C*	−517.8	206.5
CD	N 55° 26′ E	550.0	312.0		452.9		*D*	−205.8	659.4
DE	N 26° 02′ E	386.0	346.8		169.4		*E*	+141.0	828.8
EF	N 86° 10′ W	407.8	27.3			406.9	*F*	+168.3	421.9
FA	S 68° 16′ W	454.2		168.2		421.9	*A*	0.1	0.0
			686.1	686.0	953.2	953.2			

638(*e*). SUGGESTIONS. (1) On large maps it is often advisable to subdivide the reference rectangle into squares of any convenient size, say 10 in., and to use these squares for measuring off the coordinates. If all these squares are drawn at the outset, any shrinkage or expansion which may take place in the paper will not affect the relative positions of stations subsequently plotted, provided the squares are taken as the basis of measurement instead of the original scale. It is sometimes expedient not to draw the entire reference rectangle, but only those supplementary squares needed for plotting, especially if the survey does not close but zigzags in one general direction. (2) It is frequently advantageous to assume the reference rectangle so that no side is horizontal or vertical. (See directions for starting the map, Art. 655(*b*).) In that case it may be expedient to use a T-square with adjustable head, and to set the blade to coincide with the lower and upper sides of the rectangle. If a drafting machine is used, a similar adjustment can be made. (3) If in testing the reference rectangle it is found that the diagonals are not exactly equal, each right angle may be tested by laying off 10 in. along each side and measuring between the two points thus obtained to see if the distance is 14.142 in. (See Art. 627(*e*).) (4) Sometimes instead of plotting all stations from the same origin, each station is plotted by its latitude and departure from the preceding station. The only advantage in this is that by checking the *total* latitude and departure of the last station from the first station all lines between these two stations are checked. This check is of the most value when the traverse does not form a closed figure. (5) Let a fine dot represent the position of each station, draw a small freehand circle around the dot, and draw no lines of any kind

inside this circle. Then there will always be a definite point to measure from, unobscured by lines.

638(*f*). CHECKS. In addition to the check afforded by measuring distances between stations, it is well to test with the protractor the angle between two lines when either of these lines is nearly parallel to one of the coordinate axes. This is because a mistake made in measuring perpendicular to that axis does not greatly affect the length of the line. If preferred, the test in such a case may be made by checking the bearing of the line. (See also check in suggestion 4 above.) It should be borne in mind, however, that the directions of lines cannot be checked even with vernier protractors, with a high degree of accuracy.

639. Methods of Plotting Traverses Compared. The methods of plotting angles by the protractor, by tangents, by cosine and sine, and by chords have already been compared in Art. 629. In a test carried on with a large number of students the author has found that, apparently, there is little difference between the tangent method and the chord method, either in the time required to plot a closed traverse or in the accuracy attained. If anything the chord method seemed to be the quicker, and when the "home-made" beam compass (Art. 627(*c*)) was used the results were fully as accurate. This is, perhaps, contrary to the general impression.

The chief advantage in plotting *direct angles* or *deflection angles* lies in the fact that the check (Art. 634(*c*)) applied every few stations checks all previous courses, whereas if lines are plotted by *bearings* or *azimuths* no such single check can be used. On the other hand, if a mistake is made in laying off the direction of one course and it is not discovered until a number of succeeding courses have been plotted, the direction of each of these courses must be changed, whereas if they have been plotted by bearings or azimuths the correction may be made by moving each line parallel to itself. For the same reason, when plotting by direct angles or deflection angles small errors in direction are likely to accumulate, though this may be remedied by plotting every fifth line or so by its bearing (Art. 634(*c*)).

The method of bearings or the method of azimuths is best adapted to plotting surveys which close, for it is difficult to check lines plotted by either of these methods if the traverse does not form a closed figure.

The method of plotting by coordinates (or latitudes and departures) is generally recognized as the best method for plotting the average map. The preliminary computations are somewhat longer than those required for the other methods. On the other hand, they may be used for supplying missing data and for calculating areas. The draftsman knows before he begins to plot whether or not a traverse closes; in the other methods if it fails to close he is uncertain as to whether the error is in the field work or in the

plotting. Since stations are plotted independently, errors cannot accumulate, but on the other hand intermediate stations cannot be checked by checking the last station unless suggestion 4, Art. 638(*c*), is carried out. One of the chief advantages of the method is that the size of the map may be ascertained before plotting is begun, since in the majority of cases it is approximately the size of the reference rectangle.

CHAPTER 47

WORKING UP FIELD NOTES PREPARATORY TO PLOTTING

THE details of a survey are usually plotted directly from the field notes with little if any preparatory work. On the other hand more or less preliminary computation is necessary before the traverse lines can be plotted. Moreover the notes pertaining to traverse lines are often scattered through many pages of the notebook. It saves time, therefore, to get together all the data needed for computation and plotting, to make all the preliminary calculations and to arrange results in such a systematic manner that when plotting is begun the undivided attention may be given to the mechanical portion of the work.

640. The Method of Working Up Notes depends largely upon the method to be used in plotting. In any event it is well to make a rough freehand sketch of the main lines or skeleton of the survey, and to put on this sketch all lengths and angles which will be needed for computation and plotting. It is better to take the time for this at the beginning of the work than to stop in the midst of a computation to search the field notes for necessary data.

Having thus gathered all the data that in any way affect the main lines, it may be necessary to correct or adjust values. The adjusted values of lengths and angles (or bearings) may be written in red ink on the original sketch, or a new sketch may be made showing only such data as will be needed for computation or plotting. From this sketch the draftsman can see what angles are to be plotted (they are often the complements or supplements of the angles measured in the field), or if the lines are to be plotted by latitudes and departures this sketch will be very useful in the preliminary calculation.

When the protractor is used, no further preparation is necessary, as the lines can be plotted directly from the data on the sketch. When, however, the more accurate methods are employed, additional data will be needed before the work of plotting is begun. The additional data required for each of the different methods of plotting are indicated in Art. 646.

Remark. It is well to keep all sketches, computations, and results in a separate-leaf notebook, such as that described in Art. 542.

641. Correcting Measurements. Measurements may be *corrected* in order to eliminate known sources of error. Measure-

ments may be *adjusted* in order to fulfill known conditions, as, for example, known geometric conditions. Paradoxical as it may seem, the more accurate the field work the more important it may be to correct the field measurements, for in order to obtain results of a high degree of precision, certain refinements are necessary which are entirely omitted in ordinary work. The corrections given below may or may not be necessary, depending upon the nature of the work. They are usually so small that they are not needed for purposes of plotting, but they may be needed for purposes of calculation or for permanent record. Moreover, it may be desired to show these corrected measurements on the map. Errors in measurements due to known sources are often corrected in the field.

641(*a*). LINEAR MEASUREMENTS may be corrected for error in the length of the tape, Art. 68, for error due to temperature, Art. 71, and occasionally for sag and pull, Arts. 74 and 73. If a distance is chained along a slope instead of in horizontal stretches, the calculated correction is applied, Art. 60. In the case of inclined stadia readings, corrections are made as explained in Art. 645.

641(*b*). ANGULAR MEASUREMENTS. Few corrections can be applied to angular measurements. In compass surveying bearings affected by local attraction may be corrected as explained in Art. 214.

642. Adjusting Measurements means changing the angles and distances given in the notes so that they will conform more nearly to conditions which are known to exist. Adjustments are usually made for purposes of calculation, not for plotting.

642(*a*). LINEAR MEASUREMENTS. If more than one value is given for a distance, the mean may be taken, throwing out any value that is obviously incorrect. When only one value is given it is not changed unless some obvious mistake is detected, such as an even hundred feet in chaining or reading from the wrong end of the tape, as, for example, 54 instead of 46. Sometimes several consecutive measurements are made along a line, as in taking offsets. When their sum does not exactly equal the total measurement, if it is worth while correcting the error it may be distributed among the partial measurements in proportion to their lengths.

642(*b*). ANGULAR MEASUREMENT. The most common adjustment of angles is in the case of a closed figure, when the error in the sum of the interior angles (Art. 298) may be distributed equally. When a number of angles are clustered about the same point, the same adjustment may be made in order to make their sum equal 360°. When several angles at the same station are

checked by measuring another angle, the error is distributed among all the angles, including the check angle. (See Art. 41(*d*).) One should not be tied down by rules, however, for it is often better to change some of the angles without changing others. An illustration given below shows how, by merely changing six of the twenty angles of a survey, five conditions were fulfilled.

642(*c*). ILLUSTRATION. In the survey on p. 641 each distance and each angle were originally obtained by taking the mean of several measurements. No further adjustment of distances was possible. The angles, however, were adjusted on the following assumptions:

1. In the two five-sided polygons the sum of the interior angles must equal 540°.
2. In the one four-sided polygon the sum of the interior angles must equal 360°.
3. In the two triangles the sum of the interior angles must equal 180°.
4. The sum of the four angles at *Y* must equal 360°.
5. In the ten-sided polygon *NOPQRSTUVWN*, the sum of the interior angles must equal 1440°.

Since it was not desired to carry any angle to seconds, the adjustment was made by changing six of the twenty angles as indicated in the figure. In no angle did the change exceed one minute.

When several values are given for each angle, the mean value may be used. The mean value may be further modified if necessary in order that the interior angles of a polygon may add up correctly, or that some other condition may be fulfilled. Thus, for example, in class work, where several parties work independently but use the same stations, mean values may be found, these values adjusted, and the survey plotted from the composite values thus obtained.

642(*d*). REMARK. It must be remembered that in many cases nothing is gained by adjusting the measurements given in the field notes. The purpose of the survey and the limit of error will determine whether or not adjustment is necessary.

642(*e*). IN WORK OF GREAT PRECISION adjustment of observed values may be made by the method of least squares, a theoretical method of deducing the most probable results of a set of measurements. If some of the measurements are thought to be more accurate than others, they are given weight and treated accordingly. Equations are formed from conditions known to exist, probable errors are calculated, and theoretical values of angles and distances are deduced. The method should not be used unless the degree of precision in field methods was such as to reduce greatly systematic errors.

642(*f*). ADJUSTMENT OF A SURVEY AS A WHOLE. A triangulation net may be adjusted as explained in Art. 418. A traverse may be adjusted (balanced) as explained in Arts. 570–572. Unless the adjustments of a survey are great enough to affect the plotting, it is, of course, not necessary that they should be made before beginning to plot the map. It may be desired, however, to show these adjusted measurements on the map.

643. Calculating Bearings. If the bearings of lines are needed for plotting and they are not given in the notes, they may be calculated by the method explained in Chapter 37, provided the bearing of one line is known. For purposes of plotting, however, it is not essential that the bearing of even one line should be given, for

the *relative* positions of stations will be correct if the bearing of any line be assumed, and the bearings of the other lines be calculated from that assumed value. In such a case it makes little difference what bearing is assumed at the start.

When a true meridian is established in a survey (Chapters 34 and 35) the true bearing of some main line is found by measuring the angle it makes with the meridian. The true bearings of the other lines can then be calculated. (See p. 642 for an example.)

In calculating all bearings the final adjusted values of angles are used. Make sure that bearings are consistent, i.e., all forward bearings or all back bearings.

In compass surveys the magnetic bearings, or true bearings derived from magnetic bearings, are usually plotted, for, of course, no calculated bearings are possible.

644. Supplying Missing Data. When the length of a line or the value of an angle is not given in the field notes it can often be calculated by the methods explained in Chapter 39. Notice, however, that in such cases there is no check on the field work, and hence, as pointed out in the note at the beginning of Chapter 39, closing angles and distances should be measured whenever practicable.

645. Reducing Stadia Notes. In order to plot the map, inclined stadia distances must be reduced to corresponding horizontal distances and, if elevations are required, to corresponding vertical distances as well. Such reductions of horizontal distances may not be necessary if the vertical angle is too small to take into account (ordinarily 3° or less, see Art. 426(*d*)). These reductions are usually made in the office although in some types of work they are made in the field. Reduction formulas derived in Art. 425 are:

$$\textit{Horizontal distance} = R - R\sin^2 a + C\cos a$$
$$= R\cos^2 a + C\cos a$$

$$\textit{Vertical distance} = R(\cos a \sin a) + C\sin a$$
$$= R\tfrac{1}{2}\sin 2a + C\sin a$$

When the telescope of the instrument is of the internal focusing type, the constant C becomes zero.

For reasons give in Art. 425(*b*), the following approximate formulas may often be used: $H = R\cos^2 a$ and $V = R\tfrac{1}{2}\sin 2a$. For internal focusing telescopes, these formulas are practically exact.

When there are many readings to be reduced, the work involved in substituting in these formulas would become a serious objection

to the entire stadia method. Fortunately, the terms $R \sin^2 a$, $R \cos^2 a$, and $R\frac{1}{2} \sin 2a$ are independent of the instrument or the linear unit used; hence their values for different vertical angles can be computed once for all and tabulated. There are several forms of stadia reduction tables.

Unless inclined stadia distances are reduced by some mechanical device, such as the Beaman stadia arc (Art. 429), they must be reduced either by computation or by a graphic method. There are in common use three different devices for obtaining easily and quickly the values of $R \cos^2 a$ and of $R\frac{1}{2} \sin 2a$ for different vertical angles, namely: (1) reduction tables, (2) stadia diagrams, and (3) special forms of the slide rule.

645(*a*). Reduction tables. In some tables the values of $R \cos^2 a$ and $R\frac{1}{2} \sin 2a$ are computed on the assumption that the actual intercept on the rod (R) is unity, as, for example, one foot. Then any inclined distance, as *Fm* in Fig. 425(*a*), p. 442, may be reduced by multiplying that distance by the tabular value corresponding to the vertical angle. In other tables the values are based on the assumption that $R = 100$; hence the inclined distance is divided by 100 (two decimal places being pointed off), before multiplying by the tabular value. In ordinary work the values thus obtained are the final values used, but when work is of such an order that precautions are taken to eliminate sources of errors, the quantity $C \cos a$ should be added to the value obtained for the horizontal distance and the quantity $C \sin a$ to the value obtained for the vertical distance (unless an internal-focusing telescope was used). Usually these two additional quantities are given in tables that supplement the main tables. (See Table X, p. 1115.)

Since for average stadia shots of 100 ft. or over the measurement is not exact beyond the figure in the third place, i.e., the figure in the fourth place is uncertain (see Art. 546), not more than four places of figures should be retained in any result. (See Art. 546(*e*).) Frequently three places will suffice. Hence an ordinary slide rule may be used for multiplying tabular values. The next best method is that of abridged multiplication (Art. 545(*d*)), writing the extreme right-hand figure of the reversed multiplier under the fourth place of figures in the tabular value regardless of the position of the decimal point. (See Art. 546(*g*).)

645(*b*). Illustration. Given a stadia reading = 234, vertical angle = 10° 22′, and an instrumental constant = 1.25. From Table X, p. 1115, the horizontal distance per 100 ft. for 10° 22′ is 96.76 ft., and the vertical distance is 17.70 ft. Each of these values is multiplied by 234 (abridged method).

Accurate Method		Approximate Method	
Horizontal	*Vertical*	*Horizontal*	*Vertical*
96.76	17.70	96.76	17.70
432	432	432	432
19352	3540	1935	354
2903	531	290	53
387	71	38	7
226.42	41.42	226.3	41.40
1.23	.23	1.23	.23
227.6	41.65	227.6	41.63

Notice that for most purposes the approximate results on the right are accurate enough, that is to say, the extreme right-hand figure of the reversed multiplier may be written under the third place of the multiplicand instead of under the fourth.

645(c). Stadia diagrams. It is an easy matter to construct a diagram that will give fairly accurate values for $R \cos^2 a$ and $R\frac{1}{2} \sin 2a$ for vertical angles up to, say, 8° or 10°, but when the diagram is extended to cover 15° or 20°, either the scale must be reduced at a sacrifice in accuracy, or the diagram made inconveniently large. Nevertheless such diagrams are very useful, especially when two men work together, one with the notebook and the other with the diagram. Results can be obtained much more quickly than by the use of tables, and for most purposes they will be sufficiently accurate. Usually separate diagrams are prepared, one for horizontal distances and one for differences in elevation.

645(d). Diagram for horizontal distances. The simplest form may be constructed on cross-section paper, on which there are, say, 10 lines to the inch each way. (1) Assume the origin or zero point in one corner of the paper, say the lower left-hand corner. (2) Along the lower edge, number every tenth (or heavy) vertical line, the numbers running 100, 200, etc., up to the longest sight taken in the field, say 1000 ft. (3) Along the right-hand edge number every tenth (or heavy) horizontal line, the numbers running 10, 20, etc., thus making the vertical scale ten times as large as the horizontal. (4) Find the values of $1000 \sin^2 a$ for different vertical angles 1°, 2°, etc., up to say 15° or 20°, using stadia reduction tables or computing the values by trigonometry. Lay these values off along the ordinate marked 1000, using the vertical scale, and draw radial lines from the origin to the points thus obtained. Mark each radial line with its degree number.

645(e). Illustration. *Use of the Diagram.* Given a stadia reading $R = 410$, vertical angle = 13°, instrumental constant $C = 1.2$. Find the vertical line next beyond the one numbered 400, follow up this line to its intersection with the radial line marked 13°, note that the nearest horizontal line is numbered 21, and estimate the reading as 20.7. Hence $410 - 20.7 = 389.3$ is the corrected (horizontal) distance, and adding $C = 1.2$, the final result = 390.5 ft.

645(*f*). Suggestions. (1) The scale adopted depends upon the accuracy required, the maximum length of sight, and the maximum vertical angle. If, for example, the length of the longest sight taken in the field is less than 500 ft., there is no advantage in extending the diagram to 1000; it is better to double the scale. (2) In large diagrams, intended for accurate work, intermediate radial lines may be drawn at intervals corresponding to 10′ of arc. If desired, an arithmetical table giving values of $C \cos a$ and $C \sin a$ for several values of C at 5° intervals may be printed in one corner of the sheet. (3) The reason that the vertical ordinates are plotted for values of $R \sin^2 a$ instead of $R \cos^2 a$ is that the former is a small quantity which changes rapidly for small angles, whereas the latter is a large quantity which changes slowly. Notice that a quantity taken from such a diagram must be *subtracted* from R, and C added to get the final result unless an internal focusing telescope is used, then C is zero.

645(*g*). Diagrams for obtaining differences in elevation may be constructed in a similar manner, the vertical ordinates representing values of $R \frac{1}{2} \sin 2a$. The two diagrams may be drawn on the same sheet without conflicting, by laying off the abscissas and ordinates in the second diagram along the two edges of the paper not used in the first.

645(*h*). Remark. A number of modifications of the diagrams just described have been devised with a view to saving space and increasing accuracy. In one of these, the values are laid off along the sides of triangles, equilateral triangles being used in the diagram for elevations. Various forms of diagrams, including combination diagrams for obtaining horizontal distances and elevations, have been described in engineering publications.

645(*i*). Stadia slide rules may be used in place of tables or diagrams. Several types of such slide rules can be purchased. Accurate results can be obtained with great rapidity after a few minutes' practice in the use of these slide rules. The few simple directions necessary are furnished with the slide rules.

645(*j*). Methods of reducing stadia notes compared. Reductions can be made more rapidly by means of diagrams than by the use of tables. The relative accuracy of the two methods depends largely upon the scale of the diagram. For ordinary work diagrams are sufficiently accurate. Reductions can be made by means of a stadia slide rule more conveniently and rapidly than by either of the other two methods. By one setting of the slide, the horizontal distance and the vertical height can be obtained at once, and, within the usual range of vertical angles, values thus found are correct to the nearest tenth of a foot and sometimes closer.

646. Getting Together Data for Plotting—Summary. The first five or six steps are practically the same, regardless of the method of plotting that will be used.

1. Make a rough sketch, showing approximately the relative positions of stations.

2. Mark on the sketch the lengths and angles given in the field notes for the main lines. If the stadia was used, reduce the inclined readings and put the *horizontal* distances on the sketch.

3. If necessary, correct the field measurements for errors from different sources (Art. 641).

4. If desirable, adjust measurements to conform to known conditions, as explained in Art. 642. It may be desirable also to adjust a survey as a whole (Art. 642(*f*)).

5. Mark on the sketch the final values of angles and distances. This is sometimes done with red ink or a colored pencil.

6. If bearings are to be used either in the calculations or for plotting, start with the bearing of one line and calculate the bearings of the others (Art. 643). (If the survey was made with the compass, this, of course, is not done.) Use the final adjusted values of angles in calculating bearings. Enter the calculated bearings on the sketch, making sure that they are all forward bearings.

The remaining steps will differ somewhat according to the method used in plotting.

646(*a*). PROTRACTOR METHOD. No additional data are required. Work directly from the sketch. Angles, bearings, or azimuths may be plotted.

646(*b*). TANGENT METHOD. (1) It is well to indicate on the sketch the actual angles or bearings to be plotted, as many of them may be the complements or the supplements of those previously marked on the sketch. (2) Prepare a table similar to that in Art. 634(*a*), or if bearings are to be plotted, similar to that in Art. 636(*a*).

646(*c*). CHORD METHOD. (1) As in the tangent method, indicate on the sketch the actual angles or bearings to be plotted. (2) Prepare a table similar to that in Art. 634(*b*), or if bearings are plotted, similar to that in Art. 636(*c*).

646(*d*). LATITUDE AND DEPARTURE METHOD. (1) Compute the latitudes and departures, using, of course, final corrected values of lengths and bearings. (See pp. 644 and 645 for forms.) (2) If desired, adjust the values thus obtained, i.e., balance the survey (Art. 642(*f*)). (3) Tabulate for plotting (p. 752), and find the size of the circumscribed rectangle. (A complete illustration of working up notes for plotting by this method is given on pp. 641–645.)

646(*e*). AZIMUTHS. If azimuths are given they may be plotted by chords or tangents, tables similar to those on pp. 748 and 749 being prepared, or latitudes and departures may be calculated without reducing azimuths to bearings. (See Art. 565.)

647. Efficiency in Working Up Notes can be acquired only gradually and from a continual effort to learn or to devise the shortest methods consistent with the results required. Some of the elements of efficient procedure are: short cuts in computation; elimination of worthless refinements; systematic methods of computing; quick but adequate methods of checking results; use of tables; use of graphic methods; use of mechanical methods; the development of skill in the use of all office methods and office instruments. General explanations of these elements are to be found in Chapters 36 and 44.

Many special computation tables, diagrams, graphic methods, and devices for mechanical methods are described from time to time in engineering papers and periodicals. A good engineer is on the alert not only to adopt any of these which he can use to advantage, but also continually to devise his own short cuts adapted to the special work in hand.

CHAPTER 48

PLOTTING THE MAP

In plotting the map, something more is needed than a mere knowledge of the use of drawing instruments and of the different methods of plotting traverses. Practical questions arise, such as the choice of scale, the general arrangement of the map, the order of plotting, the precautions necessary to insure accuracy, the methods of checking the drawing, of plotting details, of interpolating contours, and of copying maps. These questions are discussed in this chapter.

General Methods and Suggestions

648. Equipment. (*a*). Paper. If maps are to be preserved a long time or are likely to receive rough usage, the durability of the paper is an important quality; moreover, if distances are to be scaled with accuracy, the paper should contract and expand as little as possible under atmospheric changes. These two qualities are best combined in mounted paper or paper backed with muslin. The best paper for general use in all kinds of mapping is a strong, clear white paper of highest grade, 100 per cent rag stock, and hard sized. The erasing quality of paper should be such that sharp ink lines can be drawn over repeated erasures. In many cases a buff paper of equal quality may be used; it is less trying to the eyes than white paper. Cheap paper is likely to be expensive in the end. For penciling, preparatory to tracing, almost any *white* paper will answer, provided fine lines can be drawn and its surface will stand erasing. A tough bond paper is best for maps which will be folded (creased) and carried in the pocket.

For maps which are to be tinted, mounted paper is recommended. If an unmounted paper is used it should be of high grade, with a surface adapted to either lines or wash—a pebbled or egg-shell surface is often preferred. For photo reproduction Bristol board or a hard white paper with a smooth glossy surface is the best, although excellent reproductions can be made from tracings. Tracing cloth should be of the best quality; otherwise ink is likely to strike through. Tracing paper may be used when only a few prints are needed and it is not necessary to preserve the tracing for a long period of time. For blueprints thin tough paper will usually stand rough handling better than the smooth thick paper,

but prints are likely to be somewhat inferior. The more sensitive a paper is, the quicker it deteriorates, and the more care is required in handling it in a strong light. "Medium" printing paper is recommended for most purposes. In ordering paper, be careful to state whether quick-printing, medium, or slow-printing paper is desired. (See Art. 622(*a*).)

It would be an advantage if engineers could agree, within reasonable limits, upon standard sheet sizes for maps. Although there is no such general agreement, various engineering organizations, such as those engaged in highway, railway, and municipal work, have adopted, as far as possible, standard trimmed sheet sizes. The engineer in private practice can do likewise, and, by reducing to a minimum the number of different sizes used, he can simplify his office work, particularly his system for filing maps. The standard sizes adopted in any case will depend partly on the character of the engineering work and partly on the dimensions of the sheets of drawing paper that can readily be purchased.

648(*b*). Drawing instruments. The edges of the straightedge, the T-square, and the triangles should be straight. This is more essential in map drawing than in almost any other kind of drafting. (See Art. 605 for test.) The angles of a triangle should be true, especially the right angle. (See Art. 606 for test.) Straightedges, T-squares, and triangles made of steel are usually the most accurate. The pencil should be hard (4-H or 6-H) and properly sharpened. (See Art. 604.) The ruling pen should be of the very best quality, and the points should be kept in good condition. (See Art. 608.) The scale most used is a decimal scale, and its graduations should be accurate and definite. The surface of the drawing board or table should be a smooth plane. These are the most essential requirements for instruments used in mapping.

A good drafting machine expedites the manual work and, at the same time, makes it less difficult to attain accuracy in mapping (Art. 603(*a*)).

649. Precautions Necessary in Accurate Plotting. As the edges of a drawing board are seldom exactly straight, lines drawn along the T-square blade, as it is moved up and down the board, are seldom exactly parallel. A steel straightedge fastened to the left-hand edge of the board remedies this to some extent, but even then there is likely to be some play in the blade of the T-square; hence, in very accurate work, it is better not to rely upon this method. As comparatively few lines in maps are verti-

cal or horizontal, the T-square is not so necessary in plotting as it is in other kinds of drawing. A steel straightedge about 3 ft. long may be used in its place. All important reference lines, such as traverse lines, can then be plotted entirely by measurement without depending upon parallel lines drawn with the T-square or upon perpendiculars erected with the triangles. (See the method given in *Caution,* Art. 638(*b*).) Such precautions, however, are not necessary in much of the plotting done in ordinary work.

The shrinkage of the drawing paper may be another source of error. If the map is 4 or 5 ft. wide the change in the paper is frequently as much as ¼ in., and sometimes ½ in. The more seasoned the paper the less it will shrink; hence it is well to keep paper in the office several months before using it. In plotting new details on old maps test the map first to see if shrinkage has occurred; and if it has, make proper allowance for it in plotting. On the other hand, in maps which have been rolled up, test for expansion, as rolling tends to stretch the paper—a good reason for filing maps flat when possible.

650. General Methods. A large part of the work in mapping is the reverse of field work. Distances that were measured with a tape or chain are laid off with a scale, and angles that were measured with a transit or compass are laid off with a protractor. Such work involves no calculation. When, however, the plotting is done by *indirect* measurements, that is by measurements which are not the same as those taken in the field, these indirect measurements must be obtained from the field measurements by calculation, as, for example, when angles are plotted by latitudes and departures or by any of the trigonometric methods.

A common method in field work is to run traverses and locate details simultaneously, but in mapping, the traverse lines are almost always plotted first, before work is begun on the details. The plotting is done in pencil, usually on paper, but sometimes directly on tracing cloth, and after the penciling is completed, the lines are inked. If only a single copy of the map is desired, the pencil lines are inked directly on the paper. If more than one copy is required, the lines are usually traced and process prints made from the tracing.

651. Method of Procedure Outlined. (1) Choose a suitable scale for the map (Art. 653), decide upon the method of plotting traverse lines (see Art. 636), and work up the field notes, making sketches and arranging data so that no delay in plotting will be caused by searching through field books (Chapter 47). (2) Select

a suitable drawing paper (Art. 648(*a*)) and stretch it on the drawing board until it lies smooth and flat (Art. 613). (3) Sharpen a 6-H pencil to a round point at one end and a wedge-shaped point at the other (Art. 604). (4) Wipe the triangles and T-square *clean,* and brush the surface of the paper. Throughout the work, take precautions to insure neatness (Art. 614). (5) Follow the directions for starting the map given in Art. 655. (6) Plot the traverse lines in pencil (Chapter 46), marking each station with its letter or number. Use hairlines and prick the points. Observe correct methods of penciling (Art. 615) and of laying off measurements (Art. 618). Apply checks continually as the work progresses (Art. 658). (7) Plot details in pencil (Art. 659). (8) Ink the map or trace it. Observe the directions in Arts. 608 and 616 for inking and those in Art. 621 for tracing. (9) Finish the map (Chapter 49), making sure that it meets the general requirements given in Art. 664. (10) If maps are to be reproduced, one of the processes described in Art. 622 may be used.

652. Scales. (*a*). THE MOST COMMON SCALES for ordinary maps are those in which the number of *feet* represented by one *inch* is some *multiple of ten.* In general these scales are as follows:

Nature of the Map	*Number of Feet per Inch*							
Small territory—many details	10	**20**	30	**40**	50	60	80	**100**
Large territory—few details	**100**	**200**		400	500		800	**1000**

The numbers in black-face type are probably the scales most frequently used. Notice that 70 and 90 are the only two multiples of ten below 100 that are not included. The first six (from 10 to 60 inclusive) are the scales usually found on the six sides of a triangular scale. The scales of 80 and 100 can be obtained by using the 40 and 50 sides, respectively, but 70 and 90 have no such corresponding sides.

652(*b*). THE SO-CALLED NATURAL SCALE is one in which a given distance on the map is some round fraction of the distance it represents, such as 1/10,000, 1/50,000, etc. Such scales are used for maps of large territories, e.g., those of the United States Coast and Geodetic Survey and of the United States Geological Survey. Maps of this kind are often plotted to a scale of 1/10,000 or to a scale of 1/30,000 and then reduced for the purposes of printing to 1/62,500 or 1/125,000 (about 1 in. and ½ in. to the mile, respectively).

652(*c*). SCALE EQUIVALENTS for the more common ratios may be obtained either directly or by simple arithmetic from the following table:

SCALE EQUIVALENTS FOR VARIOUS RATIOS

Feet to One Inch	Miles to One Inch	Meters to One Inch	Kilometers to One Inch	Ratio (Number Inches)
100	0.019	30.480	0 030	1 : 1,200
400	0.076	121.920	0.122	1 : 4,800
500	0.095	152.400	0.153	1 : 6,000
800	0.151	243.840	0.244	1 : 9,600
833⅓	0.157	254.000	0.254	1 : 10,000
1,000	0.189	304.801	0.305	1 : 12,000
5,280	1.000	1,609.2	1.609	1 : 63,360
10,560	2.000	3,218.4	3.218	1 : 126,720

653. The Choice of Scale. Primary considerations in choosing the scale are: (1) the use to which the map will be put, and (2) the extent of territory to be represented. Thus, for example, if distances are to be scaled on the map to the nearest foot, the scale should be at least as large as 1 in. = 100 ft., and, if possible, as large as 1 in. = 40 ft. For a small map with many details, such as that of a city lot and buildings, a scale as large as 1 in. = 20 ft., or even 1 in. = 10 ft., is preferable. For large maps with few details, such, for example, as the average topographic map, scales from 1 in. = 100 ft. to 1 in. = 1000 ft. may be used. In some cities, for example, a scale of 1 in. = 200 ft. is used for topographic maps and 1 in. = 40 ft. for property maps.

Preliminary surveys for railroads are usually plotted to a scale of 100 to 500 ft. per inch; maps of mines and mining claims are more often to a scale of 100 or 200 ft. to the inch. Of course maps of railroads and mines made for special purposes are often drawn to larger scale. Architects usually prefer a scale of 1 in. = 64 ft., or 1 in. = 32 ft., or 1 in. = 16 ft., or, occasionally, 1 in. = 4 ft., to correspond to the scale which they use for the plans of buildings.

The scales of 1 in. = 20 ft., 1 in. = 40 ft., and 1 in. = 80 ft. have the common advantage that distances may be scaled approximately with a carpenter's rule or an architect's scale, since ¼ in. equals 5, 10, and 20 ft. respectively. Similarly in the scales 200, 400, and 800 ft. per inch, ¼ in. equals 50, 100, and 200 ft. respectively.

Frequently the choice is restricted to some scale small enough so that the whole map will fall within a rectangle of given size. The dimensions of this rectangle are sometimes determined by the size of the blueprint frame, or by the size of the drawer in which the map will be filed, or by the size of sheet most convenient for handling, or by the standard size adopted for similar maps. The arrangement of the map must be taken into account, for if it

is desired to have the north at the top and the border lines square with the compass points, it may be necessary to adopt a smaller scale than would otherwise be used (see next article). When it is desirable to choose as large a scale as a given rectangle will permit, it is often expedient to plot the traverse lines roughly to any scale in order to determine the shape and extent of the survey.

653(*a*). TWO GENERAL RULES. I. *Choose a scale large enough so that in plotting or in scaling distances from the finished map it will not be necessary to read the scale closer than 1/100 in.* (Allowance should be made for expansion and contraction due to atmospheric changes.)

II. *Choose as small a scale as is consistent with a clear delineation of the smallest details to be plotted,* due regard being paid to Rule I. In topographic maps, especially, it is an advantage to adopt as small a scale as practicable, for not only is a larger area shown on each sheet, but by bringing contour lines closer together a more vivid impression of the real nature of the surface depicted is given.

It is frequently necessary to draw maps of the same territory to different scales, first one map to small scale showing the region as a whole, then other maps to a much larger scale, each sheet corresponding to a portion of the original small-scale map.

654. Arrangement. It is desirable to arrange a map with the *north* at the *top,* making the top and bottom border lines *true* east-west lines, and the side border lines *true* north-south lines. Such an arrangement, however, is seldom economical, the border lines forming a larger rectangle than is necessary; hence some other arrangement is followed when it is more desirable to confine the map to a small space than it is to have the border lines square with the compass points. Moreover it often is best to have the natural approach to a piece of property at the bottom of a map, even though this brings the south at the top. For example, in the case of a house and lot, it is natural to look for the street side and the approach to the house at the bottom of the map, even though this may bring the south end of the lot at the top.

There are many questions of arrangement, such as where to put the title, the meridian needle, and the explanatory notes, which do not have to be settled until the map is plotted and ready to finish. (See Chapter 49.)

655. How to Begin a Map. Usually the first step is to plot the traverse to a small scale, using the protractor for laying off angles. If latitudes and departures have been calculated, it is quicker to

plot by coordinates laid off on cross-section paper. This preliminary sketch is drawn roughly and quickly merely to give an idea of the general shape of the plot and the space that it will occupy.

655(*a*). How to choose the first line to be plotted. Unless the traverse is to be plotted by coordinates, the position and direction of some traverse line must be assumed to start with, and its length laid off to scale. In a closed figure it is well to begin at the first station and work halfway around the polygon in either direction to lessen cumulative errors. For the same reason, in an open traverse it is well to begin near the middle and plot toward either end.

655(*b*). How to assume the position of the first line. A starting point and a direction for the first line should be assumed in such a way that when the other lines are plotted they will not run off the paper. In this the preliminary plot referred to above is of the greatest assistance. If a meridian is to be used in plotting, it is convenient to make it a vertical line. In that case the direction of the first line cannot be assumed, but must be found by laying off its bearing or its azimuth. It is by no means essential, however, that the meridian should be a vertical line. On the contrary, there are many cases in which if it were assumed vertical the map would not fall within the limits of the paper. It is then necessary to assume the first line in the proper position as judged by a preliminary sketch, and to lay off its bearing or azimuth backward to establish the direction of the meridian. This will bring the east-west line oblique to the T-square blade, but if the head of the T-square is adjustable the blade can usually be made to coincide either with the east-west line or with the north-south line.

For the reasons just given, it is often expedient to lay off the reference rectangle, used in plotting by coordinates, so that no side is vertical or horizontal. The inclination required in order that the lines may not run off the paper is estimated by means of the preliminary plot referred to above. In this case it is often inexpedient to draw the entire reference rectangle. (See Art. 638(*c*).) Notice that the size of the reference rectangle is known from the computations (Art. 638 (*a*), 5), and that if this rectangle can be drawn to the desired scale without running off the paper there is no need of making the preliminary plot.

655(*c*). Remark. No matter how the map may be arranged with respect to the edges of the paper, after the plotting is finished border lines may be drawn in any desired direction and the sheet trimmed accordingly. (See Art. 666.)

656. Plotting the Triangulation Net. In extensive surveys where primary stations have been established by triangulation, the first step in plotting is to lay out the triangulation net. The lengths of the sides of the triangles having been computed, the lines may be plotted by any of the methods of plotting traverses. The method of coordinates, however, is generally the best. Unless this method is used, it is well to plot first those stations lying on the outside, i.e., in the perimeter of the survey. Sometimes the triangulation net may be plotted with sufficient accuracy by simply swinging intersecting arcs with radii corresponding to the lengths of the sides of the triangles, establishing each triangle in turn by this method. Occasionally a station may have to be plotted by the three-point method (Art. 630).

657. Plotting Traverses. (A summary of Chapter 46.)

1. For every method of running traverse lines in the field there is a corresponding method of plotting. In addition there is the method of coordinates or latitudes and departures.

2. In accurate work the ordinary protractor should not be used for plotting traverses. Use it for checking angles. Even vernier protractors will not give as accurate results as can be obtained by other methods.

3. The tangent method, the cosine and sine method, and the chord method are used for laying off direct angles, deflection angles, bearings, and azimuths. Latitudes and departures are used for accomplishing the same results indirectly.

4. Traverse lines must be plotted with more than ordinary care, since upon them the accuracy of the whole map depends. Use hairlines and prick points.

5. Before beginning to plot, get together all data that will be needed, including the tangents or chords of all angles to be plotted if corresponding methods are employed (see Art. 646), or the two columes of coordinates if latitudes and departures are used.

657(*a*). Five methods of plotting traverse lines

1. *Each Line Plotted from the Preceding Line.* (See Art. 634.) The tangent method, the cosine and sine method, or the chord method is used for laying off the angle (or its complement or supplement). A base of 10 in. or a radius of 10 in. is convenient to use. *Check.* Test the bearing of the last line, and if there are many lines apply a similar check at every fourth or fifth station.

2. *Each Line Plotted by a Deflection Angle.* (See Art. 635.) A modification of the preceding method. Used in railway work and other similar surveys. Checks same as in (1).

3. *Each Line Plotted from Its Bearing.* (See Art. 636.) (1) Draw a 20-in. square, subdivide into four 10-in. squares to correspond to the four quadrants of bearing, plot each bearing in its proper square by measuring ten times the tangent along the side of the square. A better method still is to make one 10-in. square answer for all four quadrants. (2) Draw a circle 20 in. in diameter, divide into quadrants of bearing, and plot each bearing in its proper quadrant by its chord. Four 90° arcs, the four centers of which are corners of a 10-in. square, may be used in place of a 20-in. circle, but no advantage is gained except in economy of space. *Check.* Test the angle between traverse lines at each station. There is no single check which checks all previous courses.

A modification of this method is to draw parallel meridians through the stations, and to lay off bearings from these meridians. Another method is to draw a series of 10-in. squares. (See Art. 636(*c*).)

4. *Each Line Plotted from Its Azimuth.* (See Art. 637.) Practically the same methods can be used as those given for bearings.

5. *Traverse Lines Plotted from the Latitudes and Departures.* (See Art. 638.) The most westerly station is chosen as an origin, and each station is plotted by its two coordinate distances: (1) north or south, (2) east from the origin. The coordinates are computed by adding algebraically first the latitudes, then the departures, starting with 0 and 0 as the coordinates of the most westerly station, and finding the coordinates of each successive station in order. As a convenient method of measuring the coordinates a rectangle is drawn, the sides of the rectangle passing through the extreme east and west, north and south stations. It is convenient but not necessary to assume the east-west sides to be horizontal (Art. 655(*b*)). If preferred, two coordinate axes through the most westerly station may be used instead of the rectangle. For very accurate work do not rely upon the T-square and triangles, but observe the caution in Art. 638(*b*). *Check.* Measure the length of each traverse line, and in addition test the angle between two lines if one line is nearly parallel to a coordinate axis.

6. Methods of plotting traverses are compared in Art. 639. The method of latitudes and departures is the best for general work.

658. Checks Used in Plotting. Precautions and checks are used constantly in field work; similar precautions and checks

should be used in plotting. A draftsman can accomplish more in the long run if he will check his work as he goes along. No reliance can be placed upon a map which has not been checked.

Some of the methods given in this article are only approximate checks intended to disclose blunders in plotting, but blunders, perhaps, are more common than small errors.

658(*a*). CHECKS FOR LINEAR MEASUREMENTS. (1) Any straight line may be divided into two or more segments, each segment scaled separately, and the sum of the results compared with the length of the line given in the field notes. (2) Several lines may be scaled, the results recorded on paper, and then compared with the corresponding distances in the field notes. This is better than to look up a distance in the field notes and then scale the corresponding line, for in the latter method the mind is prejudiced, and any mistake made originally in laying off a distance may be repeated. (3) To check lines of a traverse, lay a paper straightedge on the drawing, mark off the lengths of the lines consecutively along the edge of the paper, each line beginning at the end of the preceding line, and scale the total distance to see if it equals the sum of the lengths of all the traverse lines. This method is particularly useful when the plus-station system of numbering (Art. 337) is employed, since the station number of each mark as read from the scale may be compared with that given in the notes. (4) When two points are plotted by *indirect* measurements, always scale the distance between them as a check. Thus, for example, when two sides of a triangle have been laid off, scale the third side to see if it agrees with the length given in the field notes. This method, which is often a check on angles as well as distances, is especially useful in checking lines plotted by latitudes and departures.

658(*b*). CHECKS FOR ANGLES. (1) To check an angle which has been laid off with a protractor, measure its complement or some other related angle which requires a shifting of the protractor, instead of checking the angle itself. (2) Two or more adjacent angles may be tested by measuring the sum angle. (3) Never plot an angle by the chord method or the tangent method without checking it roughly with a protractor. (4) In plotting traverse lines, always compute the azimuths or the bearings of all the lines. If the lines are plotted by azimuths or bearings, check the angle between the traverse lines at each station, and, vice versa, if the traverse-line angles have been plotted directly, check the azimuths or bearings, as more fully explained in Art. 634(*c*). The aim

should be to check one angle by measuring an entirely different angle in order to avoid repeating mistakes.

If the traverse lines form a closed figure, the check angles may be measured with the protractor. Otherwise measure them by some method as accurate as that used for plotting the original angles.

Question. What check often employed in the field corresponds to the check used for traverses plotted by direct or deflection angles?

658(*c*). COMBINATION CHECKS. (1) Check the reference rectangle, used for plotting a traverse by coordinates, by measuring the opposite sides and also the diagonals to see if they are equal. (2) If a traverse closes, not only should the closing line pass through the first station but also the distance from the last station to the first should agree with the distance measured in the field. (3) When an extra distance or angle has been measured in the field as a check, this distance or angle should be tested on the map.

658(*d*). TO "RUN DOWN" AN ERROR. If traverse lines forming a polygon do not close when plotted, the error may often be "run down" without going over all the work again, by making the following tests. (1) If the error of closure is in a direction *parallel* to some line in the traverse, test the length of that line; if it is *perpendicular* to the line, test the bearing. (2) If a traverse has been plotted by direct angles or by deflection angles, calculate the bearing of the last line and see if the angle between this line and the meridian agrees with the calculated value; if it does, the *directions* of all the lines are probably correct. In that case test the lengths as directed in Art. 658(*a*). (3) If by the test in (2) an error in the direction of the last line is found, apply the same test to some preceding line, say three or four stations back. Continue to make the test until a line is found whose direction is correct and then work forward, applying the test at each station. (4) When lines have been plotted by bearings or azimuths no single test can be made. It is well to begin at the last station and to work backward, testing the direct angle or the deflection angle at each station. (5) If a point is obviously wrong and the mistake is not easily found, try to arrive at the same point from some other direction. (6) Keep in mind the effect of slight changes in the bearings of long lines, and of lines nearly north and south or east and west. (See (3) and (4), Art. 571 (*c*).)

659. Plotting Details. In Chapter 22 are given different methods of locating details in the field corresponding to the seven

methods of locating a point given in Art. 10. The work of plotting details is simply the reverse of that done in the field, enough points being established to determine the outlines of the details.

659(*a*). Points located by angles and distances. Usually the distances from the station to the points located are comparatively short, frequently less than 100 ft. Moreover, if a slight error is made in plotting a point, it does not affect the entire survey as does an error in plotting a station. Hence in plotting details the protractor is generally used for laying off angles. One of the most convenient forms is shown in Art. 624(*c*). Usually a number of points are located from the same station, and as fast as they are plotted they should be numbered in such a way that there will be no difficulty or confusion in drawing the outlines. Frequently, only two or three points of a building or other rectangular object are given, the perpendicular sides being completed with the triangles and scale.

659(*b*). Points located by angles only. Angles are laid off and the lines numbered so that the required points of intersection may be found without confusion. The method of plotting a point located by the *three-point* method is explained in Art. 630.

659(*c*). Points located by linear measurements only. Frequently the right angles for perpendicular offsets may be estimated by eye, especially in plotting indefinite details, such as edges of streams and roads. A point located by two intersecting distances may be plotted by swinging two arcs with the compasses, or better still by swinging one arc and using the scale. This applies especially to all tie-line methods. To test the straightness of a line located by offsets, such for example as a fence line, plot the offsets on another paper (cross-section paper will often answer the purpose), using an exaggerated scale for offsets just as heights are laid off to an exaggerated scale in plotting profiles. This is especially useful in determining boundaries.

659(*d*). General suggestions.

1. Keep the drawing free from unnecessary lines. Thus in plotting by angle and distance it is not necessary to draw radial lines clear to the station; draw only the portions near the points located. Likewise draw only short portions of intersecting arcs.

2. Test all check distances or angles. For example, if two houses are plotted independently and the shortest distance between them is given in the notes, always check this distance on the map. Likewise test all the lines given as checks. If an angle is given to a point not for the purpose of plotting but as a check, the field work in measuring the angle goes for nothing unless the angle is tested on the map.

3. Make it a practice to plot all the important points at a station first, such for example as the main corners of a building or of a street.

4. Do not waste time in trying to establish some one point which does not come right. After a reasonable effort to locate it leave it and plot the near-by

points. Frequently when they are plotted the location of the doubtful point will be evident.

660. Interpolating a Contour means plotting a number of points through which the required contour passes, by interpolating each point between two points already plotted, one higher, the other lower than the contour. Assuming the ground to slope uniformly between any two points used for interpolation, there are a number of methods of finding the position of the required point, some of which are more accurate than others. Whatever method is selected, the results should be at least as accurate as those attained in the field.

660(*a*). By ESTIMATION. Since contours are often located in the field more or less roughly, contour points can often be interpolated in plotting with sufficient accuracy by eye alone. Careful estimation may involve more or less mental calculations, but one soon becomes skillful in the process and can then interpolate contour points rapidly.

660(*b*). GRAPHIC METHOD. When the surface of the ground has been divided into squares or rectangles and the height of each corner is given, points may be interpolated, as explained in Art. 598.

Remark. Any three points on the surface of the ground will lie in some plane, but it is not often that *more* than three points lie in the same plane. Hence if a point is found by interpolating between two given points on the map it cannot be checked by interpolating between two other points. Thus, for example, if a point has been interpolated between two diagonally opposite corners of a square, it is not likely that it will coincide with a point interpolated between the other two corners, for it is not likely that all four corners lie in the same plane. This applies especially to the above method of interpolation, but it should be kept in mind in interpolating by any of the other methods.

660(*c*). TRACING-CLOTH METHOD. Let A and B, Fig. 660(*c*), be two points on a map whose elevations are 30.3 and 32.7 ft.

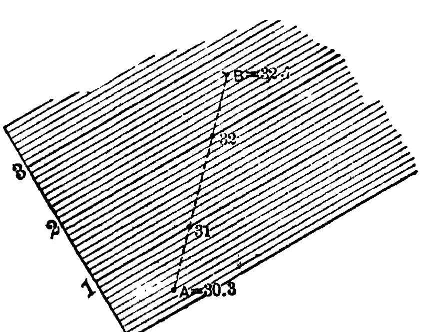


FIG. 660(*c*).

respectively. It is desired to interpolate two points whose elevations are 31 and 32 ft. A piece of tracing cloth is ruled with horizontal lines, any convenient distance apart, each space representing 0.1 ft., and the lines representing whole feet being numbered from 0 to 10. Any point on the line representing 0.3 ft. is set at *A* and held there by a prick point. The tracing is then swung on this point as a pivot until the 2.7-ft. line passes through *B*. The points where the 1-ft. line and the 2-ft. line cross *AB* are pricked through, for they are the required points. Thus the tracing cloth can be used over and over for interpolating any number of points.

Suggestions. (1) The number and spacing of the parallel lines will depend upon the scale of the map and upon the greatest differences in elevation or the maximum distances apart of any two given points used in interpolation. Thus, for example, if the scale of the map is 1 in. = 20 ft., and elevations are given to tenths at distances not exceeding say 40 or 50 ft., then 40 to 60 lines say 1/20 in. apart may be ruled. If there are 60 lines and each space represents 0.1 ft. in elevation, differences in elevation may be interpolated up to say 5 ft. This range may be doubled by calling each space 0.2 ft., quadrupled by calling each space 0.4 ft., and so on. Thus a piece of cloth the size of a postal card will often be large enough. (2) Having ruled lines to suit the conditions, number the lines along the edge to correspond. It is well also to rule every tenth line heavy or in red ink to aid the eye in reading. (3) If, occasionally, two points are too close together and of too great a difference in elevation to use the lines as numbered, let each space represent two, four, or even ten times its usual value, as suggested above.

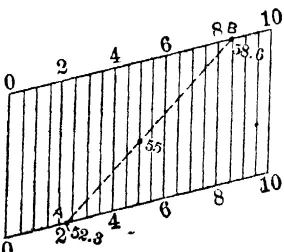


FIG. 660(*d*).

660(*d*). HOMEMADE DEVICE. Let the parallelogram in Fig. 660(*d*) represent the inside edges of a jointed frame with parallel threads stretched across, such as that shown in Fig. 660(*e*). Let it be required to interpolate between two points, *A* and *B*, whose elevations are 52.3 and 58.6 respectively. Set the 2 3 point on one edge at *A* and the 8.6 point on the other edge at *B*. Then thread 5 will cross *AB* at a point on the 55-ft. contour. Likewise if points on the 53-, 54-, 56-, 57-, and 58-ft. contours are desired, they will be where the corresponding threads cross *AB*.

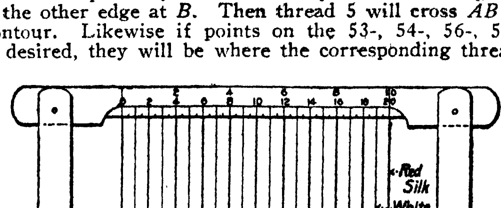


FIG. 660(*e*).

In making this device the number and spacing of threads will depend upon the conditions named in the suggestions given above for drawing lines on tracing cloth. Threads of different colors can be used to aid the eye in reading. This is one of the best of a number of devices in common use, all depending upon the principle of similar triangles.

661. Accuracy Required in Plotting. The same considerations which determine the accuracy required in field work (Art. 18) and govern the choice of scale (Art. 653) will determine also the accuracy required in plotting. As a rule, distances should be laid off as accurately as the scale adopted will permit. Not all parts of a map, however, should be plotted with the same accuracy. Plot carefully those points and lines which have been located accurately in the field; plot less carefully those which have been located less accurately in the field. Thus, for example, traverse lines and boundary lines should be plotted accurately, but indefinite or unimportant details such as river banks, paths, wood roads, and contour lines may be plotted with much less care.

In certain types of maps, accuracy in plotting may not be essential even when the field measurements are made with a high degree of precision. This is particularly true when all linear and angular measurements that will be required are shown clearly on the map. For example, it may be desirable but not essential to plot accurately a map for a deed such as that shown in Fig. 798(*f*). It is often necessary, however, to plot accurately even the small details. For example, on a large-scale topographic map made for an architect, it may be necessary to show to as true a scale as possible all well-defined details and contours which may affect the preliminary location on the map of buildings, walks, driveways, and various landscape features.

661(*a*). TESTS AND SPECIFICATIONS FOR MAP ACCURACY. The accuracy of a map depends upon (1) the accuracy of field measurements and (2) the accuracy in plotting. Tests and specifications for field work have been treated in Part I.

Tests for a planimetric map consist in comparing measurements scaled from the map with corresponding measurements made on the ground. Such tests check the accuracy in plotting but not, as a general rule, the accuracy in field work. Many of the checks used in plotting (Art. 658) can be used also in testing a map. Tests should be made on the original map except when maps are reproduced by a process which causes no appreciable distortion. A graphical scale on the map should render negligible any effect of the shrinkage of the paper.

Specifications for accuracy in a planimetric map vary considerably. One type of specifications requires that the distance between any two well-defined points as scaled on the map and the corresponding distance as measured on the ground shall agree within a specified amount. This specification is often modified by

stating that it applies to a certain percentage of the distances tested, as, for example, 90 per cent. (See Art. 844(*b*).)

Another method of testing the position of a point on a map is to scale its coordinates with respect to a fixed origin, such as a control station, and then to compare the results with the corresponding coordinates computed from field measurements. A typical specification for this test is: "The maximum error in the coordinates of any control station as scaled on the map should be 1/200 in." and "95 per cent of well-defined details shall be plotted on the map in correct horizontal coordinate position within 1/50 in." This method of testing provides a means of checking the direction as well as the length of the line between any two points, and the corresponding form of specification is particularly effective when a high degree of accuracy is required.

Specifications for map accuracy may determine to a large extent the choice of scale (Art. 653). For example, assume that the purpose of a survey will be met if the average error in scaled distances does not exceed 4 ft. and if the average error in plotting the distance between two definite points does not exceed 1/50 in. These conditions are satisfied by a scale of 1 in. = 200 ft. (See the accompanying table.)

TYPICAL MAXIMUM PERMISSIBLE ERRORS IN MAPS

Error in Scaling	Equivalent Errors for Typical Scales			
	1 in. = 200 ft.	1 in. = 100 ft.	1 in. = 40 ft	1 in. = 10 ft.
1/100 in.	2 ft.	1 ft.	0.4 ft.	0.1 ft.
1/50 in.	4 ft.	2 ft.	0.8 ft.	0 2 ft.
1/10 in.	20 ft.	10 ft.	4.0 ft.	1.0 ft.

Tests for a topographic map consist (1) of the tests for a planimetric map and (2) tests for the accuracy of contour lines. A common test for contour lines is to run selected test profiles and then to compare the elevations thus determined by leveling with the corresponding elevations read from the map.

Specifications for Accuracy in a Topographic Map. A common specification for topographic maps (in addition to the specifications for a planimetric map) requires that errors in elevations read from the map shall not exceed one-half the contour interval in a specified percentage of the points tested. For example, "90 per cent of all elevations interpolated from the map shall be correct within one-half the contour interval of 5 ft. and the remaining 10

per cent shall not be in error more than 5 ft." (See Art. 844.) In a code of recommended practice for standards of accuracy of maps, issued by a state technical council, the percentage of points tested which shall be correct within one-half the contour interval is specified for surveys of different degrees of precision as follows: For surveys with a traverse closure of 1/10,000, 90 per cent of the points; for accuracy of 1/5000, 80 per cent of the points; for accuracy of 1/1000, 70 per cent of the points.

Instead of specifying a maximum error of one-half the contour interval, a maximum permissible error may be specified in feet as, for example, 5 ft. or 8 ft.

Note. The specifications for accuracy in maps given in this article are merely typical for certain conditions. The engineer must, of course, draw on his own experience in specifying map accuracy and in following procedures in field work and plotting as well as in map testing by which the specified accuracy may be attained. In this he may be guided by standards of accuracy required by municipal, state, or federal agencies, or by standards recommended by engineering societies or councils.

662. Speed. The suggestions concerning speed in field work (Art. 19) hold good for plotting also. Perhaps nothing contributes to speed more than working up the field notes properly, and getting data together before beginning to plot (Chapter 47). Freehand sketches of transit lines with angles and distances marked on the sketches will often save delay in looking up notes. Skill in using the drawing instruments should be supplemented by an economical method of procedure. *Like operations should be grouped.* Thus, for example, all lines that can be drawn with one position of the T-square and triangles should be finished before beginning another set of lines. When the scale is in hand, all measurements which can be made at that time should be laid off. Make sure by systematic checks that the traverse lines or other lines which form the framework or skeleton of the survey have been plotted correctly before beginning to plot details (Art. 658).

663. Copying and Transferring Maps. Several methods are given below.

1. Pin the map to be copied on the drawing board with a clean sheet of paper underneath, and prick through the important points with a sharp fine needle. Work systematically in order not to omit any points. After the upper sheet is removed find first the points which determine the principal lines and then work up the less important details. Several copies can be pricked at once, and the holes in the original map may be almost obliterated by rubbing them on the back of the sheet with a smooth hard substance.

2. Copy the map with a pantograph (Art. 612).

3. Trace the map and from the tracings make blueprints, brownprints, or blackprints, by the methods described in Art. 622.

4. Rule squares of any desired size, preferably not over 10 in. on a side, on the original map or on tracing paper placed over the map. Draw an equal number of squares on another sheet of paper, and plot each important point by its coordinate distances from two sides of the square in which it is located. Unimportant points may be located by eye. If it is desired to reduce or enlarge the map draw the squares on the clean sheet to a smaller or larger scale, and use this scale or proportional dividers in plotting.

5. Trace the map on thin tough tracing paper, rub the reverse side of the paper along all the lines with a soft black pencil, pin the paper over a clean sheet of drawing paper, blackened side downward, and go over the lines with a sharp hard pencil under moderate pressure. This will leave a faint tracing on the clean paper which can be inked afterward. While rubbing the lead on the back of the tracing paper, lay the latter on a hard smooth surface, as a steel triangle, so that the lead will spread evenly.

6. Maps may be copied by the aid of light from beneath, just as small tracings on thick paper can be made by holding the two sheets against a window pane. The map with the clean paper over it should be laid on a sheet of glass and a strong light reflected through from beneath. A frame modeled after a photographer's retouching frame but larger can readily be constructed. If necessary, a cloth may be thrown over the head to exclude all light except that which comes through the glass. As a rough-and-ready expedient for camp use, a light placed in a box or barrel with a piece of glass across the top can be used.

7. Maps may be photographed, a method often used for producing small copies for printed reports and for other purposes. There are a number of processes based upon the method of photo reproduction. Directions for drawing and finishing maps intended for photo reproduction are given in Art. 684.

663(*a*). Suggestions.

1. Only methods 2, 4, and 7 can be used for copying to a different scale from that of the original map. Method 3 is the most common, since any number of prints can be made from one tracing. Method 4 is used often for drawing portions of government maps to larger scale. Method 5 is particularly useful in city work, such as transferring plots of buildings from one sheet to another.

2. When a map is transferred by methods 2 or 4, the important controlling points should be transferred first. In accurate work, the positions of these points with respect to each other should be checked by comparing various distances between them with the corresponding distances on the original map. After this has been done, details may be transferred.

3. In reducing or enlarging a map by means of ruled squares and proportional dividers (method 4), it is advantageous to use one instrument for transferring the H coordinate and another instrument for transferring the V coordinate of each important point. This combined use of *two* proportional dividers greatly expedites the work of copying a map and reduces the number of mistakes. When the most important points have been transferred in this manner, details can generally be filled in by the use of a single proportional divider.

4. When method 4 is used for transferring a map that is controlled by a "grid" system, it is not necessary in most cases to add to the squares already on the map.

CHAPTER 49

FINISHING THE MAP

FINISHING the map includes such work as inking, tracing, shading, coloring, lettering, water lining, putting on conventional signs, designing titles, drawing meridian needles, ruling border lines, and cleaning the drawing. Directions and suggestions for this work are given in this chapter.

664. Requirements for Maps. Certain requirements common to all good maps are given below, followed by additional requirements to be kept in mind when drawing maps for special purposes.

664(*a*). SPECIFICATIONS FOR WORKMANSHIP. Clean-cut ink lines accurately drawn (Art. 616)—plain unmistakable figures—good lettering (Art. 671)—symmetrical and orderly arrangement (Art. 666)—neat appearance.

664(*b*). WHAT IN GENERAL SHOULD APPEAR ON A MAP. (1) A neat, explicit title. (2) The scale given in figures and also often drawn on the map. (3) A meridian needle or arrow showing the direction of the *true* meridian, or two needles showing the direction of the true and of the magnetic meridian, *with the magnetic declination given in figures.* In some cases an arrow shows the direction of the plane or "grid" north of a state plane coordinate system. (4) Important data such as the date and the name of the surveyor. (5) A simple border line. (6) Explanatory notes. (7) A key to all symbols used.

664(*c*). ADDITIONAL REQUIREMENTS FOR AN ORDINARY MAP will depend, of course, upon the use to which the map will be put. Assuming that this consideration governed the field work, it is probable that any data which appear in the field notes (in addition to those required for plotting) should appear also on the map. The most important requirements for an ordinary property map are these. (1) Property lines indicated as such. (2) Lengths of all boundary lines and character of boundaries, i.e., kinds of fences, etc. (3) Angles at the intersections of boundary lines or else the bearings of all boundaries. (4) Points of intersection (boundary corners) properly referenced with respect to permanent objects, and all monuments marked as such. On some maps it is necessary to show the rectangular coordinates (such as those of a state system) of all important points, such as boundary corners

with reference to a fixed origin. (5) Street lines, curb lines, and building lines clearly indicated as such. (6) Property lines of adjoining land which intersect the boundaries of the tract surveyed shown and angles given. (7) At least one boundary line properly referenced. (See remark, Art. 276.) (8) Names of the proprietors of adjoining land. (9) Names of streets, streams, etc. The requirements for a property map, particularly if it is intended for public record, are given more fully in Arts. 390(*b*) and (*c*); also in Arts. 798(*c*) and (*d*).

664(*d*). REMARK. Incomplete maps are sometimes as bad as inaccurate maps, and are frequently the cause of boundary disputes. In some states and municipalities laws exist prescribing what shall appear on a map before it will be received for public record (Arts. 390(*c*) and 798(*d*)).

664(*e*). REQUIREMENTS FOR TOPOGRAPHIC MAPS in addition to those already given are: (1) contours shown and plainly marked with elevations (Art. 673); (2) different topographic features indicated by conventional signs (Art. 674); (3) datum (actual or assumed) given in a note (Art. 682); (4) different portions of the map tinted with different colors (Art. 676). This last requirement is for a limited class of maps only.

664(*f*). REMARK. Maps for special purposes must meet special requirements. Thus, for example, in Art. 391 are given the requirements for a map made for an architect. Entirely different data must be given on maps for use in landscape work, mining, railroad work, and court proceedings. The aim should be to make maps for such purposes explicit and complete like a working drawing, with all necessary dimensions and explanatory notes.

665. The General Method of Procedure in finishing a map after it has been plotted in pencil may be outlined as follows. (1) Decide upon the general arrangement of border lines, lettering, etc., and draw the working-limit line. (See the next article.) (2) Ink in the main lines such as streets lines and boundary lines. (Art. 667.) (3) Ink in details such as buildings and streams. (4) Letter the map, putting on dimensions and the small single-stroke lettering first, but sketching the names of streets, streams, property owners, etc., in pencil if there is any likelihood of the small lettering interfering with or crowding the principal names (Art. 671). (5) Finish the conventional signs if there are any. (6) Print the title and explanatory notes (Art. 678). (7) Draw the meridian needle and the scale (Art. 679). (8) Draw the key or legend, if one is necessary. (9) Ink the border lines.

Note. The above method of procedure may be followed whether the map is traced or inked directly on the drawing paper. If, however, the map is to be tinted, the tints may be laid on before inking, though many engineers prefer to use waterproof ink and to lay on the tint after inking.

666. Arrangement. (*a*). BORDER LINES. Draw the border lines in pencil. If they are to be square with the points of the compass (a question discussed in Art. 654) one side of the rectangle is either drawn parallel to a meridian line or else plotted from a line of known bearing. If the directions of border lines are immaterial, draw the smallest rectangle consistent with a good arrangement of the map as a whole (Art. 655(*b*)). Space must be provided for a title, a meridian needle, and the necessary

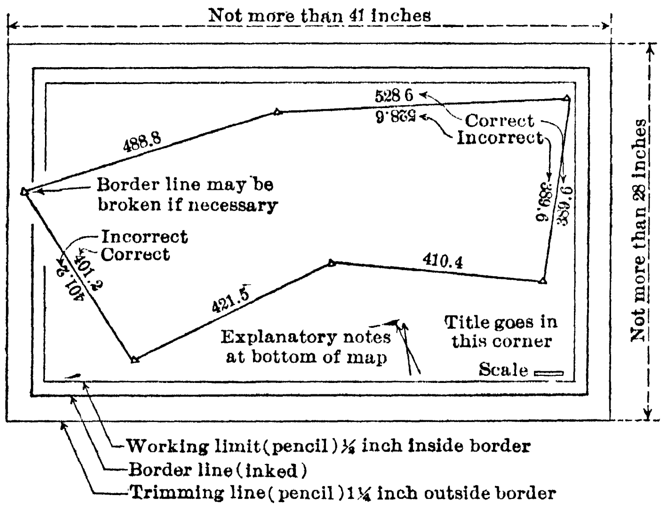


FIG. 666.

explanatory notes. Draw a pencil guide line about ½ in. inside the border line; no line of the map should extend beyond this working limit. Draw another pencil line 1 to 1½ in. outside the border line which will be the edge of the finished drawing after it is trimmed. Avoid making a map square, or with its vertical edge much longer than its horizontal edge. In Fig. 666 the overall dimensions are such that the tracing will fit inside a standard blueprint frame designed for sheets of "double elephant" size. (See Art. 677 for inking border lines.)

666(*b*). LETTERING. Having decided which edge of the map is to be the top (Art. 654), print words and dimensions so that they can be read when either the bottom or the right-hand edge of the map is held next to the body, i.e., print from left to right and from the bottom upward. In only one case should the printing read downward—when the line extends in the general direction of a

diagonal from the upper left-hand corner to the lower right-hand corner of the map. (See Art. 672.) It is a very common mistake for beginners, working from different edges of the drawing board, to put on dimensions and words upside down, especially dimensions. (See Fig. 666.)

666(*c*). Titles. The best place for a title is in the lower right-hand corner, for in searching through a pile of maps it is convenient to raise simply the lower corner of each map without disturbing the pile until the title of the map sought for is exposed to view. Frequently, however, there is no room for a title in the lower right-hand corner, or there is a much larger vacant space elsewhere in which it is desirable to place the title for the sake of a symmetrical appearance. For the same reason the meridian needle and explanatory notes are placed in the most open space, though as a rule the explanatory notes should be near the title, and the meridian needle near the bottom of the map. *Never place the title inside the property mapped.* (See Art. 678 for size of title.)

666(*d*). Names of streets, streams, property owners, etc., are placed in vacant spaces where they will catch the eye, but not where they will obscure any part of the map. If possible, place the names of a street in the center of that street, and the name of a property owner parallel to the boundary line between his land and the property mapped.

666(*e*). Final suggestions for arrangement. (1) Put on necessary dimensions before printing words, and do not put on either until the lines of the map have been inked. (Note an exception in section lining, Art. 675.) (2) Ink the border line last—it may have to be changed slightly. (3) When working on the miscellaneous small lettering, plan to leave space for the names which require large letters—it is frequently expedient to sketch the latter roughly in pencil first.

667. Inking. (1) Make sure that the ruling pen is in good working order before beginning to ink (Art. 608). (2) Make the lines clear cut and fairly heavy especially on tracings—the general tendency is to make lines too light (Art. 621). Widths of lines may vary according to their importance, but use the same width of line for like details. (3) Be careful to follow the pencil lines exactly. Of what avail is it to plot accurately in pencil only to lose accuracy in inking? (4) Follow the general directions for inking (Art. 616). (5) Ink the lines in the order of their importance, the most important lines first. (6) Edges of streams

and similar indefinite lines should not be ruled but should be drawn freehand zigzagging. Use a ruling pen or a contour pen for freehand work to secure a uniform width of line. (7) The map will "stand out" better if portions of the outlines of details are drawn heavy, i.e., if outline shading is employed, as in working drawing. (See the next article.) (8) It is not customary to ink the traverse lines, but if they are shown at all they may be drawn in red ink in order to make them less conspicuous on the paper or on the blueprint. If it is desired to show the location of stations this may be done without inking the traverse lines by merely drawing a very small circle in red ink around each station point; small equilateral triangles are used for triangulation stations. In some maps such as those of mines it is important to show all stations. (9) The dashes in broken lines should be short and close together. (See Fig. 666.)

668. Outline Shading. A certain portion of an outline may be drawn heavier than the remaining portion. City blocks, buildings, bridges, outlines of land against water, and similar objects may be shaded as *projections;* streams, lakes, streets, and roadways should be treated as *depressions.* Thus, for example, if the top of the map is north, the *south* and *east* sides of a building are the heavy lines; in the case of a stream it is the *north* and *west* banks that are shaded. The practice of outline shading is by no means generally followed.

669. Use of Colored Inks. Lines which represent existing features are almost invariably drawn in black, but proposed changes, such as a new layout for a road or a change of grade for a street, may be shown in red. Contour lines are often drawn in sepia or seal brown, and water lines in blue (except on tracings). These are the principal colors used unless a map is tinted. On special maps additional colors are used to bring out different features, as, for example, when the workings in the different levels of a mine are shown on the same plan and a different color is used for each level. Dimension lines may be fine black lines, or black broken lines, or, if it is desired to make them distinctive, they may be drawn in red. Red dimension lines with black arrow heads are sometimes drawn on tracings in order to make the lines faint but distinct.

669(*a*). TESTING INKS. Prepared colored inks furnished in bottles are sometimes unsatisfactory and deteriorate rapidly. Test by drawing several lines as wide as will be required on the map. If the ink seems thin and gathers at the end of the stroke, leaving the end of the line a more pronounced color than other portions, it should be rejected.

670. Tracings. General directions for tracing are given in Art. 621. Most of the directions for finishing a map in ink (Art. 667) apply also to tracing. If colored inks are used it should be remembered that colors known as "earths," such as deep yellows and browns, will give the whitest lines on blueprints; blues will almost vanish in printing as they offer little resistance to the light rays. Many of the prepared red inks will give only faint lines, but a scarlet vermilion color should give fairly white lines. For lettering tracings by shifting letters underneath see Art. 671(*d*). For tinting tracings see Art. 620(*b*).

671. Lettering.* Good lettering does not depend to such an extent as is generally supposed upon ability to handle a pen. If letters are properly formed, well proportioned, and well spaced the general effect will be pleasing even though the strokes are made freehand and are somewhat ragged. A knowledge of the essentials of good lettering is all the more important when the ruling pen and straightedge are used, for the mere fact that the letters are not drawn freehand indicates that an extra effort was made to do good work; hence extra good results may be expected. Letters of different styles used in the same word and letters too wide or too narrow for their heights are defects which show ignorance, and no mere skill in execution can atone for this. A knowledge of the forms and proportions of the letters in at least one good standard alphabet, such as the Gothic, may be gained in a very short time and will go far toward helping the beginner to do satisfactory lettering.

671(*a*). STYLE OF LETTERS. Letters on maps should be simple. Ornate letters indicate extra effort to produce good effects, and when they are poorly executed, as they frequently are, the failure is more pronounced than if only a simple dignified style had been attempted. Poor lettering may ruin the appearance of a map, but it is not necessary to go to the other extreme and to give to lettering an undue prominence. It too often happens that time is wasted in ornamental lettering that could have been used to advantage in plotting the map more accurately, and occasionally such lettering is used to impress the uninitiated with the excellence of a very poor map.

* The single-stroke system of lettering has already been explained in Art. 29, and hence this article deals mainly with letters which are not made with single strokes. The subject of lettering is well treated in standard books on engineering drawing and in inexpensive books on mapping and lettering. It is not the intention, therefore, to treat the subject in detail here, but rather to guide the student in his study of lettering in other books, with special emphasis on the requirements in mapping.

There are two standard alphabets which form the basis of map lettering, i.e., *Gothic* and *Roman*. Gothic capitals are the simplest but most severe in outline of all the classic alphabets. There are several styles of Gothic alphabets; that most used for maps is often called *Block Letters* (p. 792). The style of Roman letters most used for maps is called *Modern Roman* (p. 798). Capital letters of the Roman style are simple and very effective. They are perhaps the most beautiful of all letters, but the spurs, delicate serifs, and hairlines are difficult to draw, and if not executed well, they are especially noticeable because the eye is so accustomed to the comparatively perfect forms in ordinary type. The student is, therefore, advised to begin with the Gothic style. However, as soon as he has become skillful enough to undertake the Roman style, he should do so, for it is extensively used for maps. Letters in both styles may be upright or inclined. Slight variations from the standard slope of inclined letters are less noticeable than equal variations from the vertical in upright letters. On the other hand, vertical lettering is considered by many to produce work which is more uniform in appearance and more legible, especially in blueprints.

Different styles of alphabets, as, for example, Gothic and Roman, should not be used indiscriminately on the same map. It is permissible, however, to use both vertical and inclined lettering, one style for certain features on the map, the other for certain other features (Art. 671(*l*)).

671(*b*). THE HEIGHT OF LETTERS depends upon the importance of the object described, the scale of the drawing, and the size of the map. For ordinary maps, the height of the *largest* capitals varies from 0.3 to 0.6 in. Gothic capitals 0.5 in. high are large enough for most maps, and often 0.3 in. is a better height. Another rule sometimes given is that the largest capital letter should be one-sixtieth to one-fiftieth of the length of the shorter border of the map. Thus, if the shorter border is 30 in. long, the height of the largest capitals will be 0.5 to 0.6 in. When a combination of large and small capitals is used in any word, the small capitals should be two-thirds to four-fifths as high as the large capitals.

The height of lower-case letters may be three-fifths to two-thirds of the height of capitals; six-tenths of the height is recommended. The portion of a vertical stem that extends above or below the main part of a letter, as in *b* or *p*, is not included in this height.

The height of numerals may be as great as that of capitals or as little as that of lower-case letters, depending upon the use to be

made of them. The tendency is to make them too small to be clearly legible, an inexcusable fault. It may be possible to guess what an indistinct word is, but it should never be necessary to guess what a numeral is. The height of a fraction is about twice that of a whole number. The numerals in the numerator and denominator are each about two-thirds the height of a whole number, but in any case they must be large enough to be legible.

Notice that the limits given for both capitals and lower-case letters are *maximum* heights. Letters of much less height are, of course, used on maps. For example, in explanatory notes the height of capitals may be as little as 0.1 in., or even less, and the height of lower-case letters as little as 0.06 in.

All the limits given are intended merely as a rough guide. The tendency is to make letters too large. "They shouldn't stare one in the face upon picking up a map."

671(*c*). PROPORTIONS OF LETTERS. The best proportions for letters are, to some extent, a matter of individual taste; nevertheless, some guiding principles can be given. The proportions which determine very largely the general appearance of letters are (1) the width of the letter and (2) the thickness of the stems, both measured by a unit which is a fraction of the height of the letter. For example, in Fig. 671(*c*), the height of the capital letters is divided into seven equal parts, and one part, i.e., one-seventh of the height, is taken as the unit of measurement. The numeral under any letter indicates the number of these units in its width. For example, the width of the letter B is six-sevenths of its height, and the width of the letter W is eleven-sevenths. The thickness of the letters is the same as the width of the letter I, i.e., one-seventh of the height.

The letters shown on the next page are one type of Gothic letters.* From inspection it is seen that the normal width of capitals is six units, i.e., six-sevenths of the height. There are only two letters narrower and six letters wider as shown below.

Normal: **B, C, D, E, F, G, H, K, L, N, O, P, Q, R, S, U, Y, Z** (6).

Narrower: **I** (1); **J** (5).

Wider: **A** (7); **M** (8); **T** (7); **V** (7); **W** (11); **X** (7).

* The drawings of letters on pp. 792 and 798 are reproduced from the excellent book, *Technical Drawing*, by Professors Frederick E. Giesecke, Alva Mitchell, and Henry Cecil Spencer, of the Agricultural and Mechanical College of Texas. The book is published by The Macmillan Company of New York.

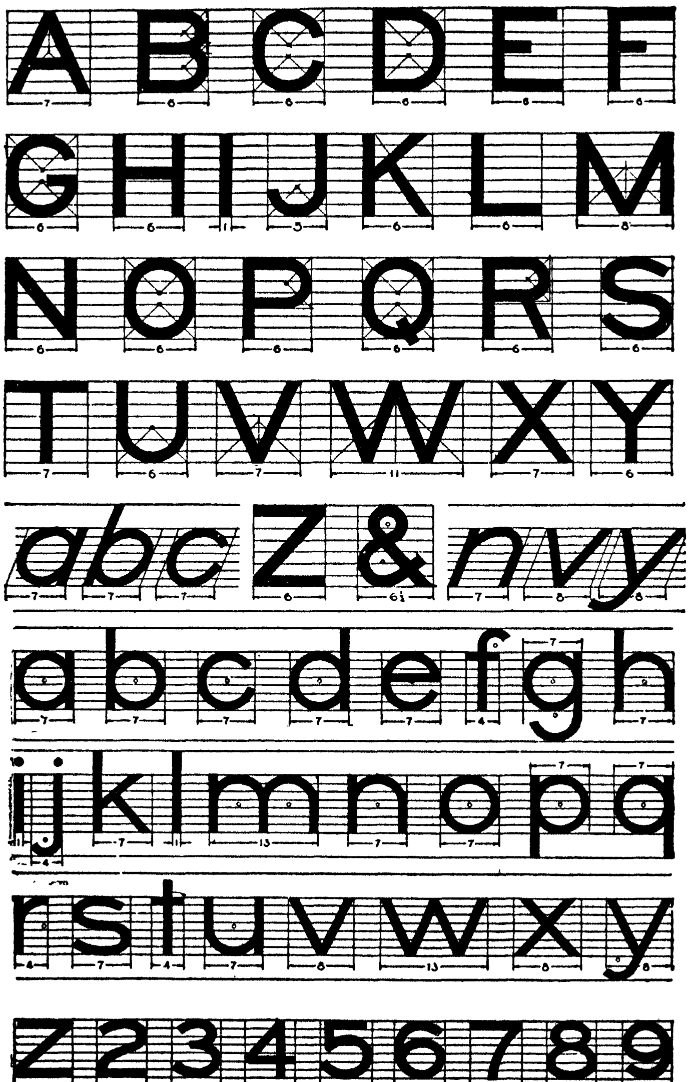

ABCDEF
GHIJKLM
NOPQRS
TUVWXY
abc Z & nvy
abcdefgh
ijklmnopq
rstuvwxy
z23456789

FIG. 671(*c*).

In another type of Gothic letters, the unit is one-sixth of the height, and this also is the thickness of the stems. This results in thicker stems than those in Fig. 671(*c*) and consequently in letters which are heavier in appearance. The proportions of widths to height are also somewhat different from those given on that page.

On p. 798 are shown the proportions of one type of Modern Roman letters. It is to be noted that the widths in this alphabet vary more than do the widths of Gothic letters.

If the upper portions of the capital Gothic letters **B**, **E**, **G**, **K**, **R**, **S** are made equal in width to the lower portions, the letters will appear top-heavy. To correct this, the widths of the upper portions are made somewhat less than the widths of the lower portions. If the central lines of the horizontal bars of the letters **B**, **E**, **F**, **H**, and **R** are at midheight, the bars will appear to be too low; consequently, they should be drawn with their central lines above midheight. On the other hand, the central line of the horizontal bar of **P** is lower than midheight.

In studying the proportions of letters, the *critical points* for each letter should be observed. Among such points for capital letters are those that mark the extreme width, top and bottom; the thickness of stem; the position of the horizontal bar; the position of an inclined stem, particularly the points in which it intersects another stem; and the beginning and end of any curve.

For every vertical style of lettering, there is a corresponding inclined style. A common inclination is two horizontal units to five vertical units; another inclination often used is three horizontal units to eight vertical units. The proportions for inclined letters are practically the same as for the corresponding vertical letters, provided the height is measured vertically and the width is considered to be the horizontal distance between the two inclined lines that bound a letter on either side.

While guiding principles are useful, only a careful study of well-proportioned letters can give one an adequate knowledge of correct proportions. From such a study and from intelligent practice one can gradually acquire a *sense of proportion.**

Remark. Since poor proportions are more evident in large letters than in small, it is well to begin practice by making pencil outlines of capital letters approximately one inch in height, and then, when some skill has been acquired in proportioning, to reduce gradually the height to the average height of capitals used in mapping.

The proportions of lower-case letters are shown on pp. 792 and 798. The *ascenders* in the letters b, d, f, h, k, and l extend to the horizontal top

* In an excellent textbook on mapping and lettering by Mr. Malcolm Lloyd, published by the Blakiston Co., Philadelphia, is a set of plates which show the proportions of letters, both vertical and inclined, for different styles of alphabets. These proportions were adopted by Mr. Lloyd after an exhaustive study of the best examples of lettering found on European and American maps.

line of the capital letters, and the *descenders* in the letters g, j, p, q, and y extend an equal distance below the bottom line.

671(*d*). SPACING. Aim to space the letters in any word in such a way that the blank areas between adjacent letters *appear* to be equal whether they really are or not. This is the first and most important rule, *but it does not mean that letters should be spaced equal distances apart*, for if they were, letters like **L** and **A** which do not entirely fill the limiting rectangle will often appear separated from other letters. Thus, for example, one of the most difficult words in which to space letters correctly is **LAW**. Whenever the letter **L** is adjacent to a letter like **A** or **W** it is necessary to close up the space, and sometimes to shorten slightly the horizontal stem of the **L**. Likewise when any letter which does not fill its rectangle occurs in a word it may be necessary to close up slightly the space on either side. The closer letters are together, the more this is necessary. When the space between letters is two or three times the width of a single letter the distances between the centers of letters may be made nearly equal without offending the eye. As a general rule, however, the best results are attained by spacing letters as close as may be but without crowding; occasionally it may be necessary to space them farther apart in order to give prominence to one or more words. When in any word there is a series of letters each of which is bounded on either side by a vertical line, as, for example, the letters **H**, **M**, and **N**, the spacing may be slightly greater than normal to avoid a crowded appearance. Frequently a name must be strung out over a considerable area on a map, but letters should not be so far apart that it is difficult to read the name.

Ordinarily the space between two words should be at least 1½ times the normal width of the letters. When a punctuation mark is used, this space should be measured from the mark, thus increasing slightly the space between words. The space should be still larger if the punctuation mark is a period.

The clear space between two successive lines of capital letters should preferably be at least two-thirds the height of the letters, but may vary from one-half to the entire height of the letters. Spaces between lines may be nearly equal regardless of whether the height of letters in one line differs from that of letters in another. This rule does not hold good, however, when, as in title, some lines of letters are long and some short; in such a case blank areas above and below a line should be balanced.

Satisfactory spacing, either of letters or of lines of letters, cannot be achieved by any mechanical method. It is best accomplished by eye alone after one has attained a *sense* of spacing.

In lettering tracings it is often best to sketch the outlines on paper first, and then, by shifting the paper back and forth underneath the tracing, to correct any defects in the spacing of letters or words. This method is

especially useful when several lines of unequal length are to be centered with respect to a vertical axis of symmetry, as in printing titles. (See Art. 678(*b*), 6.)

Remark. Rules for proportioning and spacing should not be followed blindly. They are intended, chiefly, to guide the beginner to a proper sense of harmony in lettering, without which no set of rules is of much avail.

671(*e*). THE GENERAL PROCEDURE IN LETTERING is to draw top and bottom horizontal guide lines in pencil, to block out the letters with a sharp pencil, and then to ink them with a pen. Even an expert usually requires a bottom guide line, and the beginner will need one at the top also. When lower-case letters as well as capitals are used, a third guide line for the tops of the lower-case letters may be drawn. When letters are inclined short guide lines at the angle of inclination may be drawn near enough together to assist in keeping the slope of all letters uniformly the same. Letters may be drawn freehand, or with mechanical aids, or partly freehand and partly with mechanical aids. Letters with a height of ¼ in. or greater can be drawn with mechanical aids, but for letters of less height such aids are impracticable.

671(*f*). PENCILING LETTERS. An expert can often letter directly in ink with comparatively little preliminary work in pencil to guide him, but until such skill is acquired, letters should first be drawn in pencil. For letters that will be inked with single strokes of the pen, single strokes in pencil may be sufficient, but for larger letters it is usually better to outline in pencil all the edges of each letter. This penciling should be done with care, for if the letters are not well proportioned and well spaced in pencil they are seldom improved in inking. A 2-H pencil is recommended; the point should be kept sharp.

Before outlining a letter it will be helpful to place dots at critical points of the letter (see Art. 671(*c*)). The positions of these critical points are judged by eye; the correct proportioning of the letter will depend largely on how well the dots are placed.

In penciling a letter, the sequence of strokes used in single stroke lettering may be followed (Art. 29). It is well to draw the outside edges of a letter first—vertical strokes downward, horizontal strokes from left to right. Make all lines light but sharp and clear. Complete one letter before beginning the next. When three or four successive letters have been outlined, observe the spacing. It may have to be corrected before proceeding by re-drawing one or more of the letters. When letters are inclined,

draw at least one inclined line, an inch or two long, in some convenient position where it can be used as a guide to the eye in keeping the inclination of the letters uniform. Many of the straight lines, particularly in large Gothic letters, may be drawn by means of triangles and T-square about as quickly as they can be sketched freehand.

671(*g*). INKING LETTERS. Letters may be inked freehand or with drawing instruments. There are three freehand methods. (1) Ink entirely by single strokes. (2) Draw first single strokes and build up the thickness of stems by drawing along the sides of the single strokes until the required thickness is obtained. (3) Draw the outline of the edges of the letters and then fill in between. This last method, used for large capitals with thick stems, may be outlined as follows.

Begin by inking the pencil outline, taking care to keep the outside edges straight. If the inside edges are ragged it does not matter, provided the space between outlines is to be filled in solid. *Keep inside the pencil lines.* The direction and sequence of strokes are the same as for single-stroke lettering (Art. 29). In filling in, use long strokes. After a letter is dry, corners may be made square and other imperfections touched up, although very little of this should be necessary. Wipe the pen frequently, and do not overload it with ink. Keep a piece of paper tacked to the board to try the pen on and thus start the even flow of ink. Shields are useful in erasing portions of letters (Art. 617(*b*)). *Keep the stopper in the ink bottle* to exclude dust, to prevent the ink from deteriorating, and to save the drawing in case the bottle is upset.

When letters are over ¼ in. in height and highly finished lettering is desired, the outlines may be inked with ruling pen and small spring bow compasses. Special effort should then be made to secure clean-cut corners and smooth curves. *Do not ink portions of the letters freehand and other portions by mechanical means*—the contrast will not be pleasing. Highly finished letters in many cases are not desirable; on maps of a temporary character they are unsuitable and they make lettering needlessly expensive.

671(*h*). PENS. There is a great variety of pens used in freehand lettering. The choice of pen should be determined by the size and style of letters, the kind of paper, and the character of the surface. For lines varying in width from very fine to ordinary thickness, Gillott's pens *tit-quil, 303,* and *404* are recommended; for coarser lines, ball-pointed pens are often used. The best pens for lettering are those with which it is easy to make a stroke of uniform width. There are many special pens that are designed to meet this requirement, some of which produce fine lines and others produce lines as wide as will be required in ordinary lettering. It is well to try a number of different kinds of pens until those most suitable are found.

671(*i*). Defects in Lettering. (1) The beginner almost always makes Gothic letters too heavy and clumsy. The letters outlined in ink seem correct, but after the outlines have been filled in solid black, the stems are too thick. This is because the thickness of the outlines is added to the space between them. Avoid clumsy letters by making stems slightly narrower in pencil than seems necessary. (2) Another common tendency is to make stems which are curved or inclined thicker than those which are vertical and horizontal; hence such letters as **O**, **U**, **M**, and **W** are likely to be too heavy. (3) Great care is necessary to avoid a too heavy appearance at acute angles, as, for example, at the bottom of the **V**, and at the junctions (top and bottom) of **M**, **N**, and **W**.

Other common defects are poor proportions; poor spacing; letters not uniform in height; stems not uniform in thickness; letters not uniformly vertical or uniformly inclined; letters of different types in the same word; letters too large or too small for their purpose.

671(*j*). Roman Letters. Proportions of Roman letters are indicated in Fig. 671(*j*). The general directions already given for choosing the height of letters and for spacing letters hold good for the Roman style of lettering as well as for the Gothic. However, the general directions for penciling and inking thus far given apply particularly to Gothic letters in which the thickness of stem is uniform. In Roman letters, the combination of hairlines with thick stems, serifs, and spurs makes both penciling and inking more difficult. Methods of drawing Roman letters will not be given here; they may be found in most of the standard books on engineering drawing and on mapping.*

671(*k*). Mechanical devices for lettering are sometimes criticized as producing too mechanical an effect. Nevertheless, for ordinary work they are practical and efficient aids and are quite extensively used. Among such devices are these.

Lettering Angle. A piece of celluloid, the lower edge of which is straight. In the upper edge are cut V-shape openings with edges at the various inclinations commonly used in lettering. When the lower edge of the celluloid is held in a horizontal position, the edges of the openings are used for ruling the inclined portions of the letters.

Wrico Lettering Guides and Pen. Each guide is a strip or template of transparent material in which there is a line of openings cut in the shapes of the letters of an alphabet and of numerals. The special pen is moved in contact around the edges of these openings to form the letters. Different special pens to produce lines of different widths and different templates, some with small openings and some with large, make it possible to draw letters of various thicknesses and heights. The *Normograph* is a similar device.

Leroy Scriber and Templates. Each template is a strip of laminated material on the surface of which are well-defined black letters on a white background. Each letter is outlined by a tracing groove. The scriber is a special form of pantograph equipped with a Leroy pen held rigidly in a vertical position. As the tracing point is moved in the groove around a letter, the pen draws the letter. Each template contains a full alphabet in capitals and a set of numerals; some templates have, in addition, the lower-case letters. By

* Particularly good directions for the construction of Roman letters are given in *Mapping and Lettering* by Malcolm Lloyd, published by the Blakiston Co., Philadelphia.

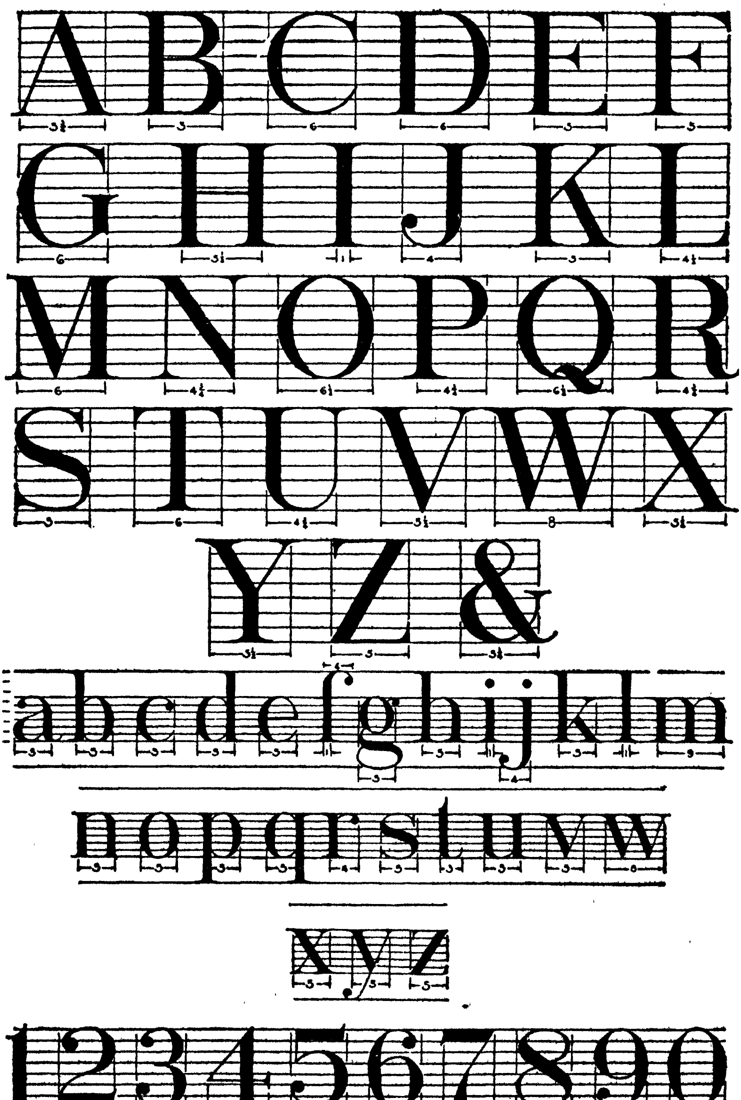

ABCDEF
GHIJKL
MNOPQR
STUVWX
YZ&
abcdefghijklm
nopqrstuvw
xyz
1234567890

Fig. 671(*j*).

the use of different templates and different pens, letters of different heights and thicknesses can be drawn. The *fixed* scriber will produce vertical letters from vertical templates and inclined letters from inclined templates. The *adjustable* scriber will produce either vertical or inclined letters from vertical templates—a decided advantage. This device for lettering is quite extensively used.

671 (*l*). Uses of different types of lettering. Attempts have been made to standardize, to some extent, the uses of various types of letters on maps, but most engineers prefer to use their own judgment. It may be said, however, that for highly finished maps, Modern Roman letters are quite generally used. The following uses of capitals and lower case letters have been recommended: *

(1) *Vertical Capitals.* Names of states, countries, townships, capitals, large cities, and titles of maps.
(2) *Vertical Lower-Case Letters* (first letter of each word a capital). Names of small towns, villages, post offices, etc.
(3) *Inclined Capitals.* Names of oceans, bays, gulfs, sounds, large lakes, and rivers.
(4) *Inclined Lower-Case or "Stump Letters"* (first letter of each word a capital). Names of rivers, creeks, small lakes, ponds, marshes, brooks, and springs.
The names of prominent land features, such as mountains, plateaus, and canyons, are lettered in vertical Gothic; the names of small land features, such as small valleys, islands, and ridges, are lettered in vertical lower-case Gothic. Names of railroads, tunnels, highways, bridges, and other public structures are lettered in inclined Gothic capitals.

672. General Suggestions for Lettering Maps.

1. The general arrangement of the lettering (Art. 666) requires that words be printed from left to right or from the bottom up except when the general direction is that of the line *AB* in Fig. 672. The aim should be to have every word lettered in such a direction that it can easily be read from the middle of the bottom edge of the map without turning the head or the sheet. It is permissible to letter a vertical line in the central portion of the map either upward or downward, though the former is usually the better direction.

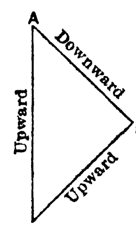


Fig. 672.

2. Aim to make as much of the lettering as possible parallel to the bottom border line or, if this is impracticable, parallel to the side border lines. There are many exceptions, however, as for example, property lines, highways, streams, and other details, for which the lettering should usually follow the general direction of the lines that represent the detail.

3. In general, words should be in straight lines, although there are many exceptions as, for example, the name of a curved street

* See *Technical Drawing* by Professors Giesecke, Mitchell, Spencer, published by The Macmillan Company, 1937, New York.

or of a winding river. Words greatly extended like the names of towns are often in long sweeping curves. When a word is on a curve the axes of the different letters should be radial lines unless the letters are of the inclined type.

4. When there are two lines of words and it is not desired to center them, it is often better to start the second line so that it will extend beyond the first, provided the second line is not too short. This applies especially to descriptive matter in single-stroke lettering which is a little too long for a single line, but not long enough to make two full lines.

5. Names or descriptions should be placed in such a position that there will be no doubt as to where each belongs, either between the boundary lines of an object, as in the case of a street or a river, or just above, as in the case of a narrow stream, or just to the right of a conventional sign, as in the case of a small building. Avoid a crowded appearance.

6. In printing values of angles *never omit zeroes.* Thus 0°–08′ is correct, but not 08′ or 8′. Likewise in linear measurements 0.8 ft. or 0′.8 is correct, but not .8′.

7. There is danger of reading angular measurements for linear measurements, or vice versa. To avoid this, always place a dash between degrees and minutes. Thus, for example, 129°–41′ cannot be mistaken for 129′.41 (feet). When there is no possibility of mistaking an angular measurement for a linear measurement the degree and minute signs are sometimes omitted. They are used less than formerly.

8. Since every line has two bearings, in printing the bearing along any line (especially a property line) it is well to draw a short arrow parallel to that line and pointing toward the end which is the *forward* end for the bearing given.

9. Prominence should be given to lettering according to the importance of the object described. Prominence may be gained in several ways, viz.: (*a*) by increasing the width of letters; (*b*) by increasing the weight; (*c*) by increasing the height; (*d*) by increasing the spacing; (*e*) by changing the style; and (*f*) by italicizing.

672(*a*). NOTE. Letters may be applied to some types of maps by the transparent adhesive method now in use by U.S. Government mapping agencies.

673. Contour Lines.

1. Contour lines may be drawn in black ink, but to avoid confusion with other black lines on the map, they are often drawn

in brown ink or burnt sienna water color. The last reproduces well in blue printing or photo reproduction. Occasionally contours are represented by broken lines in black ink (short dashes close together).

2. Contours should be represented by fine, clean, sharp lines of uniform thickness; too often they appear ragged. When an ordinary pen is used, such as Gillott's 303, it should be applied with little or no pressure to yield a uniform line. An ordinary ruling pen will produce such a line if manipulated carefully, but a swiveled contour pen makes an even better line, and it is easier to use.

3. Every fifth or tenth contour, counting from the datum plane, should be heavier than the others and *also darker in color,* unless all contour lines are black. There should be a decided contrast between these accentuated contours and those drawn between them.

4. Each accentuated contour should be marked with its elevation above or below datum, and if the map does not contain at least two such contours then every contour line should be so marked. This contour elevation may be placed just above the line or preferably in a break left for that purpose. The elevations should not be scattered haphazard, but should be placed nearly under each other, or, better still, in a line normal to the contours (usually a winding line). Where the figures are to be inserted in breaks, before inking draw *two* parallel guide lines in pencil, about ½ in. apart and normal to the contours. When inking the *accentuated* contours stop each one at these guide lines to give a uniform appearance to the breaks. Elevations may be marked in the color used for the contour lines, and they should be repeated in different parts of the map wherever necessary, especially where contours change direction abruptly. Aim to place them where they will be of the greatest help in ascertaining the elevation of any contour on the map without delay or mental confusion.

5. Contour lines should not be drawn across buildings, roads, or streams, and they should be turned slightly uphill just before ending at the edges of roads or banks of streams. (If a road is in "fill," the contour lines turn downhill.)

6. Intermediate contours, represented by broken lines, are sometimes inserted to show the outline of some feature of the surface not shown by contours at the regular interval, such, for example, as a summit of a hill or the bottom of a depression.

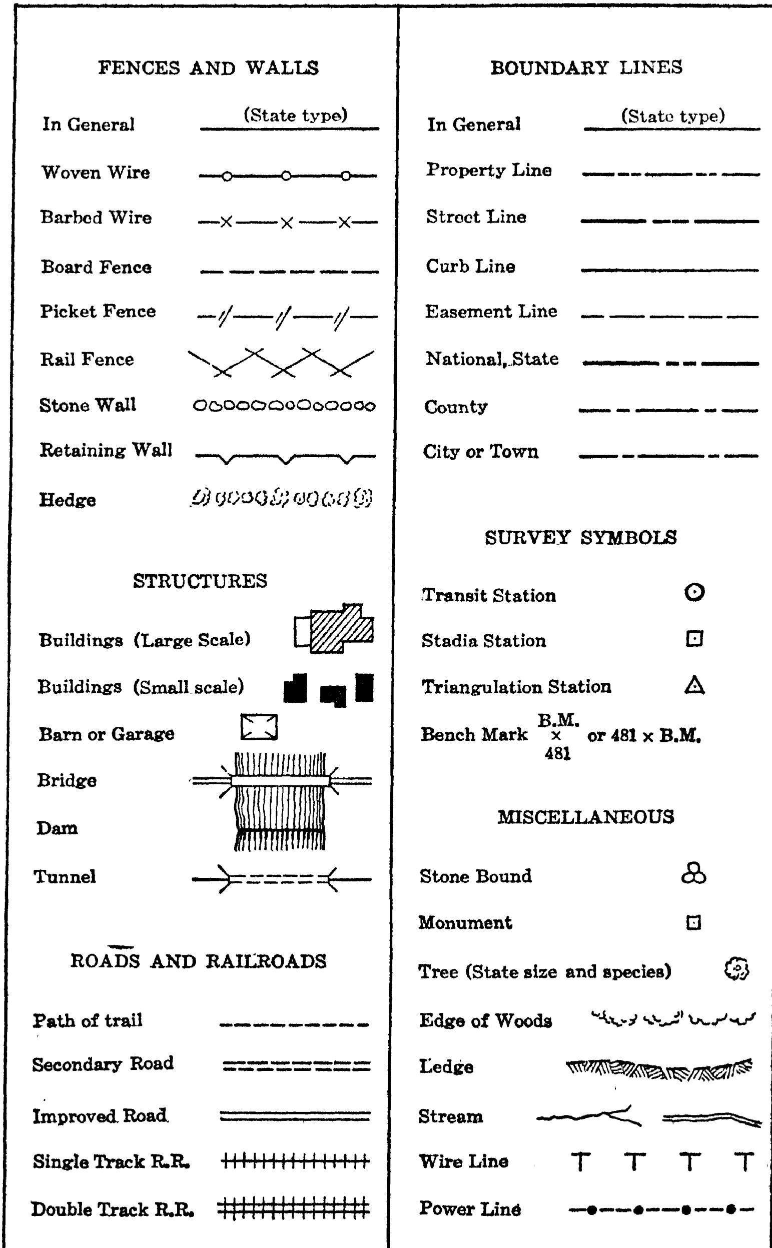

FENCES AND WALLS
In General
(State type)
Woven Wire
Barbed Wire
Board Fence
Picket Fence
Rail Fence
Stone Wall
Retaining Wall
Hedge
STRUCTURES
Buildings (Large Scale)
Buildings (Small scale)
Barn or Garage
Bridge
Dam
Tunnel
ROADS AND RAILROADS
Path of trail
Secondary Road
Improved Road
Single Track R.R.
Double Track R.R.
BOUNDARY LINES
In General
(State type)
Property Line
Street Line
Curb Line
Easement Line
National, State
County
City or Town
SURVEY SYMBOLS
Transit Station
Stadia Station
Triangulation Station
Bench Mark
B.M.
×
481
or 481 × B.M.
MISCELLANEOUS
Stone Bound
Monument
Tree (State size and species)
Edge of Woods
Ledge
Stream
Wire Line
Power Line

FIG. 674(1).

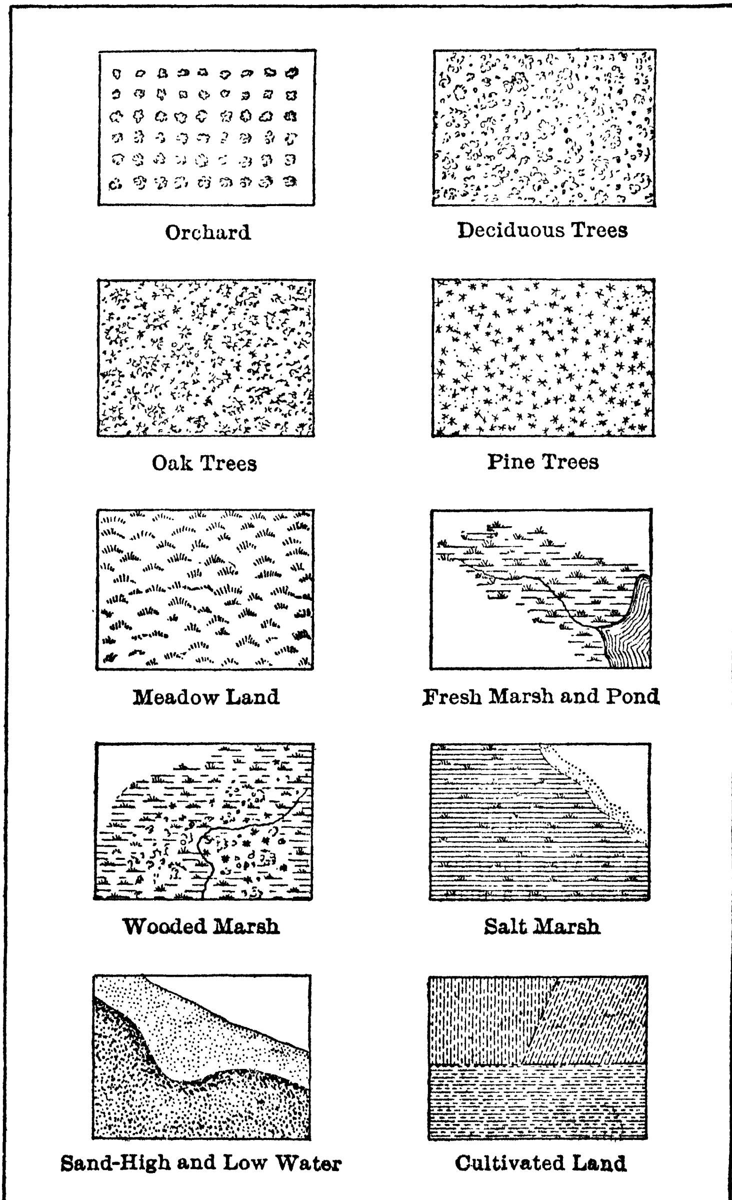

Orchard
Deciduous Trees
Oak Trees
Pine Trees
Meadow Land
Fresh Marsh and Pond
Wooded Marsh
Salt Marsh
Sand-High and Low Water
Cultivated Land

FIG. 674(2).

7. Common mistakes, such as drawing contours of different elevations across each other or merging contours of same elevation, may be avoided by remembering fundamental principles (Art. 452).

Remark. In small-scale topographic maps hills and mountains are sometimes shown by hachures or short disconnected lines. (See books on mapping and books on topographic surveying.)

674. Conventional Signs. In Fig. 674(1) are shown some of the conventional signs used on ordinary maps, drawn to a large scale, and in Fig. 674(2) are some of those used on topographic maps. Most of these conventional signs may be considered standard, though some of them, especially those pertaining to fences, are subject to more or less variation. A common fault is to make conventional signs too prominent by drawing them too large and heavy. Ordinarily, very light strokes should be used for trees and grass, and ruled lines, such as those for marshes and cultivated land, should be drawn almost as fine as the pen will work. Conventional tints are sometimes used in connection with conventional signs. (See Art. 676(*a*).)

674(*a*). SUGGESTIONS. (1) Instead of using conventional signs for fences it is quicker and often better to draw full lines or broken lines as preferred, and then to mark the character of the fence along each line. (2) Small water courses, where indicated by a single line, should be drawn toward the source so as to end there with as finely tapering stroke as possible. Aim to produce a wavy effect as shown in the right-hand side of Fig. 674(2), avoiding the flat meaningless style of line shown in the left-hand side of the same figure. (3) Outline shading (see Art. 668) may sometimes be used to advantage. (4) The tufts of grass in "meadow land," Fig. 674(2), should begin and end in short light strokes of the pen, the lines radiating from a common center some distance below. (5) For additional conventional signs, used on topographic maps, drawn to a small scale, see the chart of standard symbols, published by the U.S. Geological Survey; also books on topographic drawing. (6) Conventional signs may be applied to some types of maps by the transparent adhesive method now in use by U.S. Government mapping agencies

674(*b*). WATER LINING when well done adds much to the general appearance of a map, but when poorly done it is very offen-

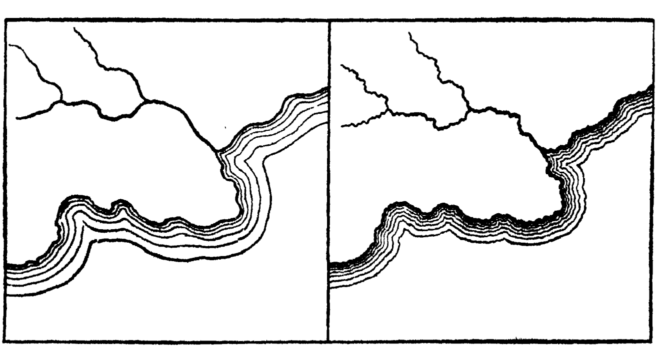


FIG. 674(*b*).

sive to the eye. The beginner is inclined to draw water lines too hurriedly. Thus the effect in the left-hand side of Fig. 674(*b*) is that of work carelessly done. In the right-hand side, on the contrary, every bend and irregularity of the shore line was carefully followed. The shore line is made heavy, and the first water line drawn as close to it as practicable. The spaces between the water lines gradually widen, and the little irregularities grow less and less pronounced. Six to ten lines will usually suffice; those farthest from shore should be drawn very lightly. Prussian blue is often used for water lines, even on maps where everything else is drawn in black. This does not apply to tracings, for blue lines practically disappear in making process prints.

675. Section Lining. The outline of a building on a map of large scale is frequently emphasized by section lines; on small-scale maps the outline is filled in solid black. Suggestions for section lining are given in Art. 619. Leave open spaces for words and dimensions so that they will not be obscured by section lines drawn over them.

676. Tinting. General directions for tinting in water colors are given in Art. 620. These are additional suggestions. (1) Either tint before inking or use waterproof ink. (Note, Art. 665.) (2) The tendency is to use tints that are too vivid. The beginner will do well to mix his tint to what seems a desirable shade, and then to dilute about half. (3) Tinting on ordinary maps is usually confined to filling in outlines of buildings and other structures. Occasionally roads and streams or bodies of water are also tinted. In many cases, maps may be tinted satisfactorily with colored pencils (Art. 620(*a*)). For tinting tracings with such pencils, see Art. 620(*b*).

676(*a*). Conventional tints

Buildings. If of wood, sepia; for masonry, crimson lake. The outlines of wooden buildings are drawn and shaded with India ink; outlines of masonry with a very strong tint of crimson lake. To distinguish brick from stone, make the outlines of the latter heavier or else use Payne's Gray for stone and light red for brick.

Roads. The pencil outlines are filled with a flat tint of yellow ocher, the edges being subsequently drawn in India ink.

Bridges. The outline is filled with yellow ocher, and the same distinction as to wood and masonry is observed as in buildings.

Fences are represented with India ink as in pen drawings (Fig. 674(1)); *stone walls* by their outlines filled with crimson lake.

Water. Prussian blue, a flat tint. If a lower tone is desired for the map, use indigo instead of Prussian blue. The surface is also outlined with a strong blue.

Remark. There are a number of conventional tints used on elaborate topographic maps to represent trees, marshes, cultivated land, sand, rocks, etc. It takes much practice to do this kind of work, and it is better to leave a

map untinted than to produce crude effects in gaudy colors. The student can easily learn, however, to lay on the flat tints given above.

677. Border Lines. (1) Avoid elaborate borderlines. A single heavy line or a heavy line and a light line close together will suffice. Avoid fancy designs at the corners. (2) In drawing the border, keep in mind the size of the blueprint frame and other requirements (Art. 666(*a*)). (3) The heaviest line for a border should rarely exceed 1/20 in., and if there are two lines the space between may be about the width of the heavier line. (4) Follow the directions in Art. 616(11) for drawing heavy lines. For a width of 1/20 in., this means drawing two outside lines and filling in between with a third.

678. Map Titles. The title is usually printed after all the other lettering on the map is finished. The best place for it is in the lower right-hand corner unless it is desired to fill up some vacant space elsewhere (Art. 666(*c*)). The size of the rectangle within which the title is inscribed should be *in proportion to the size of the sheet* (see step 1, Art. 678 (*b*)), and the size of the largest letters depends to some extent upon the scale used for the map (see Art. 671(*b*)). The general arrangement of the title should conform to the space available. The first consideration in designing a title is *simplicity. Ornate letters, scrolls, and flourishes are to be avoided.* The subject matter is usually arranged in several horizontal lines, words which are closely related being put in the same line. The most important words in a title should catch the eye first. Prominence may be gained in several ways (see Art. 672(9)). The lines should be centered, i.e., each line made to extend equal distances each side of a common vertical line. Sometimes lines are arranged in curves, but this is to be avoided unless necessary to save space. The spaces in the clear between lines may vary, although often they are made nearly equal. If too much space is left above and below a line of small lettering, this line will appear separated from the rest of the title. Avoid making two lines of the same length. The fewer styles of lettering used the neater will be the title. The tendency to use a different style of letter for each line is bad, for it gives the impression that the draftsman is trying to show what he can do. Capital letters are used almost exclusively, lower-case letters seldom appearing in well-designed titles. It is probably better to incline all the letters or to make them all upright than to have inclined letters in one line and upright letters in another. The slope of inclined letters should

be about 2 in 5 or 3 in 8 from the vertical and it should be uniform throughout the title. Single-stroke lettering may be used for words whose letters are comparatively small. Very effective titles can be drawn freehand, and instruments should be used only when the extra time required is warranted. The mere fact that instruments were used in drawing a title leads one to expect superior results, whereas slight imperfections may be overlooked in a freehand title. All parts of a title should be finished with equal care; even the smallest lettering should be done as carefully as the larg-

MAP OF PROPERTY

SURVEYED FOR

THE VALLEY GOLF CLUB

NEW HAVEN, CONN.

SCALE 1" = 50' JUNE 7, 1942

FIG. 678. The title in Fig. 678 (reproduced here one-half its original size) illustrates the difficulty in spacing certain combinations of letters. In the third line, the horizontal stem of each letter L was made slightly shortened (Art. 671(*d*)). Nevertheless, the letter causes an apparent break in each of the three words in which it occurs. Likewise the letter T causes an apparent break in the word "Property" and, to some extent, in the word "The." These defects may be made less evident by spacing letters somewhat farther apart, but the area allotted to the title may not be sufficiently large to permit the increased lengths of lines.

est. It is also inconsistent to take more pains with the letters in a title than with the other letters on the map. It is best not to abbreviate any word in a title.

It takes considerable time to execute a good title, and to save time and expense various mechanical devices are sometimes used, such as a stamping outfit or a small printing press. (See also Art. 671(*k*).)

678(*a*). WHAT SHOULD APPEAR IN A TITLE OF A MAP. (1) The name or description of the tract or object or feature which has been mapped. (2) The location of the area mapped. (3) Scale, either numerical, or graphic, or both. (4) Date of completion of the survey or map. (5) Name of the engineer or engineering firm responsible for the survey and the mapping. (Not always required.)

Among other items of information which may or may not appear on the map are: (1) type of map, i.e., property map, preliminary map, topographic map, profile map, etc.; (2) organization or company, or the individual for whom the map is made; (3) place or

office where the map is drawn; (4) Drawn by ....; (5) Checked by ....; (6) Approved by .....

If a map is made for a special purpose this should be stated, as, for example, *proposed location of a highway* or *plan of building lots.*

678(*b*). METHOD OF PROCEDURE IN CONSTRUCTING A TITLE.

1. Decide upon the place of the title and upon the approximate size of the limiting rectangle within which it should be drawn. The *maximum* dimensions for this rectangle suggested are: length = one-fifth of length of longer border lines of the map; height = one-fourth of length of the shorter border lines. Frequently the limiting rectangle should be made smaller than these proportions allow. The tendency is to make titles too large and conspicuous.

2. Write out the title, separating the subject matter into lines; it is well to place such connectives as "of the," "in the," and "showing the" on separate lines. Use as few of these connectives as practicable.

3. Number the lines in the order of their importance and prominence. Usually the name or description of the tract which the map represents is the most prominent. The connectives should be the least prominent.

4. Choose the style of letters. This style should be consistent with that used on the body of the map—if single stroke on the map then single stroke in the title; if Gothic on the map, then Gothic in the title; if Roman on the map, then Roman in the title. Use the same style throughout the title; all variations in prominence of words may be effected by varying the heights and thicknesses of letters or in other ways (Art. 672(9)).

5. It is well to outline the title on a separate sheet of paper as follows. Draw a rectangle equal in size to that of the limiting rectangle for the title and through its center draw a vertical line as an axis of symmetry. Decide upon the height of the largest letters (Art. 671(*b*)); the smallest letters, used for connectives, may be about one-fourth of this height. Estimate the height which will be used for each line of the title that was written in (2). Estimate the spaces between successive lines of lettering. These spaces may be approximately equal except where there is a short line printed in small letters. For example, when a connective is followed by two long adjacent lines, the space between the two long lines should be about one and one-half times the space between the upper of these two lines and the connective. The heights of letters and the spaces between lines of lettering must be adjusted so that the total will come within the height of the limiting rectangle. Draw the top and bottom guide lines for each line in the title, using the adjusted values for the heights of letters and the spaces between lines of lettering. Each line of the title may now be lettered, but with more attention paid to the widths and spacing of the letters than to the perfect execution of letters. Poor spacing may spoil the appearance of a title. Before beginning to letter a line, the width of each letter may be noted in terms of its height from a knowledge of proportions (Art. 671(*c*)), and the spacing between letters may be estimated by a knowledge of spacing (Art. 671(*d*)). By quickly adding up the widths of all the letters in a line and the width of spaces between letters and words, the total length of any line can be found. One half of this length can be used in starting a line at such a point that the line will be symmetrical or nearly so with respect to the vertical axis of symmetry. If a line is too long, it may be shortened by decreasing the width of spaces or by compressing the letters; if a line is too short, it may be lengthened by increasing the spacing. Avoid, however, the common defect caused by spacing letters or words too far apart. Avoid also two successive lines of the same length. In this preliminary outline of the title, it is not necessary to have each line exactly symmetrical with respect to the vertical axis, but the more nearly symmetry is attained, the easier it is to judge whether the appearance of the title as a whole is satisfactory. It often expedites the work if the preliminary outline is drawn on profile paper. The horizontal lines on the paper can be used in estimating the heights of letters and the spaces between lines of letters. When the preliminary outline of the title has been completed, draw the limiting rectangle in the correct position on the map sheet, and then within this rectangle copy the title in pencil.

6. Titles for tracings may be penciled on a small sheet of paper *without attempting to center the different lines* with respect to an axis of symmetry, and then, by shifting this paper back and forth under the tracing cloth, each line can be centered on the tracing. Similarly, a title can often be drawn on tracing paper without centering the lines, and then transferred to the map surface by pricking through a few limiting points for each line of letters after the line has been centered by shifting the tracing paper.

7. When the title has been penciled on the map, examine it to see if as a whole it is well spaced and well balanced. In inking, follow the suggestions in Art. 671(*g*).

679. Meridian Needles. Simple arrows like those in Fig. 679(1) are best. On highly finished maps, somewhat more elaborate arrows may be drawn similar to those in Fig. 679(2). The second combination in Fig. 679(2) has the advantage that there can be no confusion as to which line is the magnetic north since it is shown by an outline of a magnetic needle. Avoid too elaborate needles and compass points. As with border lines and titles, simplicity should be the first aim. The true north is usually represented by a full arrowhead and the magnetic north by a half arrowhead. The direction of one of these arrows is found either from the meridian used in plotting the map or from some line of known bearing. If magnetic bearings were used, plot the axis of the magnetic needle and then lay off the declination to find the axis of the other needle; and contrariwise if true bearings were used, find the axis of the true needle first. *On many maps nothing is gained by showing the magnetic meridian.* In such a case, only the arrow representing the true meridian is drawn. It should be marked *True Meridian* lest it be mistaken for a magnetic meridian. When both the true meridian and the magnetic meridian are shown, the magnetic declination should always be given in numerals marked along the arrow or between the two arrows as in Figs. 679(1) and (2). The needle should be of such length that its direction can be transferred accurately to other parts of the map—usually about 4 or 5 in. long. Both the arrowhead and the feather end should be drawn with care,

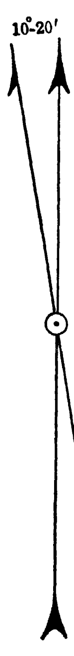


FIG. 679(1).

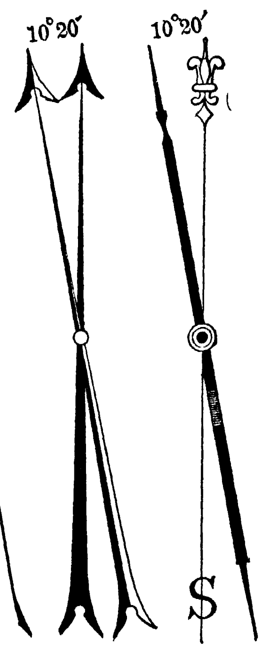


FIG. 679(2).

slim and graceful—the tendency is to make them too clumsy. The needle may be placed near the title or in a vacant space near the top of the map.

679(*a*). PLANE NORTH OR "GRID" NORTH. When a survey is connected with a state system of plane coordinates, a single arrow is often shown on the map. This arrow is drawn parallel to the central meridian of the system (Art. 770(*a*)) and is marked *Plane North*, or *Grid North*. (See Art. 770(*b*); also the sample plat, Fig. 798(*f*), p. 1010.)

680. Scales. The scale of a map is usually given in figures as, for example, "Scale: 1 in. = 40 ft." or "Scale: 1/10,000." Frequently in addition a small portion of a scale is drawn as illustrated in Fig. 680. This is most necessary for maps that are to be reproduced

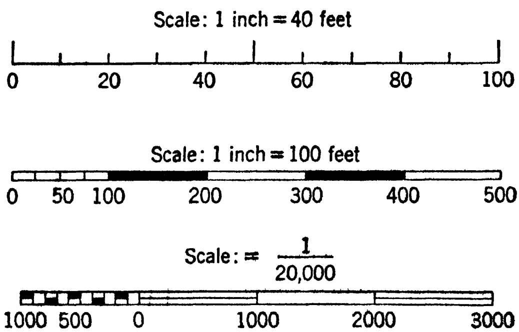


FIG. 680.

by photo reproduction or by other processes which change the scale; otherwise it would be difficult to scale the reproduction. Such a scale also indicates the shrinkage or expansion in maps and blueprints, but if designed for this purpose it should be long enough to show the change—say 6 in. Distances scaled from tracings and blueprints are most unreliable, no matter what method is used. When in the process of reproduction the scale of a map will be changed appreciably, it may be well to show only the drawn portion of the scale, and to omit altogether any reference to what one inch represents; otherwise the original scale may inadvertently be used in scaling distances on the reproductions of the map.

681. Keys or Legends. Where conventional signs are used it is customary to draw a key near the title explaining the signs. This is especially necessary when the signs are not those commonly employed.

682. Explanatory Notes. Maps, like field notes, should be self-explanatory. Anything not clear from the drawing or the dimensions should be made so by explanatory notes. The general directions in Art. 27 hold good. Notes giving general information concerning the map, such, for example, as the datum used in a topographic survey, may be placed near the title. Occasionally notes should indicate the extent to which measurements may be depended upon by giving the instruments used in the survey, the limits of error adopted, and similar information bearing on the accuracy of the work.

All explanatory notes should be terse and clear. Typical examples are the following: (1) Surveyed as occupied, or according to evidence of occupation. (2) Surveyed as the fences now exist. (3) Surveyed as found in possession. (4) Boundaries shown are those indicated on the ground by John Doe on April 8, 1945. (5) In the opinion of neighbors, this monument has not been disturbed since 1900. (6) Encroachments below grade, if any exist, are not shown hereon. (7) Elevations refer to mean high water. (8) Grade elevations at *A, B,* and *C* were furnished by the City Engineer.

683. Cleaning Maps. Use a soft eraser for removing pencil lines, and a sponge eraser or bread crumbs for taking off dirt. For tracings use gasoline or benzine. (See Art. 621(12).)

684. Maps for Photo Reproduction. Better results will usually be attained if a map intended for photo reproduction is drawn larger than the reproduction is to be; it is then photographed down to the desired size or scale. Before drawing such a map, therefore, it is necessary to determine what the size of the reproduction is to be and then to decide how much larger to make the original drawing. Perhaps the most common practice is to make the drawing twice the size of that of the intended reproduction. By that is meant that any line on the original drawing will be twice the length of the corresponding line on the reproduction. When photographed, the drawing is then said to be reduced one-half, although any given area of the reproduction will be only one-fourth of the corresponding area on the original drawing. In other words, the rate of reduction is expressed not in terms of area but in terms of linear measurement. Reductions greater or less than one-half are also common, as, for example, reduction to one-fourth size or three-fourths size. Frequently the reduction must be such that the reproduction will fit within a given space, as, for example, within the limits of a page of a given size. When the reproduction of a map will not be used for the purpose of scaling distances, the original map may be drawn to any convenient scale that will make it the desired size for reproduction, and the amount it is to be reduced is indicated as explained later. In such a case, it is not necessary to know the exact ratio of reduction.

To obtain the best results in photo reproduction, the map should be drawn on a smooth white surface, as nearly glossy as possible. Every part of the drawing should stand out clear and distinct.

In order to compare the size of an original drawing with the size of its reproduction, it will be helpful to construct two rectangles as illustrated in Fig. 684. Assume that the larger rectangle represents the border line of a map which is to be reduced one-half. If AB' is made equal in length to $\frac{1}{2}AB$, the perpendicular at B' will intersect the diagonal at a point C', which will be a corner of the smaller rectangle, and the length of the side $B'C'$ is

equal to one-half of the length of the side BC. Note that the area of the smaller rectangle is one-fourth and not one-half of the larger rectangle. Obviously this general method can be used for any ratio of reduction. For example, if AB' had been made one-third of AB, $B'C'$ would have been one-third of BC.

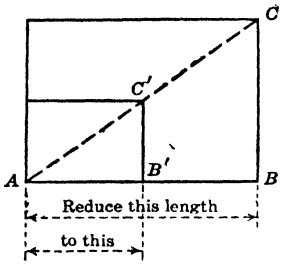


FIG. 684.

A good method of indicating how much the original drawing is to be reduced is shown in the figure. Another method is to draw only one dimension line, that from A to B, and to mark along this line the length to which it is to be reduced. For example, if the length from A to B is 36 in. and it is to be reduced one-half, the lettering along the dimension line will be: "Reduce this length to 18 in."

684(*a*). FINISHING A MAP FOR PHOTO REPRODUCTION. In finishing a map intended for photo reproduction, it is necessary to keep in mind the extent to which every detail of the map will be reduced and how it will appear after reduction. For example, if the map is to be reduced to one-half size, every line when reduced will be not only one-half as long but also one-half as thick. Likewise, every letter will be one-half as high, one-half as wide, and one-half as heavy. A fairly light line will reduce well if it is black and sharp, but too light a line may almost disappear when reduced. Precautions should be taken to avoid any ragged or imperfect reproductions. For example, if lines and stems of letters are of uneven width in the original drawing, they will be uneven in the reproduction, though the imperfections may not appear to be so great. Avoid also too small a space between lines. For example, if the space between two parallel lines is too small, they may appear as one when reduced; or if the space inside the loop of a letter, such as the space in the loop of the letter "e," is too small, it may appear as a black blotch in the reproduction. Some idea of how the drawing will look after reproduction may be gained by looking at it through a biconcave reducing glass held in different positions between the eye and the map. Such inspection may reveal that certain lines should be made heavier or lighter, or certain letters larger or smaller, or that the reduction should be less or greater.

685. Surveyor's Certificate. In filing maps for record the surveyor may be required to furnish a certificate, the exact form of which will of course vary. One form is as follows:

....
(City or Town, State, Date)

I hereby certify that the plot hereon delineated is a correct one and that permanent monuments, consisting of ...., have been set at points
(Description of the monuments)
marked MONUMENT as thereon shown at all angles in the boundaries of the land plotted, and at all intersections of streets and alleys.

.... Surveyor.
(Signature)

Another form sometimes furnished a client but not intended for public record is as follows:

....
(City or Town, State, Date)

This is to certify that I have this date made a survey of ....
.........................., and have plotted the lines thereof on the
(Description of tract surveyed)
accompanying map, which to the best of my knowledge and belief is correct.

..................Surveyor.
(Signature)

For an example of a surveyor's certificate on a plat for a deed see Fig. 798(*f*), p. 1010.

Some state boards of registration require an imprint of the registration seal of the surveyor.

686. Filing Systems. Although it is not within the province of this book to explain the various systems of filing and indexing drawings and notebooks, no discussion of office work would be complete without reference to this important subject. Whatever system is used, it should be such that maps can be kept flat and *easily accessible*. It should not be necessary to search through thick piles of maps, for to do so results in a waste of time and in wear and tear on the maps.

687. Topographic Relief Maps and Models. A good topographic relief model is the best representation of a relatively small portion of the earth's surface. To persons not accustomed to reading topographic maps and even to the surveyor himself, contour lines alone do not make the picture stand out in its full meaning. For this reason, landscape architects and engineers are using relief maps and models more and more. It is not so difficult as it may seem to construct such maps and models.

687(*a*). Scales. The horizontal scale depends largely upon the desired size of the model and upon the character and extent of the details to be shown. The vertical scale is usually greater than the horizontal. On small-scale models of mountainous regions, considerable exaggeration of the vertical scale is generally necessary, but on large-scale models of small, relatively flat areas, such as the engineer is likely to make, vertical exaggeration is much less. It should be sufficient to render the difference in elevation between successive contours clearly apparent but not so great as to result in a distorted model that conveys a false impression. The purpose for which the model is to be used also affects the choice of the vertical scale. If it is desired to bring out in sharp relief differences in elevation, the vertical exaggeration should be sufficient to do this. The choice of the vertical scale may also depend partly upon the difference in elevation between the highest and lowest points on the map and partly upon the height of the contour interval. If the model is made of layers of cardboard, the thicknesses of cardboard obtainable may affect to some extent the choice of the contour interval.

Examples. In a model, about 30 in. square, made to represent a cloverleaf highway intersection, the scales used were: horizontal, 1 in. = 100 ft., vertical 1 in. = 30 ft. The difference in elevation between the highest and lowest point on the ground represented was about 100 ft. The cardboard used measured fifteen layers to the inch; hence one layer represented a contour interval of 2 ft.

In a model 3½ ft. × 5 ft., made for use in a real estate development, the scales used were: horizontal, 1 in. = 50 ft.; vertical, 1 in. = 40 ft. The maximum difference in elevation of points on the ground was about 100 ft. The cardboard used measured 24 layers to the inch; hence one layer represented a contour interval of 1⅔ ft. On the original survey map, the contour interval was 5 ft.; but in order to use the thickness of cardboard desired, contours at the unusual interval of 1⅔ ft. were interpolated.

In a third model, the vertical scale used was 1 in. = 100 ft., the cardboard measured 5 layers to the inch, and each layer represented a contour interval of 20 ft.

687(*b*). First method of construction. A common method of constructing a relief model is this. If a contour-line map of the size required for the model is not available, one is drawn, or an enlargement or reduction of an existing map is made, usually by the photostatic or some other photographic process. Every contour line of this map is then traced on a separate piece of cardboard, by means of carbon paper. As each contour is traced, it

is well to mark on the same piece of cardboard enough guide points of the next higher contour to determine the position of the next layer of cardboard. Another method of tracing contours, particularly if blueprints of the maps are made, is to cut the map along the contour nearest the edges, lay the map on a piece of cardboard, and trace around the edges that coincide with the contour line. Then cut along the next contour, trace this contour in the same way on another piece of coardboard, and so on, cutting back contour by contour from the edges toward the center of the map, until all contours have been traced.

It is well to use a wooden base for the model; a sheet of thick plywood is excellent for that purpose. The cardboard with the lowest contour is glued or tacked to the board. The cardboard with the next higher contour is then fastened down on top of the first after it has been placed in the correct position by observing the guide points marked on the first cardboard. Thus the model is built up, layer by layer, to represent successive contour lines. The cardboards may be glued and stapled down with a stapling machine, or tacked down.

Suggestions. (1) It may be desirable, particularly if comparatively thick cardboard is used, to smooth the surface of the model with fine sandpaper. (2) The model should be sprayed with shellac to prevent its edges from curling. (3) Instead of smoothing down the edges of the cardboard, an adhesive plastic material may be applied to form the surface that represents the ground. (4) The best method of cutting the cardboard is by means of a jigsaw with round spiral blades such as are available in dental supply stores. If heavy cardboard is used, the jigsaw should be equipped with a broad blade. Good results may also be obtained with a sharp knife or razor blade. (5) Cutting exactly along contour lines requires some planning as well as skill. For example, if there are deep narrow embayments in a contour line, it is better to leave them until the entire sheet of cardboard has been cut and then to cut them from both sides or directions. (6) Certain problems are caused by knolls, depressions, and other features that may result in more than one contour of a given elevation. A contour around any such feature should be cut at the same time and from the same board as the corresponding contour of the main map. (7) Cleats may be fastened to the bottom of the wooden base to prevent warping.

687(*c*). Second method of construction. This method is a modification of the first. (1) Fasten the map temporarily to a sheet of cardboard with thumb tacks or by some other means and cut along the lowest contour. Mark this sheet No. 2 and lay it aside. (2) The portion of the map thus cut away from the main portion represents the area at the lowest level. Paste this portion of the map upon the board (No. 1) that forms the base of the model; it should be placed in the proper position with respect to the edges of the board. (3) Fasten the remaining portion of the map temporarily to a new sheet of cardboard marked No. 3, and cut along the next higher contour. This will cut from the map a strip which represents the area between the lowest contour and the next highest. One edge of this strip will fit exactly the edge cut in sheet No. 2. Paste the strip in this position upon sheet No. 2. (4) Fasten the remaining portion of the map temporarily to sheet No. 4, cut along the next higher contour, and paste the narrow strip of map upon sheet No. 3. (5) Repeat the procedure until the map is used up and each strip of map is pasted upon a previous board where it belongs. The different layers of cardboard are then used to build up the model as in the first method. The finished model appears, especially when viewed from a point directly above, as a reproduction of the map with all its cultural and topographic features but with the added advantages of the third spacial dimension of a relief map model.

687(*d*). Third method of construction. (1) Prepare a wooden base and fasten the topographic map temporarily to the base or transfer its contours to the base by means of carbon paper. (2) At controlling points on each contour, i.e., points where there is a marked change in direction, drive brads or wire nails until their tops are at a uniform height above the base. The height should be such as to represent, to the vertical scale adopted for the model, the elevation of the contour. Thus there will be as many different heights of nail heads as there are different contours, but each contour will be outlined by nails all of the same height. (3) Construct the model of plastic material, molding the surface to conform to the elevations marked by the nail heads. The topographic map may be removed from the base before

modeling is begun and used as a guide in making the surface of the model a true representation of the terrain, or another copy of the map may be used for that purpose.

Suggestions. (1) There are various plastic materials used for modeling, such as clay, plastilene, and wood pulp or papier-mâché. A molding clay can be made of two parts plaster of Paris and one part whiting (the whiting is added to slow the setting process of the plaster). Water is added slowly as the mixture is thoroughly stirred until the right consistency for molding is obtained. This mixture is allowed to stand until it begins to thicken or set. (2) The plastic material is first built up roughly to the tops of the nails and then modeled by hand with tools made of wood or wire. (3) Small but stiff pieces of wire can be used to mark points which are too high above the base to be marked by wire nails. (4) If moist molding clay is placed directly on the wooden base, it may cause it to warp even when cleated plywood is used; hence the base should be shellacked or painted. It is sometimes well to construct a model on a temporary base and then, after it is dry, to take it off and glue it to a permanent base.

687(*e*). Casting relief models. For casting a relief model, plaster is mixed in the proportion of about 4 lb. of plaster to 3 pt. of water. The mixture should be stirred thoroughly and allowed to thicken slightly before it is poured into the mold. There are two common methods of making the mold:

I. (1) Construct a cardboard relief model as explained in the first method, and finish the surface to give every detail desired in the cast model. (2) Build a wooden box, open at the top, with sides somewhat higher than the highest point of the relief model. The box should be larger all around than the model. (3) Set the cardboard model in the box to serve as a positive mold. (4) Pour the plaster mixture into the box and over the mold until it fills the box. Allow the mixture to harden for a day or more. (5) Remove the sides of the box, lift the positive mold, formed by the plaster, off the cardboard model, remove the latter, and place the plaster mold bottom side up on the bottom of the box. (6) From this negative plaster mold, any desired number of plaster of Paris relief models can be cast exactly as the negative mold was cast.

II. A second method of casting a relief model is as follows. (1) Instead of constructing a cardboard relief model, make a cardboard negative of the model. First, cut out from coardboard the *highest* contour, set it aside, and lay the remainder of the sheet of coardboard upside down on the base, so that its inner edges form the contour. (2) In a similar manner, cut out the next lower contour, set it aside, and lay the remainder of the sheet on top of the first sheet in such a position that its inner edges form the contour. (3) Continue the procedure until the entire cardboard *negative* mold is completed. (4) From this mold any desired number of plaster of Paris relief models may be cast. If better detail is desired, a model so made can be further tooled and then used for casting another and better negative mold from which the final model is produced.

Suggestions. (1) The wooden box, used in the first method of casting, should be made practically watertight and its inner surfaces should be oiled. (2) A cardboard mold may be soaked in oil to render it more impervious to the wet plaster; otherwise it may become distorted. (3) A plaster mold should be shellacked and its inner surface rubbed with oil or Vaseline to prevent the sticking of the plaster model when it is removed from the mold. (4) Strong cloth or other reinforcement may be embedded in the plaster, before it is set, in order to strengthen it. (5) During the casting process, the horizontal lines of the mold should be kept truly horizontal. (6) Any defects in the inner surface of a plaster mold may be touched out with plaster which has begun to harden.

687(*f*). Finishing a model. A plaster model should be allowed to dry for at least two or three days before beginning the finishing process. The surface of a cardboard model may be smoothed with a plastic material or sandpapered if it is desired to eliminate the edges of the cardboard contours; frequently it is better not to do this. In any case, the surface of the model should be shellacked.

A model may often be greatly improved by painting the surface, to represent, as nearly as may be, the natural colors of the cultural and topographic features. For example, a light gray for concrete highways, blue for streams, silver for lakes, brown for cultivated lands, green for lawns, and so on. With a little skill and ingenuity prominent features may be added if desired.

For example, bits of sponges dyed green for tree tops, sand sprinkled on paint before it is dry, coarse twine painted and glued down for stone walls, miniature wooden fences, telephone or power lines, and miniature models of buildings secured from a toy store. Such additions, however, detract from the general appearance unless they are well made and are approximately to the correct scale.

In constructing a relief model and in finishing it, it is helpful to observe it from time to time through reversed opera glasses held a few feet away from the model.

CHAPTER 50

PROFILES AND CROSS SECTIONS

IN this chapter are given the few directions and suggestions necessary for plotting and finishing profiles and cross sections.

688. Profile. (*a*). TRUE PROFILE. Suppose that in Fig. 688(*a*) *A, B, C,* and *D,* representing four points in a *straight* line on the ground, were plotted to a given *horizontal* scale. Let the points *a, b, c,* and *d* be obtained by laying off *A'a, B'b, C'c,* and *D'd* equal, respectively, to the elevations of *A, B, C,* and *D* above a given datum. If the *vertical* scale used in laying off these elevations is the same as the horizontal scale used in plotting *A, B, C,* and *D* the line *abcd* is a *true* profile, bounding a *true* vertical section through the given line, and showing the rise and fall of the ground.

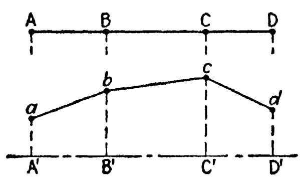


FIG. 688(*a*).

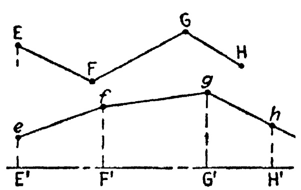


FIG. 688(*c*).

688(*b*). EXAGGERATED PROFILE. As the rise and fall of the ground between two points is usually very small in proportion to the distance between those points, it is customary in plotting profiles to use a much greater scale for laying off elevations than for plotting horizontal distances, in order to make the irregularities of the surface more apparent to the eye, and to render subsequent use of the profile easier and more accurate.

Suppose that in Fig. 688(*a*) the vertical scale is ten times as large as the horizontal. The line *abcd* is still called a profile though it is no longer a *true* profile, and the section which it bounds is no longer a true section.

688(*c*). PLAN AND PROFILE. Frequently in practice the line leveled over is not straight, but consists of a series of connected straight lines, as, for example, a traverse line. In Fig. 688(*c*) *EFGH* is the plan of a traverse line, and *efgh* is the corresponding profile in which the vertical scale is larger than the horizontal.

The distances $E'F'$, $F'G'$, and $G'H'$ are equal respectively to EF, FG, and GH, but F', G', and H' cannot be obtained by projecting from F, G, and H respectively. In Fig. 688(*a*), on the contrary, A', B', C', and D' were projected from A, B, C, and D respectively. Hence in Fig. 688(*a*) *abcd* represents a profile cut by *one* vertical plane, but in Fig. 688(*c*) *efgh* represents a profile cut by a *series* of vertical planes, each of which passes through two stations at consecutive bends of the traverse line.

688(*d*). REMARK. A profile is likely to be very misleading to persons unaccustomed to the methods of plotting. Unlike a map a profile seldom represents points projected on the same plane. Moreover, when the vertical scale is much greater than the horizontal scale, slopes are greatly exaggerated, and the surface of the ground appears to be much more irregular than it really is. Likewise all objects such as manholes, culverts, rivers, and roads which the profile crosses are shown distorted. It is a common experience among engineers to receive protests against a proposed grade because on the profile it appears to be too steep.

689. Profile Papers. A profile is usually plotted (from the left toward the right) on paper specially designed for the purpose. On this paper horizontal and vertical lines are printed in order that points may be plotted by eye without the use of a scale. Paper may be obtained with lines printed in pale green, pale blue, or pale orange; it comes in rolls so that it may be cut to any desired length. There are several styles of profile papers, distinguished from each other by the number of lines or spaces to the inch. That known as Profile Plate A, 4×20 is most commonly used. The lines on this paper are of three degrees of coarseness: fine, medium, heavy. Vertical *medium* lines are ¼ in. apart and horizontal *fine* lines 1/20 in. apart, thus making four spaces to the inch horizontally and twenty spaces to the inch vertically. Every fifth horizontal line is accentuated by making it a *medium* line, thus forming ¼-in. squares with the medium vertical lines. Every fiftieth horizontal line and every tenth vertical line are accentuated by making them *heavy* lines, thus forming 2½-in. squares bounded by heavy lines. These accentuated lines aid in estimating both horizontal and vertical distances. Another style of paper is known as Profile Plate B, 4×30; this is similar to Plate A except that the fine horizontal lines are 1/30 in. apart. Since the vertical lines are ¼ in. apart, there are four spaces horizontally and thirty spaces vertically to the inch. Plate A and Plate B rulings are also furnished on tracing paper and tracing cloth.

Profile-plan drawing paper, tracing paper, and tracing cloth may be obtained on which the profile ruling is confined to the lower half of the paper or cloth, the upper half being left blank for sketching

or mapping. Sheets intended for flat profiles may also be obtained on which there is profile ruling in the lowest quarter, a blank space in the second quarter, profile ruling in the third quarter, and a blank space in the top quarter.

Cross-section paper is sometimes used for plotting profiles. It may be obtained with various rulings to the inch, such as, 5×5, 8×8, 10×10, 12×12, and 16×16.

The rulings on profile papers are not always exactly correct, but usually the errors are so small that they may be ignored in plotting ordinary profiles, especially if the relative elevations are read directly from the paper and not by scaling the drawing.

690. Scales Used in Plotting. In choosing the scale for a profile, consideration should be given to the spaces per inch on the profile paper. On both Plate A and Plate B papers, vertical medium lines are ¼ in. apart. Consequently if a scale of 1 in. = 40 ft. is used, each space between adjacent lines represents 10 ft., and the space between two adjacent heavy vertical lines represents 10 spaces $\times$ 10 = 100 ft.; for a horizontal scale of 1 in. = 50 ft., the distances are respectively 12.5 ft. and 125 ft.; for a horizontal scale of 1 in. = 100 ft., the distances are respectively 25 ft. and 250 ft. It is obvious that a horizontal scale of 1 in. = 50 ft. is not so well suited to the paper as either of the other two scales. On Plate A paper the fine horizontal lines are 1/20 in. apart. Consequently, if a vertical scale of 1 in. = 4 ft. is used, each space between adjacent lines represents 0.2 ft., and a space between two successive heavy horizontal lines represents 50 spaces $\times$ 0.2 = 10 ft.; for a vertical scale of 1 in. = 5 ft., the distances are respectively 0.25 ft. and 12.5 ft.; for a vertical scale of 1 in. = 10 ft., the distances are respectively 0.5 ft. and 25 ft. Obviously the vertical scale of 1 in. = 5 ft. is not suited to the paper, whereas the other two scales are. When it is desired to adopt a vertical scale of 1 in. = 6 ft., Plate B paper should be used. Each space between two adjacent fine horizontal lines will then represent 0.2 ft., and a space between two successive heavy horizontal lines will represent 10 ft.

690(*a*). COMBINATIONS OF HORIZONTAL AND VERTICAL SCALES. Whatever scale may be adopted, the ratio of the vertical scale to the horizontal scale must be such that elevations can be read from the profile with sufficient accuracy to serve the purpose for which the profile is used. In order to bring the profile within the limits of the paper, the maximum difference in elevation between the highest and lowest points must also be considered, as well as the

maximum length of profile which it is desired to plot in a given length of paper.

The following are some of the most common combinations:

COMBINATIONS OF HORIZONTAL AND VERTICAL SCALES

	Horizontal Scale	*Vertical Scale*	*Smallest Space between V Lines*	*Smallest Space between H Lines*
(*a*)	1 in. = 40 ft.	1 in. = 4 ft.	10 ft.	0.2 ft.
(*b*)	1 in. = 40 ft.	1 in. = 8 ft.	10 ft.	0.1 ft.
(*c*)	1 in. = 100 ft.	1 in. = 10 ft.	25 ft.	0.5 ft.
(*d*)	1 in. = 200 ft.	1 in. = 10 ft.	50 ft.	0.5 ft.
(*e*)	1 in. = 400 ft.	1 in. = 20 ft.	100 ft.	1.0 ft.
(*f*)	1 in. = 40 ft.	1 in. = 6 ft.	10 ft.	0.2 ft.
(*g*)	1 in. = 100 ft.	1 in. = 12 ft.	25 ft.	0.1 ft.
(*h*)	1 in. = 400 ft.	1 in. = 30 ft.	100 ft.	1.0 ft.

For combinations (*a*) to (*e*), Plate A profile paper is used, and for combinations (*f*) to (*h*), Plate B paper is preferable.

When a profile is to be used as the principal basis of earthwork calculations, as in the case of a sewer or a pipe line, a combination should be selected in which both the horizontal and the vertical scales are relatively large, such, for example, as combinations (*a*) and (*f*). For the sole purpose of fixing grades, it is not usually necessary to have scales as large.

When a surface is relatively flat and uniform and it is desired to show clearly variations in elevations, the vertical scale should be large in proportion to the horizontal scale.

Common combinations for highway profiles are (*a*), (*b*), (*c*), or (*d*), depending upon the purpose for which the profile is to be used. For fixing the grades of a railway, one of the combinations (*e*) or (*h*) is commonly employed; for profiles of the finished railroad, a horizontal scale of 1 in. = 100 ft. combined with a suitable vertical scale may be used.

690(*b*). CHOICE OF PROFILE PAPER AND SCALES. It is desirable so to choose the paper and scale that the smallest space between vertical lines will represent some multiple of 5 ft., and the smallest space between horizontal lines will represent some multiple or some decimal part of 1 ft. It is also an advantage to have the distance between two heavy vertical lines represent a distance which is a multiple of 5 ft. or, better still, if conditions permit, of 10 ft., and to have the distance between two heavy horizontal lines represent a difference in elevation which is a multiple of 1 ft., 5 ft., or 10 ft. Thus, for example, in combination (*c*) above, if one 100-ft. station is on a medium vertical line, the next 100-ft. station will be on the next medium line, and the station which falls on a

heavy vertical line will be 1000 ft. from the station that falls on the preceding heavy line. The vertical distance between two medium horizontal lines represents a difference in elevation of 5 ft., and the vertical distance between two heavy horizontal lines represents a difference in elevation of 50 ft.

For scales used in plotting cross sections, see Art. 701.

691. Working Up Profile Notes. Frequently the elevations of points other than turning points and bench marks are not computed in the field, and, therefore, the elevations of points on the profile must be calculated in the office. In doing this, observe whether the level notes have been checked; if not, those on each page should be checked as the work progresses by the method explained in Art. 134. This is not a check, however, on the elevations of *intermediate* stations, and in some cases it may be well to work the notes up by the method of Art. 133(*d*) (Form D). Generally, however, any appreciable error in a profile becomes evident when the profile is plotted, as, for example, when an abrupt break appears where there seems to be no reason for it. The arrangement in Form F, p. 132, is very convenient in plotting profiles, since the only two columns needed (El. and Sta.) are side by side.

692. Plotting Profiles—General Method of Procedure.

1. Choose the combination of paper and scale best suited to the purpose (Art. 690(*a*)), and pin the paper to the board. It is slightly more convenient in finishing the profile to have the horizontal lines true with the T-square, though this is not of much advantage in plotting.

2. Glance through the level notes to ascertain the elevations of the highest and lowest points of the profile.

3. Select one of the accentuated (heavy) horizontal lines for a base line or datum line. When practicable, its position should be such that the highest and lowest points of the profile will fall on the paper. In order to allow room for lettering, this base line should be taken at least 2 or 3 in. from the lower edge of the paper even when no part of the profile extends below it. It may be necessary to assume the elevation of this base line as some whole number of feet above or below datum in order to get the profile on the paper. For example, if the highest elevation is, say, 128 ft. and the lowest is above 100 ft., the elevation of the base line may be assumed as 100 ft.

4. The profile should begin near the left-hand edge of the paper and extend towards the right. If 100-ft. stations are used, assume station 0 on one of the accentuated vertical lines, or, if the profile begins at some station remote from zero, assume instead a station

whose number is some multiple of 10. For example, if the profile begins at 48 + 21 assume station 50 on an accentuated vertical line near the left-hand edge. Similarly for other horizontal scales aim to have the accentuated vertical lines stand for some multiple of ten (Art. 690(*b*)).

5. Along the lower edge of the paper, number in pencil the accentuated or heavy vertical lines to correspond to *distances* or stations, and along either the left-hand or right-hand edge, or both edges, number the accentuated, heavy horizontal lines to correspond to *elevations*.

6. Plot each point on the profile, first by finding the vertical line nearest to the position of the point and then by following up or down this line, or parallel to this line, until the correct elevation is reached as judged by the nearest horizontal line. It is well to connect the points as fast as they are found by straight lines drawn *lightly* in pencil, freehand.

7. Check points by reading their elevations directly from the profile to see if they agree with those in the level notes. Do not check a point by reading from the level notes first and then looking at the profile, for if a mistake was made in plotting, the mind is likely to repeat it.

692(*a*). SUGGESTIONS. (1) Two men can work to advantage—one reads the notes while the other does the plotting. In checking they change places. The man who did the reading calls off elevations directly from the profile, while the man who did the plotting sees if they agree with the notes. (2) Use a fine pencil point or needle point for marking points. (3) When the profile is about to run off the top or bottom of the paper it may be stopped (preferably at some accentuated vertical line), and then continued by beginning some whole number of feet below or above the point where it was broken off. The amount that the profile line is thus lowered or raised should be some multiple of the interval between two accentuated horizontal lines, and the horizontal lines from the break on should be renumbered accordingly. (4) Turning points, of course, are not plotted unless they are points of the profile. Benches are not plotted, though they are often described by notes near the points on the profile where they would be found. (5) To avoid the common mistake of plotting the elevations of turning points and benches, go through the notes, crossing off elevations which should not be plotted. (6) Another common mistake is to plot a plus station on a vertical line as if it were a regular station instead of plotting it between vertical lines by estimating the distance. (7) When very accurate profiles are required a base line may be laid out on drawing paper of good quality, and the distances and elevations plotted by scale. (8) When several copies of a profile are required it may be plotted on profile tracing paper or tracing cloth and process prints made, or it may be transferred by pricking through, or by carbon paper, or by some other method of reproduction (Art. 663).

693. Alignment Diagram. Frequently an alignment diagram is drawn below a profile to indicate the changes in the *horizontal* direction of the line on which the profile is taken. No attempt is made to draw a plan. On the contrary the alignment line may be nearly straight. It is drawn near the bottom of the sheet in such a way that each point at which the alignment changes is indicated directly below the corresponding point on the profile. Thus from this alignment diagram it is easily seen what points on the profile are at the beginning and the end of each tangent, and what portions of the profile are on curves. The direction of each tangent is given, usually by its

bearing or azimuth, and the direction of each curve is indicated together with the most essential information concerning the curve. Sometimes other features, such as boundary lines, roads, and streams, crossed by the alignment line are indicated.

694. Profiles of Several Related Lines. Several related lines are often plotted on the same sheet, one above the other. Thus, for example, the profile of the center line of a street may be plotted about halfway between the top and bottom edges of the paper, the profiles of the right and left gutter lines being plotted near the bottom and top respectively. In this case a different base line is used for each profile. Another method is to plot the three profiles from a common base. If only a single copy is desired, these three profiles may be distinguished by lines of different colors, as, for example, black for the center line, red for the right gutter line, and blue for the left gutter line. When profiles are to be reproduced, different profiles may be distinguished by different kinds of lines formed by different combinations of dots and dashes.

695. To Establish a Grade Line. The principal purpose of a profile is generally to enable the engineer to establish a satisfactory, economic grade line. This line consists of a series of straight lines connected usually by vertical curves. The requirements that should be met in fixing the grade line are, in general, those which have to do with "limiting grades" and those which relate to the removal and placement of earth material.

For highways and railways, limiting grades are the *maximum grades* that are permissible under given requirements pertaining to traffic. For sewers, drains, and other pipe lines, limiting grades are the *minimum grades* that are permissible under given requirements pertaining to flow.

The grade line is fixed within narrow limits by certain control points. In highways for example, the grade line is fixed by terminal points, and by stream, highway, and railway crossings; in sewers and water mains, by the location of other pipe lines and by the depth to which buildings extend underground.

The grade line between governing points should be so laid out on the profile that, other things being equal, the cost of earth removal and placement will be a minimum. This usually means that the volume of earthwork in cuts will equal approximately that in near-by fills. The rate of grade of any part of the grade line should preferably be such that it can be expressed in an exact decimal to not more than two places, as 3.5 per cent or 0.75 per cent.

696. Progress Profiles are used to show graphically the progress in construction during a given period, as, for example, during each month. Estimates of earthwork removed and placed, of pavement constructed, of sewers built, etc., are made at the end of the month. The work done during the month may then be indicated on the progress profile by tinting the portions completed. The progress, month by month, may be indicated by the use of different colors for different months, and the amount of work during any month may be shown by giving within the corresponding colored area, or close to it, the volumes or other quantities involved.

697. Profile and Plan. It is common practice in certain types of work such as surveys for highways, sewers, and pipe lines to show the plan and profile on the same sheet. When this is done, plan-profile paper, described in Art. 689, is used. The plan is drawn on the unruled upper half of the paper, and the profile is plotted on the ruled portion below. Thus may be seen at a glance the general relation between plan and profile. The plan may be quite elaborate as in the case of plans for the construction of highways, and both plan and profile may be continued over many sheets.

698. Plotting a Profile from Contours. For any line drawn across a contour map a profile may be plotted, since at each intersection of a contour by the given line an elevation is known. The distances between these points of intersection may be scaled from the map, or, if the given line is straight, they may be projected from the map to the profile as illustrated in Fig. 450, p. 483. Thus, for example, a road may be laid out on a topographic map and a close approximation to its profile may be obtained no matter how crooked or winding the road may be.

699. Finishing the Profile.

1. When blueprints or other reproductions of the profile are desired, all lines and lettering are in black ink; when only a single copy is desired, the grade line and all lettering pertaining to it may be in red.

2. The profile of the existing surface may usually be inked freehand with a line of even width but slightly wavering to represent irregularities. When, however, the vertical scale is so small that the little irregularities disappear, the profile should be ruled; otherwise the surface of the ground would be represented as more irregular than it really is. The grade line should always be carefully ruled; it is often made heavier than the original surface profile in order to be readily distinguishable.

3. Each of the heaviest vertical lines of the profile paper or of the profile tracing cloth is marked with the corresponding station number at the bottom of the sheet. Each of the heaviest horizontal lines is marked with the corresponding elevation. It is convenient to have these elevations at each end of the sheet and, when the sheet is long, at suitable intervals between the ends.

4. At every point of the grade line where the rate of grade changes, draw below the grade line a vertical line and mark along this line the station number and the grade elevation of the point thus: Sta. 40 + 53, Elev. = 114.25. All the lower ends of these vertical lines should be in the same horizontal line, preferably in one of the accentuated horizontal lines of the profile paper. Lines may be full or broken. When such a line falls exactly on one of the printed vertical lines of the profile paper, it is usually drawn heavier than that line. If the grade line is drawn in red, the lettering and numerals along the vertical lines are usually in red. Among the points where the station number and grade elevation should be marked are: at the beginning and end of the profile; at the beginning, at the end, and at the highest or lower point of a vertical curve, and, if the curve is to be staked out from the profile, at such intermediate points as may be needed in setting the stakes; at control points, such as at each side of an intersecting street and at each end of a bridge; and frequently, when there is no alignment diagram, at the points where each horizontal curve begins and ends. Points of change on the grade line are frequently enclosed in small circles.

5. The rate of grade may be written along the grade line between the two points where the grade begins and ends, or it may be placed below along the horizontal line which runs through the lower ends of the vertical lines through these two points as described in (4). In the latter case, the horizontal line should have arrow points like those on a dimension line to indicate where the grade begins and ends. A plus sign indicates a rising grade, and a minus sign a falling grade.

6. Sometimes, in order to make a profile stand out more prominently, a light flat tint is laid on, extending half an inch or so below the line which represents the original surface, or, in the case of a tracing, colored crayon is rubbed on the back of the cloth instead. Sometimes pencil is rubbed on a proposed profile and erased as fast as the construction is completed, thus showing the progress of the work. Occasionally it is desired to show different materials, as, for example, the different strata determined by borings. Some of the conventional signs are shown in Fig. 699.

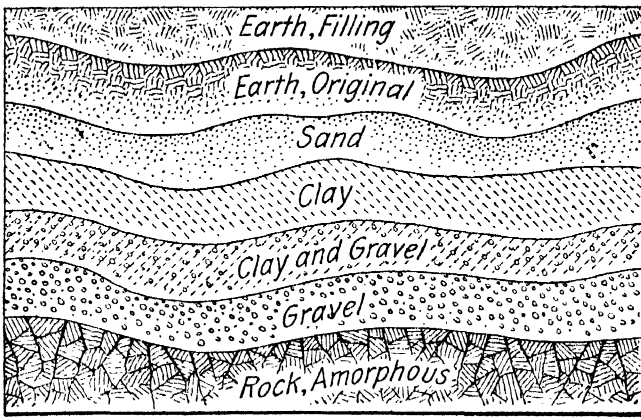


FIG. 699.

700. Lettering the Profile. Names of streets, streams, etc., and descriptions of benches, bridges, culverts, etc., are usually printed *along vertical lines* directly above the corresponding points on the profile. Explanatory notes are sometimes printed along vertical lines and sometimes along horizontal lines. When printed along vertical lines, words and numerals should read from the bottom up, and all notes should be at least one inch above the profile line, except the rate of grade, which may be printed along that line.

Remark. On long profiles, such as those used in railway work, some engineers prefer to have the vertical lines of lettering read from the top downward so that, in consulting a partly unrolled profile, the zero or left-hand end may be held toward the body as if the eye were looking along the general line represented by the profile.

Notes should be definite but brief, as, for example, "B.M. 36. Elev. = 126.192. Tack in root of Oak 30 ft. to left of Sta. 36." When a note describes some portion of the profile as, for example, "Proposed Grade," or "Profile on Center Line," an arrow may be drawn from the note to indicate which line is described. The elevations of the accentuated horizontal lines, and the station numbers of the accentuated vertical lines, should be in heavy black figures.

700(*a*). THE TITLE should give information similar to that specified in Art. 678(*a*). The datum should always be given near the title. Practically all the directions for constructing a title for a map (Art. 678) hold good for constructing a profile title. For sheets of ordinary size, the title is usually placed either in the

lower right-hand corner or in the center near the top of the sheet. For longer sheets that may be rolled, the title may be near the end of the sheet where the profile begins. It is convenient also to place brief titles on the back of the sheet at each end in order to identify the profile without unrolling.

701. Plotting Cross Sections. Cross sections, taken for the purpose of calculating earthwork, are frequently plotted to scale, particularly when they are irregular or when their areas are to be determined by means of a planimeter. They are usually plotted on cross-section paper (Art. 689). The proper scale depends upon the accuracy with which it is desired to scale distances, or, if the planimeter is to be used, upon the requirements to be met in the use of that instrument (Art. 589). A common scale for both horizontal and vertical measurements is 1 in. = 10 ft.; for this scale cross-section paper with rulings 10×10 to the inch is suitable. For highway cross sections, a scale of 1 in. = 5 ft. is often used; the corresponding paper is ruled 5×5 to the inch with additional horizontal lines $\frac{1}{25}$ in. apart. When, as is sometimes the case, cross sections are plotted on the same sheet with the profile, the scale is greater, as a rule, than that of the profile. Cross sections can be plotted from contours by the same method as that already explained for profiles (Art. 698).

The cross profile of the original surface is usually drawn as an irregular line of medium weight; the cross grade line is a smooth line drawn somewhat heavier. Several cross sections may be plotted on the same sheet, so arranged that successive cross sections are one above the other or one below the other, as may be preferred, with their centers in the same vertical line. Frequently there is room on the sheet for two or more sets of such cross sections arranged side by side. Just below each cross section should be printed its station number; the elevation of the horizontal line, used as a base line for the section, should also be printed just above that line. These station numbers and base elevations should be one under the other, approximately in a vertical line through the centers of the cross sections. Frequently the area of each cross section in square feet is printed just below the center of that section. The volumes of cut and fill between successive cross sections are also often shown, each volume being recorded in the space between the two plotted cross sections from which the volume was determined.

Cross sections are generally used only in the office and are of temporary value; hence they are commonly left as they were drawn in pencil, without inking. A simple descriptive title giving the location, the number of the sheet, the total number of sheets, and other necessary information may be placed on each sheet or at the ends of a roll. (See directions in Art. 700(*a*).)

PART III: SURVEYING INSTRUMENTS

INTRODUCTION

702. The Study of Instruments. When the student is first learning to use an instrument it is not necessary that he should study it in detail; on the contrary, it is better for him to confine his attention to the few parts which he has to manipulate. These parts are visible and need no description. When, however, he has become somewhat accustomed to the use of an instrument, he should begin a careful study of the different parts, some of which are not visible after the instrument has been put together. Parts thus concealed are shown in the following pages by drawings or photographs. Thus in the photograph on p. 841 a telescope is shown taken apart, and on p. 867 a transit is shown in a similar manner. The student should not be content, however, with merely observing the construction of the different parts. He should investigate the properties of lenses, of magnetic needles, of level tubes, the theory of the telescope, the defects of construction, and similar subjects. In this way he will become familiar with important facts and fundamental principles which are not evident from inspection. A careful study of this portion of the subject (Chapters 51, 52, and 53) should precede the study of the adjustments of instruments (Chapter 54).

"The mode of using any instrument is best learned from an examination of the instrument itself, with the instruction of a skillful teacher." *

* From J. M. Sganzin's American translation of *Elementary Course in Civil Engineering*—a work originally written in French previous to 1818.

CHAPTER 51

PRIMARY PARTS OF SURVEYING INSTRUMENTS

THERE are certain primary parts, such as the vernier, the magnetic needle, the level tube, and the telescope, which are more or less common to all surveying instruments. These parts are discussed in this chapter preparatory to a study of the instruments themselves in the next chapter. In some cases the discussion is little more than a review of certain fundamental principles of physics.

703. The Vernier, invented in 1631 by Pierre Vernier, is a device for measuring the fractional part of one of the smallest divisions of a graduated scale. It usually consists of a small sliding scale running parallel to and in contact with a fixed scale or a graduated arc. The divisions of the vernier, instead of being equal to the divisions on the graduated scale or arc, are either just a little smaller or a little larger. The amount by which the smallest division on the vernier differs from the smallest division on the graduated scale or arc determines the smallest reading or *least count* of the vernier.

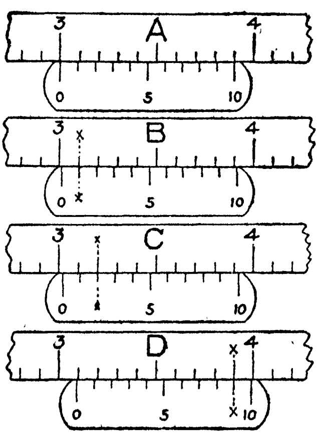


FIG. 703(*a*).

703(*a*). ILLUSTRATION. Figure 703(*a*) shows at *A* a portion of a scale divided into inches and tenths of an inch, with a vernier attached. The same scale is shown at *B*, *C*, and *D*, but the vernier has been moved in each case. At *A* the vernier zero is at the 3-in. mark of the scale, and it is seen that ten divisions of the vernier cover a space equal to nine divisions on the scale; hence one division on the vernier is equal to 0.9 ÷ 10 = 0.09 in. long, and is 0.01 in. shorter than the smallest divisions on the scale. If, therefore, the vernier is moved along the scale until the first mark on the vernier (not counting the zero) is exactly opposite to the corresponding mark on the scale (as shown at *B*), the vernier zero must be 0.01 in. from the 3-in. mark on the scale. If moved until the second mark on the vernier is opposite to the corresponding mark on the scale (as shown at *C*), the zero will be 0.02 in. from the 3-in. mark. Thus by moving the vernier 0.01 in. at a time, each mark in succession will coincide with a mark on the scale until, when the ninth mark is opposite to the corresponding mark on the scale (as shown at *D*), the vernier zero is 0.09 in. from the 3-in. mark. A movement of another 0.01 in. will then bring the zero of the vernier opposite the 3.1-in. mark, and the process will begin again.

703(*b*). READING THE VERNIER. From the preceding illustration it is evident that when the zero point of the vernier is between any two adjacent marks on the scale, its position may be determined by observing which mark on the vernier is exactly opposite some mark on the scale. For this reason the zero mark of a vernier frequently serves as an indicator for the movable part of an instrument to which the vernier is fastened. For example, when a vernier is attached to the target of a leveling rod (as shown on p. 101), the zero point of the vernier is placed opposite the horizontal center line of the target, so that when this center line (and the vernier zero) stops between two adjacent graduation marks on the rod, the *fractional* reading may be obtained from the vernier. There are always two steps involved in the *whole* reading, *viz.:*

1. Note the position of the vernier zero with respect to the main (stationary) scale, paying no attention to the other marks of the vernier.

2. Observe which mark on the vernier is opposite or most nearly opposite a mark on the scale, and add this reading to that obtained in (1).

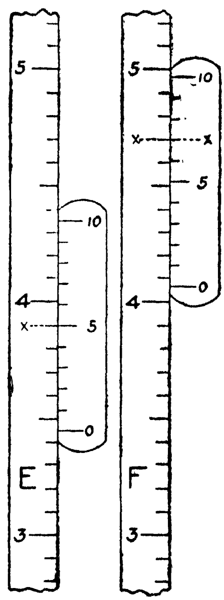


FIG. 703(*d*).

703(*c*). REMARK. Notice that the only use made of the vernier is in getting the small fractional reading in step (2) and that no attention is paid to the value of the mark on the scale which happens to be opposite a mark on the vernier. In reading any vernier for the first time it will be necessary to determine its least count as explained later.

703(*d*). ILLUSTRATION. In Fig. 703(*d*) the smallest division on the scale is 0.01 ft., and the divisions on the vernier are each equal to 0.009 ft., or 0.001 ft. less than the smallest division on the scale.

On the scale marked *E* the zero of the vernier is just beyond the 0.34-ft. mark, and the fifth mark of the vernier is opposite a mark on the scale; hence the reading is 0.34 ft. + 0.005 = 0.345 ft., i.e., the vernier zero or indicator is just halfway between 0.34 and 0.35 ft. On the scale marked *F* the reading is 0.40 + 0.007 = 0.407 ft. A common mistake would be to call it 0.47 ft.

Remark. In reading a vernier there are certain *danger points* where mistakes similar to that indicated above are likely to occur. These points for verniers on leveling rods are pointed out on pp. 100–102, and for transit verniers on pp. 231–235.

703(*e*). DIRECT AND RETROGRADE VERNIERS. A *direct* vernier is one in which the smallest division on the vernier is shorter than the smallest division on the main scale. In a *retrograde* vernier a division on the vernier is longer than a division on the scale. Most

verniers on surveying instruments are direct, such as those shown in Figs. 703(*a*) and 703(*d*).

703(*f*). REMARK. In reading a retrograde vernier proceed exactly as if it were a direct vernier, observing, however, that the divisions are numbered backward, or in an opposite direction from the numbers on the main scale.

Retrograde verniers are used on leveling rods which read *downward* from the center of the target instead of upward. (See p. 101 for target verniers which are direct and hence read upward.) Retrograde verniers are also used in special forms, such as those described in Art. 239.

703(*g*). TO DETERMINE THE SMALLEST READING or least count of a vernier: (1) *Observe how many divisions on the scale correspond to the same number plus one (on a direct vernier) or minus one (on a retrograde vernier).* (2) *Divide the value of the smallest division on the scale by the number obtained in* (1).

In algebraic terms let d = length of the smallest division on the scale, v = length of smallest division on the vernier, n = the number of d's which added together equal $(n+1)$ v's for direct or $(n-1)$ v's for retrograde vernier. Then $nd = (n+1)v$ (direct) or $nd = (n-1)v$ (retrograde).

Hence the smallest reading of the vernier is

$$d - v = \frac{d}{n+1} \text{ (direct)} \quad \text{or} \quad d - v = \frac{d}{n-1} \text{ (retrograde)} \qquad (1)$$

On many surveying instruments verniers are so constructed that the above rule may be simplified to:

Divide the value of the smallest division on the scale by the number of divisions on the vernier.

703(*h*). VERNIERS FOR LEVELING RODS AND TRANSITS. The principal use of the vernier in surveying is in taking target readings in leveling, and in reading angles in transit work. The verniers on targets are similar to that shown in Fig. 703(*d*) and are fully illustrated on p. 101. They should not be confused with the ordinary scale used on some targets, and illustrated on p. 99.

The vernier on the transit is divided into spaces corresponding to angular measure instead of linear. The principle however is exactly the same. Illustrations of transit verniers are given in Chapter 18.

704. The Magnetic Needle. (*a*). DIP OF THE NEEDLE. A needle which is perfectly balanced before it is magnetized will, after it is magnetized, dip towards the pole, and, in the northern hemisphere, the farther north the needle, the more its northern end will be attracted downward. In order to counteract this tendency, a small wire coil is placed around the needle to act as a weight on

its southern half. As the dip varies in different localities this little weight may be slipped in or out along the needle to preserve a perfect balance.

704(*b*). MAGNETIC DECLINATION. The magnetic needle does not point directly north except in a few localities. In some places it points west of north; in other places east of north. The meridian in which a needle properly magnetized and properly balanced comes to rest at any given place and time is called the *magnetic meridian* for that place and time, and the angle between a magnetic meridian and a true north and south meridian is the corresponding *declination of the needle* or the *magnetic declination.* In some places this declination is practically nothing; in other places it is large. Thus, for example, in certain portions of the central part of the United States it is zero, or nearly zero, but in portions of Maine it exceeds 22° west and in portions of the State of Washington it exceeds 24° east. (See the isogonic chart on page 835.)

704(*c*). CHANGES OF DECLINATIONS.* The magnetic meridian at any given place is continuously changing; consequently there are corresponding changes of declination. The changes may be classified according to their character as: (1) secular change; (2) annual change; (3) diurnal variation; (4) irregular changes; (5) lunar changes. The last-named change is so small that it may be ignored. The *secular change* in declination is the change that is continually taking place through the years and centuries. The *annual change* is the amount of the secular change in one year. The *diurnal variation* is the systematic departure of the declination from its mean value during a period of 24 hours. *Irregular changes* are irregular fluctuations due to magnetic disturbances which, when large and rapid, constitute a *magnetic storm.*

704(*d*). THE SECULAR CHANGE of declination appears to be periodic in character; at least it does not go on indefinitely in one direction. The compass needle at any given place, after moving continuously for a period of years in one direction with respect to the true north, gradually comes to a standstill, and then begins to move in the opposite direction. For example, at Portland, Maine, the declination appears to have been 12° west in 1700, then decreasing to only 8° west in 1780, then increasing again until now it is more than 17° west. Secular change from year to year is not uniform for any given locality, and it is different for different places. In the work of the surveyor, secular change is very important, and, unless otherwise specified, it is the change commonly referred to. Tables of secular changes over a long period of years are given for the various states in *Magnetic Declination in the United States.*

* The terms in this paragraph are those adopted by the U.S. Coast and Geodetic Survey and defined in *Magnetic Declination in the United States.* This government publication contains much valuable information concerning magnetic declination.

704(*e*). THE ANNUAL CHANGE of declination at a given place is not a constant quantity, as sometimes assumed by the surveyor. It is greatest near the middle point of the movement of the needle in one direction, described in 704(*d*), and becomes less and less as the end of the movement in that direction is approached until, at a place where the needle has reached its maximum declination, east or west, the annual change may be almost zero. The annual change is also different for different places, and varies in continental United States from 0° to more than 2°. (See isogonic chart.)

704(*f*). THE DIURNAL VARIATION is a well-defined daily periodic change. During the night the departures from the mean are usually small. In the morning the needle moves eastward, with an extreme between 8 and 9 A.M., then there is a westerly motion until about 1 P.M., after which the needle returns to its mean position in 3 or 4 hours. Although diurnal variation can ordinarily be neglected in compass surveying, it should be taken into account when greater accuracy is desired. The difference in declination between morning and afternoon is often as much as 10′ of arc.

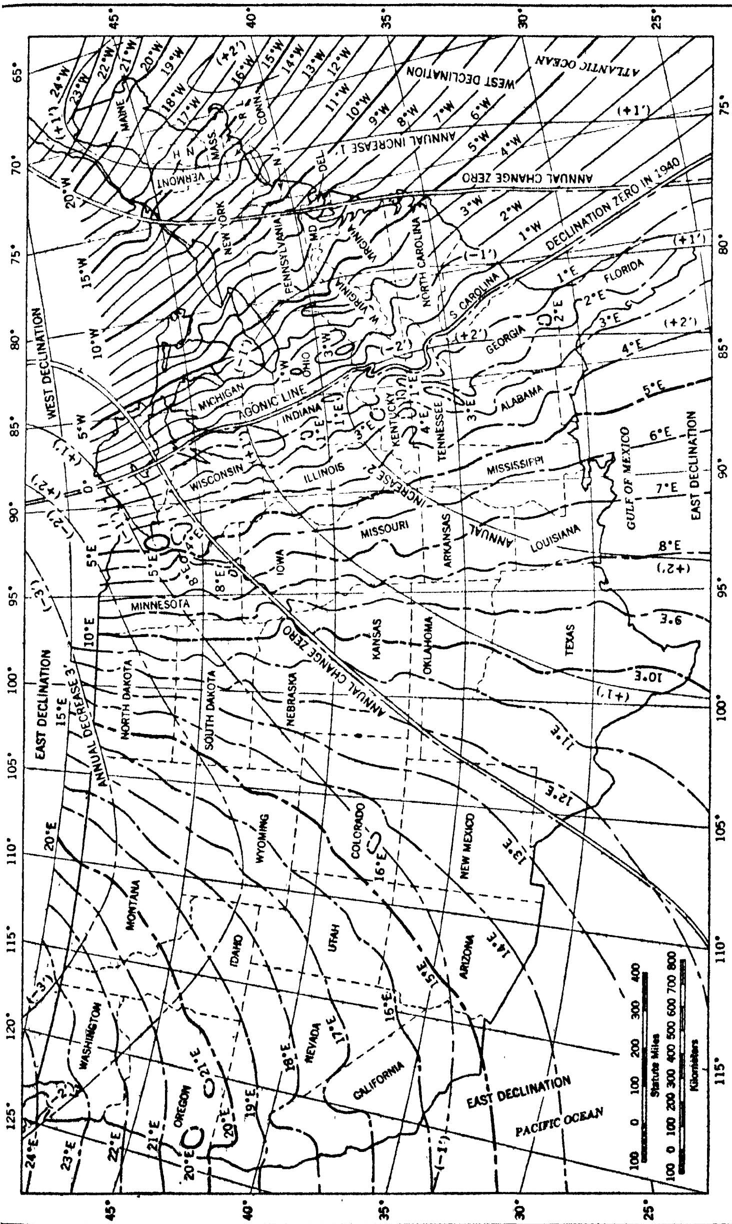
704(*g*). IRREGULAR CHANGES, due to magnetic disturbances or storms, may occur at any time and cannot be predicted. Changes of this kind amounting to more than a degree have been observed.

704(*h*). DETERMINATION OF THE MAGNETIC DECLINATION. The declination of the magnetic needle at a given place and for a given year may often be obtained with sufficient accuracy from tables of declinations or from an isogonic chart. Such tables or charts represent, however, *averages* of conditions obtained from observations at a *limited* number of places. Moreover, the changes of declination just explained are continually taking place. For these reasons, it is often best for a surveyor to determine for himself the declination at the place where he is surveying. This may be done by determining the true meridian of that place by one of the methods explained in Chapters 34 and 35. The field work is simple and little calculation is involved, particularly when the meridian is determined from observations on Polaris.

704(*i*). ISOGONIC LINES are imaginary lines passing through places where the magnetic declination is the same at a given time. From inspection of the isogonic chart given on the opposite page the isogonic lines are seen to be very irregular.

704(*j*). THE AGONIC LINE passes through places where the declination at a given time is zero. In the United States this line at present extends from Michigan to South Carolina, passing through portions of some of the central states. (See isogonic chart.) It is moving slowly westward. For places *east* of this line the declination is *west* and for places *west* it is *east*. This fact is a help in remembering whether the declination for a given place is east or west.

704(*k*). LOCAL ATTRACTION is a term used to denote any influence which prevents the needle from pointing to the *magnetic* north in a given locality. In its strict sense it does not include avoidable attraction due to things about the person or to other sources not connected with the place where the needle is read. Some of the sources of local attraction are: magnetite in the ground, wires carrying electric current, steel structures, railroad rails, underground iron pipes, etc. Contrary to a common belief, iron ores do not affect the needle unless they contain magnetite or pyrrhotite. Some of the avoidable sources of attraction are: keys,



Isogonic Chart. (Courtesy of U.S. Coast and Geodetic Survey.)

knives, steel-bowed spectacles, spectacle cases (which though covered with leather may be solid steel inside), wire in a hat brim, metal buttons, and other bits of iron or steel about the person; also axes, chains, steel tapes, etc., which may be lying on the ground near by.

Note. In retracing an old survey made many years ago, it was found that the bearings of all the courses were incorrectly given on the original map, and the errors were so large as to indicate that something was wrong with the compass originally used. Fortunately the instrument was still in existence, and an examination disclosed the presence of a steel screw in the compass box.

705. The Level Tube. The interior surface of the glass tube of a spirit level should be ground so that a longitudinal section will show a circular arc. The more uniform this curvature is throughout the length of the tube the more regular will be the motion of the bubble, and the greater the radius of curvature the greater the sensitiveness of the bubble. Within reasonable limits the more sensitive the bubble the more perfect the work, though a very sensitive bubble may be too unsteady for many purposes. On the other hand a sluggish bubble, though it may give the appearance of steadiness to an instrument and an impression that it "keeps" its adjustment, cannot be relied upon in accurate work.

The fluid used in spirit levels is usually a mixture of alcohol and ether. The length of the bubble changes with variation in temperature, and as the bubble grows shorter it becomes less sensitive. If the whole tube becomes overheated as it sometimes does when left in the sun, the bubble may become so short as to be nearly useless. If one end of the tube becomes warmer than the other, the tendency is for the bubble to move toward the warmer end. (See Art. 167(*a*).)

The line tangent to the circular arc of the tube at its middle point, or a line parallel to this tangent, is called the *axis of the level tube.* When this axis is horizontal the bubble will be in the center of the tube. Should the axis become slightly inclined the bubble will move toward the higher end of the tube in proportion to the angle made by the axis with the horizon. The glass tube is usually graduated on top by marks 2 mm. apart. The value of a level tube is usually expressed by the angular change which takes place in the inclination of the axis when the bubble moves over a single division. For example, if the change is 40″ of arc when the bubble moves 2 mm. (one division), the sensitiveness of the level tube is expressed as 40″ of arc per 2 mm., or simply as 2 mm. = 40″ of arc. The engineer can test for himself the sensitiveness of a level tube by the method explained in Art. 705(*a*).

When a level tube is attached to a telescope the relation between the sensitiveness of the bubble and the magnifying power of the telescope should be such that the smallest vertical movement of the cross hairs visible to the eye is followed by an appreciable movement of the bubble. For high-precision telescopes, the sensitiveness is usually from about 2 mm. = 10″ of arc to 2 mm. = 25″ of arc. For general surveying, the sensitiveness of the telescope level is usually from 2 mm. = 30″ of arc to 2 mm. = 60″ of arc.

Circular spirit levels are used for plumbing levels attached to leveling rods, for plane-table alidades, and occasionally for other surveying instruments.

705(*a*). To TEST THE SENSITIVENESS OF A BUBBLE. Assuming that the level tube is attached to the telescope of a level, the angular value of the bubble may be found as follows. (1) Set up the level and hold a leveling rod on some point a known distance from the instrument. (2) Take two readings, one with the bubble two or three divisions to one side of the center of the tube, the other with the bubble about the same number of divisions the other side of the center. (3) The difference in readings divided by the distance to the rod is the natural tangent of the angle between the two lines of sight. Divide this tangent by the tangent of 1″ of arc (0.00000485), thus obtaining the value of the angle in seconds of arc. Divide the result by the number of divisions between the two positions of the bubble; the result is the average value in seconds of a single division.

Example. A rod is held 300 ft. from the center of the instrument. The telescope is revolved through a vertical arc until the bubble has moved from three divisions on one side of the center of the tube to three divisions on the other side. The difference in the two rod readings is 0.269 ft.

$$\frac{0.269}{300} \times \frac{1}{0.00000485} \times \frac{1}{6} = 30.8''$$

Suggestions. (1) Set the level up in the shade with two diagonally opposite leveling screws in the line to the leveling rod. Change the position of the bubble by means of these screws. Probably better results will be obtained if an extra man acts as bubble tender. (2) In measuring off a base line to the rod it saves arithmetical work to make it an even 200 or 300 ft. (3) Repeat the test, holding the rod at different distances and displacing the bubble by different amounts. (4) Any level tube other than a telescope level may be tested in a similar manner by attaching it rigidly to a telescope, with its axis approximately parallel to the line of sight. (5) There are small instruments called level triers especially designed for testing levels. (6) Draw a figure and show that if D is the distance from the instrument to the rod, R the radius of curvature of the level tube, d the length of one division of the scale, and h the difference in two rod readings when the bubble is moved over n divisions of the scale, then $R = \frac{nd}{h} D$.

706. The Theory of Lenses. In order to understand the construction of the telescope it is necessary first to study some of the most important properties of lenses. The different forms of lenses

most used in telescopes are: (1) *double-convex* (convex on both sides); (2) *plano-convex* (plane on one side, convex on the other); (3) *concavo-convex* (concave on one side, convex on the other). No one of these forms is used singly, but two or more are combined to correct certain defects which could not be overcome in a single lens. It is unnecessary to give here a complete statement of the theory of lenses; it will suffice to point out the principles which apply to the passage of light through a double-convex lens. These principles are in general the same for the other forms.

706(*a*). Double-convex lenses. Each of the two surfaces of the *double convex* lens is a portion of a sphere, and the line passing through the centers of these two spheres is the *principal axis*. If the radii of these two surfaces are equal, the *optical center* is at the geometrical center of the lens. If the lens is very thin it may be assumed that any ray of light passing through its center undergoes no deviation. The *principal focus* of a convex lens is the point of its principal axis where rays of light parallel to that axis are converged (approximately) after passing through the lens, and in the ordinary *double-convex lens* the two principal foci are on the principal axis equal distances from the optical center. Thus in Fig. 706(I), *AA* is the *principal axis*, *C* the *optical center*, and *F* and *F′* the *principal foci*. The distance *FC*, equal to *F′C*, is the *focal length* which, for a given lens, is always constant. The focal length depends upon the curvature, not upon the size of the lens, but as large lenses usually have longer radii of curvature than small lenses their focal lengths are usually greater.

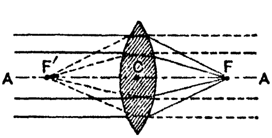


Fig. 706(I).

Proposition I. *A cone of rays from a point located at either focus becomes a cylinder of rays after passing through the lens, the axis of the cylinder coinciding with the principal axis of the lens. Conversely rays of light parallel to the principal axis converge at a principal focus after passing through the lens.*

Illustration. In Fig. 706(I), the full lines converge at *F* after passing through the lens and the dotted lines at *F′*.

Note. The figures in this article are not drawn to scale; on the contrary, they are necessarily much distorted. The distance represented by f_2 in Fig. 706(II) is so great compared with the diameter of the lens that the rays from *A* are nearly parallel, and the rays which converge at *B* are much more nearly parallel than they appear to be in the figure. Moreover, the image of a distant point is not focused by the objective lens as a point but as a *diffraction disk.*

Illustration. The rays from the sun are practically parallel; and in the familiar experiment of burning a hole in paper with a reading glass, the rays are converged at the principal focus of the glass.

Proposition II. *Rays diverging from any point converge approximately at a* real *focus after passing through the lens, provided the distance from the optical center to the point is* greater *than the focal length of the lens.*

Illustration. In Fig. 706(II), a cone of rays from any point A beyond the focus F becomes another cone of rays after passing through the lens, converging at a point B *beyond the other focus* F'. Conversely, rays from B converge at A.

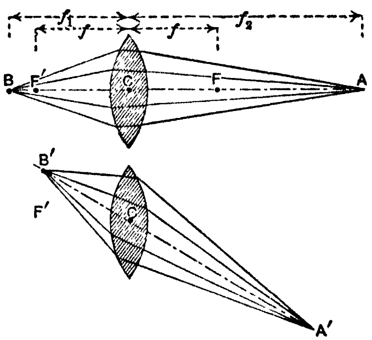


FIG. 706(II).

Conjugate Foci. If rays of light from any point converge at a second point after passing through a lens, and if, vice versa, rays from the second point converge at the first, the two points are called *conjugate foci.* Thus, for example, A and B in Fig. 706(II) are conjugate foci as are also A' and B'. The line joining conjugate foci passes through the optical center of the lens. When conjugate foci are not in the principal axis, the line joining them is called a *secondary axis,* e.g., $A'B'$. If f_1 and f_2 are the distances from the optical center to conjugate foci, and f is the focal length of the lens, then from a principle of optics

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f} \tag{2}$$

From the above equation it is evident that *both f_1 and f_2 must be greater than the focal length.*

Proposition III. *Rays diverging from any point continue to diverge after passing through the lens as if they came from a so-called* virtual focus *behind the point, provided the distance from the optical center to the point is* less *than the focal length of the lens.*

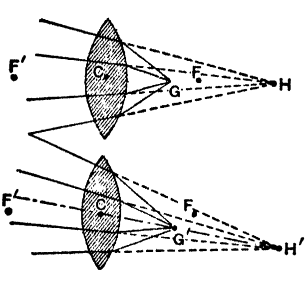


FIG. 706(III).

Illustration. In Fig. 706(III) rays diverging from any point G at a distance g *less* than f continue to diverge after passing through the lens, though less rapidly than before. They will therefore appear to radiate not from G but from H, a point behind G. H is a conjugate focus, but the rays do not actually pass through it as in the other cases; hence it is called a *virtual conjugate focus.*

706(*b*). FORMATION OF IMAGES. If the source of light is an extended surface, then the cone of rays emanating from each point

forms its own focus after passing through the lens; and the collection of foci constitutes an image which is *real* and *inverted* if the foci are real, but virtual and erect if the foci are virtual. The relative sizes of the object and image are sensibly proportional, if the lens is thin, to their respective distances from the optical center.

From proposition III, if an object is between a lens and one of its principal foci, the image is virtual and erect, and is larger than the object. This is the fundamental principle upon which magnifying glasses, microscopes, and eyepieces of telescopes are constructed.

From proposition I, if an object is at either of the principal foci, no image is formed (unless it is at infinity) for the cone of rays from any point become parallel after passing through the lens.

From proposition II, if an object is beyond one of the principal foci, its image is real and inverted, and lies on the opposite side of the lens beyond the other principal focus. This is the fundamental principle upon which the object glass of a telescope is constructed.

From equation 2, if the distance from the object to the lens is less than *twice* the focal length, the image is larger than the object and at a greater distance from the lens. If the distance is equal to exactly twice the focal length, the object and its image are of equal size and equally distant from the lens. If the distance is more than twice the focal length, the image is smaller than the object and less than twice the focal length from the lens. In all three cases the object and its image are on opposite sides of the lens, and the image is real and inverted. The farther an object recedes from a lens the nearer its image approaches the lens, until with the object at infinity the image is at a principal focus (proposition I).

707. The Telescope. The photograph of the telescope of a level shown in Fig. 707 will serve to illustrate the construction of the telescopes in general use on surveying instruments. The cross hairs are placed in a vertical plane *DD* at the principal focus of the objective lens at *A*. This lens gathers the rays of light from an object and forms a very small *inverted* image of that object in the plane of the cross hairs at *DD*. The eyepiece, extending from *F* to *G*, acts as a microscope to magnify the cross hairs and the image of the object. In some telescopes the image appears to be erect; in others it is inverted. The difference lies wholly in the combination of lenses in the eyepiece, as explained in Art. 708.

The *erecting* eyepiece, shown by itself at *F′G′*, is similar to a microscope. When in place, it can be moved toward or away from the cross hairs in order to bring both the cross hairs and the

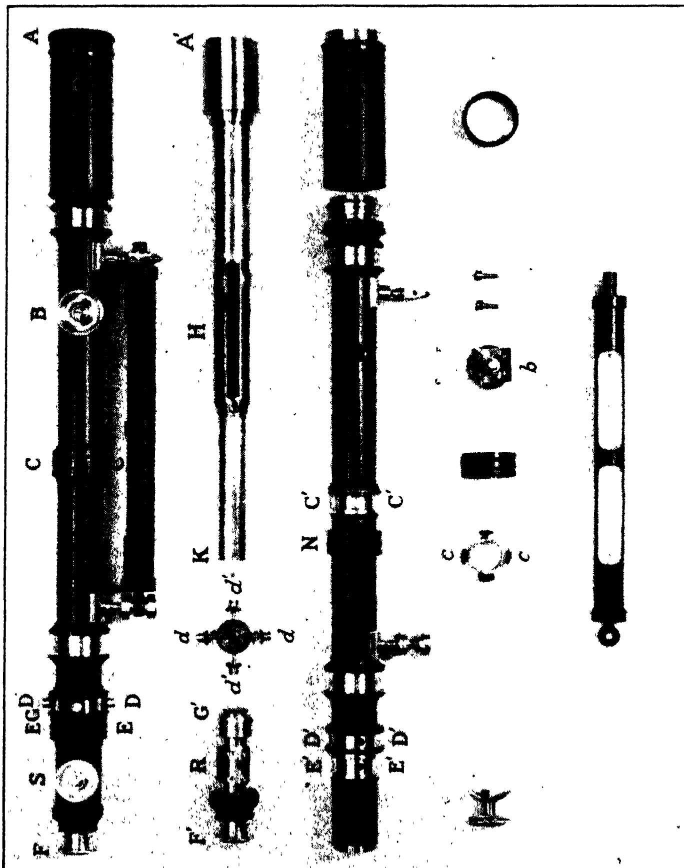


FIG. 707. (Courtesy of W. & L. E. Gurley.)

image of an object into sharp focus. The movement, in or out, is controlled by turning the large-headed focusing screw at *S*. At the inner end of this screw is a pinion, similar to that shown at *b*, which meshes with a rack shown at *R*. Some eyepieces are focused by rotating the entire eyepiece so that the projecting head of a screw, sliding in a spiral slot, causes the eyepiece to move in or out.

The objective slide with its lens at *A* is shown by itself at *KA'*. The end *K* slides through and beyond a close-fitting bearing ring inside the telescope at *CC*. (The ring is shown at *cc*.) To focus on a distant object, the objective slide is moved in or out by means of a focusing screw at *B*. The pinion for this screw is shown at *b* and the rack at *H*.

The cross-hair ring or, as it is often called, the *reticule* or *diaphragm*, is shown in a separate view at *dd*. It is held in place by two pairs of capstan screws which pass through smooth, *unthreaded* holes in the shell of the telescope. (See *D'D'*.) The ring may be moved up or down by loosening one of the screws marked *d* and tightening the other. Likewise, it may be moved to the right or to the left by loosening one of the screws marked *d'* and tightening the other. The cross hairs, one vertical the other horizontal, are either spider webs or filaments of platinum about 1/10,000 of an inch in diameter. They are almost invisible and therefore cannot be seen in the photograph. In some instruments two fine scratches, one vertical the other horizontal, are ruled on the face of the ring to aid in replacing broken hairs. In addition to the two cross hairs just described, there may be stadia hairs, or diagonal hairs, or a combination of both to meet some particular purpose. There is a possibility that fine lines on glass may replace gradually spider webs and platinum wires.

707(*a*). ADJUSTMENT OF THE CROSS-HAIR RING. It is very important for purposes of adjustment to understand clearly how the capstan screws control the movement of the cross-hair ring. The cross hairs must be fastened to this ring at right angles to each other before the ring is placed in the telescope, as they cannot be changed in this respect afterward. Notice, however, that the holes in the telescope through which the screws pass are not threaded. On the contrary, they are a little larger than the screws, so that when the latter are loose the whole ring may be turned slightly by moving the four capstan heads simultaneously around the outside of the telescope until one cross hair is vertical and the other horizontal. When the capstan screws are tight, each screw presses a curved washer (shown in the photograph) against the outside surface of the telescope. When one screw is loosened and the opposite screw tightened, the whole ring is drawn toward the tightened screw (since the holes in the shell of the telescope are smooth) until the loose screw and its washer are brought into contact again with the outside of the telescope. Notice that before tightening one screw the opposite screw should be loosened, otherwise the ring cannot move and the screw thread may be stripped. By loosening the lower screw and tightening the upper screw, the whole ring may be drawn upward, or by reversing the

process it may be drawn downward. Likewise by working the side screws in a similar manner, it may be drawn to one side or the other. All this may be done without turning the ring, i.e., one hair may be kept vertical, the other horizontal. Obviously no strain is brought on the cross hairs by turning the capstan screws since the hairs are attached to the solid cross-hair ring and move with it.

Complete procedures for adjusting the cross hairs, including tests, are explained in Chapter 54.

707(*b*). ADJUSTMENT OF THE OBJECTIVE SLIDE. The bearing ring at *CC* through which the end of the objective slide moves is shown by itself at *cc*. The ring is held in place by four slotted, flat-headed screws which pass through the shell of the telescope. (See *C'C'*.) The ring may be moved up or down, right or left, by loosening one screw and tightening the opposite screw with a screwdriver. This procedure is similar to that followed in the adjustment of the cross-hair ring. Turning the screw clockwise draws the ring toward the screw, *provided the opposite screw has been loosened.* In order to expose the screw slots, it is necessary to unscrew the cover ring and move it to one side. (The photograph shows the ring moved to *N*, leaving the four screws at *C'C'* exposed.) The provision for adjusting the objective slide may differ for instruments of different make or even for different instruments of the same make. In a well-made instrument the adjustment should seldom be necessary, and in many instruments there is no provision for it. The adjustment is a delicate one and is best made in the factory. (The complete procedure for the adjustment of the objective slide is explained in Art. 739(*q*).)

707(*c*). TO CENTER THE EYEPIECE. The purpose of this adjustment is to make the cross hairs appear in the center of the field of view. In the telescope shown in the photograph, the inner end of the eyepiece slides through a bearing ring at *EE*, similar to the bearing ring shown at *cc*. By removing the cover ring at *EE* (see *E'E'*) and turning the slotted, flat-headed screws, the ring may be moved up or down, right or left, by a procedure similar to that just explained in Art. 707(*b*). Before tightening any screw be sure to loosen the opposite screw. In some telescopes, it is necessary to unscrew and remove the entire eyepiece from the telescope in order to get at the flat-headed adjusting screws. In still other telescopes the movement of the bearing ring is controlled by capstan screws similar to those used for moving the cross-hair ring. It is well to remember that the bearing ring is always between the cross-hair ring and the rear end of the telescope. (The complete procedure for centering the eyepiece is explained in Art. 739(*n*).)

708. The Theory of Lenses as Applied to a Telescope may be studied more in detail by means of Fig. 708. Let the upper figure represent a telescope with a non-erecting eyepiece, and the lower figure one with an erecting eyepiece. The former will be considered first. Let *GG* represent the objective lens and *AC* any object sighted at. Usually the distance from the objective lens to the object is many times the length of the telescope; hence the latter is necessarily shown to a greatly exaggerated scale. Indeed, the length *d* is so great that the rays from *A* would appear nearly parallel if the figure were drawn to scale.

Note. For reasons to be explained later, the objective lens *GG'* is usually a combination of two glasses, but for purposes of explanation this combination may be represented by a single glass as shown in the figure.

1. The cone of rays from *A* becomes another cone of rays after passing through the lens, converging at *a* (proposition II, Art. 706(*a*)). Likewise rays from *C* converge at *c*, and rays from all points of the object converge at corresponding conjugate foci

between *a* and *c*. Thus a very small *inverted* image of the object is formed at *ac* (Art. 706(*b*)).

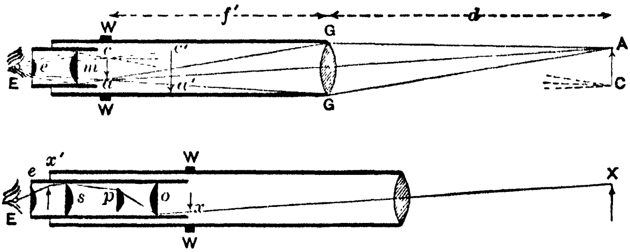


FIG. 708.

2. The distances *d* and *f'* are approximately distances to conjugate foci, both being greater than the focal length of the objective. As *d* is increased *f'* must decrease and vice versa, i.e., the farther away the object the nearer must be the inverted image to the objective lens. (From equation 2, Art. 706(*a*).) Since, however, it is necessary that the image should lie in the plane of the cross hairs *WW* (in order to eliminate parallax see Art. 708(*a*)), the distance *f'* is changed for objects at different distances by moving the objective lens *in* (nearer to *WW*) for distant objects, and *out* (away from *WW*) for near objects. This explains the rule for focusing in Art. 103(2). The focus changes rapidly for objects 10 to 30 ft. from the telescope, but very slowly after a distance of 100 ft. is reached, the focus for 100 ft. being nearly the same as for greater distances.

The inverted image *ac* is between the eyepiece and its principal focus; hence, from proposition III, Art. 706(*a*), the image will appear to be larger than it is, as if the rays of light came from virtual foci behind *ac* in some such position as *a'c'*.

In the *erecting eyepiece,* shown in the lower figure, extra lenses are inserted which reinvert or erect the image at the cross hairs before it is magnified. Thus rays from *X* converge at *x* in the plane of the cross hairs, the lenses of the eyepiece form another image reinverted at *x'*, and finally this image as seen by the eye is magnified by the lens at *e*.

708(*a*). PARALLAX. If the image formed by the objective lens is not in the same plane with the cross hairs, any movement of the eye is likely to cause an apparent movement of the image with respect to the cross hairs. This is called *parallax.* (See Art. 737 for the method of eliminating parallax.)

708(*b*). Optical terms. The term *ocular* is a general term for the eyepiece of a telescope. It is also used to designate specifically the combination of the two lenses at *e* and *s* in Fig. 708 in contradistinction to the combination of *erectors*, i.e., the two lenses at *o* and *p*.

The *entrance pupil* is the *aperture* through which the rays enter the telescope. Its diameter is usually that of the mount of the objective lens. The *exit pupil* is the round image of the objective formed behind the eyepiece by the ocular. Its position may be found by directing the telescope toward a uniformly bright area, such as the sky, and then moving a piece of ground-glass screen behind the eyepiece until the bright disk formed on the screen becomes as small as possible and has a sharp outline. The objective must be focused for infinity, and the eyepiece must be in its normal focusing position. *Eye distance* is the distance from the exit pupil to the nearest lens surface of the eyepiece.

The *line of collimation* of a telescope is usually defined as the line from the optical center of the objective lens to the intersection of the cross hairs. Some writers maintain that this definition is correct only when the cross hairs are in perfect adjustment. The *line of sight* is determined by the intersection of the cross hairs and is, therefore, a less ambiguous term, but for all practical purposes the two terms may be considered synonymous.

708(*c*). The two types of eyepieces compared. The non-erecting or inverting type of telescope is superior in practically every respect except one, namely, the inconvenience to the observer in seeing objects inverted. Thus, for example, a flagman appears to stand on his head, to move to the right when he really moves to the left, and vice versa. But the use of two additional lenses in an erecting eyepiece causes a loss of light which may amount to 15 per cent or more, and this loss of light is a far more serious objection than the slight inconvenience caused by inversion. Additional advantages of an inverting telescope are these. (1) For any desired magnifying power the length of an inverting telescope can be about 2 in. shorter than a telescope with an erecting eyepiece. (2) The definition is certain to be better because the image does not have to be erected but instead is formed by an achromatic eyepiece.

Although the telescopes with erecting eyepieces are in general use for ordinary surveying, inverting telescopes are used almost exclusively on high-precision instruments.

709. Aberrations. *Aberration* is a deviation of the rays of light when unequally refracted by a lens so that they do not converge and meet at a focus but separate, forming an indistinct image of the object or an indistinct image with prismatically colored edges.

709(*a*). Chromatic aberration. The focal length of any *single* lens is different for each different color of light, consequently a beam of white light, composed as it is of a mixture of all the colors of the spectrum, instead of converging at a focus after passing through a single lens, is distributed along the axis in a series of focal points. This chromatic aberration is of opposite sign in positive and negative lenses so that a combination of a positive and a negative lens of the *same kind* of glass can be free from chromatic aberration but it will have no power. Fortunately, optical glasses of widely dispersive powers are available so that it is possible to make a combination lens in which the chromatic aberration of a negative lens of relatively low power is sufficient practically to neutralize the chromatic aberration of a positive lens of relatively high power. Such a combination lens is known as an *achromatic lens*. The combination used in most surveying instruments is composed of an outer double-convex lens of crown glass and an inner concavo-convex lens of flint glass.

709(*b*). Spherical aberration. Rays of light passing through a *single* lens near its edge converge at points slightly nearer the lens than those in which rays closer to the center meet so that, even if white light were of one color instead of a mixture of colors, there would still be a series of focal points instead of one clear image. This imperfection or blurring arises from the form of curvature of the lens. Aberration due to this cause is known as *spherical aberration*. The spherical aberration of a negative lens tends to neutralize that of a positive lens; consequently the positive and negative elements of an achromatic lens, designed to eliminate chromatic aberration, can be so shaped as practically to eliminate spherical aberration also.

The elimination of aberrations is only one of the problems in the design of telescopes, but the extent to which a telescope is free from aberrations determines to a considerable degree the quality of a telescope.

710. The Internal Focusing Telescope differs from the ordinary telescope in three important respects. (1) The objective lens is fixed immovably in the end of the telescope, the objective slide is thus eliminated, and the objective lens cannot, therefore, be used for focusing. (2) The cross hairs are placed not at the principal focus of the objective lens but between that focus and the eyepiece. (3) A movable negative focusing lens is interposed between the objective lens and the plane of the cross hairs. This additional lens, mounted in a focusing slide, changes the direction of the rays which pass through the objective lens so that, by moving the focusing lens toward or away from the objective lens, the image of an object at any distance may be focused on the plane of the cross hairs instead of on the plane at the principal focus of the objective lens.

The *advantages* claimed for internal focusing are these. (1) Since both ends of the telescope are practically scaled, the cross hairs, the interior surfaces of lenses, the focusing-lens slide, and the entire interior of the telescope are kept more nearly free from dust and moisture. (2) A more accurate seating of the focusing-lens slide is provided throughout its entire range. Moreover, since this slide is nearly free from grit, there is much less wear to cause lateral displacement than there is on the objective slide of an ordinary telescope. (3) The lateral displacement, with its effect on collimation, is a principal source of error. It cannot be avoided altogether, but it can be greatly reduced by internal focusing, first because there is less wear on the slide, and second because, whatever lateral displacement of the focusing lens may be, the effect on collimation is much less than that due to an equal displacement of the objective lens—possibly not more than one-third as much. (4) After a telescope has been brought to a level position, any movement of an internal focusing slide is less likely to throw the telescope out of level than is an equal movement of an objective slide. This is because an internal focusing slide is lighter and nearer the center of the instrument than an objective slide. (5) In stadia surveying, the instrumental constant is eliminated. (6) The negative focusing lens serves a useful optical purpose because the focal length of the combined objective and negative lens is greater than the distance between the objective lens and the focal plane. This extra equivalent focal length can be used to increase the power, or an eyepiece with long focus can be adopted. Internal focusing also gives the optical designer an extra lens to work with in order to reduce the aberrations of the system and to increase the diameter of the objective lens.

The *principal disadvantages* of an internal focusing telescope are: (1) The interior is less accessible for cleaning or for other purposes. (2) The extra lens reduces the amount of light that reaches the eye. (The loss of light, however, is only about 8 per cent or only about one-half as much as that due to the two erector lenses in an erecting eyepiece.)

Remark. Some of the advantages claimed for the internal focusing telescope are of little consequence except in work of the highest precision. For example, the effect of wear in the objective slide of any telescope is generally negligible, both in transit work and in leveling, in all but high-precision instruments. Nevertheless, there is a decided trend in favor of the use of internal focusing telescopes.

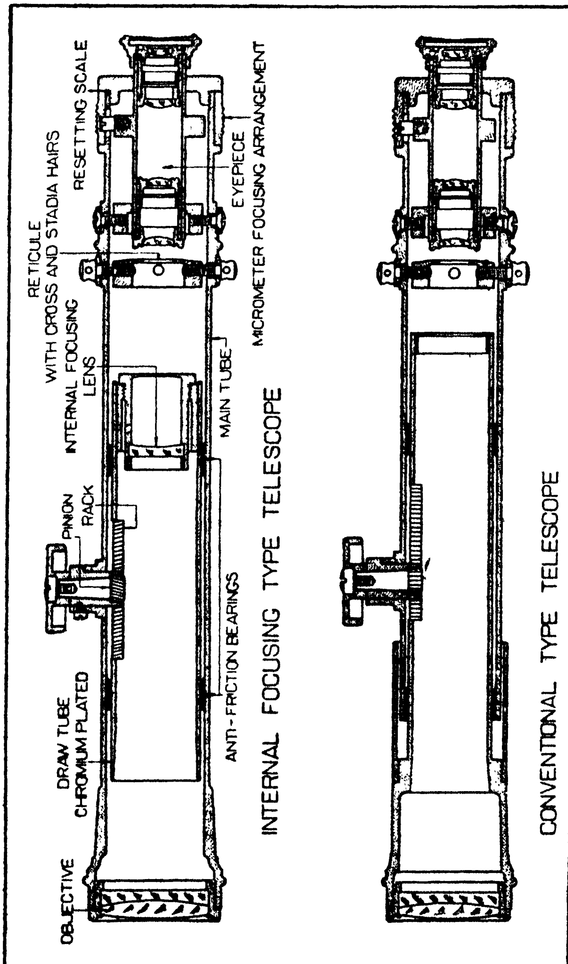


Fig. 710. (Courtesy of Keuffel and Esser Co.)

711. Optical Characteristics of the Telescope.* The desirable optical characteristics of the telescope of a surveying instrument depend upon the purpose for which the instrument is used. Obviously, therefore, in order to select a telescope best suited to a particular need, a general knowledge of the optical characteristics is necessary.

711(*a*). MAGNIFICATION depends upon the ratio of the focal length of the objective lens to the focal length of the eyepiece. For example, if the focal length of the objective is 9 in. and that of the eyepiece 0.3 in., then the magnifying power is 30. The magnifying power should be proportional to the aperture, i.e., to the amount of light which enters the telescope. If the magnifying power is too high for the aperture, ordinary objects will appear too faint, especially under unfavorable atmospheric conditions. If the magnifying power is too low, the objects will appear too small for accurate sighting. The magnifying power should be such that the least movement of the level tube or change in the vernier causes an easily perceptible movement of the cross hairs over the object in the field of view. Except perhaps in stadia work, a higher power than this is a disadvantage since there is a needless loss in the brightness of the image, and a lower power does not develop the full capacity of the instrument. The loss of light in high-power instruments is often mistaken for cloudiness due to poor lenses.

For rapid working the exact focusing required for a high-power telescope is a drawback, a change in focus being required for a small change in distance. For example, in a low-power telescope the focus for distances between 300 and 400 ft. is practically the same. In high-power telescopes, however, the change of a few feet requires a change in focus, and if the telescope is not properly focused the image is likely to be less distinct than in low-power telescopes. Low power is better also for focusing upon objects very near the instrument.

Non-erecting eyepieces are not more powerful than the erecting eyepieces, but, being shorter, they allow for a given length of telescope a greater focal length of objective lens, and thus increase the power. They, however, transmit much more light to the eye and have a much larger field.

For transits the average magnifying power is 20 to 24 diameters; for levels it is 28 to 38, running to 45.

To Determine the Magnification. Divide the diameter of the entrance pupil by the diameter of the exit pupil. Another method is to focus on a coarsely graduated rod held 15 or 20 ft. from the instrument. Look through the telescope with one eye and along the outside of the telescope with the other eye. One space viewed through the telescope will appear to cover a certain number of spaces seen with the naked eye. This number is approximately the magnifying power of the telescope.

711(*b*). THE DIAMETER OF THE ENTRANCE PUPIL, which is ordinarily equal to that of the objective, should be as large as the other requirements in design will permit in order that the objective lens may form an image that is as bright and sharp as possible.

The focal length of the objective plus the length of the eyepiece determines the length of the telescope. For any given focal length of the objective, an increase in diameter above a certain maximum diameter increases the aberrations and the definition suffers. Consequently the desirable diameter of the entrance pupil is limited by the length of the telescope.

711(*c*). THE DIAMETER AND THE POSITION OF THE EXIT PUPIL. All the light which emerges from the telescope must pass through the circle of the exit pupil. If the pupil of the observer's eye is placed at the plane of the exit pupil and if the eye pupil is at least as large as the exit pupil, *all* the light emerging from the telescope will be utilized in forming the observer's retinal image. The ideal condition is to have the diameter of the exit pupil equal to the diameter of the eye pupil, but other factors enter into the con-

* In writing this article on the optical characteristics of the telescope, the author has made use of portions of a paper, *Surveying Instrument Telescopes,* by Carl W. Keuffel of Keuffel and Esser Co., Hoboken, N. J.

trol of the size of the exit pupil to make it somewhat smaller than the eye pupil.

The *eye distance* must be such that the observer can place his eye comfortably at the plane of the exit pupil. The eye distance of any telescope may be determined as explained in Art. 708(*b*).

711(*d*). THE RESOLVING POWER of a telescope is the power to form distinguishable images of objects separated by small angular distances. Since the image of a distant point is a diffraction disk, the image of two distant points close together will consist of two diffraction disks which may overlap and become indistinguishable, depending upon the resolving power of the objective. This power is usually expressed as the angular separation in seconds of arc of two points which can just be resolved. The resolving power of the unaided eye is about 60″, varying with different observers. The resolving power of any telescope, if the aberrations of the optical system are not excessive, should be at least 140 divided by the diameter of the entrance pupil in millimeters. Thus the resolving power in a properly designed telescope depends directly upon the diameter of the objective but is independent of the focal length of the objective. If the diameter of the objective is too small, no amount of magnification or of light will result in sufficient resolving power. If the resolving power is insufficient, the *definition* will be poor, i.e., images will not be clear and distinct. The importance of resolving power should, therefore, be emphasized.

To test the definition or resolving power of a telescope, focus on small, clear, well-illuminated print 40 or 50 ft. away. If the definition is good, the print will appear clearly defined and legible. The circles in Fig. 711(*d*),

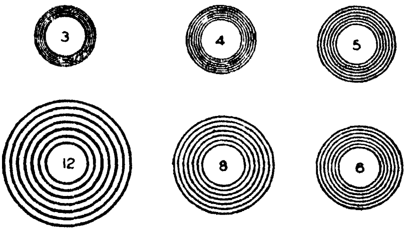


FIG. 711(*d*). (Courtesy of W. & L. E. Gurley.)

when set at a distance of 25 ft. and well illuminated, can be used to measure the resolving power of a telescope in seconds. The circles also afford a check on the astigmatism of the optical system.

711(*e*). ILLUMINATION. The illumination or brightness of the image of a telescope depends upon the magnifying power, the size of the entrance pupil, and the number and quality of the lenses. If the illumination is poor, the image will appear faint, though it may be sharp and definite. Lack of illumination means simply lacks of light. The *absolute* brightness of the image is inversely proportional to the second power of the magnification and directly proportional to the second power of the entrance pupil. Thus, for example, a 10 per cent increase in the entrance pupil will give over 20 per cent increase in brightness, whereas a 10 per cent increase in magnification will result in about 17 per cent loss in illumination. The *apparent* illumination seen through the telescope is dependent upon the relation of the diameter of the exit pupil and the diameter of the pupil of the eye; the ideal condition is to have the two diameters equal (Art. 711(*c*)).

The essential facts to remember are these. (1) Increase in magnification results in decrease in illumination. (2) Increase in the entrance pupil results in increase in illumination. (3) Increase in the number of lenses results in decrease in illumination. (Extra lenses, such as the erecting lenses in the eyepiece, or the internal focusing lens, decrease illumination.)

711(*f*). THE FIELD. The *real field* of the telescope is the total angular field of view in the object space. This real field is magnified by the eyepiece, and the field emerging from the eyepiece and converging through the center of the exit pupil is called the *apparent field.* Approximately, the real field multiplied by the magnification is equal to the apparent field. The *field*

of view of the telescope, i.e., the whole circular area seen by the observer when looking through the telescope, is limited by the *stop* in the focal plane of the objective. In the telescope of a surveying instrument, this stop is usually the hole in the cross-hair reticle. The field of view not merely is dependent upon the size of this hole or upon the size of the objective, but also increases as the magnification of the telescope decreases. It is increased by the use of two plano-convex lenses in the eyepiece in place of a single lens. A wide field of view is desirable as it enables the observer to work more easily and rapidly. In stadia surveying, *flatness of field* is also desirable. If there is distortion of the field of view so that it does not appear flat, there is difficulty in observing stadia readings. When the stadia rod is brought into focus on the intersection of the cross hairs, the division marks on the rod which appear to be at the stadia hairs will not be in sharp focus. This is because the image of the rod does not appear in a plane (flat) but in a curved field of view. This curvature of the image is due principally to spherical aberration.

To determine the field of view, set two range poles 200 or 300 ft. from the instrument at such a distance apart that one is at one edge of the field of view and the other at the other edge. Multiply this distance by 57.3 and divide by the distance to the poles, 200 or 300 ft. as the case may be. The result is the angular width in degrees of the field of view.

712. The Selection of a Telescope. In selecting a telescope for any surveying instrument, it is usually best to consult instrument makers who, as a rule, are well qualified to discuss the various characteristics of the telescope. Among general considerations are general type of work for which the instrument will be used and the maximum degree of precision required in that work; section of the country; general atmospheric and climatic conditions.

In discussing the characteristics of the telescope, it is well to keep in mind the limitations in the attainment of the most desirable characteristics, and the interrelations between different characteristics as explained in the preceding article. Since the attainment of one characteristic to the highest degree often affects adversely some other characteristic, the aim in the selection of a telescope is not to have each characteristic as nearly perfect as possible but to obtain the most suitably balanced combination of characteristics.

Among the important considerations to be kept in mind are:

1. An erecting eyepiece is convenient but it adds about 2 in. to the length of a telescope and may reduce the illumination 15 per cent or more. For a given length of telescope all characteristics can be improved by the use of an inverting eyepiece (Art. 708(*c*)).

2. The diameter of the entrance pupil (objective) is limited for a given length of telescope (Art. 711(*b*)).

3. Magnification greater than is necessary results in a needless loss of light, is hard on the eyes, tends to decrease the field of view, may be a disadvantage in sighting and focusing, and may increase the length of telescope with a corresponding increase in the size and weight of the upper part of the instrument (Art. 711(*a*)).

4. Illumination is highly desirable but it is limited by the size of the entrance pupil (objective). Moreover, since an increase in magnification causes a decrease in illumination, it may be necessary to sacrifice illumination to some extent to obtain a desired high magnification.

5. Resolving power is also highly desirable, but that also is limited by the diameter of the entrance pupil.

6. A wide field of view is desirable but, like illumination, it is decreased by an increase in magnification.

The general method of selecting a telescope will be illustrated by considering the desirable characteristics, first of a telescope for a transit, and then of a telescope for a level.

712(*a*). Telescope of a transit. The longer the telescope, the higher must be the supporting standards in order that the telescope may be plunged; and the higher the standards, the larger must be the upper plate in order that the standards may be spaced a sufficient distance apart. The length of the telescope should, therefore, be as short as possible. The length seldom exceeds 12 in., and on very small light transits it may be as short as 6½ or 7 in. The design of any transit depends to a large extent on the size and type of telescope adopted.

In ordinary work the telescope of a transit is used mainly for pointing in the measurement of angles. The pointing or aiming accuracy is proportional to the square root of the magnification. The probable pointing error at 16 magnification is about 1″ of arc, at 20 it is about 0.9″, at 25 it is about 0.8″, and at 30 it is about 0.75″. From these values it is obvious that a magnification of 20 is ample for the ordinary degree of precision, and even in work of high precision little is gained in pointing accuracy by increasing the magnifying power much above 25. When visibility is poor, owing to atmospheric refraction or haze, the pointing error may even increase with an increase in magnifying power. In telescopes for small light transits, magnifying power may be as little as 12 or 14; in special types of precise theodolites, such as those used in first-order triangulation, magnification may be as great as 40.

A length of telescope of 11 to 12 in. and a magnification of about 20 permit the design of a telescope in which the other characteristics will be satisfactory for ordinary work. For example, the following is *one* combination of characteristics that is satisfactory in ordinary transit surveying. Erect image; overall length 11 in.; magnification 20; entrance pupil 36 mm. (about 1.4 in.); exit pupil 1.8 mm. (about 0.07 in.); resolving power 3.9″; relative brightness 3.25; probable pointing error 0.9″; field of view 1° 30′.

An increase in magnification would increase the pointing power very little and would decrease the illumination. An increase in the objective diameter would be advantageous but would be limited by the ability to design a better objective. An increase in the overall length would help all the optical characteristics but would make the transit top-heavy. By changing the eyepiece to the inverting type, the length of the telescope could be decreased by at least 2 in. and all characteristics would be improved. By the use of an internal focusing lens, a still more efficient telescope could be designed (Art. 710).

When a transit is to be used in work which involves a good deal of leveling or a large amount of stadia surveying, the magnification should be as high as atmospheric conditions and other limitations will permit.

712(*b*). Telescope of a level. In leveling, it is not the pointing error but the ease and accuracy with which graduations on the leveling rod can be read that are the determining factors in deciding the desirable magnification of the telescope. Consequently, the magnification of the telescope of a level is, in general, greater than that of a transit telescope. This increase in magnification is practicable because the telescope of a level can be longer and both the objective diameter and the focal length can be increased to give additional brightness and greater magnification. When, however, the magnification is too great, legibility and ease of rod reading decrease, particularly when visibility is poor owing to refraction and haze. Moreover, the longer the telescope the more it may be thrown out of level by the movement of the objective slide in focusing; for a given length of telescope, the effect of internal focusing on the balance of a telescope is considerably less than that due to focusing the objective. The length of telescope seldom exceeds 18 in. in a dumpy level or 22 in. in a wye level, and may be as short as 12 in. in small levels. A common length is 18 in. for a dumpy or a wye level and 14 or 15 in. for a precision level.

The magnifying power of the telescope of a level varies from about 20 in a small level, such as an "architect's level," to about 45 in a precision level. Common values range from 30 to 36 for ordinary dumpy and wye levels and from 36 to 40 for precision levels. Effective apertures of telescopes on most levels range from about 1.3 to about 1.6 in.

CHAPTER 52

CHAINS, TAPES, AND RODS

In this chapter, the essential characteristics of chains, tapes, leveling rods, and stadia rods are discussed. Methods of standardizing tapes are explained and permanent standards used for this purpose are described.

713. Chains. A surveyor's chain usually consists of 100 pieces of wire bent into rings at the ends and joined together by rings. Each link consists of one straight piece and a separate ring at each end; hence there are 300 separate pieces in one chain. The *engineer's chain* is 100 ft. long with links 1 ft. long. *Gunter's chain* is 4 rods or 66 ft. long, with 100 links each 0.66 ft. or 7.92 in. long. The 66-ft. length was originally adopted as a convenient one for land measurement since 10 square chains then equal one acre. This gives rise to the special units of measurement given in Art. 8(*g*). Although the chain is still used, it has been quite generally superseded by the chain tape.

714. Tapes. *Cloth tapes* of closely woven linen, treated to resist moisture, are light, flexible, and surprisingly durable, even under severe conditions. Because a cloth tape has no metal in it, it can be handled with safety, in places where there is danger from electric current. Since, however, cloth tapes stretch easily, they should be used only for comparatively rough measurements.

714(*a*). Metallic woven tapes of waterproofed fabric interwoven with small brass or bronze wires do not stretch as easily as cloth tapes; nevertheless, under a strong pull, they will sometimes stretch as much as 0.3 ft. in 100 ft. They stretch permanently under continued use and are affected by moisture. Since metallic tapes are light, flexible, and not easily broken, they are particularly useful in cross sectioning, in some methods of taking topography, and in similar work where small errors in the length of the tape are of no consequence.

714(*b*). Steel tapes vary in quality and accuracy of graduation, but even a poor steel tape is generally superior to a cloth or a metallic tape for most of the linear measurements that are made in surveying. There are two common types of steel tapes. (1) The *band* (*or chain*) *steel tape* is comparatively narrow and thick and is intended for heavy duty where a high degree of precision is not required. (2) The *ribbon steel tape* is generally wider and thinner than the band tape and is more accurately graduated; it is almost invariably used when more than an ordinary degree of precision is required. A first-class ribbon steel tape is one of the most accurate devices known for linear measurement. *Pocket steel ribbon tapes* in lengths of 3 ft. up are very useful not only in the field but often in the office as well. The *invar tape* is made of a nickel and steel alloy which has such a low coefficient of thermal expansion that the changes in the length of tape due to changes in temperature are much smaller than those in a steel tape. Invar tapes are used mainly for linear measurements of a high degree of precision, such as measurements of base lines. They are so easily bent and distorted that they are not suitable for ordinary work.

714(*c*). Sizes of tapes. Steel band tapes vary in width from ⅛ to 5⁄16 in.; ribbon tapes from ¼ to ½ in. Either type may be obtained in lengths of 50, 100, 200, 300, and 500 ft.; a Gunter's chain tape may be had in lengths of 1, 2, 3, 5, and 8 chains; steel tapes graduated according to the metric system may be obtained in lengths of 15, 25, 30, 50, and 100 meters. Cloth tapes and metallic woven tapes vary in width from 7⁄16 to ⅝ in. and may be obtained in lengths of 25, 33, 50, 66, 75, and 100 ft., or in lengths of 10, 15, 25, and 30 meters.

714(*d*). Standard subdivisions. (1) *Steel tapes* in feet, inches, and eighths of an inch; or in feet, tenths, and hundredths of a foot; or in half centimeters and millimeters. (2) *Woven tapes* in feet, inches, and one-half inches; or in feet, tenths, and half tenths of a foot; or in half centimeters. (3) *Spring winding pocket tapes* in inches and sixteenths of an inch; or in tenths and hundredths of a foot; or in millimeters.

714(*e*). Common graduations on tapes. (1) Graduated to feet throughout, with the first and last foot subdivided to tenths of a foot, or to inches. (2) Graduated to feet throughout, with each foot subdivided to tenths of a foot, or to inches. (3) Graduated to feet throughout, with each foot subdivided to tenths and hundredths of a foot, or to inches and eighths of an inch. (4) Graduated to centimeters throughout, with the first decimeter subdivided to millimeters. (5) Graduated to half centimeters throughout.

714(*f*). Special graduations. In addition to the combinations of graduations just given there are many other similar combinations, some of which can be obtained only by special order. These special graduations are described in catalogs issued by dealers. In addition, special graduations include those in units of vara measurements, Spanish measurements, Russian measurements, etc.

Tapes may be had which are graduated on both sides. One combination of English units may be on one side of the tape and a different combination of English units on the other side; or the tape may be graduated in English units on one side and in metric units on the other side.

714(*g*). Graduation numbers. Common methods of numbering graduations are: (1) every fifth foot; (2) every foot; (3) every foot and every inch or every tenth of a foot. The first two methods are common on band tapes, the third on ribbon tapes.

On the so-called *ready-reading tape* or *instantaneous-reading tape* the last preceding foot number is repeated at each tenth of a foot, or at each inch, in small yet easily read figures. This keeps the total reading directly before the eye, eliminates the chance of error in looking back to the last preceding foot, and saves time.

714(*h*). Graduation marks. Common methods of marking graduation lines and numbers on steel tapes are: (1) stamped on short brass sleeves or on lumps of solder at graduation points; (2) etched on the tape; (3) raised lines and figures in natural steel against a dead black background; (4) black lines and figures (not etched) against a permanent white background. The first two methods are used mainly for band tapes. Etching in the third method may affect the full temper and strength of the tape. Unless otherwise stated, graduation *points* are where the center lines of the graduation marks intersect that edge of the tape to which the shortest marks extend. If all marks extend entirely across the tape, the graduation points are where the lines intersect that edge of the tape that is the farther from the observer when the zero of the tape is at his left hand. (See "Comparison of Tapes with the Standards," Art. 715(*d*).)

On band tapes the zero point and the final or terminal graduation point may not be marked by any of the methods given. Instead, they may be marked by metal shoulders, or by the ends of the loops or end fasteners that hold the handles to the tape.

714(*i*). A tape thermometer scale is a scale marked on a tape at the terminal point; it is graduated to correspond to the contraction or expansion of the tape. Each mark on the scale is the terminal point of the tape when the prevailing temperature at the time of measurement is the same as that indicated by the graduation number at that mark—Fahrenheit units for tapes graduated in feet, centigrade units for tapes graduated according to the metric system.

714(*j*). Tape accessories, described in catalogs issued by dealers, include *handles* for gripping any part of the tape; *tension handles* for determining the tension applied; *thermometers* for determining the temperature of the tape; *detachable hooks*, one of which when placed on the end of a tape enables one man unassisted to take measurements; *detachable end rings* and *leather thongs* to serve as handles; steel *chain pins* or *arrows; tape wipers; reels* of various types; *tape repair kits* and outfits.

715. Standardization of Steel Tapes by the Bureau of Standards. A tape is *standardized* or *tested* by comparing its

length, under specified conditions of support, temperature and tension, with a length known to be standard. Fractions of the length of a tape may be tested as well as the total length. The United States Department of Commerce, National Bureau of Standards, in Washington, will standardize a tape for a small fee and will furnish a certificate of comparison. The *conditions of support* may be: (1) supported throughout; (2) supported at the ends only; (3) supported at the ends and at one or more intermediate points.

715(*a*). THE STANDARD TEMPERATURE is 68° F. (equal to 20° C.). The *standard tension* when the tape is supported throughout on a horizontal flat surface is 10 lb. (4.5 kg.) for tapes 25 to 100 ft. or 10 to 30 meters in length; for tapes longer than 100 ft. or 30 meters it is 20 lb. (9.0 kg.). For tapes supported at the ends only, there is no standard tension. The tension required varies with the unsupported length, the weight of the tape, and the character of the steel. The variation may be from about 13 lb. for a light 50-ft. ribbon tape to about 24 lb. or more for a heavy chain tape. The standard coefficient of expansion for steel tapes is 0.00000645 per degree Fahrenheit or 0.0000116 per degree centigrade. This amounts to 0.000645 ft. or 0.00744 in. per degree Fahrenheit on a 100-ft. tape; on a 50-ft. tape to just one-half; and on a 200-ft. tape to twice that distance per degree.

The Bureau of Standards is not prepared to test tapes supported at points more than 150 ft. or 30 meters apart, nor can it test a tape at any point which is not a point at which the tape is supported.

The following information concerning specifications, certification, comparison of tapes, and the different tests that are made is furnished by the National Bureau of Standards.

715(*b*). SPECIFICATIONS FOR STANDARD STEEL TAPES

A steel tape is considered as standard when it has been tested by the National Bureau of Standards and found to conform to the following specifications. It shall be made of a single piece of metal ribbon, and none of the graduations shall be on pieces of solder or an sleeve attached to the tape or wire loops, spring balances, tension handles, or other attachments liable to be detached or changed in shape. The error in the total length of the tape, when supported horizontally throughout its length at the standard temperature of 68° F. (20° C.) and at standard tension, shall not be more than 0.1 in. per 100 ft. (2 mm. per 25 meters).

715(*c*). CERTIFICATION OF TAPES

Tapes conforming to the above specifications will be certified by the National Bureau of Standards and a precision seal showing year of test will be placed on each tape. For tapes not conforming to the specifications, a report will be issued without the official certification. The Bureau's serial number on a tape simply signifies that it has been tested by the Bureau and either a certificate or a report issued.

715(*d*). COMPARISON OF TAPES WITH THE STANDARDS

(*a*). Unless otherwise stated, the comparisons of this tape with the Bench Standard have been made at the center of the lines on the edge to which the shortest graduations are ruled. If all the graduations extend entirely across the tape, the ends farthest from the observer when the zero of the tape is at his left hand are used.

(*b*). On tapes which have been cut off at the zero mark, the extreme end of the steel ribbon has been taken as the zero point and not the center of any line that may be at that point.

(*c*). On tapes which have the zero point on a loop attached to the steel ribbon at the end, the zero has been taken at the outside of this loop unless noted to the contrary.

715(*e*). SCHEDULE OF STANDARD TESTS FOR STEEL TAPES *

214*a*. *Determination of the correction to the total length of the tape when supported throughout at standard tension and at standard temperature, for each 150 feet or 50 meters or fraction thereof.* This is the regular standard test required by the Bureau of Standards, which must be made (and charged for) in each case in order to determine whether or not the tape is entitled to the Bureau's certification.

214*b*. *Determination of the correction to the total length of the tape when supported throughout at any tension other than the standard tension for each 150 feet or 30 meters or fraction thereof.* The tension desired must be specified.

214*c*. *Determination of the correction to the total length of the tape when supported at the ends only.* Standard tension will be used unless another tension is specified. Length limited to 150 feet or 50 meters.

214*d*. *Determination of the correction to the total length of the tape when supported at the ends and at one or more intermediate points, for each 150 feet or 50 meters.* Standard tension will be used unless another tension is specified.

214*e*. *Determination of the correction to the length of a subinterval under the same conditions as to tension and points of support as for the total length.* The points at which these measurements are made must be points at which the tape is supported, and these points must not be more than 150 feet or 50 meters apart.

214*f*. *Determination of the correction to the length of a subinterval under specified conditions as to tension and points of support which are different from those used for the total length.* See item 214*e*.

214*g*. *Determination of the tension to the nearest integral half pound or quarter kilogram at which the correction to the length of an interval is most nearly zero under a specified condition of support, for each 150 feet or 50 meters or fraction thereof.*

214*h*. *Determination of the correction to a subinterval at the tension at which the correction to the total length is most nearly zero, and under the conditions of support used in the test under item* (214*g*).

214*i*. *Determination of the coefficient of expansion of a tape for each 150 feet or 50 meters or fraction thereof.* This test is expensive and advisable only when a tape is to be used for work in which a very high degree of precision is required.

214*j*. *Determination of Young's modulus of elasticity for each 150 feet or 50 meters or fraction thereof* [Art. 73(*a*)].

214*k*. *Determination of the weight per foot or per meter of a tape.*

214*l*. *Test of the spring balance accompanying a tape.*

214*m*. *Additional charge for each tape sent without a reel, for each 150 feet or 50 meters or fraction thereof.*

214*n*. *Determination of AE* [Art. 73(*a*)].

214*o*. *Computed values.*

214*x*. *Copies of certificates or reports previously issued, or reissue of worn or damage certificates or reports which are returned.*

* Issued by the United States Bureau of Standards. A similar schedule for invar tapes may be obtained on application to the Bureau.

Information concerning special tests not covered by the above schedule will be furnished upon request.

Caution. A tape should be restandardized at intervals, the length of the interval depending upon the amount of use, the care with which the tape is handled, and the character of the work in which it is used. During continuous work of high precision a tape should be tested before and after field measurements of fundamental importance, such as measurements of base lines.

716. Permanent Standards and Equipment for Testing Tapes. When it is necessary to test tapes frequently, it is well to establish a permanent standard equipped for making the tests. The standard should be established in a place as free as possible from disturbing elements that might affect the test. It is often difficult to obtain space of sufficient length within a building or space in which the atmospheric temperature would be uniform throughout the length of the tape; consequently many standards are established out doors, preferably on the north side of a building where the maximum number of hours of shade are secured.

For testing a 100-ft. tape, the following standards and equipment are recommended.

716(a). A 100-FT. STEEL TAPE STANDARDIZED AND CERTIFIED BY THE U.S. BUREAU OF STANDARDS, used only for testing tapes, and restandardized by the Bureau from time to time as may be required. In many cases it is desirable to have two such tapes which may be compared with each other in order to detect any change in the standard length of either.

716(b). A PERMANENT STANDARD consisting primarily of (1) a horizontal, plane, smooth surface upon which a tape may be supported uniformly throughout its length while it is being tested; (2) a metal plate at one end

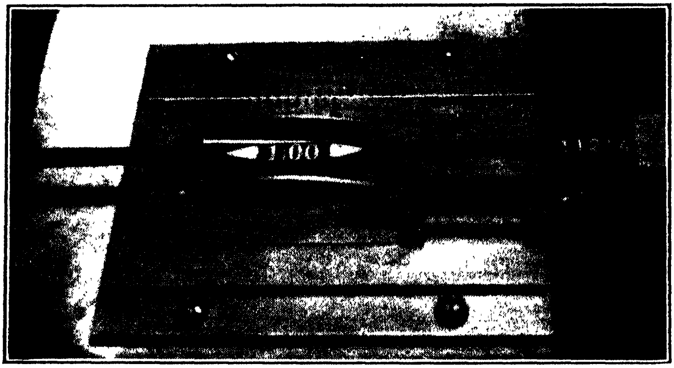


FIG. 716(*b*).

of the standard on which the zero graduation line is engraved; (3) a finely graduated scale at the other end of the standard, 100 ft. from the zero line. In some cases, a vernier is used with the scale, as illustrated in the device used by engineers of the Tennessee Valley Authority. (See Fig. 716(*b*).)

The following is a typical outdoor permanent standard. (1) A supporting concrete surface, smooth, flat, and level. (2) Two monuments set 100 ft. apart in holes through the supporting surface and free from contact with it. The monuments should be deep enough in the ground to resist disturbance by frost. (3) A special bronze plate or disk set in the top of each monument. One of these plates has engraved upon its surface a single line perpendicular to the line along which the tape will be stretched. This line is the *zero* graduation mark. The other plate carries a scale graduated at intervals of 0.002 ft. and numbered arbitrarily in thousandths of a foot.' It is well to place the scale so that a graduation mark near the center of the scale will be exactly 100 ft. from the zero graduation mark on the other plate. This is done as follows. Keep the zero mark of a standardized tape at the

zero mark of the standard and the tension in the tape at standard (10 lb.). Determine the temperature of the tape and the correction for temperature. From the certified length of the tape and the correction for temperature, determine what distance should be measured from the 100-ft. mark of the tape in order to determine the 100-ft. point on the standard. For example, the certified length of the tape is 100.007; the temperature is 60° F. From the table on p. 1103 the correction for temperature is $+100 \times 0.000052 = +0.0052$ ft., i.e., the tape at a temperature of 60° is 0.0052 shorter than it is at the standard temperature of 68°. If the tape were at standard temperature, the required 100-ft. mark on the permanent standard could be found by measuring back from the 100-ft. mark of the tape a distance of 0.007 ft., but for a temperature of 60° this measured distance should be $0.007 - 0.005 = 0.002$ ft. It is well to determine the position of the 100-ft. point of the standard when the temperature is as near as practicable to 68° and to verify this position by comparison with the standardized tape on days of different temperature. The position of the 100-ft. mark should also be checked periodically in order to be sure that it has not been disturbed.

For ordinary work, a somewhat less elaborate permanent standard can often be established. A smooth level floor can be used as a supporting surface; or better still, a line of planed planks, about 6 to 8 in. wide, laid end to end at the same level, and supported 2 to 3 ft. above the floor or the ground. Frequently a satisfactory degree of precision may be obtained when the tape is stretched across the tops of supports set at 5-ft. intervals. One standard of this type consists of wall brackets set 5 ft. apart with their tops on a level. If the brackets are hinged in such a way that they can be turned back against the wall, a tape can be tested when it is supported by all the brackets, or by the end brackets only, or by the end brackets and one or more intermediate brackets. In every type of permanent standard thus far described, it is essential that the zero mark at one end and the scale at the other should be kept in a fixed position so that the distance between them remains constant.

For standardizing a tape supported only at the zero point and the 100-ft. point, an outdoor standard may be constructed as follows. Two steel pipes filled with concrete are set 100 ft. apart, center to center, in concrete bases. In the top of each pipe is set a bronze disk. One of these disks bears the zero line and the other the graduated scale. Two similar pipes, but somewhat taller, are set in line, one a short distance beyond the zero point, the other a short distance beyond the 100-ft. point. These two pipes are used, one for anchoring the device that holds the zero end of the tape, and the other for anchoring the device that applies the tension at the 100-ft. end.

716(*c*). AN ADJUSTING DEVICE FOR KEEPING THE ZERO GRADUATION MARK ON A TAPE EXACTLY OPPOSITE THE ZERO MARK OF THE STANDARD. The simplest device is a turnbuckle, one end of which is anchored and the other connected to the zero end of the tape. Another device consists of a wooden or metal lever, 8 or 10 in. long, and a turnbuckle. One end of the lever is hinged at a point an inch or two below the supporting surface and a short distance from the end of the tape. The point should be in line with the tape, and the hinge should permit the lever to move in the vertical plane through that line. One end of the turnbuckle is connected to the upper end of the lever and the other end is anchored. A horizontal wire connects the end of the tape with the lever at a point an inch or two above the hinge. Any movement of the upper end of the lever caused by screwing or unscrewing the turnbuckle will result in a much smaller movement of the zero point of the tape, thus making possible a fine adjustment of the position of the zero point.

716(*d*). A DEVICE FOR KEEPING AN UNVARYING TENSION IN THE TAPE EQUAL TO THE AMOUNT DESIRED FOR THE TEST. The device should be at the 100-ft. end of the standard in order that the scale reading and the tension reading may be taken by the same observer. The simplest device is composed of a spring balance and a turnbuckle. This device requires frequent observation and adjustment during a test; moreover, a spring balance is likely to be inaccurate unless tested frequently.

A device that requires no attention during a test consists of a pulley and a weight. The pulley is placed a short distance beyond the 100-ft. end of the standard, set in such a position that if the tape which is being tested were extended in a horizontal line, it would be tangent to the top of the pulley. An unvarying tension is kept in the tape by a cord attached to its

end and passing over the pulley to a suspended weight equal to the desired tension. The pulley should be such that the friction is negligible, as, for example, a bicycle wheel from which the tire has been removed.

A bell crank is sometimes used in place of a pulley. This is a form of lever with two arms at right angles similar to a carpenter's square; like the pulley, it is set in a vertical plane in line with the tape. It is pivoted at the intersection of the two arms. An inclined brace from one arm to the other may be desirable. The bell crank should be in such a position that when one arm is vertical, a horizontal cord or wire extending from the 100-ft. end of the tape may be attached to that arm; from the horizontal arm a weight is suspended. When the distance from the pivot to the cord attached to the tape is the same as that from the pivot to the cord that holds the weight, the tension in the tape will be equal to the weight. Obviously, the ratio of one of these two distances to the other may be made to conform to any desired ratio of the tension in the tape to the suspended weight. For example, if the distance from the pivot to the cord that holds the weight is three time that to the horizontal cord attached to the tape, the tension in the tape will be three times the suspended weight. Thus a comparatively large pull may be obtained by means of a comparatively small weight.

716(*e*). Tape thermometers for determining the temperature of a tape. It is frequently sufficiently accurate to determine the air temperature, but better results may be obtained by attaching thermometers to the tape, usually one near each end.

A tape thermometer should be so designed that the bulb can be brought in contact with the tape. If only the metal case of the thermometer is in contact, the difference between the temperature indicated by the thermometer and that of a tape used for field work on a hot sunny day may be as great as 5° F. A thermometer may be held to the tape by adhesive tape or other device. Thermometers should be compared with a standardized thermometer throughout a range of 0° to 120° F., and discarded if not accurate within 2° F.

716(*f*). Miscellaneous equipment. (1) A finely graduated scale unless one is permanently fixed at the 100-ft. mark of the permanent standard. (2) Two magnifying glasses, for the use of the two observers, one at each end of the tape. (3) Some means of illumination may be needed, such as a spot light or an ordinary flashlight at each end of the tape, set in a position where it can be focused on the zero mark or the 100-ft. mark. (4) A book in which to record tests.

717. Methods of Procedure in Testing Tapes. Three general methods of testing a 100-ft. tape supported throughout its length will be explained. The standard temperature is assumed to be 68°.

717(*a*). By direct comparison with a standardized tape

The test is carried on by two observers, one at each end of the tape.

1. Clean the tape.

2. Stretch the tape and the standard tape side by side on the supporting surface of the permanent standard, and allow them to remain in this position until the temperature is the same for both tapes and for the supporting surface. (It may be convenient to set up the tapes at the end of a day's work and then to make the test on the following morning.)

3. Anchor both tapes in such a position that the zero mark of

each can be brought to the zero mark of the standard by the turnbuckle device already described.

4. Bring each tape to the standard tension by the pulley device, by the bell-crank device, or by a turnbuckle and spring balance.

5. Bring the zero mark of each tape exactly opposite the zero mark of the standard as observed through a magnifying glass.

6. By means of the graduated scale at the 100-ft. mark of the permanent standard, read through a magnifying glass the distance between the 100-ft. mark on the tape and the 100-ft. mark on the standardized tape. While this is being done, the two observers, working together, should make sure that the zero marks of both tapes are kept exactly opposite the zero mark of the standard. Be careful to note whether the tape tested is too long or too short.

7. Record the test in some such manner as this: Tape No. 18 tested (......Date......) by (........Names of the observers ....). Length 100.005 at 68° F. Sometimes these data are also given on a tag attached to the tape.

Comments. In this method of testing, temperature may be ignored except that precautions should be taken to have the temperature in both tapes the same. Moreover, when a permanent standard is not available, the test can be made on any suitable supporting surface. Any error due to a change in the length of the standardized tape since it was tested will cause the same error in the observed length of the tape tested. Such an error can be detected, however, if the zero and 100-ft. marks on the permanent standard have been accurately established. Otherwise it is well to have *two* standardized tapes reserved for the sole purpose of testing field tapes and to have them restandardized from time to time.

A *special form of permanent standard* designed for this method of testing consists of a steel strip a little more than 100 ft. in length, laid in a wooden trough with a cover to protect it. Graduations are on silver inserts. This steel strip replaces a standardized tape. A tape laid on it for testing expands or contracts, as the temperature changes, the same amount as the strip itself.

717(*b*). By direct comparison with a permanent standard

It is assumed that the standard is of the usual type in which the zero mark and the graduated scale remain fixed in position. The method of procedure is similar to that just outlined for method 1, except that the discrepancy measured is that between the 100-ft. mark of the tape and the 100-ft. mark of the standard. The measured discrepancy, however, is corrected for the temperature of the tape unless the temperature happens to be 68°.

Example. The temperature is 74°. The length of the discrepancy is +0.01 ft. (tape that much too long). Correction for temperature from the table on p. 1103, 100 × 0.000039 = 0.004. Hence the length of the tape at 68° is 100.001 − 0.004 = 99.997 ft.

Comments. The distance between the zero and the 100-ft. marks of the standard is less likely to change over a long period of time than the corresponding distance on a standardized tape. This is an advantage, but the necessity for temperature correction is a disadvantage.

717(*c*). By indirect comparison with a standardized tape

In this method, the standardized tape as well as the tape tested are compared with a permanent standard, and from the results thus obtained the standardized length of the tape tested is computed. In each comparison, the tape is pulled to standard tension and the zero mark of the tape is kept opposite the zero mark of the standard, as explained in method 1.

1. Measure and record the discrepancy between the 100-ft. mark on the standardized tape and the 100-ft. mark (or any other convenient mark) on the graduated scale of the permanent standard. Observe and record the temperature of the tape.

2. Measure and record the corresponding discrepancy for the tape that is being tested. Observe and record the temperature of the tape. (The temperature may have changed since it was observed in step 1.)

When there are several tapes to be tested, repeat step (2) for each tape, and when this has been done repeat step (1) for the standardized tape. The mean of the two corrected lengths obtained in the first and last test of the standardized tape is used in calculating the standardized lengths of the tested tapes, as illustrated by the following example.*

Computations for Standardizing Tapes

Tape Number (1)	Temperature in Degrees F. (2)	Reading (3)	Temperature Correction (4)	Add (5)	Tape Length (6)
1 *	60	0.052	+0.0052	....	100.007 *
2	60	0.036	+0.0052	99.9493	99.990
15	61	0.068	+0.0045	99.9493	100.022
28	61	0.033	+0.0045	99.9493	99.987
46	62	0.065	+0.0039	99.9493	100.018
1 *	63	0.055	+0.0032	....	100.007 *

* Length of Tape No. 1 as standardized and supplied by the U.S. Bureau of Standards.

The length of the standardized Tape No. 1, as found by taking the mean of the lengths obtained from the first test at 60° F. and the last test at 63° F. is: 100.007 − (0.052 + 0.0052 + 0.055 + 0.0032) ÷ 2 = 99.9493 ft. The length of each of the tested tapes is found by adding the quantities in columns 3, 4, and 5. For Tape No. 28, for example, the standardized length is: 0.033 + 0.0045 + 99.9493 = 99.987 ft.

The temperature corrections in column 4 are obtained from the table on p. 1103.

* Taken from Manual 20, entitled *Horizontal Control Surveys*, published by the American Society of Civil Engineers.

717(*d*). Special tests

In addition to the basic test to determine the length of a tape when it is supported throughout, there are a number of other tests, as is evident from the list in Art. 715(*e*). The ones most commonly required are these.

1. *To test specified intermediate graduations,* as, for example, the 5-ft., 10-ft., 25-ft., 50-ft., and 75-ft. graduations. This may be done by comparing the tape directly with a tape that has been standardized for the corresponding intervals measured from zero. Corrections must be made for errors in the length of intervals between graduations on the standardized tape. Such errors may be unevenly distributed so that some graduations are correct and some are not, as illustrated by the following lengths of intervals obtained by the U.S. Bureau of Standards in testing a certain tape: 0 to 5 ft. = 5.000 ft.; 0 to 10 ft. = 10.000 ft.; 0 to 25 ft. = 25.001 ft.; 0 to 50 ft. = 50.001 ft.; 0 to 75 ft. = 75.001 ft.; 0 to 100 ft. = 100.001.

2. *To test a tape supported at the ends only.* (*a*) Under a specified tension: When, in chaining, a tape is supported at the ends only, the pull exerted on the tape is usually 20 lb. or more, and the specified tension should be the same. (*b*) To determine the tension required to bring the zero and the 100-ft. graduations 100 ft. apart.

Any one of the three methods of testing a tape already explained, if slightly modified, may be used for either (*a*) or (*b*).

3. *To test a tape supported at the ends and at one or more intermediate points.* The conditions of support and tension should be as nearly like those in the field as practicable. Any one of the methods of testing a tape may be adapted for use in this test.

Note. For ordinary work the test in either (2) or (3) can be made without temperature correction by direct comparison with a tape that has been standardized for the same conditions of support as those under which the test is made. (This corresponds to the method explained in Art. 717(*a*).)

Note. In Arts. 71, 73, and 74 much will be found that is pertinent to the whole subject of testing tapes, such as the correction for temperature, the method of calculating the change in the length of a tape due to a change in pull, and the compensating pull required to offset the sag of a tape for any specified unsupported length.

717(*e*). Choice of procedure and degree of precision. When a tape is to be used in work of high precision, as, for example, in the measurement of a base line for triangulation of the first order, the best standards for comparison obtainable should be used and corresponding care should be exercised in the choice of the method of procedure and in carrying out that procedure. On the other

hand, in testing tapes used in ordinary work, there is likely to be a tendency to attain a degree of precision much higher than that required in the field. It is not always easy to decide to what extent refinements should be carried in testing a tape, but it will be helpful to keep in mind the method of chaining that will be used in the field, the character of the terrain, the climatic conditions, and the degree of precision required.

718. Leveling Rods may be *target rods* or *self-reading rods.* The pattern of graduations on a self-reading rod must be such that the levelman can take direct readings through the telescope without the aid of a target. Experiments show that the difference in accuracy between target readings and direct readings is not so great as is generally supposed and that it is a waste of time to use a target in a large part of ordinary leveling. However, a target on a self-reading rod may be used when a reading of a relatively high degree of precision is required or when the length of sight is too great for an accurate direct reading.

Common lengths of leveling rods are 12 and 13 ft. A single-piece rod is best for leveling when little transportation is involved; a two-piece rod, however, is more commonly used. The joint may be a slip joint, a sliding joint, or a hinged joint. A three-piece rod or telescoping rod is most convenient for transportation but least desirable in all other respects. Every good rod has a metal rod shoe at its base. When specified, a steel ball or a steel plug to serve as a turning base can be inserted in the rod shoe. The cross-sectional area (and corresponding weight) of a rod varies from that in a light rod, such as one that is designed for the use of architects and builders, to the larger cross section in a so-called heavy rod.

Graduations are generally marked or stamped on the face of the rod, either directly upon the enameled surface of the wood or upon the enameled surface of a metal strip held fast to the wood. Contrary to the general belief, the length of a wooden rod can vary appreciably with average changes in humidity. If graduations are on an invar strip, changes in length due to changes either in humidity or in temperature are negligible. Such a rod, however, is mainly for precise leveling.

Graduations should be distinct, durable, and of uniform accuracy. For the protection of the graduations, the face of the rod is generally recessed, so that anything coming in contact with that side of the rod will strike the edges of the recessed face and not the graduations. The smallest subdivision is generally 0.10 ft. or 0.05 ft. or 0.01 ft., depending upon the type of pattern used for the graduations. The most widely used pattern, both for target and for self-reading rods, is that on the *Philadelphia rod* (see lower part of p. 99), although some of the more recent types, such as the Chicago rod, offer certain advantages. The smallest subdivision on a metric rod is generally one centimeter. Rods graduated in feet, inches, and fractions of an inch are convenient for certain kinds of work. On most rods, the background is white and the graduations are black. Numbers indicating feet are generally red, and those indicating tenths are generally black. On some rods "instantaneous" foot numbers are furnished at extra cost. Near each black number that marks a tenth of a foot is a small red number which indicates the number of the foot mark just below that tenth, i.e., the foot number with which the tenth should be joined in the final reading. Graduations on any rod may be tested by means of a good steel tape, finely graduated and standardized. A test that should be made at intervals of two-piece and three-piece rods is that for extended rod readings (Art. 111).

718(*a*). TARGETS. The *quadrant* target shown on pp. 99 and 101 is of the type most used. Another form called the *diamond* target gives better results for long sights. There are several other patterns in use. Targets are provided with a *vernier* as shown on the New York rod, p. 101, or with a *scale* as shown on the Philadelphia rod, p. 99. The *angle* target, in which the right-hand half is at right angles to the left-hand half, is a special form devised to insure the rod's being held perpendicular to the line of sight. The horizontal dividing line of this target is carried over the surfaces of the two mutually perpendicular halves of the target so that it will appear as a con-

tinuous unbroken line only when the rod is held in a vertical position. A *micrometer target* has a device for rapidly and accurately setting the target. This device may be used either on a plain target or on an angle target.

718(*b*). SPECIAL FORMS OF LEVELING RODS. *A precision leveling rod or* a *geodetic leveling rod* is generally a one-piece rod, wider than an ordinary rod, with refinements in construction and graduation that make possible exceptionally accurate vertical measurements. A rod on which the graduation numbers are upside down is convenient for use with an inverting telescope. (2) An *automatic leveling rod* or *self-computing rod* is described in Art. 172. (3) A *flexible* or *pocket leveling rod* is made of a prepared, waterproof, closely woven, durable tape, graduated like self-reading rods. For use they are tacked to a straight board, preferably through eyeleted holes. When not in use, the tape is rolled up and carried in a metal box made for that purpose. (4) A *hand leveling rod* is generally a rod 5 ft. long, with block or rectangular graduations that enable an observer, looking through the telescope of a hand level, to estimate a vertical distance on the rod in feet and fractions of a foot. The upper end of the rod has a metal inverted arc in which to rest the hand level when sighting.

718(*c*). A ROD LEVEL OR PLUMBING LEVEL is a device used to determine when a leveling rod is plumb in every direction. In its simplest form it consists of a circular spirit level fastened to a metal plate. This plate is held against the rod or attached to it in such a position that when the rod is plumb, the bubble will be in the center of the circular vial. In a more accurate form of rod level, there are two level bulbs of the ordinary type with their axes at right angles to each other.

719. Stadia Rods. Most stadia rods are self-reading. For great accuracy or for exceptionally long sights a rod with two targets may be used. Since

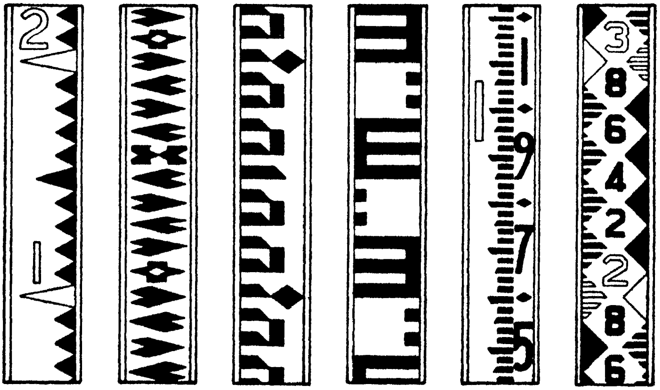


FIG. 719.

both targets must be set for every sight taken, such a rod is much less convenient than the self-reading rod and it is seldom used in ordinary stadia surveying.

Stadia rods are usually 2½ to 6 in. wide, ½ to ⅞ in. thick, and 10 to 15 ft. long. A rod is generally of one piece, but may be in two or more sections with hinged or slip joints. The rod is usually graduated in feet and decimals of a foot. The smallest subdivision is generally one-tenth of a foot, but hundredths may easily be estimated by means of the sloping lines which are characteristic of most stadia-rod patterns. Rods are also graduated in decimals of a yard or of a meter. Self-reading leveling rods are often used as stadia rods for short sights not exceeding, say, 400 ft. in length.

Stadia rods can be purchased in a variety of standard designs, and rods of special design can be obtained made to order. In Fig. 719 are shown six of the more common stadia-rod designs or patterns. Various patterns may also be obtained on so-called flexible stadia rods made of waterproofed,

woven tape that may be rolled up and carried in the pocket. For use, the strip is fastened to a straight board by means of thumb tacks.

The principal requirement in selecting or in designing a stadia-rod pattern is to secure graduations on the face of the rod which can easily be read at a distance and yet be fine enough to indicate definitely and accurately the distance of the rod from the stadia wires. For short sights, fine graduations are best; for long sights, larger, heavier graduations are required. In some patterns fine and large graduations are combined. Whatever may be the pattern, there should be sufficient white background at any point on the rod to make a stadia wire covering that point appear clear and distinct to an observer looking through the telescope. The pattern should also be such that the observer can easily estimate or interpolate distances less than the smallest subdivision of the pattern.

719(*a*). Directions for making a stadia rod. It is comparatively easy to make a stadia rod, and descriptions of many homemade rods, designed to meet special needs, are to be found in engineering periodicals. The rod should be preferably 4 to 6 in. wide, ½ to ¾ in. thick, and 10 to 14 ft. long. The wood should be light, well-seasoned, free from knots, straight-grained, and tough. It should be planed smooth, and the surface used for the face of the rod should be sandpapered. A thin wooden strip fastened securely to each edge and projecting slightly in front of the face of the rod serves to stiffen the rod and to protect the graduations. If the side strips are left off, the corners should be rounded slightly and a stiffening strip about 1½ in. × ¾ in. should be screwed to the back of the rod from top to bottom. For protection and convenience in transportation, some engineers prefer a two-section, hinged rod so that the upper and lower sections may be folded together, faces inward. A metal shoe may be screwed to the bottom of the rod and thin metal strengthening plates fastened on the back, top, and bottom.

At least two good coats of white paint should be applied to the rod. It is better to give the rod a preliminary coat of shellac and then several coats of paint, rubbing each coat down (after it is dry) with powdered pumice or similar abrasive. The paint should be of a kind that will resist the wear of ordinary use and the effects of the weather. A good lacquer paint meets these requirements. Whatever paint is used, the final coat on the face of the rod should be rubbed to a smooth, dull finish.

The graduations and the pattern can be laid out by means of a scale, beginning either at the bottom of the metal shoe or at a short distance from the bottom of the rod. The pattern should first be penciled, then ruled with a drawing pen filled with paint, and finally filled in with a camel's hair brush. A dull black paint may be used throughout, although some engineers prefer to emphasize certain division marks, as, for example, the whole-foot marks, by painting them red. If these marks are numbered, the numbers are also painted red. If desired, the whole rod may be coated with a durable, elastic varnish rubbed down. The paint should be allowed to dry thoroughly before the rod is used. For a special method of graduating a stadia rod see Art. 423(*b*).

720. Illuminated Rods. For night work, range rods and leveling rods may be illuminated by a variety of methods. One of the best methods is by means of a battery-operated lighting system. A range rod illuminated by electric bulbs, arranged about one foot apart along the rod and wired in parallel, can be seen distinctly for several miles. Mining and tunnel leveling rods are equipped with "slit" targets. A light placed behind the target of one of these rods throws a line of light through a narrow slit which is coincident with the horizontal division line of the target.

CHAPTER 53

TRANSITS, LEVELS, PLANE TABLES, SURVEYOR'S COMPASS, SEXTANT

SINCE no person is likely to study a surveying instrument unless he has one to look at, no attempt is made in this chapter to describe those parts of the transit, the level, the plane table, the surveyor's compass, and the sextant which can readily be seen. On the other hand, a special effort has been made to describe important parts that cannot be seen and to explain how those parts function. The essential characteristics of each instrument are discussed, and methods of testing these characteristics are given.

721. Surveying Instruments. Desirable Characteristics. Among the characteristics generally desired in surveying instruments, particularly in the transit and in the level, are the following.

721(a). PERMANENCE OF ADJUSTMENT. Constant improvements in the design and construction have resulted in such a high degree of permanence of adjustment, that a good modern surveying instrument is more likely to get out of adjustment because of abuse in field adjustment and use than because of defects in its manufacture.

721(b). DURABILITY. Durability depends not only upon adequate structural strength, but also upon protection against dust and moisture, upon proper lubrication, and upon the care with which the instrument is cleaned and otherwise cared for.

721(c). SIZE AND WEIGHT. In general, a larger transit or a larger level is required for work of high precision than for work of low precision. The length of telescope is an important element since it often determines the relative size and weight of an instrument and of its carrying case. Adequate size without excessive weight and short telescopes are characteristics of the better modern transits and levels. There should be a proper proportion between the weight of an instrument and the weight and stiffness of its tripod. The trend in recent years has been toward stiffer tripods. For stiffness, tripod legs of fixed length are better than extension legs. The shoes on tripod legs should be firmly fixed. If they become loose, they are a source of error.

721(d). DESIRABLE OPTICAL CHARACTERISTICS OF THE TELESCOPE. These have been discussed in Art. 711. The modern tendency is toward large objective apertures.

721(*e*). Convenience of manipulation. This convenience depends largely upon the proper arrangement of parts, particularly upon the location of verniers and of clamp and tangent screws; also upon the ease with which all screws work.

721(*f*). Adequate provision for clamping. Clamps and set screws should hold without forcing. There should be no "back lash" or lost motion in tangent screws.

721(*g*). Accuracy of graduation. The automatic dividing engine has made possible accurate graduation. Considerable skill, however, is still required in the centering of the limb and in the matching and centering of the verniers.

722. The Transit. There are upward of four hundred parts in an ordinary transit, but for purposes of explanation the instrument may be divided into the four parts shown in the photograph in Fig. 722, namely, the *telescope* with its supporting transverse axis, (2) the *upper plate* (circular) to which are attached the upright standards, (3) the *lower plate* (circular), and (4) the *leveling head*. The supporting axis of the telescope fits at either end into the bearing at the top of the corresponding standard. The spindle of the upper plate is known as the upper spindle (also as the inner spindle). The spindle of the lower plate is known as the lower spindle (also as the outer spindle).

The *upper spindle* fits into a socket in the *lower spindle*, and the latter fits into a socket in the leveling head. The *upper clamp screw* is used to clamp the upper and lower plates together; the *lower clamp screw* to clamp the lower plate to the leveling head. When the upper clamp screw is set and the lower clamp screw is loose, both plates, clamped together, can revolve on the lower spindle. When the lower clamp screw is set and the upper clamp screw is loose, the upper plate can revolve on the upper spindle, but the lower plate will remain in a fixed position. When both clamp screws are loose, either plate can revolve on its spindle independently of the other. When the upper clamp screw is set, the upper plate can be made to turn slowly on the upper spindle by means of a slow-motion or tangent screw, even though the lower plate remains clamped to the leveling head. When the lower clamp screw is set, the lower plate can be made to turn slowly on the lower spindle by means of a similar slow-motion screw even though the upper and lower plates are clamped together.

The standards are rigidly attached to the upper plate; consequently when the telescope is turned horizontally (sidewise) the upper plate must always turn with it. This upper plate is sometimes called the *vernier plate* because it has a vernier attached to

it. The zero graduation mark of the vernier serves as an indicator in measuring horizontal angles. Many transits have two

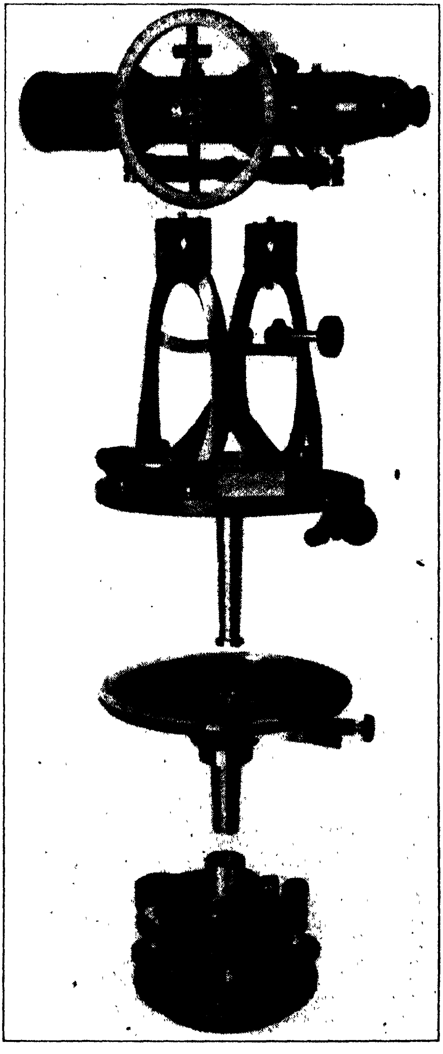


FIG. 722.

plate verniers 180° apart. The lower plate or *limb* has around its outer edge a graduated circle.

When the transit is set up properly, the leveling head is stationary on the tripod, both spindles theoretically are vertical, and

both plates theoretically are horizontal. There are three steps in measuring an horizontal angle, and it is important to know what parts of the transit are involved in each step. The three steps are these. (1) Revolve the upper plate until the indicator of the vernier is nearly opposite the zero graduation on the limb, clamp the upper and lower plates together by setting the *upper* clamp screw, and, by means of the corresponding (upper) tangent screw, bring the indicator of the vernier exactly opposite the zero of the limb. (2) Loosen the *lower* clamp screw, turn the telescope toward the point used as a backsight (both plates clamped together will revolve on the lower spindle), and when the line of sight is nearly on the backsight, clamp the lower plate to the leveling head by setting the lower clamp screw. By means of the corresponding (lower) tangent screw bring the line of sight exactly on the backsight. Now neither plate can turn on its spindle, and the indicator is still opposite the zero on the limb. (3) Loosen the upper clamp screw, and turn the telescope toward the point used as a foresight. The upper plate will revolve on the upper spindle, but the lower plate will remain stationary since the lower clamp screw has been left set. When the line of sight is nearly on the foresight, set the upper clamp screw, and, by means of the corresponding tangent screw, bring the line of sight exactly on the point. The value of the arc between the zero on the limb and the point on the limb at which the indicator of the vernier stops when the line of sight is brought exactly on the foresight is the value of the horizontal angle between the line of sight to the backsight and the line of sight to the foresight. Obviously in measuring this angle the indicator of the vernier could have been set at *any* graduation of the limb. The value of the angle would then be the difference between this initial reading on the backsight and the final reading on the foresight.

722(*a*). NOTE. No tangent screw will function until the corresponding clamp screw has been set. The special construction which permits the upper or the lower plate, after it has been clamped, to be turned by the corresponding tangent screw will become evident from inspection.

722(*b*). NOTE. When a transit is in adjustment and properly set up, only vertical and horizontal angles can be measured with it, for when the telescope is turned up or down it revolves in a vertical plane and when it is turned sidewise it causes the indicator of the vernier to move around a graduated circle that lies in a horizontal plane.

723. Different Types of Transits. (1) Small lightweight transits, such as those designed for the use of contractors, builders, and architects, or for use on expeditions. (2) Somewhat larger and heavier transits, such as mountain and mining transits. (3) Transits of average size, such as the ordinary engineer's transit and the city transit. (4) Still larger and heavier transits such as those used for precise traversing and triangulation.

724. Theodolites. The theodolite antedates the transit. In its original form its telescope had no axis on which it could be reversed or transited. The

transit was formerly called a *transit theodolite* because it could be transited. This distinction has now disappeared since some theodolites are now made with telescopes that can be reversed. The modern theodolite is designed for work of high precision. The diameter of the graduated circle or limb is usually 7 to 8 in., and the circle is more finely graduated than on most transits. The *repeating theodolite* has an upper and a lower revolving plate as in the transit, but the *direction theodolite* has only one revolving plate (the upper). The lower plate, with its graduated circle, is rigidly fixed to the lower part of the instrument. The direction theodolite, as its name implies, is used mainly for the measurement of horizontal angles by the direction method.

725. Special Parts for Transits. (1) A *compass needle* and *box*, generally but not always furnished on transits. (2) *Stadia wires* should be in every transit intended for general work; they are sometimes omitted, however, unless specially ordered. (3) *Diagonal* or *prismatic eyepiece* for sighting at exceptionally high points. (4) A *reflector* for illuminating the inside of the telescope, particularly the cross hairs, when working in the dark. (5) A *striding level* for use when it it desired to bring the supporting axis of the telescope into a horizontal position more accurately than can be done by the less sensitive plate levels. (6) A *gradienter* for running grades, etc. (See Art. 181.) (7) A *Beaman stadia arc*, a mechanical device for obtaining the horizontal and vertical distances that correspond to inclined stadia distances. (See Art. 429.) (8) A *solar attachment* for use in solar observations.

726. Desirable Characteristics of a Transit. In addition to the characteristics generally desired in all surveying instruments (described in Art. 721), the following are the most important requirements to be met in an ordinary transit.

1. The axis of the spindle of the upper plate should be exactly perpendicular to the plane of the graduated circle on the limb.

2. The axes of the upper and lower spindles should coincide, the surface of the upper spindle and the inner surface of the lower spindle should be parallel and in close contact, and the instrument should revolve on either spindle without play and without undue friction.

3. The graduations on the limb should be accurate, the lines should be clear, even, and definite, and the center of the graduated circle on the limb should be in the axis of rotation.

4. Since it is impossible to fulfill perfectly all the above requirements, generally there should be two verniers, 180° apart, to be used in reducing errors due to eccentricity.

5. When any graduation line on the limb is exactly opposite a line of the vernier, the space between the adjacent ends of the two lines should be as nearly imperceptible as practicable in order that the lines may appear continuous. In order to avoid parallax in observing two such lines, some provision should be made on precision instruments for bringing the eye into the vertical plane passing through the lines. A mounted reading microscope with an alignment index is a common device for reducing the error due to parallax. By its use, vernier readings may be observed with much greater accuracy than with an ordinary reading glass.

6. The sensitiveness of the plate levels and of the telescope level should be adequate. The sensitiveness (as defined inArt. 705) of plate levels varies about as follows. For small light transits, 2 mm. = 80″ to 2 mm. = 100″; for ordinary transits, 2 mm. = 60″ to 2 mm. = 80″; for precision transits and theodolites, 2 mm. = 8″ to 2 mm. = 80″. Only in work of an exceptionally high degree of precision is a sensitiveness greater than 2 mm. = 30″ required.

7. The relation between the magnifying power of the telescope and the least count of the vernier should be such that the least perceptible movement of the vernier will cause a clearly perceptible movement of the cross hairs on a distant object, and vice versa (Art. 711(*a*)). A similar relation should exist between the magnifying power of the telescope and the sensitiveness of the telescope level and of the plate levels. The desirable characteristics of a transit telescope are discussed in detail in Art. 712(*a*).

8. Tripods should be rigid, and there should be no looseness or play in the tripod shoes.

727. Tests for the Transit. *Graduation.* Graduating engines have been perfected to such an extent that no errors in graduation are likely to occur

which can be detected by any simple test that the surveyor can make. Errors frequently run in periods, so that one or two repetitions, should they fall within these periods, give the same though false results. A second class of errors of graduation, called *accidental* errors, follow no regular law. The surveyor for the most part must depend upon the instrument maker to eliminate as far as possible both periodic and accidental errors of graduation.

Eccentricity. The two opposite verniers should preferably read exactly alike in any position. Test by reading both verniers in several positions until the circle has been turned through at least 90°. If readings differ by a constant amount the verniers are not set exactly 180° apart. If they differ by a variable amount and the graduation on the limb is assumed to be correct, then the center of the upper plate (vernier plate) is not in the center of the graduated circle, i.e., the vertical axes of the inner and outer spindles do not coincide. The total difference in readings between two opposite verniers is generally due to a combination of the two causes. A similar test can be applied to the vertical circle if it has two opposite verniers.

Note. For tests of the levels see Art. 705(*a*), and for the telescope see Art. 711.

728. The Selection of a Transit. (*a*). THE SELECTION OF THE TELESCOPE, the first step in the selection of a transit, is discussed in detail in Art. 712(*a*). When the most suitable combination of telescope characteristics has been determined, the choice between an external focusing and internal focusing telescope may be made (Art. 710).

728(*b*). THE DIAMETER OF THE HORIZONTAL LIMB varies from about 5 in. on a small light transit to about 8 in. on a large precision instrument. Common diameters for ordinary work are 5½ and 6¼ in. The choice of diameter depends largely on the least count of the vernier desired.

728(*c*). THE LEAST COUNT OF THE VERNIER. Common least counts are 1′, 30″, 20″, and 10″; on precision instruments, 5″ and 10″. The choice of least count should be consistent with the characteristics of the telescope, since most of the considerations that govern the selection of the telescope govern also the choice of least count. Nothing is gained by greater accuracy in reading angles than can be obtained in pointing or vice versa.

728(*d*). NOTE. In ordinary work, the accuracy in vernier reading depends more upon the clarity or definition of the graduations than upon the diameter of the limb, the spacing of the graduations, or the least count of the vernier. Experiments seem to indicate that as between the different sizes of circles from 5⅛ in. to 6½ in., there is little if any difference in accuracy of vernier readings,* and that as long as the vernier is read with the usual reading glass and the transits are comparable, a 20″ instrument is no more accurate than a 1′ instrument. For example, in one hundred measurements of the same angle with each of several transits, the angular error in the reading of a 1′ vernier exceeded 4.6″ in about 50 per cent of the readings, and 11.4″ in about 10 per cent; yet with a 20″ vernier the corresponding values were 5.3″ and 12.8″.† In this case the clearest graduations not the least count produced the most accurate readings. Consequently, when selecting a transit for ordinary use (that is when angles are not to be repeated many times), the graduations should be observed with care to make sure that they are clear and definite.

728(*e*). THE SIZE AND WEIGHT OF THE INSTRUMENT are determined largely by the size of the telescope and the diameter of the graduated circle. In a lightweight instrument, the choice of telescope characteristics and of limb and vernier graduations is much more restricted than in a heavier instrument; but this generally does not matter since lightweight instruments are designed for a type of work in which the required degree of precision is relatively low.

728(*f*). NOTE. In the selection of a transit it is particularly desirable to consult instrument makers, but this cannot be done intelligently without a knowledge of the optical characteristics of telescopes (Art. 711), of the characteristics desired in surveying instruments in general (Art. 721), and of the characteristics particularly desired in transits (Art. 726).

* See University of Illinois Bulletin, Vol. XXV, No. 75, by William H. Rayner.

† See article in *Engineering News-Record,* April 10, 1941, by Philip Kissam.

729. The Level in its simplest form is used merely to establish a horizontal line of sight at any point on the earth's surface. All forms except *hand levels* are mounted on tripods similar to those used for transits.

There are two distinct types of tripod levels, the *wye* level and the *dumpy* level. Each of these has but one vertical spindle, and is much less complex than the transit. Each has a leveling head similar to that used on the transit, and the upper part carries a telescope with a level bubble tube attached. The most essential qualities in the telescope are definition and magnifying power. (See Arts. 711(*a*) and (*d*).) The length of the telescope is not limited by the height of standards as it is in a transit; consequently a higher magnifying power may be obtained without sacrificing illumination or field of view. Moreover, the bubble tube can be made longer than on a transit telescope with a corresponding increase in the radius of curvature; thus the sensitiveness of the bubble can be increased to correspond to the increase in the magnifying power of the telescope.

An important characteristic of a good level is stability, and this depends mainly on the leveling head and the rigidity of the tripod and its shoes. Four leveling screws are furnished on ordinary levels, although three give greater stability and delicacy. Leveling heads with only three leveling screws are found mainly on precision levels. The spindle on which the level turns should be relatively long and well fitted to the socket. The whole upper part of the instrument should be well balanced when the telescope is focused for the average length of sight.

729(*a*). The wye level, so called because the telescope rests in two Y-shaped supports, has two distinguishing characteristics. (1) The telescope may be revolved about its own axis as it rests in the wyes, and (2) it may be taken out of the wyes and replaced, turned end for end. The object of this construction is to facilitate adjustment (see Art. 743(*f*)), but unless the diameters of the rings or collars are exactly the same, perfect adjustment is impossible by the indirect method. Even if the rings are alike to start with they are subject to wear that is seldom the same on the two rings. Hence if the level attached to the telescope is adjusted in the usual manner by reversing it end for end in the wyes, it may be adjusted parallel to the bottom of the collars and not parallel to the line of sight. Most wye levels are subject to this uncertainty of adjustment unless the "peg method" is used. (See Art. 742.)

729(*b*). The dumpy level is one in which the telescope is attached to the bar in such a way that it cannot be revolved about its own axis or taken out of its supports and turned end for end. As the line of sight is adjusted generally by the "peg method," the uncertainty of adjustment just described in connection with the wye level is avoided.

729(*c*). Special types of levels. *Levels of precision* are simply wye levels or dumpy levels made with the greatest care, and having some modifications and attachments designed to increase the accuracy of the work. *Hand levels,* such as *Locke's Hand Level,* are for rough work and are used mostly for locating contours in topographic surveying. To this class also belongs the *Clinometer* or *Abney Hand Level,* which is a modification of the Locke level, having a supplementary telescope and graduated arc for reading vertical angles. For work of relatively low precision, there are small lightweight levels, such as the *Architect's Level,* which are simply cheaper and less accurate forms of the wye level or the dumpy level. There are a number of special forms of precision levels, such, for example, as the *precision tilting level* of the reversible type. Certain refinements, such as micrometer screws, an extra sensitive bubble tube, and a mirror or a supplementary side telescope in which the bubble is reflected, as well as refinements in manufacture, make precision levels all that the name implies.

729(*d*). Homemade levels. By exercising a little ingenuity homemade levels may be devised which can be used for rough work. For example, a right angle may be constructed of two wooden pieces and pivoted to a staff. The vertical leg is made vertical by a plumb line attached to the upper corner, and the line of sight directed along the horizontal leg. Another form is the well-known *A* level in which the plumb line is attached to the apex of the *A* and the line of sight directed along the horizontal bar. This form may be adjusted by placing the two bottom points of the *A* on two rigid supports, marking the point at which the plumb line crosses the horizontal bar, reversing the *A* on the supports, marking a second point on the bar, and taking a final point halfway between. If the *A* level is held so that the plumb line passes through this final point, the horizontal bar will be truly horizontal. There are also forms of water levels, such as a U-shaped tube with water in it, or even a bucket of water with a wooden float and two upright sticks to serve as sights. A carpenter's square and plumb bob are frequently used for establishing a horizontal line.

729(*e*). Tests for the level. (1) Sensitiveness of the bubble. (See Art. 705(*a*) for test.) (2) Magnification: The least displacement of the bubble should cause a perceptible movement of the cross hair, otherwise the magnifying power is too low. (See Art. 711(*a*).) (3) The optical center should move in the line of collimation as the objective is moved from the shortest to the longest focus. (See *Suggestion* 6 (Art. 743(*d*), for test.) (4) The diameter of the rings on a wye level should be exactly equal; otherwise one of the chief advantages of the wye level is sacrificed. (See Art. 743(*h*).) (5) For other tests of telescope see Art. 711.

730. The Selection of a Level. The two parts of a level to consider first are the telescope and the bubble tube. (*a*). The optical characteristics of the telescope are discussed in detail in Art. 711. (See Art. 712(*b*) for the selection of the telescope of a level.) After the desired characteristics have been determined, a choice may be made between an external and an internal focusing telescope.

730(*b*). The sensitiveness of the bubble tube (defined in Art. 705). Many of the considerations that govern the selection of the telescope govern also the required sensitiveness of the bubble tube. For small lightweight levels, the sensitiveness is generally 2 mm. = 50″ to 2 mm. = 60″; for ordinary levels, 2 mm. = 15″ to 2 mm. = 30″; for precision levels, 2 mm. = 10″ to 2 mm. = 20″.

730(*c*). Choice between a dumpy level and a wye level. Engineers differ as to the relative merits of the dumpy level and the wye level. The chief advantages claimed for the dumpy level are that there are fewer parts subject to wear, fewer adjustments to make, and that this type of level stays

in adjustment longer. The principal advantage claimed for the wye level is that by merely taking the telescope out of the wyes and replacing it turned end for end, the levelman, unaided, may determine at any time whether or not the axis of the bubble tube is parallel to the line of sight. It is claimed, in addition, that the uncertainty of this adjustment and the errors that may occur from this uncertainty are in most cases more theoretical than real. Moreover, the adjustment can always be checked, when thought desirable, by the direct or peg method of adjustment. Despite the trend toward the exclusive use of the dumpy level, many engineers still prefer the wye level. In obtaining accurate results, a well-made instrument of either type is likely to be "better than the man."

731. The Plane Table. A complete plane table consists of a drawing board mounted on a tripod and an alidade consisting of a thin flat metal strip or *blade* upon which is mounted a telescope with a vertical arc attached.

731(*a*). THE BOARD. The common sizes of plane-table drawing boards are 15 in. × 15 in., 18 in. × 18 in., 18 in. × 24 in., and 24 in. × 31 in. Special sizes, such as 19 in. × 19 in. and 24 in. × 24 in., may be ordered. Resistance to warping and a smooth, plane upper surface are desirable characteristics. Boards made up in narrow sections, connected by glued tongue and grooved joints and reinforced at each end by a hardwood strip, will resist warping except under extreme climatic conditions. For tropical use, cleats are screwed to the underside, the screws passing through oblong slots which allow the board to expand or contract freely.

There are various devices for fastening the paper to the board such as thumb tacks and spring clamps. On the better boards, the paper is held firmly by brass screws, passing through the paper into brass sockets set near the edges of the board. These are slightly below the surface of the board so that there is no obstruction to the movement of the alidade when used at an edge of the board.

The drawing board is connected to the tripod by a mechanism which supports the board, yet leaves it free to revolve in a horizontal plane. If the diameter of this support is small, the board is likely to be unstable; if the diameter is large, the instrument is likely to be heavy and cumbersome. The maximum of rigidity and strength is probably obtained by the use of radiating arms which support the board. For ordinary work, however, a lighter and more portable instrument with a quick leveling device is desirable. An excellent form is that adopted by the U.S. Geological Survey, known as *Johnson's* plane table movement. It consists of two cups accurately fitted together; one of these, attached to the board, is free to revolve in any direction inside the other, attached to the tripod. This, in effect, is a ball and socket joint which is arranged

so that the plane-table board can be readily leveled and rotated horizontally.

731(*b*). THE ALIDADE. The blade is generally about 3¼ in. wide; common lengths are 11½ in. and 15 in.; special length, 18 in. At least one of the longer edges of the blade should be beveled. If a blade is chromium plated on the bottom, it is less likely to blacken the drawing paper. Either or both of the longer edges may be ordered graduated to any special scale. If both edges are graduated, the scale on one may be different from the scale on the other. The blade generally has attached to it either a circular spirit level or a pair of tube levels at right angles to each other, and a trough compass with a 4-in. needle; also a lifting knob at each end.

There are two types of alidades. In the *fixed-tube* type, the telescope is rigidly attached to the transverse axis as it is in a transit. In the *tube-in-sleeve* type, the tube of the telescope fits accurately within a sleeve which is attached to the transverse axis. The telescope may be rotated in this sleeve just as a telescope of a wye level may be rotated in its wyes. In the fixed-tube type the telescope level is attached to the telescope by a device similar to that used on the transit. In the tube-in-sleeve type a striding level, extending *lengthwise* along the telescope, is used instead of the common form of telescope level. The bubble tube is supported by inverted wyes resting on accurately turned shoulder rings or collars. The striding level can be removed from its supports and replaced, turned end for end.

The telescope of an alidade should be selected with the same care as that of a transit. Many of the characteristics that are considered in the selection of a transit telescope should be considered also when ordering an alidade telescope. (The selection of a transit telescope is discussed in detail in Art. 712(*a*).) Common lengths of alidade telescopes vary from about 8 in. to about 12 in. Magnification varies from about 10 diameters to about 15 diameters for inverting eyepieces; for erecting eyepieces, the magnification may be 20 diameters or more. The diameter of the vertical arc is generally about 4 in.; the vernier reads to single minutes. The axis of the telescope should be parallel to the edge of the alidade blade. For the better types of alidades, either an external or an internal focusing telescope may be chosen. Stadia wires are sometimes omitted unless specified.

731(*c*). EXTRAS AND ACCESSORIES. Striding spirit level; gradienter; Beaman stadia arc, vernier control level; attached microscope for reading any part of the vertical limb; quarter interval wire in the upper half of the

field; detachable diagonal prism which erects images of inverting telescopes, an extra eyepiece giving low magnification or an extra eyepiece giving high magnification; extra reticule with cross and stadia hairs; extra level vials; parallel ruler; plumbing bar; umbrella; carrying case.

731(*d*). TRAVERSE BOARDS are small plane tables used for rough work, such as sketching in contours in topographic surveys. In place of the telescope two slotted uprights or open sights are used.

731(*e*). TESTS FOR PLANE TABLE. The edge of the ruler which forms the base of the alidade should be straight. (See test for straightedge, Art. 605.) To see if the upper surface of the board is a plane lay a straightedge on it in all directions. If it is only a little out, it may be scraped or planed down, and then smoothed with sandpaper. Some of the tests already described for transits and levels, such, for example, as tests for telescopes and for the sensitiveness of the level bubble, may be applied to the alidade. (See also "Adjustments of the Plane Table," Art. 745, for other tests.)

732. The Surveyor's Compass in its simplest form is mounted on a single staff, but in its usual form it is mounted on a tripod. The upper part consists of a horizontal arm having two uprights or sight vanes, with a compass box between them. There are various devices for attaching this upper part to the tripod, such as a ball-and-socket joint, a leveling head similar to that used on transits, and in some instruments a combination of the two. Some compasses are leveled up by means of two bubble tubes at right angles to each other; other compasses, by a circular spirit level. The needle should be so sensitive that when it is attracted to one side by a piece of iron held a foot or so away it will settle in its original position several times in succession. Sensitiveness depends upon the sharpness and perfect form of the pivot, the form and smoothness of the cup, the shape and balance of the needle, and the strength of its magnetism. A good needle properly charged does not often lose its magnetism, and hence in the majority of cases a sluggish needle is due to the dulling of the pivot point, or to the scratching of the cup. As it is difficult to sharpen a pivot, special precautions should be taken to avoid dulling it, such, for example, as lifting the needle off the pivot when not in use. (See Arts. 212(3) and (10).) The extreme ends of the needle should come close to the graduated circle and should lie in the same plane with it to avoid parallax in reading. The center of the needle and the two ends should lie in the same vertical plane, so that if the pivot is in the center of the graduated circle the readings of the two ends of the needle will agree. The magnetic axis of the needle should coincide with its geometric axis. The plane passing through the slits of the sight vanes, in which the line of sight is contained, should pass through the zero points of the graduated circle and be at right angles to the plane of this circle. The above are the most important requirements for a good compass. Most of them depend wholly upon the process of manufac-

ture, but a few may be met by proper adjustment of the instrument. (See Art. 748.)

732(*a*). Special forms of the compass. When the *plain compass* described above has a movable vernier for setting off the magnetic declination it is called a *vernier compass*. This type of compass is the most suitable for retracing old land boundaries. The *prismatic compass* used for less accurate work has a triangular glass prism and a mirror by means of which the needle may be read at the same time that a sight is taken. This is an advantage when the instrument is held in the hand instead of being mounted on a Jacob's staff or a tripod. Compasses are sometimes provided with a graduation for reading angles independently of the needle, but the precision which can be obtained with open sights is hardly sufficient to warrant this. Telescopes sometimes replace the open sights, but if the work is of such a nature as to require a telescope, it is better to use a light transit in place of the compass. The solar compass, formerly used in land surveys for determining the true meridian by an observation on the sun, has now been replaced almost entirely by the solar attachment on the transit which is constructed on the same general principles. (See Art. 531.)

732(*b*). Tests for the compass. (1) Stretch a fine thread through the two sight vanes. The north and south points of the graduated circle and the zero of the declination-arc vernier should be directly underneath this thread. (2) Test the sensitiveness of the needle to see if it will settle several times in succession at the same reading after being attracted to one side. (3) By means of a transit set stakes about 200 ft. from a station at angular intervals of, say, 30° until a circle is completed. Set the compass over the station and sight at each stake in succession, noting the needle readings. If at any point the change in the needle reading is not 30° from that of the preceding reading, and if all sources of attraction outside the compass have been carefully removed, it is probable that the metal of the compass box contains some magnetic substance, as, for example, iron mixed with the brass. If a compass has a long declination arc and vernier, the transit may be dispensed with in the above test, the angles being turned off by means of the vernier. (4) See "Adjustments of the Compass," Art. 748, for other tests.

733. The Sextant. The distinguishing characteristic of the sextant is the arrangement of mirrors, which enables the observer to sight at two different objects *simultaneously*, and thus to measure an angle at a single observation. Whereas other instruments must be kept stationary during the time between the backsight and the foresight, the sextant may be held in the hand, since the two sights are taken at the same time, and horizontal, vertical, or inclined angles may be measured very accurately even though the instrument is in motion. For this reason the sextant is used for observations at sea. In surveying it is used for measuring angles from a boat to points on shore when locating soundings, buoys, etc., by the three-point method explained in Art. 321; it is also used in reconnaissance work, explorations, geological surveys, and for solar observations.

733(*a*). To measure a horizontal angle. In Fig. 733(*a*) let *B* represent the mirror or *index glass* which is rigidly attached to the *vernier arm BH*. Let *C* represent the *horizon glass*, the lower portion of which is a mirror, and the upper portion clear glass. The colored glasses may be turned back out of the way except

when making observations on the sun, when they will be needed to protect the eye. Usually three eyepieces are furnished: one for ordinary terrestrial work, one for astronomical work, and one without magnifying power.

Let A and D be the two points sighted at, and E the eye of the observer; then the angle required is AED.

1. Holding the sextant in one hand by the handle underneath, bring the plane of the graduated arc JK into the plane of the eye and the two points A and D.

2. Sight through the upper portion (clear glass) of the horizon glass C directly at D and swing the vernier arm BH (thus turning the index glass B) until the image of A reflected from B to the lower half (mirror) of C appears superimposed upon D.

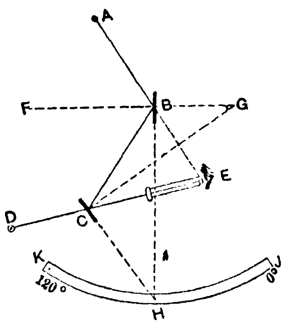


FIG. 733(*a*).

3. Clamp the vernier and by means of the tangent screw bring the images of A and D into exact coincidence.

4. Read the vernier. The reading is the angle AED, but unless these three points lie in a horizontal plane the angle is not a true horizontal angle. In most cases, however, where the sextant is used, the difference between the angle and its horizontal projection is a negligible quantity.

733(*b*). PRACTICAL SUGGESTIONS. (1) It is a little difficult at first to bring the two images into coincidence, especially if the observer is in a boat, where on account of the unsteady motion it may be necessary to make quick observations. *In such a case it is often best to dispense with the use of the eyepiece altogether.* Simply place the eye at the center of the ring from which the eyepiece has been removed and proceed as if looking through a telescope. This is a good way in which to become accustomed to the use of the sextant, and it will be found that angles can be measured by this method with surprising accuracy. (2) Sight through the telescope directly at the fainter object, so that the brighter object will be the one reflected. To do this it may be necessary to hold the sextant upside down. (3) Do not try to hold the images still in the field of view, but find the final position of the vernier arm by a series of observations, twisting the wrist to produce a regular oscillating motion, and noting the relation of the reflected image to the direct image as they come together. (4) The adjusting screw that moves the telescope to or from the plane of the arc is for the purpose of centering the two images and making them equally bright. (5) Keep the fingers off the graduation, as it tarnishes easily. (6) The degrees on the arc are numbered to correspond to twice their value, that is, each degree is marked two degrees, etc., as explained below. The sextant is so called because the arc is about 60° long, but angles may be measured to 120° or more. (7) The index correction should be determined as accurately as the angle is to be measured. (See Art. 749(IV).) (8) The vertex of an angle measured is not exactly at the eye but at the intersection of the two lines of sight, which, for small angles, is considerably behind the eye. For this reason, there may be an appreciable error in the measurement of an angle less than, say, 15° or when the lengths of sights are less than, say, 1000 ft.

733(*c*). Vertical angles may be measured by holding the sextant so that its arc lies in a vertical plane. In observations on the sun an artificial horizon, such as a shallow box of mercury, is used. The image of the sun in this reflecting surface and the sun itself are the two objects sighted at, the resulting angle being the *double* altitude of the sun.

733(*d*). The theory of the sextant. If a ray of light is reflected successively from two plane mirrors the angle between the first and last directions of the ray is twice the angle between the planes of the two mirrors.

Proof. In Fig. 733(*a*) let FG and CG be drawn perpendicular to the planes of the mirrors B and C respectively. Let the ray from A be deflected at B to C, and again at C to E. Since the angle of incidence equals the angle of reflection,

$$ABF = FBC \quad \text{and} \quad BCG = GCE$$

Also by geometry $BGC = CHB$.

$$BEC = 180^\circ - EBC - BCE = ABC - BCE$$

$$BGC = 180^\circ - GBC - BCG = \tfrac{1}{2}ABC - \tfrac{1}{2}BCE$$

Therefore

$$BEC = 2BGC = 2CHB$$

When the mirrors are brought into parallel planes CHB becomes zero and hence the angle AED will be zero, i.e., the rays from A and D are parallel.* This gives the position of the arm for a reading of 0°. If the arm is moved to any point as H, the zero of the vernier will move over the arc an angular distance equal to the angle between the mirrors, which, as shown above, is one half the required angle BEC or AED. For this reason the arc is numbered to give at once the angle required at E, i.e., each degree is counted as twice its real value.

* A ray which comes from any point to the eye through the horizon glass is not strictly parallel to a ray from the same point coming to the eye indirectly by reflection, since the index glass is several inches out of the direct line of sight. For sights more than 1000 ft., however, the error would not exceed 30 to 40″, and for sights of a mile or more it is inappreciable.

CHAPTER 54

ADJUSTMENT OF INSTRUMENTS

The actual work of adjusting a surveying instrument is comparatively simple and easy once the reasons for the various steps are clearly understood. If, on the other hand, the reasons are not clear, and the student is content simply to follow directions such as those printed in catalogs of instruments, he is not likely to work intelligently or to gain confidence in his ability to adjust an instrument. The aim in this chapter, therefore, is to lead the student to a thorough understanding of fundamental principles, so that if he forgets the methods and order of adjustments (as he almost surely will) he will be able to reason them out for himself instead of relying upon his memory or upon printed directions.

The adjustments explained are those of the transit, the level, the plane table, the compass, the sextant, and solar attachments.

734. Why Adjustments Are Necessary. When an instrument leaves the hands of the maker it should be in complete adjustment. There are certain parts, however, which are likely to be deranged by accident or by continuous use. Accurate work can often be done with an instrument out of adjustment, provided certain methods of eliminating errors are followed, as, for example, the method of double reverse (Art. 252). Such methods, however, involve extra labor; hence it saves time to keep an instrument in adjustment. Moreover, there are some errors which cannot be eliminated except by adjusting the instrument. A good instrument will sometimes stay in adjustment for months; nevertheless all instruments should be tested frequently. In accurate leveling, for instance, the level should be tested at least once a day. The more accurate the instrument and the more exacting the requirements, the more attention should be paid to the adjustments. No man therefore is a competent surveyor who is unable to adjust his instruments.

735. Favorable Conditions for Adjustments. The most important favorable conditions for adjustment are these. (1) Nearly level ground where sights of sufficient lengths are possible, and where most, if not all, of the adjustments can be made without shifting the instrument. (2) A location where "air boiling," successive areas of light and shadow, or other atmospheric conditions do not interfere with clear sights. (3) Where the instrument is set up the ground should be dry and firm so that moving around the instrument will not disturb it. If the instrument is set up on a

floor of concrete, brick, or stone, precautions should be taken to prevent any movement of the tripod points. Too often a slight, unnoticed displacement of the instrument is the cause of what appears to be an imperfect adjustment. (4) Preferably the instrument should be in the shade during adjustment. If this is not practicable, the instrument should be protected from the sun. This is particularly true of the level, since a bright sun affects the bubble. Direct rays of the sun which strike only one side of any instrument are likely to affect adjustments. Adjustments should not be undertaken where the wind is blowing. (5) Whenever practicable, sights should be taken in a direction away from the sun, not only because the conditions for sighting will be better, but also because the illumination of the range pole, target, or other object sighted at will be greater. (6) When instruments are to be adjusted frequently at any location, it is generally best to establish permanent standards for testing adjustments as explained in Art. 740.

736. General Suggestions. The following suggestions apply to the adjustment of all surveying instruments.

1. Before beginning the adjustment of an instrument make sure that it is in a proper condition for adjustment. All bearings, adjusting screws or nuts, clamps, tangent screws and tangent springs, telescope lenses, and other parts of the instrument which affect adjustments should be *clean.* There should be no undue looseness or play in movable parts or in parts that should be in close contact with each other. Make sure that the tripod shoes are tight.

2. For each instrument there are generally several adjustments. Each adjustment consists of two steps: first a *test* to determine the error, and then the *correction* of this error by turning certain adjusting screws or nuts. In most cases this is done by inserting an adjusting pin; hence the terms "capstan screws" and "capstan nuts." When there is a pair of *opposing* capstan screws or capstan nuts *one of the pair should be loosened before the other is tightened.*

3. The adjusting pin should fit snugly into the hole; otherwise the holes will become worn and out of shape. In an emergency a wire nail can be used, *but not in such a way as to injure the hole.*

4. If all the adjustments are badly out, do not perfect the adjustments one at a time, but rather bring the whole instrument to a nice adjustment by repeating the entire series as many times as may be necessary. This is because the adjustment of one part may affect some other part already adjusted. *Hence the final step*

in the adjustment of any instrument is to test each adjustment to see that none has been disturbed. For this reason select a place where the adjustments may be made in succession without disturbing the instrument.

5. In making any one adjustment the test and correction should be repeated a number of times. It will save time not to split hairs during the early part of the adjustment, but the more nearly the adjustment is perfected the more carefully it should be made.

6. An instrument will stay in adjustment longer if screws and nuts are left in perfect contact with the surfaces against which they operate. For example, after a screw has come to bearing, a slight turn of the adjusting pin of, say, 20° will "set" it so that it is not easily disturbed. Undue strain, however, should be avoided.

7. Before attempting to adjust a cross-hair ring study carefully the note in Art. 707(*a*) and the photograph to which it refers. Tightening the capstan screws which move the cross-hair ring does not affect the tightness of the cross hairs, but if too tight the whole telescope may be affected. If one pair of screws is fastened tightly while the other pair at right angles is being turned, undue strain may be brought on the first pair if the adjustment is considerable. In such a case loosen all four screws *slightly* before working either pair.

8. In this book the line between the intersection of the cross hairs and the center of the objective lens is called the *line of sight.* A distinction is made between this line and the *line of* collimation (see Art. 708(*b*)), with which it may or may not coincide.

9. Since the slightest displacement of the instrument during an adjustment may render that adjustment imperfect, great care should be taken in manipulating the instrument and in moving around it to avoid displacement. For example, resting the hand on the telescope, or on the plates, or on a tripod leg, or stepping close to a tripod shoe may cause a displacement which may be unnoticed but nevertheless sufficiently great to disturb the adjustment.

736(*a*). CAUTION. While the theory of adjustments is simple and the methods of field adjustment are easy, there is a danger that when the delicate adjusting screws are left to the mercy of some inexperienced or heavy-handed adjuster, he will not keep the same degree of tension on the screws as that put on by the maker. Once that balance of tension supplied by the maker is lost, field instruments rarely stay in adjustment for any considerable length of time. For this reason many larger organizations prefer that all adjustments be made by a trained man within their organization, or by the maker himself. In general, however, the surveyor should be competent to adjust his instrument, and to do it without undue strain on any part. He should keep in mind, however, that not only may undue strain injure the

instrument, but it may also make the instrument appear to be out of adjustment or in adjustment when it is not. For example, if, in leveling the instrument a leveling screw is forced, it may affect the plate or the spindle to such an extent as to make the plate levels appear out of adjustment when they are actually in adjustment, or, vice versa, they may seem to be in adjustment when they are actually out.

736(*b*). NOTE. Different makes of surveying instruments vary in minor details of construction; consequently there are minor differences in details of adjustment. Instrument makers generally furnish instructions for the adjustment of their instruments, and the surveyor should possess such instructions for the particular make or makes of instruments which he uses.

737. Parallax. If the image formed by the objective lens is not in the same plane with the cross hairs, any movement of the eye is likely to cause an apparent movement of the image with respect to the cross hairs. This is called *parallax*. The effect is similar to that produced in looking through a window, where any movement of the eye causes an apparent movement of objects outside. Parallax may render accurate work impossible. To remedy it the image and the cross hairs must be brought into the same plane by correctly focusing the eyepiece. Because of differences in eyesight, this must be done by each observer for himself. Parallax is one of the most common sources of error in the use of the telescope. *Before using the telescope of any surveying instrument for any purpose whatever, the user should eliminate parallax. This step should always be taken before beginning the adjustment of any instrument that has a telescope.* The procedure for eliminating parallax is as follows: (1) Point the telescope toward the sky or some bright object and move the eyepiece in or out until the cross hairs appear black and sharp. If both eyes can be left open, a better focus may be obtained. (2) Direct the telescope to some object which has sharply defined detail and is more than 300 ft. distant. Focus the objective upon the object until there is no apparent movement of the cross hairs over the object when the head is moved from side to side. It is best to run the objective pinion back and forth behind and in front of its position for a true focus before deciding where that position is. When the true position has been found, the objective lens has been focused correctly, even though the image of the object may appear to be slightly out of focus. (3) Refocus the eyepiece slightly, if necessary, to make the image (not the cross hairs) appear sharp. It will now be found that the cross hairs also appear sharp and black.

An eyepiece properly focused as just described is set correctly for the individual who did the focusing and should not require any change of focus for his use, regardless of whether the objective is focused upon a far or a near object.

Note. A young man may have more trouble than an older man in getting an eyepiece properly focused, on account of the greater "accommodation" of the younger eye. The procedure just outlined tends to produce a relaxed and normal condition of the eye when setting the final focus of the eyepiece. Greater difficulty is experienced with the simple eyepiece of the inverting telescope than with the compound eyepiece of the erecting telescope.

738. The Method of Reversion. Nearly every important adjustment is tested by the method of reversion, i.e., the adjustment is tested by reversing the instrument, or some part of it, so that an error will become apparent. This *apparent* error is *double* the *true* error, because reversion simply places the error as much to one side as it was on the opposite side before reversion. This will become evident as the adjustments are explained. The thing to remember is that a *single* reversion *doubles* the error.

738(*a*). ILLUSTRATION. *The reversion of the level tube, end for end,* is an excellent illustration of the method of reversion. Moreover, since it is the basis of three of the six principal adjustments, it is worth while to understand it perfectly.

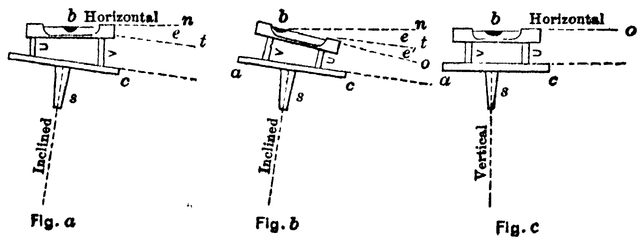


FIG. 738(*a*).

In Fig. *a* let the bubble be in the center of the tube—its axis *bn* must therefore be horizontal—but let the supports *u* and *v* be of *unequal* length. Then any surface as *ac*, upon which the supports rest, cannot be level, but *a* will be higher than *c*, since *u* is shorter than *v*. Suppose that it is desired to bring the axis of the bubble *bn* parallel to the surface *ac* by making the supports *u* and *v* of equal length. This is equivalent to making *bn* coincide with some line *bt* which has the same inclination *e* with the horizontal that *ac* has. This error *e*, which is to be corrected, is *above* the required line *bt*. Keep the surface *ac* stationary, but take the level tube off, and reverse it end for end (Fig. *b*). The supports *u* and *v* being attached to the tube, and their lengths as yet being unchanged, the new axis of the bubble *bo* still makes the angle *e* with *ac* and *bt*, but is *below bt* instead of above. In other words, reversion has simply placed the error on the opposite side, and the total error $e + e' = 2e$ is *double* the true error. To correct the error therefore change the lengths of the supports *u* and *v* until the bubble moves halfway back to the center; *bo* will then coincide with *bt*, and *u* and *v* will be of equal lengths. If now the surface *ac* is made horizontal (Fig. *c*), the bubble will return to the center and its axis *bo* will be horizontal.

738(*b*). LEVEL TUBE ON A REVOLVING SURFACE. Let the level tube be *attached* to the surface *ac* by supporting screws *u* and *v*, and let this surface be revolved on a spindle *s*. It is evident that if the axis of this spindle is at right angles to *ac the inclination of ac cannot change during revolution,* and that if the surface is revolved through 180° *the level tube will be turned end for end* just the same as if it had been taken off and reversed. Hence in this case the reasoning and the conclusions of the preceding paragraph

still hold good. It is to be noted that before adjustment (Figs. *a* and *b*) the axis of the spindle is inclined; after adjustment (Fig. *c*) it is vertical.

Remark. If the axis of the spindle is not perpendicular to *ac*, reversion and correction will still bring the axis of the bubble *perpendicular* to the axis of the spindle.

739. Adjustments of the Transit. (*a*). THE PRINCIPAL LINES to be considered in the adjustment of the transit are these. (1) The *line of sight.* (2) The *axis of the tube in each plate level.* (See Art. 705.) (3) The *transverse axis of the telescope*—the axis about which the telescope turns when it is plunged or transited. The conventional term "telescope axis" is ambiguous since it might mean the longitudinal axis of the telescope. To avoid this ambiguity, the term "transverse axis" will be used throughout the explanations of adjustments. (4) The *vertical axis* through the spindles around which the plates revolve. (5) The *axis of the telescope level tube.*

739(*b*). THE PRINCIPAL ADJUSTMENTS are six in number, namely:

I. To bring the vertical cross hair into a plane perpendicular to the transverse axis of the telescope so that any point of the hair can be used in "lining in" or in measuring horizontal angles.

II. To bring the axis of each of the two level tubes on the upper plate into a plane perpendicular to the vertical axis so that when the transit is properly leved up, the vertical axis will be truly vertical. The plane of the graduated limb will then become truly horizontal for the measurement of horizontal angles, provided the vertical axis has been made by the manufacturer perpendicular to the plane of the limb. If it is not perpendicular, leveling up the transit will not bring the limb into a horizontal plane, but the deviation will generally be so small that it will cause no appreciable error in the measurement of horizontal angles.

Note. It is not correct to think that because both bubbles remain centered during a complete revolution of the upper plate the plate must necessarily be horizontal. Nevertheless it is sufficiently horizontal for practical purposes; hence the expressions "leveling the plate" and "plate levels" are in common use.

The adjustment of the plate levels is an important one since it affects either accuracy or convenience in nearly every kind of work for which the transit is used. It is particularly important in precise work. It does not depend upon the completion of any other adjustment nor is it affected by any subsequent adjustment.

III. To make the line of sight perpendicular to the transverse axis of the telescope, so that when the telescope is revolved about its transverse axis the line of sight will move in a plane.

IV. To make the transverse axis of the telescope perpendicular to the vertical axis through the spindles so that when the vertical axis is truly vertical the transverse axis will be horizontal. The line of sight will then revolve in a *vertical* plane when the telescope is plunged, provided the third adjustment has previously been completed.

The third and fourth adjustment are particularly important in all work which involves plunging the telescope. The fourth adjustment is also important in the measurement of an *horizontal* angle when the two points sighted at differ considerably in elevation, a measurement which is also affected to some extent by the second and third adjustments.

V. To make the axis of the telescope level tube parallel to the line of sight so that the transit may be used for direct leveling, or for measuring vertical angles with reference to a horizontal plane.

VI. To make the vernier reading on the vertical circle (or arc) zero when the line of sight is horizontal. This adjustment is not essential, but it facilitates the reading of vertical angles.

739(*c*). ORDER OF ADJUSTMENTS. The six principal adjustments of the transit have just been enumerated in the order in which these adjustments are generally made, but this order may be modified. Some engineers prefer, for example, to adjust the plate levels (adjustment II) after adjustments I, III, IV, and V have been completed, but before the adjustment of the vertical circle (adjustment VI).

Throughout the adjustment of the transit keep in mind the general suggestions at the beginning of this chapter, and be particularly careful not to force any screw or to bring undue strain on any part of the instrument. (See *Caution,* Art. 736(*a*).) Before beginning the adjustment eliminate parallax—a prerequisite for any adjustment that requires the use of the telescope (Art. 737).

739(*d*). I. FIRST ADJUSTMENT OF THE CROSS-HAIR RING

To bring the vertical cross hair into a plane perpendicular to the transverse axis of the telescope.

Test. (1) Set up the instrument, sight at a sharply defined point at least 200 ft. away, and clamp both the upper and the lower plates. (2) By means of a tangent screw, set one end of the vertical cross hair exactly on the point. (3) Revolve the telescope up or down until the point appears to move from one end of the vertical cross hair to the other. If the point appears to stay

exactly on the cross hair throughout its length, the hair is in the correct position.

Correction. If the point appears to move off the vertical cross hair, change the position of the hair as follows. (1) Loosen slightly all four screws that control the movement of the cross-hair ring. (See Art. 707(*a*).) It is helpful to loosen two *adjacent* screws a definite amount of rotation (as a quarter turn or 15 minutes on the clock) and subsequently to retighten the *same two screws the same amount of rotation.* By this procedure proper tension is maintained. (2) Rotate the cross-hair ring into such a position that when the test is repeated the point appears to stay on the hair throughout its length. (3) Tighten the capstan screws and then check the adjustment.

Suggestion. To rotate the ring, hold a screwdriver or a small wooden wedge against one of the curved washers or "collets" and tap lightly against it.

739(*c*). II. Adjustment of the plate levels

To bring the axis of each of the two level tubes on the upper plate into a plane perpendicular to the vertical axis through the spindles. As already stated, the ultimate object of this adjustment is to make the vertical axis through the spindles truly vertical when the transit is leveled up.

Test. The final test is to see that each bubble comes to rest in the center of its tube when the plate is turned on its spindle to any position from 0° to 360°. During adjustment, however, repeated tests for each bubble are made as follows: (1) Level up and bring the level tube parallel to two diagonally opposite leveling screws. (2) Turn the plate halfway around until the level tube is parallel to the same pair of screws, but reversed end for end. If the bubble does not come to rest at the center the tube needs adjusting.

Correction. (1) Bring the bubble halfway back to the center of its tube by turning first one and then the other of the capstan screws which support the ends of the tube. (2) Bring it the rest of the way (i.e., to the center) by leveling the plate with the leveling thumb screws. (3) Repeat the test and the adjustment until the bubble remains in the center when the plate is turned halfway around. (4) Adjust the other level tube in like manner. (5) Apply the final test of turning the plate completely around from 0° to 360°.

Principle Involved. The reason why the bubble is brought *half-way* back after reversion is explained in Art. 738(*a*).

Practical Suggestions. (1) Set up the transit so that it is *firm.* If a place is chosen where a 300-ft. sight can be taken each way, and where the telescope can be directed to some high point, all four of the remaining adjustments can be made without disturbing the instrument. (2) It is better to clamp the lower plate and to allow the upper plate to remain unclamped throughout the adjustment. When the final test is satisfactory, clamp both plates together, loosen the lower clamp, and turn the two plates completely around. Both bubbles should remain in the centers of their tubes; if they do not, it shows that the axes of the inner and outer spindles (see photograph, p. 867) are not parallel, a defect best remedied by the maker. If the axes are not parallel, no error will be caused in the measurement of horizontal angles unless they are measured by repetition, provided the plate levels are adjusted perpendicular to the axis of the inner spindle. (3) If the bubbles are not badly out, the adjustments of the two level tubes may be carried along together, provided care is taken always to reverse each tube parallel to the same pair of diagonally opposite leveling screws to which it was parallel before reversion. (4) It will often suffice to turn one capstan screw only, lowering slightly the end of the tube toward which the bubble lies, or raising the other end. Many transits have as one support of the tube a hinged or spherical bearing block, so that all adjustment is on one screw. Many tubes are supported by two nuts, one top and one bottom. In this case, one must be loosened, the other tightened to move the tube up or down the screw. (5) Turning a capstan screw clockwise screws it into the plate and therefore *lowers* the corresponding end of the tube; counterclockwise unscrews the screw and *raises* the end of the tube.

739(*f*). III. Second adjustment of the cross-hair ring

To make the line of sight perpendicular to the transverse axis of the telescope. This is often called the "adjustment of the line of collimation." The ultimate object is to make the line of sight revolve in a plane when the telescope is transited.

Test. (1) Level up the transit and backsight on some sharply defined point *A* about 300 ft. away. (Both plates clamped.) (2) Leaving both plates clamped, plunge the telescope, and set point *B* in line at approximately the same distance and elevation as point *A*. (3) Leaving the telescope reversed and the lower plate clamped, rotate the upper plate a half turn, and again sight on *A*. (4) Again plunge the telescope (bringing it to its normal position). If the line of sight passes through *B*, the vertical cross hair is in adjustment; if the line of sight does not pass through *B*, set a point *D* in line and opposite *B*.

Correction. (1) Measuring from point *D* toward point *B*, mark a point *P*, a distance from *D* equal to *one-fourth* the distance from *D* to *B*. (2) Move the cross-hair ring sidewise (without rotating it) until the vertical cross hair is set on point *P*. This is done by *first* loosening the horizontal capstan screw on one side of the telescope and then by tightening the horizontal capstan screw on the other side of the telescope. (3) Repeat the test and correction until the adjustment is perfected. (4) When this adjustment has been completed, repeat the test and, if necessary, the correction for the first adjustment of the cross-hair ring (adjustment I).

Principle Involved. This is a *double* application of the method of reversion. Reversing (plunging) the telescope once doubles the error (Art. 738); reversing a second time again doubles the error *on the opposite side*, so that the total *apparent* error is *four* times the *true* error. This may be better understood by referring to Fig. 739(*f*).

Illustration. Let *XY* be the supporting or transverse axis of a telescope (top figure in Fig. 739(*f*)). If *AKX* is an exact right angle the line of sight

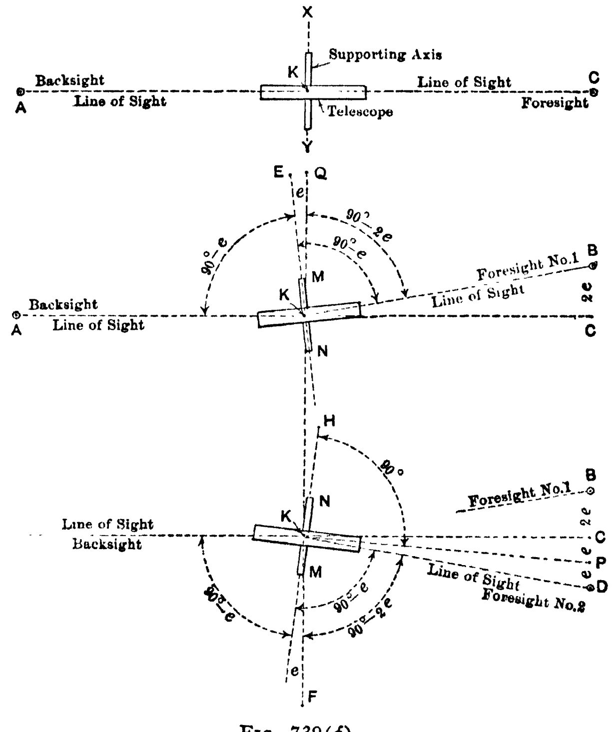


FIG. 739(*f*).

is in adjustment, because backsighting on *A* and plunging the telescope the line of collimation will generate a plane, striking a point *C*, which is in the line *AK* produced. But suppose the line of sight is out of adjustment by an amount *e*, that is, the angle *AKE* (middle figure) is not 90° but 90° − *e*. When the telescope is plunged the line of sight will generate the surface of a cone and strike a point *B* instead of *C*. From the figure it is evident that *QKB* is 90° − 2*e* and hence *BKC* = 2*e*. Leave the telescope inverted, turn, and backsight on *A* again. (Bottom figure: Notice the reversed positions of *M* and *N*, the ends of the supporting axis.) Plunge the telescope; the line of collimation will strike some point *D*, and it is evident that *DKC* = 2*e*. In practice the point *C* is not found, but the angle *BKD* equals 4*e*. Hence mark a point *P* one-fourth the distance *DB* from *D* and move the cross-hair ring until the vertical hair is on *P*. Then *HKP* is a right angle—the object of the adjustment.

Practical Suggestions. (1) Set up in some comparatively level place where a sight of about 300 ft. each way can be had. Closer adjustment can be made at this than at longer distances. If conditions make necessary a shorter sight in one direction than in the opposite direction, a point at the shorter distance should be used for a *backsight.* (Why?) (2) Select some well-defined point for a permanent backsight; if necessary set a very slender stake or establish a backsight in some other way. (3) Each foresight should be marked and numbered. The backsight and the position of the transit can often be chosen so that the foresights will fall on some vertical surface, where pencil marks can be distinctly seen through the telescope, as, for example, marks on a sheet of paper tacked to the side of a building. (4) When the man who marks the foresights has measured back one-fourth the apparent error, he should hold his pencil on the point, and *give some well-understood signal* that the point indicated is the one to which the vertical hair should be adjusted; otherwise points may be confused by the man at the transit. If the transitman is sighting at a vertical surface, confusion may be avoided by drawing a pencil line which can be seen from the transit. As tests are repeated erase all lines previously drawn, so that for each test there is only one line that the transitman can see, i.e., the line to which the vertical hair should be adjusted. An experienced transitman can often make this adjustment without help, by taking for a foresight some such object as a window pane, estimating the error each time, and repeating the test and correction until the vertical hair bisects the object used as a foresight. (5) On some transits there are three sets of capstan screws. Those nearest the eye adjust the eyepiece, the next set are for the stadia wires, and the *third* set for the cross-hair ring. In this adjustment, therefore, *use the two side screws farthest from the eyepiece.* (6) If, in looking through the telescope, which has an erecting eyepiece, the vertical hair should *apparently* be moved to the *right* in order to cover the point indicated, *loosen* the capstan screw on the *right-hand* side of the telescope and tighten the opposite screw. This really moves the vertical cross hair to the *left,* since the cross-hair ring is drawn towards the screw that is tightened (see Art. 707(*a*)), but the eyepiece inverts the movement so that it *appears* to be drawn to the *right.* Hence the rule: *Loosen the screw towards which the cross hair must* APPARENTLY *be moved, and tighten the opposite screw.* (7) Do not be too particular about getting the cross hair on the point until after two or three tests by double reverse have been made. The nearer the adjustment approaches completion, the more nearly should the hair split the point. (8) The final adjustment can often be made by simply tightening one screw without loosening the other, but screws must not be strained. (See *Suggestion 7,* Art. 736.) (9) When the line of sight is in adjustment it should coincide with the line of collimation, and this in turn should be concentric with the geometrical axis of the telescope. If these conditions are not fulfilled, errors may be introduced as the objective is moved in or out for sights of different lengths. Perfect adjustment in this respect depends largely on the maker of the instrument. (10) Since the line of sight is determined solely by the position of the intersection of the cross hairs with respect to the center of the objective lens, it is not affected by the position of the eyepiece. The eyepiece may be moved, in any direction, or even entirely removed and a new one substituted without affecting the adjustment of the cross hairs.

739(*g*). IV. ADJUSTMENT OF THE STANDARDS

To make the transverse axis of the telescope perpendicular to the vertical axis through the spindles. By the preceding adjustment the line of sight is made to revolve in a *plane perpendicular to the transverse axis.* The ultimate object of this adjustment is to make the line of sight revolve in a *vertical* plane when, by leveling the transit, the vertical axis has been made truly vertical. This can be done only when the transverse axis is perpendicular to the vertical axis.

Test. (1) Level the transit carefully, observing particularly the bubble in the plate level that is at right angles to the telescope. (2) Sight at some sharply defined point *A* that is sufficiently high to require the telescope to be turned upward through a large vertical angle. (3) Leaving both plates clamped to prevent sideways motion, depress the telescope without plunging and set a point *B* on the ground. (4) Leaving the lower plate clamped, rotate the upper plate halfway around, transit the telescope, and sight again at *A*. (5) Leaving both plates clamped, depress the telescope. If the line of sight again passes through *B*, no adjustment is necessary; if it does not pass through *B*, set a new point *C* in line and opposite *B*.

Correction. (1) Set a point *M approximately* halfway between *B* and *C*. (2) Sight on *M* and, leaving both plates clamped, point the telescope upward; the line of sight may not quite strike the original point *A*. (3) Make the line of sight pass through *A* by raising or lowering the adjustable end of the transverse axis. If the end of the axis is to be raised, *loosen first* the screws which fasten the cap to the standard before turning the capstan screw. If the end is to be lowered, turn first the capstan screw, then tighten the cap screws. (4) Repeat the test and the correction until the points *B* and *C* coincide, i.e., until the line of sight moves in a vertical plane.

Principle Involved. Since the telescope is reversed but *once,* this is a case of single reversion, and the apparent error is *approximately* double the true error, not four times the error as in the third adjustment. However, there is no easy theoretical method of determining just how much the error is or how much the end of the transverse axis should be raised or lowered; consequently *the correction is made by the "cut-and-try" method,* as explained in the following illustration.

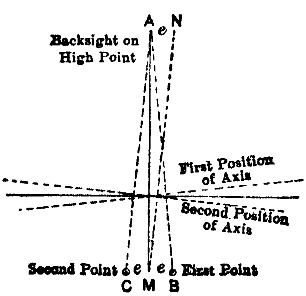


FIG. 739(*g*).

Illustration. In the first position of the axis, Fig. 739(*g*), the *right*-hand end is the higher. After backsighting on *A* and depressing the telescope the intersection of the cross hairs falls on *B*. Turning the plate halfway around brings the higher end of the axis to the *left,* so that the second point falls at *C*. The point *M* is *approximately* where the line of sight should strike when the axis is horizontal, i.e., *A* and *M* are in the same *vertical* plane.

Theoretically the figure does not truly represent the adjustment, and the point *M* is not exactly halfway between *C* and *B* except under certain conditions. Two oblique planes are generated by the line of sight, which are not truly represented by *AB* and *AC*. In practice, however, points *B* and *C* are

so near together that by taking M halfway between and making the correction $e = CM = NA$ the supporting axis is made *more nearly* horizontal, until after a few trials the adjustment is perfected.

Note. A common method of correction after points B and C have been established is to leave plates clamped with the line of sight on C and then to adjust the end of the transverse axis until the line of sight strikes M. When, however, M is nearly at the same level as the telescope it is better to begin the correction by sighting on M and then to point the telescope upward and adjust by making the line of sight strike A, as already explained. The reason for this procedure is that the higher the telescope is pointed the more the direction of the line of sight is changed by a slight adjustment of the transverse axis (why?), and the easier it is, therefore, to gauge the proper amount of adjustment.

739(*h*). AN ALTERNATIVE METHOD OF ADJUSTMENT based on the principle just stated is as follows. (1) Sight at a high point A, depress the telescope, and establish point B. (2) Rotate the upper plate 180°, transit the telescope, and sight on B. (3) Leaving both plates clamped, point the telescope upward until the line of sight passes near A. (4) Note a new point estimated to be halfway from the line of sight to A, and make the line of sight pass through this new point by adjusting the transverse axis. (5) Repeat the test and the correction until the adjustment is perfected.

Suggestions. (1) For a permanent backsight, a church spire, a point near the top of a high building, or any other sharply defined high point may be taken which will make it necessary to move the telescope through a large vertical arc. The point sighted at should be as high as possible and at least 200 ft. distant. (2) The plate bubble in the tube parallel to the transverse axis should be in adjustment and should remain in the center whenever the telescope is depressed or elevated. The telescope level may be used instead of the plate levels to attain greater accuracy in making the vertical axis truly vertical. (3) To decide whether the adjustable end of the axis is too high or too low, note in which direction the second plane generated slopes, and remember that the transverse axis is perpendicular to that plane. This means that the higher end of the axis is always on the same side of the vertical plane through the high point A as the *second* point set on the ground. (In Fig. 739(*g*) this point is C.) (4) In readjusting the bearing cap at the top of the standard, it should not be made so tight that it binds the transverse axis or so loose that the telescope, when unclamped, revolves of its own weight.

739(*i*). V. ADJUSTMENT OF THE TELESCOPE LEVEL TUBE

To make the axis of the telescope level tube parallel to the line of sight. The principal purpose of this adjustment is to make it practicable to use the telescope level whenever it is desired to bring the line of sight into a horizontal plane.

Test and Correction. For the test use either procedure A or procedure B of the "two-peg method." (See Art. 742.) For the correction, adjust the *level tube* as explained in Art. 742(*d*), except that in step 2 the telescope is inclined by revolving it on its transverse axis instead of by the use of the leveling screws. (Do

not make the line of sight parallel to the axis of the telescope level by adjusting the cross-hair ring.)

739(*j*). THE FOUR-PEG METHOD. Some engineers use the four-peg method instead of the two-peg method in the adjustment of the telescope level. The procedure is illustrated in Fig. 739(*j*). (1) Drive four stakes *A, B, C,* and *D* in line and exactly equidistant, 50 to 100 ft. apart. (2) Set up the transit at *A*. (3) Bring the bubble to the center of the telescope level by moving the telescope. Clamp the telescope in that position. (4) Obtain the elevation of the line of sight at *B* and at *C* by holding a leveling rod first on one point and then on the other. Let R_1 represent

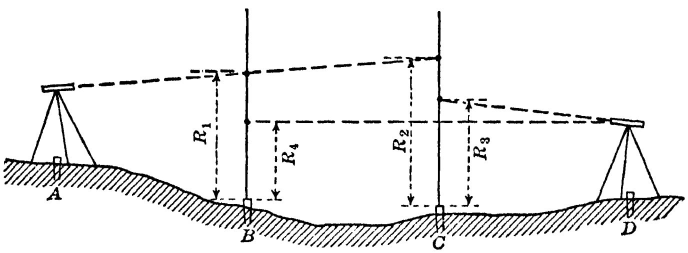


FIG. 739(*j*).

the reading at *B* and R_2 the reading at *C*. (5) Set up the transit at *D,* and with the bubble in the center of the telescope level, obtain the elevation of this second line of sight at *C*. Let R_3 represent this reading. (6) Let R_4 represent the reading at which a horizontal line of sight from the transit at *D* meets the leveling rod at *B*. Calculate the value of this unknown reading from $R_4 = R_1 + R_3 - R_2$. This equation, which may easily be derived, holds true regardless of whether the point *B* is above or below *C,* provided proper algebraic signs are observed. (7) Set the target at the reading R_4 and hold the rod at *B*. (8) By means of the transverse axis tangent motion, move the telescope until the center horizontal cross hair coincides with the center horizontal line of the target. (9) Raise or lower one end of the bubble tube by turning the capstan nuts until the bubble returns to the center. (10) Repeat the test and correction until the adjustment is perfected. (One more setup at *A* may provide a satisfactory check.)

739(*k*). REVERSION VIAL. When the telescope level vial is of the reversion type, i.e., one that can still be read when the telescope has been revolved, a procedure may be followed that is simpler than either the two-peg or the four-peg method. (1) Set up the transit, sight on a level rod about 100 ft. distant, center the telescope level bubble, and read the level rod (middle horizontal wire). (2) Rotate the upper plate 180°, transit the telescope,

again sight on the rod with the telescope level bubble centered, and read the rod. (3) Set the target at the mean of the two readings obtained in (1) and (2), set the horizontal cross hair on the target by moving the telescope up or down, and center the bubble by turning the capstan screws. (4) Repeat the entire procedure until the adjustment is perfected.

739(*l*). VI. Adjustment of the vertical-circle vernier

To make the vertical-circle (or arc) vernier read zero when the line of sight is horizontal. This adjustment is for convenience in the measurement of vertical angles with reference to a horizontal plane. Note, however, that it is not essential to the correct measurement of any vertical angle, since any initial reading may be used in place of zero.

Test. (1) Level up the transit carefully, using the telescope level. (2) Bring the bubble of the telescope level exactly to the center by the transverse-axis clamp and tangent screw. (3) By inspection ascertain if the zero on the vernier and the zero on the vertical circle (or arc) coincide.

Correction. (1) If the zeroes do not coincide, loosen slightly the screws which hold the vernier to the standard, shift the vernier until the zero of the vernier, as seen through a magnifying glass, is exactly opposite the zero of the vertical circle. Note also whether or not the graduation mark at the other end of the vernier is exactly opposite the mark on the circle with which it should theoretically coincide. (2) Tighten the vernier screws but not enough to make the vernier bind on the vertical circle as the telescope is turned on its transverse axis. The space between the vernier and the circle should about equal the thickness of thin paper. (3) Check the adjustment and correct again if necessary.

Practical Suggestion. (1) If the zeros cannot be made to coincide, note the *amount* and *sign* of the difference. This will be an *index error* to be applied as a correction to each angle of elevation or depression. (2) If there is no level underneath the telescope, make the line of collimation horizontal in step 2 of the test by the method used in the "peg" adjustment, i.e., set the transit exactly halfway between two points, clamp the telescope in as nearly a horizontal position as can be estimated, and proceed as directed in Art. 742(*a*). The difference in elevation between the two points having been found by the peg method, the telescope can be leveled without the use of a telescope level.

Movable Vertical Vernier. When a transit has a movable vertical vernier with control level, the axis of the auxiliary level should be made parallel to the axis of the telescope level when the vernier reads zero. This is done as follows. (1) Rotate the telescope about its transverse axis until the bubble of the telescope level is centered, and by means of the vernier tangent screw make the vertical vernier read zero. (2) Bring the bubble of the control level to the center of its tube by means of the capstan screws at one end.

739(*m*). ADDITIONAL ADJUSTMENTS OF THE TRANSIT that may be desirable or necessary are: VII. To center the eyepiece. VIII. To bring the horizontal cross hair into the horizontal plane through the optical axis of the telescope. IX. To adjust the objective slide. X. To balance the compass needle. XI. To straighten the compass needle. XII. To center the compass pivot. Adjustments X, XI, and XII are explained under the heading "Adjustments of the Compass" (Art. 748). The other three adjustments are required, if at all, much less frequently than the principal adjustments I to VI.

Note. For the adjustment of the Beaman stadia arc and the gradienter see Arts. 746, 747.

739(*n*). VII. To CENTER THE EYEPIECE

A transit may be in good adjustment in all other respects and still have the intersection of the cross hairs appear considerably out of the center of the field of view. This does not affect the accuracy of the work, but it does affect, to some extent, the seeing power of the telescope. It is also a source of inconvenience to the observer. The inverting (erecting) eyepiece usually has its *inner end* held in place by an adjustable ring which can be moved by four screws. (See the photograph of a telescope and its parts on p. 841.) When the intersection of the cross hairs appears out of center, observe in which direction this intersection should be moved and estimate the amount of movement. By means of opposing screws, move the eyepiece slide in the proper direction (opposite to the apparent direction) until the intersection of the cross hairs appears to be in the center of the field of view. (See Art. 707(*c*).)

Practical Suggestions. (1) In some telescopes, the screws that move the adjustable eyepiece ring have capstan heads like those that move the cross-hair ring; the eyepiece captan screws can be identified as those that are nearer to the eyepiece end of the telescope. In other telescopes, the screws are concealed under a collar, which may easily be unscrewed and slipped along the telescope. The flat-headed adjusting screws can then be turned with a screwdriver. In still other telescopes, the eyepiece must be removed from the telescope in order to get at the flat-headed adjusting screws. (2) Centering the eyepiece does not affect any of the other adjustments, but the centering should be done after the cross-hair ring has been adjusted. (3) There is generally no provision for centering a non-erecting eyepiece.

739(*o*). VIII. THIRD ADJUSTMENT OF THE CROSS-HAIR RING

To bring the horizontal cross hair into the horizontal plane through the optical axis. The optical center of the objective lens moves in this plane as the objective is moved in or out in focusing, and unless the horizontal hair

is in this plane also, the direction of the line of sight will change slightly when the focus is changed. This adjustment, if needed at all, is important only when the transit is used in leveling or in measuring vertical angles with reference to a horizontal plane, and only then when a relatively high degree of precision is required.

Test. Set up the transit at a point A and set stakes at points B and C on a straight line through A, at distances respectively of about 30 and 300 ft. from A. The points B and C should be chosen so that a line of sight (not necessarily horizontal) will intersect a leveling rod held first on C and then on B. (2) With the telescope clamped so that it cannot move up or down, take a rod reading R_1 on C, the more distant point, and with the telescope still clamped, take a rod reading R_2 on B. (3) Plunge the telescope, rotate the upper plate 180°, and with the rod held on B direct the line of sight at the reading R_2; tighten clamps to hold the telescope in that position. (4) Take a rod reading R_3 on C. If this reading is not the same as the first reading R_1 on C, adjustment is needed.

Correction. When the first and last rod readings at C (R_1 and R_3) are not the same, the horizontal cross hair should be moved up or down, whichever direction reduces the error. There is no simple method of determining the distance that the cross hair should be moved; consequently a cut-and-try procedure is followed. If the hair is considerably out of adjustment, *it may be necessary to move it a much greater distance than the apparent error would indicate.* Since the movement must be up or down, the position of the cross-hair ring is changed by turning the top and bottom capstan screws only. When the telescope has an inverting (erecting) eyepiece, *loosen* the capstan screw on *top* of the telescope and *tighten* the *bottom* screw to move the cross hair *upward* as observed through the eyepiece, and vice versa to move it downward. By observing carefully the amount that the error is reduced by each successive test and correction, it should not be difficult to bring the horizontal wire into the correct position after a few trials. When the adjustment has been completed, repeat the test for the second adjustment of the cross-hair ring (adjustment III).

739(*p*). Note. It is often stated that the error indicated by a single test is the mean of the two readings at C, i.e., $\frac{1}{2}(R_1 + R_3)$, and that consequently in correcting the error the horizontal hair should be moved until it intersects a point halfway between the first and second readings. This is a serious mistake. The procedure in a single test is not a case of single reversion as it may seem to be, because, during the reversion, the focus is changed from that on a near point to that on a far point. If the line of sight is not in the horizontal plane through the optical axis, the change of focus produces a change in the direction of the line of sight, and this may result in an actual error several times as great as the apparent error. For example, if a correction of $\frac{1}{2}(R_1 + R_3)$ is applied after each test, as much as 90 per cent of the actual error *may* remain after the first correction, 60 per cent after five successive tests and corrections, and 5 per cent after thirty tests and corrections. This indicates that the cross hair should be moved a distance greater than $\frac{1}{2}(R_1 + R_3)$, possibly several times that distance if the difference between the readings R_1 and R_2 is considerable. Consequently, instead of adjusting the cross hair to a point on the rod between R_1 and R_3, it may be necessary to adjust it to a reading considerably above or below both those readings.

Suggestions. (1) It is well to use two rods, one at C and one at B. The target setting on each rod can then remain unchanged until the first trial adjustment has been completed. Both targets are reset on each test. (2) The reading R_2 on the rod at B near the transit should be taken with great care and the *rod must be plumb* since a small difference in the apparent position of the horizontal hair on a rod at B will cause a much larger error in the position of the hair on a rod at C. For this reason, instead of using the graduations on a leveling rod, it may be well to establish the point to be sighted on at B by drawing a fine horizontal line on a piece of cardboard or paper tacked to a narrow board, or by some similar device such as that described in the suggestion in Art. 739(*q*). (3) Some engineers prefer a test which is the reverse of that previously described. A sharply defined point at a distance is taken as a permanent backsight, i.e., for the first point sighted at after setting up the transit and for the first point sighted at after plunging. The two corresponding readings at a point near the transit, as observed on

a finely graduated scale (with perhaps a vernier), are then used in estimating the distance that the cross hair should be moved.

739(q). IX. ADJUSTMENT OF THE OBJECTIVE SLIDE

To make the objective slide move parallel to the line of sight in order that the direction of the line of sight will not be changed when the focus is changed. When the transit leaves the maker's hands, the objective slide should be perfectly straight and its line of motion should be perpendicular to the transverse axis of the telescope. If the line of sight has also been made perpendicular to the transverse axis (adjustment III) the line of motion and the line of sight will be parallel. No further adjustment of the objective slide should be necessary at least for a long time. In many instruments no provision is made for this adjustment. In other instruments, however, the rear end of the objective slide moves through a bearing ring which can be adjusted by means of four screws in the same way in which the ring that holds the forward end of an eyepiece is adjusted. (See *C'C'* in the photograph of a telescope and its parts, p. 841.)

If there is an appreciable effect of wear in the objective slide, it can be corrected by the adjustment just mentioned. Results obtained in tests on the objective slides of transits seem to warrant, however, the following conclusions: * "(1) The effect of wear in the objective slide in negligible; (2) the seriousness of this source of error for both transit and level work, in all but exceptional instruments, is slight; and (3) except for work of the highest precision, the adjustment for this condition is unimportant."

Should it be necessary to make the adjustment the following procedure may be followed.

Test. (1) Bring the vertical cross hair exactly on a sharply defined, distant point and set all clamps. (2) Focus the telescope on a finely graduated *horizontal* scale about 10 ft. distant and read the position of the vertical cross hair on this scale. Let R_1 represent this reading. (3) Plunge the telescope, bring the vertical cross hair on the distant point, and set all clamps. (4) Focus the telescope on the horizontal scale and again read the position of the vertical wire. Let R_2 represent this reading. If R_1 and R_2 are not the same, adjustment is needed.

Correction. Bring the vertical cross hair to a position on the scale midway between R_1 and R_2. To do this, move the bearing ring of the objective slide laterally by turning the opposing screws on the sides of the telescope. (Loosen one and tighten the other.) (See Art. 707(*b*) for more explicit directions.) Since this adjustment and the second adjustment of the cross-hair ring (adjustment III) are inter-related, they should be repeated alternately until both are perfected.

The vertical adjustment of the objective slide can be made by a method similar to that just described for the horizontal adjustment except that the horizontal hair is set on the distant point and readings are taken on a vertical scale a short distance from the transit. It is generally better, however, to make the necessary correction, up or down, by adjusting the horizontal cross hair (adjustment VIII) unless it is reasonably certain that the vertical adjustment of the objective slide is necessary.

Suggestion. The adjustment of the objective slide is a delicate one and it should not be undertaken in the field without first taking proper precautions to insure accurate results. It is particularly important to obtain accurate scale readings at the point near the transit. An apparatus for this purpose, used in the experimental tests referred to above, consisted of a substantial wooden frame (made of 2 in. × 4 in. pieces) and two finely graduated scales. The frame was formed by two posts, about 2 ft. apart, set firmly in the ground, and two cross pieces. The cross pieces were so placed that they could hold the two scales in place, one in a truly horizontal position and the other in a truly vertical position. By means of a suitable arrangement, the scales were subject to a small amount of lateral and vertical movement.

739(*r*). ADJUSTMENT OF A STRIDING LEVEL. When a transit is equipped with a striding level, *the axis of the level should be made parallel to the transverse axis of the telescope.*

* See "Two Investigations on Transit Instruments," by William H. Rayner, *University of Illinois Bulletin,* Vol. XXXV, No. 75.

Test. (1) Place the level with its wye supports resting on the transverse axis. (2) Level the transit, bring the transverse axis over a pair of diagonally opposite leveling screws, and by means of these screws center the bubble in the striding level. (3) Take the striding level off the transverse axis, turn it end for end, and replace it on the axis. If the bubble returns to the center, the level is in adjustment.

Correction. (1) If the bubble does not return to the center, bring it halfway back by turning the capstan screw at one end of the level tube. (2) Repeat the test and correction until the adjustment is perfected.

740. Permanent Standards for Testing Adjustments, both of the transit and of the level, should be established where there is sufficient work in adjusting to warrant their use. Each of these standards consists of points permanently fixed in specified positions by monuments, or by permanent marks on a durable surface, or by some other means. The principal standards for testing adjustments of the transit are: (*A*) *Three points exactly in the same straight line,* the center point about 300 ft. from each of the other two and so located that the transit can be set up over it or under it (for adjustment III). (*B*) *Three points in the same vertical plane,* one for the transit point, one high point, and one low point (for adjustment IV). (*C*) *Two points on the ground at exactly the same elevation* and 300 to 400 ft. apart (for adjustment V). If ground conditions do not permit the establishment of two points at exactly the same elevation, two points may be fixed at different, known elevations, provided the difference is not too great for a rod reading on each point when the line of sight is approximately horizontal. Obviously, it may be possible to combine standards *A* and *C* and also to have the transit point and the low point in standard *B* coincide with two points of standard *A*.

740(*a*). SUGGESTIONS. (1) It is an advantage to have each permanent point that is sighted on during the adjustments marked so that the instrument man can sight on it without having a rod or other target held on the point. He can then make adjustments unassisted. For example, a point may be marked by one or more lines on a vertical surface, such as the side of a building, or by a target set on such a surface. When a concrete monument set in the ground is used to mark a point, a small pipe can be inserted in a vertical position to hold a slender ranging pole, or the upright support of a target. When the monument is cast, a hole larger than the pipe should be left, so that after the monument has been set as nearly as practicable at the permanent point, the pipe can be moved in the hole until it is plumb and in the exact position to mark the point. It can then be grouted in that position. If the ranging pole or target is removable, the pipe can be kept plugged when not in use. (2) In establishing permanent points for the adjustment of the transit, it is well to consider the use of some of these points in the adjustment of the level as well. For example, the two points in standard *C* can be used in the "two-peg" adjustment of the level. (3) At permanent points where instruments are set up during adjustment, it may be well to construct small platforms on which to set the tripod, particularly if the ground is not always dry and firm. Each platform may be built of heavy planks set flush or nearly flush with the surface of the ground; there should be some device to keep the tripod shoes from slipping.

740(*b*). ONE OR MORE PERMANENT PIERS may be used instead of tripods for the support of instruments during adjustment. A concrete post with sloping sides is one form of pier. The post should be set well below the sur-

face of the ground, well below the frost line in a cold climate, and its top should be at such an height above the surface that when the instrument is mounted on the top, the telescope will be at a convenient height for sighting. The flat top of the pier should have a device for holding the instrument securely in a fixed position, just as if it were on a tripod head. The position of the pier should be carefully established. For example, in standard *A* the pier should be set at the middle point exactly in line with the other two permanent points. It may be desirable to set a pier at each of these two points also, with each point marked on its pier in such a way as to be clearly visible from the middle pier. In setting permanent piers, their use in the adjustment of levels as well as of transits should be considered. Permanent piers are particularly useful in the adjustment of *precision* transits and levels.

741. Adjustments of the Dumpy Level. Throughout the adjustment of the dumpy level, keep in mind the general suggestions given at the beginning of this chapter, and be careful not to force any screw or to bring any undue strain on any part of the instrument. (See *Caution*, Art. 736(*a*).) Before beginning the adjustment, eliminate parallax (Art. 737).

The principal lines to be considered in the adjustment of a dumpy level are: (1) *The line of sight* from the intersection of the cross hairs to the optical center of the objective. (2) The *line of collimation* with which the line of sight should coincide. (3) The *axis of the level tube*. (See Art. 705.) (4) The *vertical axis* through the spindle of the level. The ultimate object in the adjustment of a dumpy level is to make the line of sight and the axis of the level tube both truly horizontal when the vertical axis through the spindle is truly vertical.

741(*a*). I. Adjustment of the level tube

To make the axis of the level tube perpendicular to the vertical axis so that when the bubble is centered the vertical axis will be truly vertical.

Test. (1) Set up the level, bring the telescope over two diagonally opposite leveling screws, and by means of these screws center the bubble. (2) Rotate the instrument 180° about its vertical axis. If the bubble does not remain in the center of the tube, the tube needs adjustment.

Correction. (1) Bring the bubble halfway back to the center by means of the two diagonally opposite leveling screws. (2) Bring the bubble through the remaining distance to the center by turning the capstan nuts at the end of the level tube—either end if both ends are adjustable. (3) Repeat the test and correction alternately over each pair of diagonally opposite leveling screws until the bubble remains in the center of the tube when the instru-

ment is rotated 360° about its vertical axis. Usually several trials are necessary.

Principle Involved. This is a case of single reversion, and the apparent error is therefore double the true error as explained in Art. 738.

741(*b*). II. First adjustment of the cross-hair ring

To make the horizontal cross hair lie in a plane perpendicular to the vertical axis so that when the instrument is leveled up, the cross hair will be truly horizontal.

Test. (1) Sight one end of the horizontal cross hair on a sharply defined point at least 200 ft. away. (2) Turn the level slowly about its spindle until the point appears to move from one end of the cross hair to the other. If the point appears to stay on the cross hair throughout its length, the hair is in the correct position. It is not necessary to level up the instrument during the test.

Correction. If the point appears to move off the cross hair, change the position of the hair as follows. (1) Loosen slightly all four capstan screws that control the movement of the cross-hair ring. (See Art. 707(*a*).) It is helpful to loosen two adjacent screws a definite amount of rotation (as fifteen minutes on the clock) and subsequently to retighten the *same two screws the same amount of rotation.* (2) Rotate the cross-hair ring into such a position that when the test is repeated, the point appears to stay on the hair throughout its length. (3) Tighten the capstan screws and check the adjustment.

Suggestion. To rotate the ring, hold a screwdriver or a small wooden wedge against one of the curved washers or collets, and tap gently against it.

741(*c*). III. Second adjustment of the cross-hair ring

To make the line of sight parallel to the axis of the level tube so that when the bubble is in the center, the line of sight will be horizontal, regardless of the direction in which the telescope may be pointed.

Test. Either the first or the second "two-peg" method, Art. 742, or the "four-peg" method, Art. 739(*j*).

Correction. Correct by adjusting the *cross-hair ring* as explained in Art. 742(*c*).

The three adjustments as just outlined are for the ordinary

dumpy level in which there is no provision for changing the position of the telescope with respect to the level bar. In such a level, after the axis of the level tube has been made perpendicular to the vertical axis (first adjustment), the level tube should be left alone; otherwise the bubble may move away from the center every time the telescope is pointed in a new direction.

741(*d*). ADJUSTMENT OF SPECIAL TYPES OF DUMPY LEVELS. In some types of dumpy levels the line of sight is made to coincide with the optical axis of the telescope by the manufacturer. The cross-hair ring is then fixed in that position and should not be disturbed. In such a level, the only field adjustment consists in making the *axis of the level tube* parallel to the line of sight. Either the first or second "two-peg" test or the "four-peg" test is used, and the correction is made by adjusting the level tube as explained in Art. 742(*d*).

In some dumpy levels the position of the telescope with respect to the level bar can be changed by raising or lowering the uprights or standards which support the telescope. For such a level the adjustments in order are as follows.

I. *To make the horizontal cross hair lie in a plane perpendicular to the vertical axis.* (First adjustment of the cross-hair ring previously explained.)

II. *To make the axis of the level tube parallel to the line of sight. Test:* Either the first or second two-peg method. *Correction.* Adjustment of the *level tube,* as explained in Art. 742(*d*).

III. *To make the line of sight and the axis of the level tube both perpendicular to the vertical axis.* The test and the correction are similar to those used in *adjustment V* of the wye level (Art. 743(*g*)) except that the provision for raising or lowering the uprights or standards that support the telescope may be different. For example, in some dumpy levels the adjusting screws are on the underside of the level bar, and are to be turned with a screwdriver.

741(*e*). ADJUSTMENTS OF THE DUMPY LEVEL OCCASIONALLY MADE

IV. *To center the eyepiece.* (See Art. 739(*n*).)

V. *Adjustment of the objective slide.* When the objective slide in the telescope of a dumpy level is adjustable, the slide may occasionally require adjustment. The procedure is as follows.

Test. (1) In adjusting the cross-hair ring by the two-peg method the difference in elevation of two points *A* and *B* (Fig. 742), was determined. After the completion of this adjustment, set up the level between *A* and *B* about 15 ft. from *A*. (2) Unscrew the metal sleeve or cover ring that protects the flat-headed screws for adjusting the inner bearing ring through which the rear end of the slide passes (Art. 707 (*b*)). (3) Level carefully and take a rod reading R_B on *B*, which is much farther from the instrument than *A*. (4) Calculate the rod reading R_A at *A* such that the difference between the readings R_B and R_A is equal to the difference in elevation of the two points *A* and *B* as found by the two-peg method. (5) Set the target at the computed rod reading R_A, swing the telescope approximately 180°, center the bubble, and sight at the rod on *A*. If the horizontal hair does not bisect the target, it indicates that in changing from the focus on the distant point *A* to the near point *B*, the axis of the objective slide did not move in or parallel to the optical axis; consequently the slide should be adjusted.

Correction. The correction is one-half the apparent error, but part of this correction is made by adjusting the *bearing ring* of the objective slide, by means of the flat-headed screws (Art. 707(*b*)), and part by adjusting the *cross-hair* ring by means of the capstan screws (Art. 707(*a*)). Just how much each ring should be moved is not readily calculated; hence the correction is made by trial. In making the adjustment, the two rings should be moved in opposite directions.

Note. Since the adjustment of the objective slide may disturb the adjustment of the cross-hair ring previously made by the two-peg method, these two adjustments should be made alternately, until both are correct.

Caution. Before undertaking the adjustment of the objective slide study adjustment IX of the transit, Art. 739(*q*). Note particularly the three conclusions in Art. 739(*q*). The adjustment should not be undertaken at all unless it is really necessary and unless it will be carried through with care.

742. The Two-Peg Method of Adjustment. This method may be used either (1) *in adjusting the cross-hair ring to make the line of sight parallel to the axis of the telescope level tube,* or, vice versa, (2) *in adjusting the level tube to make its axis parallel to the line of sight.* The test is the same in either case; only the method of correction differs.

The two-peg method may be used in adjusting the level tube on the telescope of a transit; or in adjusting the cross-hair ring in the

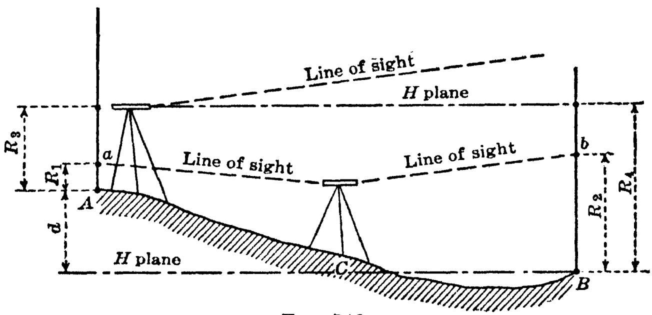


FIG. 742.

telescope of an ordinary dumpy level; or in adjusting the level tube in special types of dumpy levels; or in checking the adjustment of the level tube of a wye level. This method of adjustment is sometimes called the *direct method* in contradistinction to the *indirect method* commonly used in adjusting the level tube of the wye level.

There are two different procedures that may be followed in the two-peg method; one differs from the other only in the method of making the test. There is a difference of opinion as to which is the better procedure. The importance of the adjustment warrants the explanation of both procedures.

742 (*a*). THE TWO-PEG METHOD—PROCEDURE A

Test. (1) Select a place where the surface of the ground is nearly level and drive two stakes *A* and *B* (Fig. 742) 200 to 300 ft. apart. (2) Set up the instrument at *C* midway between *A* and *B*. (3) With the bubble carefully centered, take a rod reading R_1 at *A*. (Point *a* on the rod.) (4) Keeping the bubble centered, take a rod reading R_2 at *B*. (Point *b* on the rod.) The points *a* and *b* are in the same horizontal plane regardless of how much the line of sight may be inclined to the horizontal, i.e., to the axis of the level tube; consequently the difference $R_2 - R_1$ in the two readings will be the true difference in elevation of the points *A* and *B*

however much the level may be out of adjustment. This difference d is shown in the figure. (5) Set up the instrument close to A so that when the rod is held on A, the eyepiece end of the telescope will swing within ½ in. or so of the face of the rod. (6) Point the telescope at the rod on A with the eyepiece toward the rod, and, sighting through the objective end of the telescope, take a rod reading R_3. The cross hairs will not be seen, but the field of view will be so small that a pencil held on the rod by the rodman can be centered accurately in the apparent aperture. The reading thus obtained should be correct within two thousandths of a foot or less. (7) Determine the rod reading R_4 at B on the assumption that the line of sight from the instrument at A is truly horizontal. The difference in the elevation of the two points A and B as found with the instrument at A should be the same as that found with the instrument at C; consequently:

$$R_2 - R_1 = R_4 - R_3 \quad \text{or} \quad R_4 = R_3 + (R_2 - R_1) = R_3 + d$$

In determining the rod reading R_4, the adjuster does not know whether the line of sight slopes upward, as in the figure, or downward, if indeed it slopes at all. Moreover, the point B may be above A instead of below. The equation just given holds true in any case, provided proper algebraic signs are observed. The principle to be kept in mind, however, is that when the instrument is in adjustment, *the difference between the two rod readings R_3 and R_4 taken with the instrument at A must be equal to the difference between the two rod readings R_1 and R_2 taken with the instrument at C*, regardless of the relative elevations of points A and B. (8) Keeping the bubble centered, point the telescope at the rod on B. If the reading differs from the computed reading R_4, the line of sight and the axis of the level tube are not parallel.

742(*b*). SUGGESTIONS. (1) In setting up the instrument at C and at A it is well to have two diagonally opposite leveling screws in the line AB. (2) If, when the instrument is at C, it is turned 360° on a truly vertical axis, an inclined line of sight will generate the surface of a right cone. Since the points a and b lie in this surface equidistant from the vertex, they must be at the same elevation. (3) Check the readings R_1 and R_2 before moving the instrument from C. (4) If two stakes A and B can be driven at exactly the same elevation, R_1 and R_2 become equal and the test is simplified. (5) A *permanent standard* for the test can be established consisting of two points permanently fixed at known elevations, preferably at the same elevation. (See standard C, Art. 740.) Such a standard obviates the necessity of a setup at C. (6) Instead of setting up the instrument close to A, some surveyors prefer to set it up a few feet back of A on the line BA produced and as near to A as good focusing of the telescope on A will permit. Unless the instrument is considerably out of adjustment, this short distance will not appreciably affect the computed value of R_4.

742(*c*). CORRECTION BY ADJUSTMENT OF THE CROSS-HAIR RING. (1) With the instrument still at A, point the telescope toward the rod on B and make sure that the bubble is centered. (2) Move the cross-hair ring up or down by turning the top and bottom capstan screws (Art. 707(*a*)) until the horizontal cross hair cuts the reading R_4. The line of sight will then be horizontal and parallel to the axis of the level tube. For example in Fig. 742, the line of sight through the instrument at A is inclined

upward. By adjusting the cross-hair ring, it is brought downward until it strikes the reading R_4. It will then lie in a horizontal plane. (3) Check by repeating steps 6, 7, and 8 of the test and repeat the correction if necessary.

This method of correction is used when, as in an ordinary dumpy level, there is no adjustment of the instrument by which both the line of sight and the axis of the level tube can be made perpendicular to the vertical axis *after* they have been made parallel to each other.

742(*d*). CORRECTION BY ADJUSTMENT OF THE LEVEL TUBE. (1) After making the two-peg test as previously described and with the instrument still at A, point the telescope at the rod held on B. (2) By means of the leveling screws incline the telescope until the horizontal cross hair cuts the reading R_4. This throws the bubble out, but makes the line of sight lie in a horizontal plane. (3) Raise or lower one end of the level tube by turning the capstan nuts at the end of the tube until the bubble is in the center. (4) Check by repeating steps 6, 7, and 8 of the test, and repeat the correction if necessary.

This method of correction is used when, as in the transit and in the wye level, there is an adjustment of the instrument by which both the line of sight and the axis of the level tube can be made perpendicular to the vertical axis *after* they have been made parallel to each other.

742(*e*). THE TWO-PEG METHOD—PROCEDURE B

Some surveyors prefer a procedure for the two-peg test that is based on the principle of reciprocal leveling. In this method the level is not set up halfway between the pegs, as in the preceding method, but it is set up first very near one stake and then very near the other. Two sets of readings are thus obtained, and the true difference in elevation is obtained by taking the average of the two *differences*, as in reciprocal leveling. (See Art. 179.)

Test. (1) Drive two stakes A and B 200 to 300 ft. apart as in procedure A. (2) Set up the instrument close to one of these stakes, say A, so that when the rod is held on A, the eyepiece end of the telescope will swing within ½ in. of the face of the rod. (3) Point the telescope at the rod on A with the eyepiece toward the rod, and, sighting through the objective end of the telescope, take reading R_1. (4) Keeping the bubble centered, take a reading R_2 on B. (5) Set up in a similar position close to B and take a reading R_3 on B by sighting through the objective end of the telescope. (6) Keeping the bubble centered, take a reading R_4 on A. (7) The difference in elevation of A and B as determined with the instrument at A is the difference in the rod readings R_1 and R_2. Let d_1 represent this difference. The difference in elevation of A and B as determined with the

instrument at B is the difference in the rod readings R_3 and R_4. Let d_2 represent this difference. If the differences in elevation, d_1 and d_2 are not equal, the line of sight and the axis of the level tube are not parallel. In this case the true difference d in elevation is the average of d_1 and d_2 or $d = \frac{1}{2}(d_1 + d_2)$. Compute the value of d. (8) Let R_5 represent the reading on the rod at A when the line of sight from the instrument at B is truly horizontal. Compute the value of R_5 by adding d to or subtracting d from R_3. (See step 7 in procedure A.)

Note. If a permanent standard for this test is established, only one setup is required. (See suggestion 5, procedure A.)

Correction. With the position of the instrument unchanged at B and the bubble centered, move the cross-hair ring up or down until the horizontal cross hair cuts the reading R_5 on a rod at A. (See adjustment of cross-hair ring in Art. 742(*c*), procedure A.) If it is desired to adjust the level tube instead of the cross-hair ring, bring the horizontal cross hair (and line of sight) on the reading R_5 at A, and adjust the level tube as explained in 742(*d*).

Note. The four-peg test explained in Art. 739(*j*) may be used instead of the *two-peg* test. Whatever may be the test for establishing a truly horizontal line of sight, one of the two methods of correction explained in Arts. 742(*c*), 742(*d*), procedure A, is used in making the adjustment.

743. Adjustments of the Wye Level. These are the *principal lines* to be considered in adjusting a wye level. (1) The *line of sight* from the intersection of the cross hairs to the optical center of the objective lens. (2) The *line of collimation,* with which the line of sight should be made to coincide. (3) The *axis of the telescope collars* or *rings,* a line joining the centers of the collars. (4) The *axis of the level tube* (Art. 705). (5) The *axis through the bottom of the wyes,* a line tangent to the inner surfaces of both wyes and passing through the lowest points in those surfaces, i.e., where the bottoms of the telescope collars rest. (6) The *vertical axis* through the spindle of the level. For the sake of clearness another line perpendicular to this axis will also be considered, namely: (7) The *axis of the level bar,* a theoretical line of reference that is perpendicular to the vertical axis and extends lengthwise through the level bar. It does not necessarily coincide with the geometric axis of the bar.

743(*a*). NOTE. The term "axis of the wyes" is commonly used in connection with the adjustment of the wye level, but unfortunately it is often employed as if it were synonymous with "axis of the collars" and sometimes as if it were synonymous with "axis through the bottom of the wyes." The term is synonymous with neither and little if anything is gained by its use.

743(*b*). The principal adjustments

Let Fig. 743 represent a level that is out of adjustment; the errors, of course are shown greatly exaggerated. The three principal adjustments may be outlined as follows.

The adjustment of the cross-hair ring:

I. To make the horizontal cross hair truly horizontal.

II. To make the line of sight coincide with the axis of the collars. This adjustment will also make the line of sight parallel to

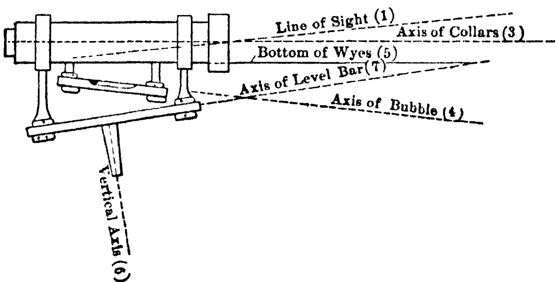


Fig. 743.

the axis through the bottom of the wyes, provided the two telescope collars are true circles and of the same diameter.

The adjustment of the level tube:

III. To bring the axis of the level tube in the same plane with the axis of the collars.

IV. To make the axis of the level tube parallel to the axis of the bottom of the wyes.

The adjustment of the wyes:

V. To make the axis of the level tube parallel to the theoretical axis of the level bar and consequently perpendicular to the vertical axis through the spindle.

In a well-made level, in which not only are the telescope collars true circles of the same diameter but also their outer surfaces fit accurately into the inner surfaces of the wyes, the line of sight, the axis of the collars, the axis of the level tube, and the axis through the bottom of the wyes are all made parallel by the adjustment of the cross-hair ring and the adjustment of the level tube. All four of these axes are then made parallel to the theoretical axis of the level bar by the adjustment of the wyes; consequently, all four will then be perpendicular to the vertical axis through the spindle which is the ultimate object in the adjustment of a level.

Order of Adjustments. The order in which the adjustments of the wye level are given here is a common one, but there is no general agreement as to what order is best.

General Suggestions. Throughout the adjustment of the level, keep in mind the general suggestions given at the beginning of this chapter. Be careful not to force any screw or to bring any undue strain on any part of the instrument. (See *Caution,* Art. 736(*a*).) Before beginning the adjustment, eliminate parallax (Art. 737).

743(*c*). I. First adjustment of the cross-hair ring

To make the horizontal cross hair truly horizontal, i.e., to lie in a plane perpendicular to the vertical axis through the spindle.

This adjustment is exactly the same as the corresponding adjustment of the dumpy level (adjustment II, Art. 741(*b*)).

Note. In some wye levels, the adjustment is made by rotating the telescope in the wyes and then fixing it in the correct position by an adjustable stop, instead of by adjusting the cross-hair ring.

743(*d*). II. Second adjustment of the cross-hair ring

To make the line of sight coincide with the axis of the collars.

Test. (1) Loosen the clips which hold the telescope, direct the line of sight to some well-defined point, and clamp. (2) Rotate the

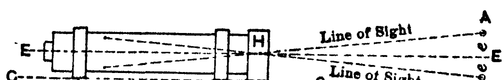


Fig. 743(II).

telescope halfway around in the wyes (not end for end) until the level tube is above the telescope. If the line of sight does not strike the point the cross hairs should be adjusted. In the final test the intersection of the cross hairs should remain on the point throughout a complete rotation of the telescope.

Correction. If both hairs are off the point, each must be adjusted by bringing it *halfway* back. This is done by moving the cross-hair ring, loosening the upper or lower capstan screw and tightening the opposite screw for the horizontal hair, loosening a side capstan screw and tightening the opposite screw for the vertical hair. (See Art. 707(*a*).) Repeat the test and adjustment until the latter is perfected.

The principle involved is that of single reverse. If the line of sight is directed to *A,* Fig. 743(II), and the telescope is then ro-

tated, the line of sight will strike some point at *B*. The angle *AHB* is twice the error. In like manner the distance that the vertical hair is to the right or to the left of the point, after reversion, is double the error.

Practical Suggestions. (1) It is not necessary that the telescope should be horizontal while making this adjustment so long as the wyes remain in a fixed position. Any two points of support could be used in place of the wyes. (2) A nail in a fence or building about 300 ft. away, or, better yet, a piece of paper tacked up with a fine cross marked on it, will serve for a point to sight at if no other definite point is visible. (3) Loosen the capstan screw *towards* which a cross hair should be moved (as observed through the erecting eyepiece), and tighten the opposite screw. (See *Suggestion* 7, Art. 736.) If the ring is to be moved a considerable distance, it is well to loosen slightly a third screw before tightening the second screw. (4) The capstan screws beyond the inner end of the eyepiece control the cross-hair ring (*DD* in the photograph, p. 841). (5) The adjustment for one hair may disturb the adjustment of the other. (6) When the adjustment has been completed for a point 300 ft. away, focus on a point near the instrument, say 20 ft. away, and repeat the test. If the instrument is out of adjustment for this second point, either the objective slide does not work parallel to the axis of the collars or the optical center is not in the axis of the collars. On some makes of levels the objective slide can be adjusted (see Art. 739(*q*)), but the latter defect can be remedied only by the maker.

743(*e*). III. First adjustment of the level tube

There are two adjustments of the level tube, neither of which, theoretically, is entirely independent of the other.

To bring the axis of the level tube into the same plane with the axis of the collars. (The axis of the level tube will then be in the same plane with the line of sight, provided adjustment II has been perfected.)

Test. (1) Loosen the clips, level up, and clamp the telescope over two diagonally opposite leveling screws. (2) Rotate the telescope a few degrees in its wyes (not end for end), until the level tube is no longer underneath the telescope. If the bubble does not remain in the center, the tube should be adjusted. In the final test the bubble should remain in the center when the tube is swung to either side.

Correction. Bring the bubble *all* the way back to the center by turning the lateral adjusting screws on each *side* of the level holder usually at the end of the tube toward the objective.

Principle Involved. There is no reversion during the test; hence the *apparent* error is equal to the *true* error, not double it.

Practical Suggestions. (1) Since in most instruments the telescope is not free to turn, being held in place by a pin in one of the clips, this adjustment is not important except as it prepares the way for the second step, in which the clips are left open. (2) If, when the telescope is rotated first to one side then to the other, the bubble travels toward the *same* end, it shows that the level tube is conical instead of cylindrical and the adjustment will not correct it. (3) Before beginning the adjustment, put on the sunshade. This is

not of much importance in this adjustment; but in some of the following adjustments it is, since the maker usually plans to have the telescope in perfect balance when the sunshade is on and the eyepiece and objective each at the average focus. (4) In some instruments it is necessary to loosen the capstan nuts used in the next adjustment before the screws used in this adjustment can be turned.

Remark. Assuming that the axis of the tube and the axis of the collar are in the same vertical plane, the bubble will still move toward one end of the tube when the telescope is rotated unless the axis of the tube is parallel to the axis of the collar and to the axis through the bottom of the wyes (the object of the next adjustment). Hence it may be necessary to repeat adjustments III and IV until the two corresponding tests are satisfactory.

743(*f*). IV. Second adjustment of the level tube

To make the axis of the level tube parallel to the axis through the bottom of the wyes. (The axis of the level tube will then be parallel to the line of sight, provided adjustment II has been perfected.)

Test. (1) Level the instrument carefully and clamp the telescope over two diagonally opposite leveling screws. (2) Loosen clips, take the telescope out of its wyes, and replace it *turned end for end.* If the bubble does not return to the center, the tube should be adjusted.

Correction. Bring the bubble halfway back to the center by lowering the end of the tube towards which the bubble has moved, or, what is equivalent, by raising the other end. This is done by turning the capstan *nuts* which hold one end of the tube to the telescope. Level up again, and repeat the test and adjustment until reversion can be made without causing any change in the position of the bubble.

Principle Involved. This is a case of single reversion, exactly like that explained in Art. 738. Before reversion the *axis of the bubble BB* in Fig. 743(IV) is horizontal, but *not* parallel to *CC,*

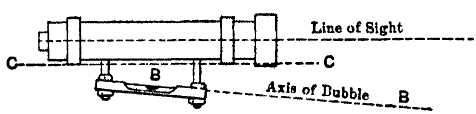


Fig. 743(IV).

the bottom of the collars; or, since the bottom of the collars and the bottom of the wyes are in contact, *BB* is not parallel to the bottom of the wyes, also represented by *CC.* The bottom of the wyes *CC* is kept stationary during the test, and therefore corresponds to *ac,* Fig. 738(*a*), p. 883. Hence turning the level tube end for end doubles the error.

Practical Suggestions. (1) Great care is necessary in taking the telescope out and replacing it in the wyes to avoid jarring or disturbing the rest of the instrument. (2) Loosen the upper capstan nut first, raise or lower the lower nut as the case requires, and finally bring home the upper nut in completing the adjustment. On some instruments the nuts are at one end of the level tube only. (3) Notice that this adjustment, like the adjustment of the cross hairs, could be perfected on any two points of support independent of the wyes. (4) Although by this adjustment the axis of the bubble is made parallel to the bottom of the wyes, it does not follow that is is also parallel to the *line of sight,* even though the latter has been made by the adjustment of the cross hairs to coincide with the *axis of the collars,* for the collars may not be true circles of equal diameter. If they are not, this method of adjustment fails and the "peg" method must be used. (See Art. 742.)

743(*g*). V. Adjustment of the Wyes

To make the axis of the level tube parallel to the axis of the level bar, and thus to make all the principal lines of the level perpendicular to the vertical axis. Practically, the object is to get the two wyes of equal height with reference to the axis of the level bar.

Test. (1) Level up, bring the telescope over two diagonally opposite leveling screws, and see that the bubble is exactly in the center. (2) Revolve the telescope on its spindle halfway around or until it is over the same pair of leveling screws, but reversed end for end. If the bubble does not remain in the center the wyes should be adjusted. In the final test the bubble should remain in the center during a complete revolution of the telescope.

Correction. Bring the bubble halfway back by turning the large capstan nuts, i.e., those at the ends of the level bar, thus changing the heights of the wyes above the axis of the level bar until they are equal. Level up the instrument again, and repeat the test and the adjustment until the position of the bubble is not changed by revolving the telescope on its spindle.

Caution. On some wye levels there are one or two small set screws in the uprights at the end of the level bar. They must be loosened before the capstan nuts can be turned.

Principle Involved. This is also a case of single reversion, exactly like that explained in Art. 738; it is also exactly similar to the second adjustment of the transit. The level tube is attached to the level bar by the supporting wyes. Before reversion the axis of the bubble is horizontal, but since the inclination of the level bar *HH,* Fig. 743(V), does not change during revolution it corresponds to *ac,* Fig. 738(*a*), and reversion doubles the error.

Note. Notice that in Fig. 743 (V) the line of sight (which coincides with the axis of the collars), the *axis YY* through the bottom of the wyes, and the axis of the level tube are all parallel (from the preceding adjustments), and that this last adjustment will make the axis *HH* of the level bar parallel to these other axes. Since, by definition, *HH* is perpendicular to the axis

through the spindle, all axes will then be perpendicular to the spindle; and when the axis of the spindle is vertical, all other axes will be horizontal.

Practical Suggestions. (1) When the other adjustments of the level are "out," it is not apparent unless tests are made; but when this last adjustment is "out" it is known at once, since the bubble will not remain in the center. For this reason undue importance is too often attached to this adjustment. In reality it does not affect the accuracy of the work, provided the bubble is brought to the center of its tube just before each sight is taken. The adjustment is made simply as a matter of convenience to save this continual leveling up. A good levelman will not depend upon it, but will watch the bubble

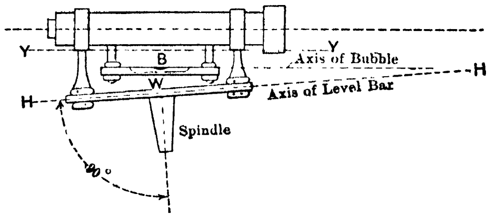


FIG. 743(V).

instinctively and always bring it to the center before taking any important sight. (2) It is well to make the first test with the telescope over one pair of leveling screws, the next test over the other pair, alternating until the adjustment is nearly perfected, when it may be completed over a single pair of screws. (3) Loosen the upper nut, then raise or lower the lower nut as the case requires. Though nuts may be provided at each end of the bar, unless the adjustment is badly out, it can usually be made wholly by the nuts at one end.

743(*h*). NOTE. When the collars or the wyes are worn or are inaccurate from any other cause, the defect may not be disclosed by the indirect test used in adjustment IV. For this reason, some surveyors prefer always to adjust the wye level as if it were a dumpy level, using the direct test of the two-peg method. To do this, however, is to sacrifice one of the chief advantages of the wye level, namely, the provision for testing the adjustment of the bubble tube at any time by merely reversing the telescope in the wyes. After a wye level has been adjusted for the first time in the field by the usual methods, it should be tested by the two-peg method. If it is found to be in correct adjustment, it may be adjusted thereafter by the usual methods, with an occasional check by the two-peg method. Should the collars or the wyes become inaccurate, the level should then be adjusted as if it were a dumpy level.

743(*i*). ADJUSTMENTS OF THE WYE LEVEL OCCASIONALLY MADE

VI. *To center the eyepiece.* See Arts. 739(*n*) and 707(*c*).

VII. *Adjustment of the objective slide.* When the objective slide in the telescope of a wye level is adjustable, it may occasionally require adjustment. The procedure is as follows.

Test. (1) Set up the level, remove the wye pins from the clips, and raise the clips so that the telescope is free to rotate in the wyes. (2) Check adjustment II, Art. 743(*d*), by sighting at a distant point. (3) Unscrew the metal sleeve or cover ring that protects the screws for adjusting the inner bearing ring through which the rear end of the objective slide passes. (4) Set the intersection of the cross hairs on a sharply defined point about 15 ft. from the telescope, and clamp to hold the axis of the telescope in this position. (5) Gently turn the telescope halfway round in its wyes. If the intersection of the cross hairs does not remain on the point, the objective slide should be adjusted.

Correction. The method of correction is exactly the same as that explained for the adjustment of the objective slide in the telescope of a dumpy

level (Art. 741(*e*)). Since this adjustment may disturb the adjustment of the cross-hair ring previously made in adjustment II, these two adjustments should be made alternately until both are correct.

Caution. Before undertaking this adjustment, study adjustment IX of the transit, Art. 739(*q*). Note particularly the three conclusions in that article. The adjustment should not be undertaken at all unless it is really necessary and unless it will be carried through with care.

744. Adjustment of the Hand Level. *To make the line of sight horizontal when the bubble is centered.*

Test. (1) Select two rigid supports 50 to 100 ft. apart on nearly level ground. (2) Hold the hand level or clinometer at a point *A* on one support (Fig. 744), center the bubble, and mark the point *B* when the line of sight strikes the second support. (3) Hold the level or clinometer at the point *B* and mark the point *C* where the line of sight strikes the first support. If *C* does not coincide with *A*, adjustment is needed.

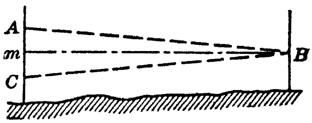


FIG. 744.

Correction. Mark a point *m* midway between *A* and *C*. The line *mB* will be horizontal. (*a*) To adjust a Locke hand level hold the level at *B*, center the bubble, and by means of the screws that move the cross hair (not the level tube), bring the line of sight on *m*. (*b*) To adjust the Abney hand level with clinometer, set the index at zero on the graduated arc, hold the instrument at *B*, sight at *m*, and adjust the level tube by raising or lowering one end until the bubble is centered.

Suggestions. (1) A nail driven horizontally part way into the side of a post, a telephone pole, a tree, or a building can be used as a support for the level, or, better still, a small wooden block can be nailed to the side of the supporting object. Two instrument tripods are sometimes used as supports. (2) A final check on either adjustment *a* or adjustment *b* is to hold the level at *m*, sight at *B*, and observe if the bubble is centered.

745. Adjustments of the Plane-Table Alidade. The adjustments of the plane-table alidade are so nearly like some of those already explained for the transit and the level that it is unnecessary to explain each in detail. The accuracy with which each adjustment should be made can be judged by considering the effect which a small error in that adjustment would have on the accuracy of a plane-table map. In general, it is unnecessary to make the adjustments of the alidade with the same refinement that is required for the transit and the level. There are two types of alidades, the *fixed-tube* type and the *tube-in-sleeve* type. (See Art. 731(*b*).) The adjustments for the two types are nearly the same

745(*a*). Adjustments of the fixed-tube alidade

I. *Adjustment of the plate levels to make the axes of the levels parallel to the plate.* This corresponds to the adjustment of the plate levels of the transit, Art. 739(*e*). There may be one circular spirit level or two level tubes. (1) Level the plane-table board by manipulating it in the metal socket of the tripod. (2) Set the alidade on the board with its center of gravity approximately over the center of the plane table, draw a guide line across each end of the alidade, and center the bubble or bubbles. (3) Turn the alidade end for end, and fit it between the two guide lines, thus using the same area of the board. (4) If a bubble is off center, bring it halfway back by turning the adjusting screws that move the tube, or, if it is a circular level, by turning one or more of its three adjusting screws. Bring the bubble the remainder of the way by leveling the board. (5) Turn the alidade 90° from its original position, level the board, and repeat the adjustment. (6) Continue to repeat the adjustment with the alidade alternately in one position and then in the other until the board is level "all around."

II. *Adjustment of the cross-hair ring to bring the vertical cross hair into a plane perpendicular to the transverse axis of the telescope.* This adjustment is the same as the adjustment of the cross-hair ring explained in *adjustment I* of the transit (Art. 739(*d*)).

III. *Adjustment of the telescope level tube to make the axis of the tube parallel to the line of sight.* This adjustment is the same as the adjustment of the telescope level tube explained in *adjustment V* of the transit (Art. 739(*i*)).

IV. *Adjustment of the vertical vernier to make it read zero when the line of sight is horizontal.* This adjustment is the same as the adjustment of the vertical circle vernier explained in *adjustment VI* of the transit (Art. 739(*l*)).

V. *Adjustment of the vernier control level.* When the bubble of telescope level is centered and the vertical vernier reads zero, the bubble of the control level should be at the center of its tube. If it is not, bring it to the center by means of the adjusting screws at the end of the tube.

745(*b*). Adjustment of the sleeve-in-tube alidade

The adjustments of the sleeve-in-tube alidade are the same as those of the fixed-tube alidade except that adjustment III (the two-peg method of adjusting the telescope level) is replaced by the two following adjustments.

III. *Second adjustment of the cross-hair ring. To make the line of sight coincide with the axis of the telescope sleeve.* This adjustment is the same as the second adjustment of the cross-hair ring of a wye level (adjustment II, Art. 743(*d*)), except that instead of rotating the telescope in the wyes it is rotated 180° in the telescope sleeve.

III*A*. *Adjustment of the striding level to make its axis parallel to the axis of the telescope sleeve and consequently to the line of sight.* (1) Attach the striding level to the telescope and bring the bubble to the center of its tube. (2) Remove the level from the telescope and replace it turned end for end. (3) If the bubble does not return to the center, bring it halfway back by means of the adjusting screws at one end of the tube. (4) Again center the bubble and repeat the adjustment until it is perfected. Note that this adjustment is similar to the second adjustment of the level tube of a wye level, adjustment IV, Art. 743(*f*).

745(*c*). NOTES. (1) Some surveyors prefer to adjust the sleeve-in-tube alidade exactly as if it were a fixed-tube alidade. It may generally be assumed, however, that the telescope fits accurately the telescope sleeve, that the telescope collars on which the inverted-wye supports of the striding level rest are circular and of the same diameter, and that consequently after adjustments III and III*A* have been completed, the objects of these two adjustments have been accurately attained. (2) The telescope of an alidade moves through such a small vertical arc that if the line of sight is not exactly perpendicular to the transverse axis of the telescope, the resulting error is generally negligible. (3) In a well-made alidade it may be assumed that the ruling edge of the blade is straight and that the transverse axis of the telescope is parallel to the plane of the blade. (4) It is desirable, though not essential, to have the line of sight as nearly as practicable in the vertical plane through the ruling edge of the blade or parallel to that plane.

746. Adjustment of the Beaman Stadia Arc. *To make the stadia arc index read 50 when the line of sight is horizontal.* (See Art. 429.) (1) Clamp the telescope in such a position that the line of sight is horizontal. (2) If the V reading is not exactly 50, loosen slightly the screws that hold the index plate and with light blows against a wooden wedge drive the index plate over until the reading is exactly 50. (3) Tighten the holding screws. Take care to set the index plate far enough away from the arc to keep it from rubbing the silver.

Note. The procedure just described is for the adjustment of the form of stadia arc generally used on the plane-table alidade. It may be necessary to modify the procedure slightly for other forms of stadia arcs, such, for example, as the full-circle arc often used on transits.

747. Adjustment of the Gradienter. (See Art. 181.) *To make one revolution of the gradienter screw cause the horizontal cross hair to appear to move exactly 1 ft. over a vertical surface 100 ft. from the supporting axis of the telescope.* (1) Set up the

instrument with the transverse supporting axis of its telescope at a measured distance of 100 ft. from a vertical surface, such as a side of a building. (2) Determine the point where the horizontal line of sight strikes the vertical surface. (3) Draw two distinct but fine horizontal lines on the vertical surface, one 6 in. above, the other 6 in. below the point established in (2). (Lines may be drawn on white paper or cardboard fastened to the vertical surface.) (4) Set the horizontal cross hair on the lower mark, using a right-hand or pushing motion of the screw. Read the gradienter drum. (5) Turn the gradienter screw until the cross hair is on the upper mark (one foot from the first mark). Read the drum again. If the drum has turned exactly one revolution and the reading is the same as before, the gradienter is in adjustment. (6) If the drum has turned more than one revolution, loosen the nut, turn the eccentric bushing in the direction to raise the gradienter screw closer to the supporting axis of the telescope, and tighten the nut. If the drum has turned less than one revolution, the eccentric nut should be turned in the opposite direction so as to lower the screw. (7) Repeat the adjustment until the drum turns exactly one revolution in moving the horizontal cross hair from the lower to the upper mark. (8) If it is necessary to readjust the space between the index scale and the drum, loosen the screw, turn the index holder until the desired space is obtained, and tighten the screw.

Suggestions. (1) A rod with two well-defined marks exactly one foot apart (or with two targets set one foot apart) may be used instead of a vertical surface. When the telescope is level it should point halfway between the two marks. (2) It may be difficult to obtain a single reading sufficiently accurate for the test. A more accurate result may be obtained by taking the average of the readings obtained from a number of settings. (3) In the test for this adjustment, as in the collimation adjustment, accurate focusing is important.

748. Adjustments of the Compass. The adjustments of the compass are four in number; they affect the following parts: I. Levels. II. Sights. III. Needle. IV. Pivot point.

I. ADJUSTMENT OF THE LEVELS. *Object. To make the axes of the level tubes perpendicular to the vertical axis.*

Test and Correction. Exactly the same as for the similar adjustment of the transit (Art. 739(*e*)), except that after bringing a bubble halfway back it is brought the rest of the way by turning the instrument on the ball-and-socket joint. On some compasses a leveling head like that on transits is furnished; then the adjustment is exactly the same.

II. ADJUSTMENT OF THE SIGHTS. *Object. To bring the sights into a vertical plane.*

Test. Level up and sight at a suspended plumb line, first through one of the sight vanes and then through the other, turning the instrument so that each sight vane in turn is nearer the plumb line than the other. If the slit in either is not vertical the bottom of the vane must be refitted to the plate.

Correction. Either file off the high side of the bottom of the sight vane or insert a piece of paper under the low side.

Remark. Ordinarily this adjustment should not be necessary. It is not essential that the plane of the sights should be exactly in the plane of the vertical axis.

III. Adjustment of the needle. *Object. To straighten the needle so that its two ends shall lie in the same vertical plane with its center.*

Balancing the Needle. Before straightening the needle, it may be necessary to balance it on its pivot. Level up the instrument, lower the needle on its pivot, and remove the compass glass. Note which end of the needle is high, remove the needle, slide the counterweight along the needle toward the high end, and replace the needle on its pivot. Repeat, if necessary, until the needle balances.

Test for Straightening the Needle. With the instrument level, the compass glass removed, and the needle balanced, bring the south end of the needle exactly opposite some prominent graduation of the circle, as the zero or the 90° mark, and read the north end. Turn the needle on its pivot end for end, bring the north end to the same graduation mark, and read the south end. If the two readings do not agree, the needle is not straight.

Correction. Remove the needle from the pivot and straighten it. The correction should be for one-half the difference of the two readings, but as this must be estimated, the needle should be replaced on its pivot and the test and correction repeated until the needle is straight.

Suggestions. (1) To remove the compass glass, press the palm of the hand flat on the glass and turn counterclockwise. If the glass is tight, tap gently around the bezel ring with a piece of wood to loosen the grip of the threads. On some compasses the glass is held in place by a spring bezel. Release the spring with the point of a screwdriver, place the palm of the hand on the glass, turn the compass upside down, and the glass should drop into the hand. (2) Instead of turning the needle on its pivot, the whole instrument may be turned if preferred to bring each end of the needle in turn to the same graduation mark. (3) In addition to the above adjustment it may be necessary to bend the ends of the needle slightly up or down, so that they shall lie in the same plane as the horizontal circle—or, better still, a trifle above. If, however, this brings the ends above or below the plane of the pivot point the ends of the needle are likely to be unsteady. Continued resharpening of a pivot may bring its point much too low; then the only remedy is a new center pin.

IV. Adjustment of the pivot point. *Object. To bring the pivot point exactly in the center of the graduated circle.* (1) Bring

the north end of the needle exactly opposite the north zero mark of the circle. If the south end of the needle does not read zero, correct for the whole error by bending the center pin slightly in a direction at right angles to the line between the north and south zeroes. (2) Rotate the needle 90° to bring its north end exactly opposite a 90° mark. If the south end of the needle does not read 90°, correct for the whole error by bending the pin in a direction at right angles to the line between the two 90° marks. Repeat the procedures in (1) and (2) alternately until the readings for the opposite ends of the needle agree for any position of the needle.

Suggestions. (1) To bend the pivot apply the little brass wrench, or the flat pliers usually furnished with the instrument, about one-eighth of an inch below the top of the pivot. (2) This adjustment should be made after the needle has been straightened in III.

749. Adjustments of the Sextant. I. ADJUSTMENT OF THE INDEX GLASS. *Object. To make the index glass perpendicular to the plane of the graduated arc.*

Test. Hold the sextant with the arc away from the body and move the vernier to a point between the 20° and 40° marks. Hold the eye just above the plane of the arc and slightly to the right of the index glass. Look obliquely into the right-hand side of the glass in such a way as to see a portion of the arc both directly and by reflection at the same time. If the direct and reflected portions do not appear to form a continuous curve the index glass is not perpendicular to the plane of the arc.

Correction. Change the inclination of the glass by means of the screws at its base, or by whatever means have been provided on that particular instrument, until the above test is fulfilled. It may be necessary to insert thin paper under the base. If the reflected image is below the arc, the glass is leaning backward; if above, forward.

II. ADJUSTMENT OF THE HORIZON GLASS. *Object. To make the horizon glass perpendicular to the plane of the graduated arc.* This must be done after the index glass has been adjusted.

Test. Point the telescope to any well-defined object and move the index arm slowly back and forth over the zero of the graduated arc. The reflected image will be seen moving back and forth. If it passes exactly over the direct image the two mirrors are perpendicular to the plane of the arc; otherwise the horizon glass should be adjusted.

Correction. Turn the screws on the back of the horizon glass until the above test is fulfilled.

Suggestions. A star is the best object to sight at, although the sun will also answer the purpose. In case the latter is used it is well to make the two images of different colors by use of the glass shades. A clear-cut line of some terrestrial object may be used—as, for example, a vertical edge of a tall chimney, half a mile or more away. In that case the vertical line should appear straight and unbroken.

III. ADJUSTMENT OF THE TELESCOPE. *Object. To make the line of sight parallel to the plane of the graduated arc.*

Test. Place the sextant face upward on a table or some other firm and level support, point the telescope at some vertical surface about 20 ft. away. Place two objects of equal height on the extreme ends of the limb to serve as temporary sights. Sight across the tops of these objects and mark a point where the line of sight strikes the vertical surface. Near this point mark another point where the line of sight through the telescope strikes the same surface. The difference in elevation between these two points should equal the difference in elevation between the axis of the telescope and the line across the top of the two temporary sights, measured at the sextant; otherwise the telescope must be adjusted.

Correction. Tip the ring or collar that carries the telescope by means of the proper screws until the above test is fulfilled.

Suggestions. If the temporary sights are made of the same height as the telescope axis the two points on the vertical surface should coincide. With some instruments two peep sights are furnished for this purpose. A discrepancy of ½ in. in 20 ft. would cause a maximum error of only 1″ of arc.

IV. INDEX ERROR. *Object. To determine the reading of the arc when the direct and reflected image of an object coincide.*

Test. Bring the direct and reflected image of a distant point into coincidence. If the vernier does not read zero the error is called the *index error.*

Correction. The error may be corrected by turning the horizon glass around an axis perpendicular to the plane of the graduated arc. Unless the error is large, however, it is better not to correct it because the horizon glass is so easily moved out of adjustment. Instead, it is safer to determine the index error from time to time and then to apply it as a correction for each reading when measuring angles. It should be noted whether an index error is to be added or subtracted from all angle readings to which it is applied.

Suggestions. If the vernier zero is to the left of the zero of the graduated arc the index error is *minus*—i.e., it must be subtracted from all readings; if to the right, it is *plus.* Since this index error must be applied as a correction to every angle reading it should be determined with great care. For terrestrial measurements it will suffice to sight at some distant line, as the edge of a chimney. (See suggestion, second adjustment.) For astronomical observations the error should be determined with greater accuracy. For example, the direct and reflected images of the sun may be brought externally tangent to each other in each of the two possible positions. Call the reading

to the right *minus*, the one to the left *plus*, and algebraically *divide by two*. The result is the index error with the proper sign. For greater accuracy, repeat the process at least two or three times and take the mean of all the results. In taking readings to the right of the zero, be careful to reverse mentally the numbering of the graduations on the vernier.

750. Adjustments of the Smith Solar Attachment.* The adjustments given here are those recommended by W. and L. E. Gurley for telescopic solar transits constructed in accordance with the specifications of the U.S. General Land Office. Since the various models of other instrument makers differ only slightly in details of construction rather than in basic principles of design, the surveyor should have no difficulty in adapting these directions to the adjustment of his own instrument. (For a photograph of a telescopic solar transit (Smith Solar Attachment), see p. 600.)

I. Transit Adjustments

1. Check adjustments of the transit itself for collimation, telescope level, and plate levels before attempting to adjust the solar attachment.

II. Focusing Solar Telescope

1. Focus on the cross wires by rotating the eyepiece cap or knurled ring nearest the eye.
2. Focus the objective by pulling out on the tube which holds the eyepiece. Use care in focusing to eliminate parallax.

III. Equatorial Wires

To make equatorial wires parallel to the axis of the reflector.

1. Set off approximate latitude, declination, and hour angle.
2. Orient to approximate meridian and view reflected image of sun.

If sun is not visible, any other well-defined distant point may be used.

3. Rotate solar telescope back and forth in hour angle, that is, in its collar bearings, and the image should follow the wires from side to side.
4. If it does not, unscrew the cover ring just back of the knurled head and lightly tap the cross-wire screws to rotate the cross-wire ring. Move ring until the image of the sun follows the wires.

IV. Collimation of the Solar Telescope

To make the line of sight pass through the axis of the collar bearings.

1. Unclamp vernier arm of declination arc and swing the mirror parallel to the line of sight, thus giving a direct view through the auxiliary telescope.
2. Sight on a distant point, then rotate the telescope 180° in its collar bearings. Note whether cross-wire intersection stays on point.
3. If not, move the *telescope* until the error seems to be one-half corrected.
4. Move the *cross-wire ring*, using each pair of opposite capstan screws successively, until the error is entirely corrected and the cross-wire intersection now covers the point.
5. After moving the cross-wire ring, check the adjustment of the equatorial wires and the collimation alternately until both are satisfied.

V. Polar Axis

To make the polar axis perpendicular to the latitude axis.

1. Unclamp vernier arm of declination arc and swing the mirror parallel to the line of sight, thus giving a direct view through the auxiliary telescope.
2. With solar telescope approximately level and with one cross-wire vertical, set on point about 200 ft. distant. Be sure the collimation adjustment, section IV, is correct.

* Through the courtesy of W. and L. E. Gurley, the directions for these adjustments are taken from the *Gurley Manual of Surveying Instruments.*

3. Transit solar telescope 180° about latitude axis and set point an equal distance to the rear.
4. Rotate transit 180° about main axis and set on point measured 8.95 in. to the left of the first point (double the offset distance of solar telescope).
5. Again transit solar telescope 180° about latitude axis and set point beside the second point.
6. Measure distance between second and last points which should be exactly 8.95 in.
7. If not, set a new point halfway between the last point and 8.95 in. from the second point.
8. Bring the cross wires to the new point by adjusting the three screws which are in the short arm on the upper end of the latitude arc. The center screw pulls the eye end closer to the transit; the outside screws oppose the center screw and push the eye end farther away.

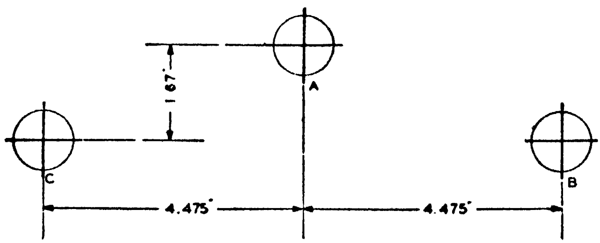


FIG. 750.

VI. LATITUDE AXIS

To make the latitude axis parallel to the main telescope axis so that the auxiliary solar telescope will trace a plane parallel to the collimation plane of the main telescope.

1. Prepare a target in accordance with the dimensions given in Fig. 750.*
2. Mount target on wall of building about 200 ft. distant, and approximately at the level of the instrument.
3. Set up transit, leveling it carefully. Make sure that the collimation of the main telescope is in adjustment, also sections IV and V of the solar telescope. Move vernier arm to give clear vision.
4. Set cross-wire intersection of main telescope on target *A*.
5. Cross-wire intersection of auxiliary solar telescope should coincide with target *B*.
6. If not, adjust the capstan nuts holding the lower corner of the triangular base to the standards until the telescope wires are in coincidence with the target.
7. Sight solar telescope on high point on wall.
8. Depress telescope and set point on wall near the ground.
9. Rotate instrument 180° in azimuth and transit solar telescope. Sight on lower point on wall.
10. Elevate telescope and sight on first point. Cross wires should intersect point. If not, correct for half the error by adjusting the capstan nuts at the upper corner of the triangular base.

VII. ZERO OF LATITUDE ARC

To make the latitude arc read zero when the collimation lines of both telescopes are horizontal.

1. Make sure that the main telescope level is in adjustment, i.e., parallel to the line of collimation. Also check the collimation of the solar telescope.
2. Level the main telescope and set the solar adjusting target on a wall

* The dimensions given are those of the U.S. General Land Office Model Corresponding dimensions on other models should be measured.

about 100 ft. distant so that the cross wires intersect target *A*, and targets *B* and *C* are level. Move vernier arm to give clear vision.

3. Set the solar telescope on target *C*.

4. If the latitude vernier does not read zero, loosen the two screws at the back of the vernier and correct it.

VIII. Zero of Declination Arc

To make the declination arc vernier read zero when the mirror is at an angle of 45° with the line of collimation of the solar telescope.

1. Turn the mirror back into position and set the declination arm to read zero.

2. Set up the solar adjusting target about 200 ft. distant and level with the instrument.

3. Through the solar telescope observe the reflected image of the target and set the cross wires on target *B*.

4. Transit the solar telescope on the latitude axis and set on target *C*.

5. If vernier does not read zero, correct for half the error. Slightly loosen the two screws holding the vernier and drive the vernier in the proper direction with light blows. Tighten the screws after correcting.

This adjustment may be checked or even made by a reading on the sun at apparent noon. Place the main telescope in the plane of the meridian. Set off the latitude on the latitude arc. Adjust the declination arm until the sun is between the equatorial wires. Follow the sun by means of the declination tangent motion until the highest declination is reached. Compare the reading of the vernier with the declination obtained from the ephemeris of the sun. Use the difference as an index correction or move the vernier the full amount of the required correction as described in par. 5 above. When making this adjustment, the transit must be carefully leveled, the latitude must be known and set off accurately, and the latitude zero must be carefully adjusted.

IX. Hour Circle

To make the hour circle read apparent solar time.

1. Set up instrument in the plane of the meridian.

2. If hour circle does not read apparent solar time, move graduated rim until it does. First loosen screw in rim on the side opposite the zero hour, make necessary corrections, and tighten screw.

Correct watch, if set on standard time, by difference in longitude between local and standard meridians and by equation of time.

751. Adjustments of the Saegmuller Solar Attachment.

1. *Check the adjustments of the transit* itself for (*a*) plate levels, (*b*) line of sight, (*c*) standards, and (*d*) telescope level before attempting to adjust the solar attachment.

2. *The polar axis* of the solar attachment must be perpendicular to the line of sight of the main telescope and to the supporting axis of the main telescope. (*a*) Level up very carefully, using the telescope level in at least two positions at right angles for the final test. (*b*) Turn the solar telescope so that it is parallel to a pair of diagonally opposite adjusting screws at the base of the solar. Bring the bubble of the level attached to the solar telescope to the center of its tube. Revolve the solar 180° on its polar axis. If the bubble remains centered, the adjustment is correct; if not, bring it halfway back by means of the adjusting screws at the base of the solar and the other half by means of the vertical tangent screw of the solar. The same test and correction should be made with the solar telescope in a position over the other pair of base adjusting screws. The adjustment is correct when the bubble will remain centered during an entire revolution.

3. *The vertical cross hair* should cover a plumb line suspended a short distance from the solar. If it does not, loosen the capstan screws slightly and tap them gently in the direction desired until a coincidence with the plumb line is effected.

4. *The horizontal cross hair* of the solar is the one which requires special attention since it is the one used mainly in connection with vertical angles. If the solar telescope is in adjustment, its line of sight should be parallel to the axis of the *solar level* and to the line of sight through the main transit

telescope when the latter is level. This can be tested as follows. Mark a point about 200 ft. away in line with the horizontal hair of the main transit telescope when the bubble on that telescope is centered. Measure upward from this mark the distance between the centers of the two telescopes and set a second mark. Set the middle horizontal cross hair of the solar telescope on the second mark. The line of sight through the solar telescope is now horizontal, and the bubble should be centered by means of the adjusting screws on the solar-level tube.

Note. In the procedure just outlined, the adjustment is really that of the *solar level* because the line of sight in the solar has probably been perfected by the maker and is less likely to be "out" than is the level itself. Conditions in this respect are similar to those encountered in adjusting the line of sight to the level tube of the dumpy level.

752. Adjustments of Burt Solar Attachment.* (For a photograph of a transit with a Burt Solar Attachment see p. 603.)

I. Solar Lenses and Lines

To make parallel the lines of sight which enter the declination arm in opposite directions.

1. The lenses and lines engraved on the silver plates which define the lines of sight of the declination arm are so adjusted in the factory that these lines of sight are parallel and should remain so unless tampered with or accidentally deranged. Should correction be necessary at any time, the solar attachment should be returned to the factory. In order to make these adjustments it is necessary to dismantle the declination arm and to use a special adjusting bar to relocate the plates.

For the field man to determine the need for this adjustment, perform section II below. If the declination arc vernier cannot be adjusted, then the lenses and lines are out and the attachment should be returned to the factory.

II. Declination Arc Vernier

To adjust the vernier so that the line through the axis of the declination arm and the vernier is parallel to the lines of sight of the declination arm.

1. Set the vernier to read zero on the declination arc.
2. Raise or lower the telescope until the sun's image appears exactly between the equatorial lines. Clamp the telescope axle in this position.
3. Revolve the declination arm until the sun's image appears on the other plate.
4. If precisely between the lines, the adjustment is complete.
5. If not, move the declination arm by its tangent screw until the image is precisely between the lines on the two opposite plates. Clamp the arm and remove the index error by slightly loosening the two screws that fasten the vernier and driving the vernier with light blows until the zeros of arc and vernier are in coincidence.
6. Tighten the screws and check.

III. Polar Axis

To make the polar axis perpendicular to the line of collimation and to the telescope axis.

This is by far the most delicate and important adjustment of the Burt Solar Attachment and it should be made with the greatest care. The adjustments of the transit itself should be carefully checked. During the course of the adjustment, the telescope bubble should be kept precisely centered at all times.

1. Level the transit carefully by the long level of the telescope until the bubble will remain centered throughout a complete revolution of the transit upon its spindle.

Level the telescope over one pair of leveling screws. Turn 180° about spindle. Correct run of bubble half by axis tangent screw and half by level-

* Through the courtesy of W. and L. E. Gurley, the directions for these adjustments are taken from the *Gurley Manual of Surveying Instruments.*

ing screws. Repeat over pair of leveling screws and continue alternately until bubble remains centered when complete revolution is made.

2. Place the solar attachment upon the polar axis and see that it rotates freely.

3. Bring the declination arm into the same vertical plane with the telescope.

4. Place the adjusting level (packed separately in the box) upon the top of the rectangular blocks, and bring the bubble of the level into the middle by the tangent screw of the declination arc.

5. Turn the arc halfway around, bringing it again parallel with the telescope, and note the position of the bubble. If still centered, the polar axis is vertical in that direction.

6. If not, correct one-half the error by the capstan head adjusting screws under the base of the polar axis, moving each screw of the pair the same amount, but in the opposite direction.

7. Again center the bubble by the tangent screw of the declination arc and repeat the procedures in 5 and 6 above until the bubble remains centered upon reversal.

8. Turn the declination arm at right angles to the plane of the telescope.

9. Again center the bubble, turn 180°, and correct any run of the bubble by the other pair of capstan head adjusting screws as explained in paragraph 6 above.

10. Repeat the above until the bubble will remain centered throughout a complete revolution of the declination arm.

IV. Hour Circle

To make the hour circle read apparent solar time.

1. Set up instrument in the plane of the meridian.

2. If the index of the hour circle does not read apparent solar time, move graduated rim until it does. First loosen the two flat-headed screws on the top of the hour circle, and, with the hand, turn the circle around until the proper reading is reached. Tighten screws and check.

Correct watch, if set on standard time, by difference in longitude between local and standard meridians and by equation of time.

CHAPTER 55

THE CARE OF INSTRUMENTS *

In the care of instruments it is more important to know how to avoid unnecessary wear and tear than it is to know how to make repairs, though the surveyor should be able to do both. The most important precautions to be taken in the care of instruments are suggested in this chapter. In addition, directions are given for cleaning instruments, for repairing broken tapes, for replacing broken cross hairs, for stopping the fretting of slides and centers, and for making such other minor repairs as do not require the services of a skilled instrument maker.

753. Introduction. By taking proper precautions the wear and tear on instruments may be greatly reduced and frequently permanent injury to some part of an instrument may be avoided. *"It must be remembered that there are many parts of an instrument which, if once impaired, cannot be restored to their original efficiency."*

Many surveyors return their surveying instruments to the factory at least once every six months for inspection and proper cleaning and oiling. Moreover, such careful users know that when an instrument has been damaged, it is generally far better to send it to the maker for repairs and adjustments than to try to do the work themselves. However, every user should know how to keep his instruments clean and properly oiled and to make at least adequate makeshift repairs when an emergency occurs. He must, however, discriminate carefully between the adjustments and repairs which he can and should make himself and those which he should not attempt. Harm may be done both by neglecting to make repairs and by trying to perform in the field delicate operations which only the maker with the proper equipment should undertake.

The proper care of surveying instruments requires an adequate supply of suitable tools such as screwdrivers, spanners, wrenches, and adjusting pins. These tools are generally a part of the equipment furnished with every instrument. In order to prevent wear

* In this chapter the author has quoted frequently from catalogs and manuals of instrument makers, as it is to be presumed that such men are well qualified to give directions for the care of instruments. Extracts reproduced word for word are placed in quotation marks; but as it is impracticable to credit each quotation, this method is taken of acknowledging the author's indebtedness.

and tear, each tool should fit the part or parts on which it is to be used. For example, a screwdriver should fit the slot in a screw head; where there are a number of instruments, it may be necessary to have a considerable number of screwdrivers of various sizes and to keep each of them filed to fit the size of screw head for which it is intended. An adequate collection of good tools (including, perhaps, some not furnished by the instrument maker), if properly used, can reduce wear and tear. Makeshift implements are very poor substitutes; they should not be used except in an emergency and then only with the greatest care.

754. Precautions. Good instruments are remarkably good servants, but they are sensitive and must be treated with care and consideration if they are to do their best work. Most of the precautions which should be taken in the care and use of surveying instruments are little more than the dictates of common sense.

754(*a*). A FEW PRECAUTIONS

1. In carrying an instrument set all clamps lightly to prevent needless wear, yet loose enough so that if the parts are bumped they will yield.
2. In carrying the transit or the level a long distance, it is better to put it in its box. However, be careful that there is no looseness in the resting blocks or in any of the parts of the instrument itself.
3. In packing the instrument for shipment, it is best to fill the hollow spaces in the instrument box with tissue paper. After the box has been wrapped in strong paper, place it in another box at least an inch larger all around and fill the space between the two boxes with excelsior.
4. When a telescope is not in use keep the cap over the lens.
5. Keep the hands off the vertical circle and other exposed graduations to avoid tarnishing.
6. In case of rain when there is no waterproof bag at hand, turn the telescope objective end up with the cap over the lens to keep the water out.
7. After an instrument has been exposed to moisture, wipe it dry before putting it away.
8. "To protect an instrument from the effects of salt water, when used near the sea-coast, a fine film of watch-oil rubbed over the exposed parts will often prevent the appearance of oxide. To remove such oxide-spots as well as possible, apply some watch-oil and allow it to remain for a few hours, then rub dry with a soft piece of linen. To preserve the outer appearance of an instrument,

never use anything for dusting except a fine camel's-hair brush. To remove water and dust spots, first use the camel's-hair brush, and then rub off with fine watch-oil and wipe dry; to let the oil remain would tend to accumulate dust on the instrument."

754(*b*). A FEW DON'TS

1. In taking an instrument out of its box, do not lift it by the telescope or with the hands under the horizontal-circle plate. Place the hands under the leveling base or the foot plate. If it is necessary to lift an instrument by its superstructure, it is advisable to grasp the base of the standards near the horizontal plate, being careful during the lifting not to exert any lateral pressure.

2. Do not leave unguarded an instrument that is standing in a street, road, or pasture, or in the close vicinity of blasting.

3. Do not expose an instrument needlessly to dust, or to dampness, or to the bright rays of the sun. A waterproof cloth hood should be kept to cover the instrument when not in use.

4. Do not let the compass needle swing needlessly; always lift it from its pivot when not in use. Every precaution should be taken to guard the pivot and to keep it straight and sharp.

5. Do not *force* any screw or other movable part of an instrument such as a focusing slide. Too strong a pressure on an adjusting pin may strip the thread of a screw or nut. It is possible through forcing a binding screw of a four-screw leveling base to apply a pressure of several hundred pounds and thus produce distortion on the socket of the leveling head and a direct thrust on the vertical spindle.

6. Do not continue to use slides or centers which are fretting. If an instrument has met with a fall, bending the centers, it should not be revolved until repaired, lest still further injury be done.

7. Do not allow oil or grease to remain on exposed parts, for it will gather dust.

8. Do not rub lenses with silk or muslin. Avoid rubbing them altogether; use a brush for removing dust.

9. Do not set an instrument on a smooth floor without proper precautions; the tripod legs are likely to open out and to let the instrument fall.

10. Do not carry an instrument on the shoulder through a doorway or in any place where it is likely to get bumped.

11. Do not store tripods in a damp place.

12. Do not take instruments apart unnecessarily. "Even when fittings are perfect it often requires care and experience to place

them together properly. A screw left loose or one tightened too hard may interfere with the correct working of the instrument."

13. Do not let a "high" level rod down on the run, or lean the rod against a tree, or leave it in any other position where it is likely to fall.

14. Do not pull on a steel tape when there is a kink in it, or jerk it needlessly, or step on it when it is on soft ground, or allow vehicles to run over it, or bend it into sharp corners. If a tape gets wet do not put it away without wiping it dry.

15. In using a plane table do not start to carry it until the accessories have been removed.

755. Steel Tapes. *Care.* Wipe tapes clean and dry after using. To remove rust, rub with oil and fine sand or emery. On the better class of tapes use a little mineral oil instead, as sand or emery is likely to efface the etchings. Some simple device for holding the tape is useful when one person does the scouring.

Since tapes are easily broken the surveyor should know how to repair them. The directions which follow apply to the repair of relatively heavy tapes, such as chain tapes, which are used for work of ordinary precision. It is doubtful if the surveyor should attempt to repair thin ribbon tapes used for high precision measurements except possibly in an emergency. A good, complete repair kit should be part of the equipment for chaining. For accurate work, a tape should be standardized after it has been repaired (Arts. 715, 716).

755(*a*). Riveting a broken tape. One of the best devices on the market for mending broken steel tapes is an eyelet tool. This tool consists of a punch for cutting holes in the tape and a set which clinches or sets the eyelets used as rivets. A piece of scrap tape an inch or two in length is needed to serve as a splice. If possible this should be of the same width as the broken tape, and with the same graduations. Lay one end of the tape on the splice, taking care to match the graduations if the tape is fully graduated. Clamping the two pieces together, punch a hole about half an inch back from the broken end and *set the eyelet or rivet as soon as the hole is punched.* Usually one additional rivet will be needed. If the tape is wide the two rivets may be set crosswise; if narrow, they may be set tandem. Make the other broken end of the tape butt against the end already fastened to the splice and rivet it in the same manner. After riveting, cut away a portion of the original tape at the splice by filing a notch crosswise close to the rivets on each side of the joint. Slip a knife under and break off the ends at the file marks. Unless this is done the tape is likely to break again at the splice. In a similar manner the ends of the splice piece may be cut off close to the last rivets. The splice should not be less than three-fourths of an inch long. In mending a heavy tape the above method must be modified by punching one thickness at a time. Punch one piece, and then clamping it to the other punch the second piece through the holes already made in the first. For thick tapes solid copper rivets fastened with a hammer may be used in place of eyelets.

755(*b*). Soldering a broken tape. Another good method of mending a tape is to wrap a small strip of copper around the broken joint and to fasten it with soft solder. The outfit required consists of a small soldering iron or

a blowpipe, some flux and solder, small flat-nosed pliers, a small flat file, and a piece of emery cloth.

Clean the broken ends with the emery cloth until the surfaces are bright, moisten with flux, melt a small piece of solder, spread it on both sides of the tape, and wipe the solder with a piece of cloth while the tape is still hot. Both of the broken ends having been tinned in this way, clean one side of the copper strip with the emery cloth and give it a similar coating. Bring the broken ends in perfect contact, put on the copper sleeve, slowly heat, and when the solder is hot enough to run clamp the joint with the pliers for a moment. When cool, trim the copper sleeve, filing off any solder that may remain on the outside, and polish the joint with the emery cloth.

Suggestions. (1) Flux may be made by putting zinc into muriatic acid, and when the bubbles cease to rise, adding all the sal-ammoniac that the acid will dissolve. Stick flux, or prepared flux, such as electricians use, is more convenient to carry in the field. Resin is also used. (2) If the work has to be done in the field, as is often the case, the method may be modified as follows. Cut a strip of tin about ½ in. to ⅝ in. wide (a piece from an old tin can will answer the purpose), and of such a length that it will reach around the tape to be mended, and just about meet on the flat side—not the edge. Using one end of the tape as a "form," bend the strip into a sleeve, insert the broken ends of the tape until by measuring between graduation marks they are known to be in perfect contact, clamp the sleeve tightly about the tape with the pliers, heat the joint, and put sufficient flux on to run under the sleeve. Hold the solder on until it has melted and run under and about the sleeve, then allow it to cool without movement of the ends of the tape. A number of sleeves may be made in advance. These, together with solder, stick flux, a candle for heating the tape, a small flat file, and a pair of pliers, form an outfit which can be carried in the pocket. Tapes can be mended in this manner in a few minutes and will stand all a man can pull. (3) Sleeves should not be more than, say, ⅝ in. wide. If the break is a long one use a piece of old tape and two sleeves. Too long a sleeve interferes with the pliability of the tape. Some makers of tapes furnish sleeves to fit the tapes. Certain types of sleeves are held by screws.

755(*c*). Emergency repairs. There are several methods of mending a broken tape in the field when no outfit for riveting or soldering is at hand.

The Stick Splice. A chain tape may be spliced temporarily as follows. Secure a round stick about 4 in. long and ½ in. in diameter, and split it lengthwise, making two half-round pieces. Across the flat side of one of these pieces and equidistant from the two ends cut two notches about ¼ in. apart, ⅛ in. deep, and at right angles to the axis of the stick. Bend ⅛ in. of each of the broken ends of the tape at right angles to fit the notches in the stick, taking care that when these ends are inserted in the notches the graduations shall be the proper distance apart. Complete the splice by placing the other half-round stick with its flat side against the tape to hold the ends in place, and binding the two halves together with stout string so that they are almost as they were before the stick was split. Trim each end of the stick to slope like a sharpened pencil, so that it will not catch in the brush. Such a splice, made in a few minutes, will frequently last several days, or until more permanent repairs can be made.

Rough Riveting. Tapes may be roughly riveted as follows. Drive a darning needle through a thick piece of wood, allowing each end to project. The needle, supported by the wood throughout most of its length, may then be used as a punch. The broken ends of the tape may be riveted to the splice piece with large pins cut off and hammered down, or they may be laced together with fine wire. The lacing should be continued until the holes are filled with the wire, and then it should be smoothed and pressed down with pliers or with two blocks of wood. Tapping the wire with a hammer is likely to break the tape.

756. Cleaning and Oiling Instruments—General Suggestions.* "It is of course necessary for a surveyor to be able to disassemble his instrument for cleaning and oiling, as well as to make the proper field adjustments. As a practical precaution, frequent cleanings seem to be the best preventive of instrument trouble. Of course an instrument should never be cleaned where

* From a paper on "The Care of Surveying Instruments," by L. H. Berger, published in *Civil Engineering*, Vol. 9, No. 7, for July 1939, American Society of Civil Engineers.

dust or dirt is present; a clean bench and a dust-free room are prime requisites. An orderly arrangement of the disassembled parts is also necessary to insure uninterrupted and proper reassembly. Particular care should be taken to place the graduated circle in such a position that no dirt will possibly get on its surface."

"When the parts needing attention have been disassembled, the first operation is to remove all traces of the old oil. The instrument should be thoroughly cleaned before any re-oiling is attempted. An absolutely clean chemical solvent that leaves no residue, such as naphtha or benzine that has not previously been used, should be employed. This should be applied with a piece of good-grade linen which has been thoroughly washed and is absolutely free of lint. It is important to clean both the spindles and the sockets, or the purpose of the cleaning is defeated. The flanges of the spindles and their upper rests should be cleaned with another clean rag to remove all traces of film left by the solvent." Sockets may be cleaned with a stick of about the same taper, around which linen, soaked in the solvent, has been wrapped.

"The proper way to oil the instrument centers is to apply a few drops to the spindles and their sockets, and their respective flanges. No harm is done if the oil is applied liberally, but there is no point in wasting it." The outer spindle of a transit will need more attenton than the inner spindle and so long as the latter turns freely in its socket, the two plates should not be taken apart unless the graduated circle needs cleaning. Fretting centers should be treated in the same way as that explained for a fretting slide of the telescope (Art. 759). If unable to stop the fretting without danger of injury to the centers, "temporary relief may be found on placing between the shoulders of the center a washer of thin paper or thin card. There will be a slight play or shake in the center, but this may be better than the temporary loss of the use of the instrument."

"The vernier clamp, the horizontal-circle clamp, and the telescope clamp should be thoroughly cleaned and oiled."

"Leveling screws should be cleaned with naphtha or benzine, applied with a stiff, clean, typewriter brush. No heavy lubricants or greases should be applied as dust will readily adhere to such substances and settle on the screws. Vaseline should not be put on screw threads as it is likely to harden, especially in winter, causing the screws to move stiffly." Miscellaneous small screws can generally be washed in soap and water applied with a stiff toothbrush, then oiled and wiped clean. The insides of nuts and other threaded holes may be cleaned by screwing through the hole a piece of soft wood with two sides thinned down, applying a little oil, and removing it in the same manner. Unless the oil is almost wholly removed, the screw threads are likely to be injured by dust and fine grit adhering to the screw.

"Use of the best oils and lubricants on the market is a prime factor in lowering the maintenance cost of surveying instruments. *Only the finest grades of watch or clock oil should be applied.* Light machine oils, however purified they may be, are not suitable and oils of the penetrating or cutting type should not be allowed on the same bench as the instrument. It is worth noting that oils and greases, formerly recommended by instrument manufacturers, have in many instances been discontinued because superior products are now available. Certain excellent watch and clock oils made by various companies work well at even 20 degrees below zero."

756(*a*). GRADUATED ARCS AND CIRCLES. When the instrument is apart, the graduated horizontal circle may be dusted with a fine camel's-hair brush. Any additional cleaning must be resorted to only when absolutely necessary, and then only with the greatest care as cleaning is likely to reduce the minuteness of the graduation and to spoil the fineness and distinctness of the graduation marks. No attempt should ever be made to scour the surface with any abrasive even as fine as optical rouge, or the highest grade of silver polish.

If *after all dust has been brushed from the graduations* additional cleaning is necessary, the first rule is to avoid rubbing the edges of the graduations. Every precaution should be taken to protect the edges, for if they become worn or injured the value of the entire instrument is greatly lessened. "To remove dirt and oxide that may have accumulated on the surface of a solid silver graduation, apply some fine watch-oil and allow it to remain for a few hours; take a soft piece of linen and rub lightly until dry, but without touching the edge of the graduation. If after cleaning the solid silver surface

should show alternately brighter spots, which would interfere somewhat with the accurate reading of the graduation, barely moisten the finger with Vaseline and apply the same to the surface; then wipe the finger dry and lightly rub it once or twice around the graduation. Avoid touching the edges as much as possible. If, during the cleaning, dirt and oil have accumulated on the inner edge of the graduations, gently wipe the edge clean before restoring the vernier plate to place."

The graduations, after cleaning, should never be coated with a film of oil or Vaseline or transparent lacquer, however thin, since any film is likely to overflow across the graduated lines and be the cause of erroneous angular readings.

757. Level Bubbles and Tubes. It frequently happens that the glass tube of a spirit level becomes loose in its brass tube, or is broken and has to be replaced. Extra tubes are often provided with the instrument. "Upon finding that the adjustment of a spirit-level in an even temperature is not as stable as desirable, the level fastenings, tube, screws, etc., should be examined, to see if any of them are loose. If the trouble is in the screws, tighten them, but if the spirit-level can be shifted in its tube by a touch of the finger, take it apart; soften the plaster of Paris in water, and remove it with a sharp-pointed stick of wood. Cautiously move the spirit-level with your finger, at first only a trifle to and fro, increasing the length of stroke little by little, until it can be safely taken out without breaking;—clean thoroughly. Cut pieces of white paper, of the width of the radius of the tube, and somewhat shorter than the length of the spirit-level, but longer than the opening in the brass tube, and insert these in sufficient quantity at the bottom of the brass tube, to fill up the space intervening between the glass and the brass tube. The uppermost layer of paper should, however, be so wide as to envelop the spirit-level up to the opening in the brass tube. Now insert the spirit-level, taking care not to touch the glass ends that are sealed, and place the division or other marks, indicating where the level has been ground to a true curvature, uppermost in the brass tube. The level must be pushed in with sufficient friction to prevent slipping in the tube, yet not so tight as to cause a crack at a subsequent low temperature, as brass will contract more than glass. No part of the spirit-level should touch any part of the metal tube. Now prepare some plaster of Paris with water, of the consistency of paste, and pour in at each end enough to fill up the space between the end-pieces and the glass, stirring it sufficiently to make a perfect contact by it and the glass and the brass, but leaving the spirit-level ends exposed. Now put the level together and adjust as described elsewhere."

758. Telescope Lenses. "The lenses of a telescope should not be cleaned too often. Too frequent wipings will scratch the glass and injure the polish, which is more injurious than a little speck of dirt. When it becomes necessary to clean the glass, take a soft, dry piece of chamois-skin or an old piece of linen which by repeated washing has become soft. If the glass is very dirty use a little alcohol, but be careful not to touch the mounting of the lens, as the alcohol dissolves the lacquer. *Dirt on the eyepiece, especially on the field-lens*, is far more objectionable than on the objective; hence the former requires more frequent cleaning. Small scratches and bubbles in the objective have no seriously injurious effect, as they only take up a very small portion of light." Sometimes a film settles between the crown and flint glasses of the objective. If this film is simply moisture it can be evaporated usually by placing the telescope in the sun or in a warm room some distance from a stove. Sometimes it is advisable to remove the eyepiece and to let the telescope stand with a cloth over the open end, in order that the moisture on the inside may evaporate more rapidly. If, however, the haziness is caused by a network of small cracks in the cement between the two glasses; or if it cannot be removed by evaporation, the telescope should be sent to the maker or to an expert. Sometimes the lenses can be cleaned and recemented by a local optical firm. Do not unscrew the lens of the objective if it can be avoided. Small specks of dust or lint should be brushed from the glass with a camel's-hair brush. The glass next to the eye, especially in high powers, frequently becomes dimmed from moisture of the eyelash. It should seldom be necessary to wipe the inside of glasses that are protected; but if the objects seem dim and the outside surfaces of the lenses seem clean, remove the eyepiece and look through it. If it is not clear, examine the inner end. Very likely the surface of the lens most remote from the eye

needs cleaning. While the eyepiece is out, look through the telescope to see if the objective lens is clear.

759. Focusing Slides. A focusing slide should need no lubrication. "If it is oiled the lubrication will harden in cold weather, so that the slide will move only under undue force; in warmer weather, the oil will thin, causing a dust-bearing and fogging medium to be freed within the telescope barrel, which will be deposited on the cross hairs and on the surfaces of the lenses."

Both the eyepiece and the objective should move freely for focusing. If the focusing slide frets, a little fine watch oil may be applied as an emergency practice, but the instrument should not be used more than necessary before being repaired. If dust or grit causes the slide to fret, deep scratches are likely to be cut in the soft metal. *Once commencing to fret, it grows worse rapidly, and oftentimes is then beyond repairing.*

When a focusing slide has been damaged and emergency repairs are absolutely necessary, proceed as follows. Clean the slide and the tube thoroughly. "Endeavor to smooth carefully the injured parts with the back of a pen-knife, and apply barely enough tallow to grease the surface of the injured part. If this does not remove the trouble, a little scraping of the roughened parts on the slide, and, if accessible, on the inside of the tube, may become necessary; then apply a mere trifle of finely powdered pumice-stone moistened with oil. Replace the slide and grind a little by moving it in and out; *clean thoroughly,* and with a piece of charcoal moistened with oil smooth the parts thus ground on the slide. This process of grinding is a most precarious operation, and generally requires the hand of a skillful workman; it should be resorted to only in case of the utmost necessity. Whenever permissible, recourse should be had to a maker. These remarks apply equally to the centers."

"To prevent the focusing slide from fretting, which is usually due to the rush of air carrying in dust and grit when the slide is being run out and is causing momentarily a rarefied space, wrap a piece of chamois-skin [or other suitable material] over the barrel in the shape of a tubular form and fasten by means of rubber bands or by sewing. In an emergency fine watch-oil may be used to grease the slide should it continue to fret, until the instrument can be sent to the maker."

"If the focusing slide seems to work too hard, everything else being right, it is generally caused by the lubricant on the pinion hardening in cold weather, and the same cause may also make the focusing slide work too freely in hot weather by softening, i.e., the slide will not stay in place when in a vertical position. If the slide moves too freely it should be tightened by running out the slide to its full length, then applying a screw-driver to the screw on top of the focusing screw and turning a very small part of a turn until the required friction is obtained. If the slide works too tightly run in the slide, unscrew the top screw one turn, gently tap it by the screw-driver handle to release it, and then tighten to the required stiffness."

"A great deal of annoyance is caused to the engineer if the eyepiece or the objective slide of the telescope moves too freely in its tube, requiring a refocusing of the cross-wires and object at every revolution of the telescope in altitude. If the eyepiece can be retained in its socket, with sufficient friction to keep it focused to the cross-wires, no matter how much it may wabble otherwise, this imperfection (in old instruments) will not lead to any inaccuracy; but if there is not sufficient friction to keep it focused to the wires, a little rendered tallow or marrow applied to its bearing-surfaces in most cases will remedy this evil. Wabbling in the objective slide, however, leading to inaccuracy of collimation, or back-lash in its rack and pinion motion, can be remedied only by a maker; but if the objective-slide moves too freely in and out of its tube only, this may be remedied by applying a little tallow to the bearing-parts of the rack and pinion, or by tightening the screw in the pinion-head. If not entirely successful, a thin disk made of parchment, or a thin leather-washer, both greased with tallow and inserted between the flanges of the pinion-head and its socket, will insure the desired result."

Whenever it becomes necessary to remove the objective-lens slide, a test should be made for the line of collimation after it is replaced. (See 741(*e*)v.)

759(*a*). Internal focusing slide. The removal of an interior focusing slide calls for a different procedure from that (explained in the next article) for the removal of the exterior type of focusing slide. "After the objective

and focusing pinion are removed, the telescope should be held in a vertical position so that the weight of the slide will force it to come out. The hand, or still better, some soft material such as a clean linen towel, should be held as a cushion under the slide to receive it and to prevent it from becoming damaged. In replacing the slide, one should be sure that both it and the lens are perfectly clean."

It should be stressed that to remove the interior lens or to make repairs to the interior focusing slide requires the services of an experienced instrument maker and should not be attempted in the field except in the utmost necessity. Fortunately the interior lens and slide are so well protected from dust and grit that they seldom need attention.

760. Cross Hairs. "A properly constructed telescope is provided with small air vents to prevent the creation of a vacuum in one end of the tube when the slide is focussed. Lack of proper ventilation in the telescope tube causes the condensation known as 'dew drops' to gather on the cross hairs as well as on the surfaces of the various lenses. This condensation should gradually disappear when temperatures have been equalized."

When there is a small amount of dust on the cross hairs it is generally best to put up with the annoyance until the instrument can be sent to the maker. Actually the cross hairs seldom need cleaning or adjusting on the cross-hair diaphragm, and an inexperienced person attempting to do so may change the telescope's line of collimation. If an engineer is going into remote sections, it is better for him to take along with his instrument an extra diaphragm. However, should it be found necessary to clean the cross hairs, the first step is to remove the diaphragm from the telescope. (In this connection study first the construction of the telescope, and particularly the way in which the diaphragm is held by its capstan screws [Art. 707(*a*)].) The following procedure for removing the diaphragm from an *erecting* telescope, whether of the exterior or interior focusing type, is advised: "(1) Unscrew the object glass by turning it to the left. (Before doing so, observe whether there are any marks to show how far the cell is to be turned to place it in the correct position. If there are none, a small mark should be made parallel to the line of collimation which will extend from the edge of the cell to the collar. When the cell is put back, it must be turned until these marks coincide exactly.) (2) Remove the focusing pinion, and carefully draw out the focusing slide. (3) Remove the telescope spirit level. (4) Remove the two side capstan-head screws and their washers at the right and left of the cross ring." (5) Loosen the two remaining screws to such an extent that the ring may be turned at right angles to its usual position so that one of the holes from which a screw has been removed is toward the objective end of the telescope. (6) Sharpen a piece of soft wood to fit the hole and screw it lightly into the hole—possibly into both holes. (7) Remove the top and bottom capstan screws, and take the ring out on the end of the stick. Leave the stick in the ring to serve as a handle while cleaning the cross hairs, and then use it afterward in putting the ring back into the telescope.

"With an *inverting eyepiece,* the diaphragm is more easily taken out through the eyepiece end of the tube. It is not necessary to detach the telescope level or to remove the object glass, the focusing pinion, or the slide. First remove the two opposite screws having slots which hold the eyepiece in its mounting in the outer barrel of the telescope. Then slightly release all four diaphragm adjusting screws. Remove the entire eyepiece mounting. Then, by taking out two opposite diaphragm screws and inserting a sharp-pointed stick, the diaphragm can be removed as in the case of the erecting telescope."

Most eyepieces of erecting and inverting telescopes now have protection guards for their slides. The correct procedure for removing such an eyepiece is as follows: (1) To take out the *erecting* eyepiece, first remove the dust cap, to which the protection guard for the slide is attached. Then remove the screw in the spiral slot to be found near the end of the main tube; take hold of the eyepiece and pull it out gently. (2) To take out the *inverting* eyepiece, first unscrew the eyepiece cap, unscrew the protection guard for its slide, remove the spiral slot screw, and then take hold of the eyepiece and pull it out gently.

The cross hairs may be cleaned by using a small wedge-shaped piece of fine-texture tissue paper, moving it gently forward and back along the hairs. This operation requires a very steady hand and the use of a reading glass to prevent cross-hair breakage. Instead of the tissue paper, a brush may be

used made of a single, straight fine hair fastened to a match or toothpick and extending about one-half inch.

When, in order to clean the cross hairs, it is necessary to remove the objective, a test should be made for the line of collimation after the objective has been replaced.

760(*a*). Replacing broken cross hairs. It is not difficult in an emergency to replace a broken cross hair temporarily with sufficient accuracy for ordinary work. Proceed as follows. (1) Take the cross-hair ring out of the telescope and clean it. The fine scratches which indicate where the cross hairs are to go should be plainly visible after the old shellac and other foreign matter have been removed from the surface. Lay the ring on a flat surface, marked side uppermost. (See *dd'dd'* in the photograph, p. 841.) (2) Procure a clean, fine, opaque spiderweb and stretch it across the ring so that it coincides with two of the opposite scratches. Fasten one end with a drop of shellac, wait for this to harden, then, making sure that the hair is stretched tight, fasten the other hair in a similar manner. Allow the shellac to harden for several minutes, cut off the two ends of the web, and then proceed in the same manner to replace the other web if that too is broken.

Suggestions. (1) Remove the objective or the eyepiece, according to whether the telescope is erecting or inverting, and take out the cross-hair ring on the end of a stick as previously explained. (2) Spiderweb can be purchased in a convenient form from which it can readily be unrolled, and it is well to have some of it in the repair kit. However, it will be assumed that the surveyor must find the web in the field. There are two ways of procuring spiderwebs—from the spider's nest or cocoon and from the spider itself. "The best spider-threads are those of which the spider makes its nest. These nests are yellowish brown balls, which may be found hanging on shrubs, etc., in the late fall or early winter. The nest should be torn open and the eggs removed; if this is not done the young spiders when hatched will eat the threads. The fibers next to the eggs are to be preferred on account of their fineness and darker color. One or two cocoons may be kept on hand, and if desired they may be wrapped in a bit of paper and carried in the pocket, where they are always ready." From the spider's cocoon select two or three webs, each about 2 in. long and of the same appearance. Attach each end of these webs to a bit of wood or wax to act as weights, and immerse them in water 5 or 10 min. Remove one web from the water, and very gently pass it between the forefinger and thumb nails, holding it vertically to remove any particles of moisture or dirt. Then stretch it across the ring as already explained. The little weights should be about as heavy as the hair will support in order to stretch it tight, and before dropping the shellac on either end examine the hair with a magnifying glass to see if it exactly coincides with the scratch. If it does not, move it gently with a knife point. If the ring is placed nearly on a level with the eye and between it and the source of light (as a window), the spiderweb will glisten and become plainly visible; otherwise it is sometimes difficult to see it. (3) When a cocoon cannot be procured, a spider can usually be found. Cut a forked stick 5 or 6 in. long, get the spider on the end of one of the prongs, and gently shake him off. He will usually begin to spin a web, and, as he spins, wind the web around the fork in such a way that no two strands will be close together. If the fork is pressed together *a very little* during the winding it will stretch the webs when released. When one strand is laid across the ring the forked stick will act as a weight to stretch it tight. (4) An extra diaphragm taken into the field is good insurance. (5) Platinum wires are stronger than spiderwebs and are used in many instruments; but if one is broken, it may be replaced temporarily by a spiderweb.

761. Care of Tripods. "Legs of tripods, if fitting too loose or too tight, and dull shoes are frequent sources of falls. The test of the proper degree of tightness of the legs is this, that if the leg is raised to a horizontal position and left free, it should gradually sink to the ground. If it drops abruptly it is too loose; if it does not sink it is too tight." The nuts will need adjustment after a period of dry or damp weather or after the tripod has been kept in the dry atmosphere of a room. The steel shoes will loosen from wear and tear unless the screws are tightened from time to time. When the wood inside the shoes is continually moist, it is likely to decay to such an extent that the screws will not hold and the tripod legs will then become useless. For this reason, tripods, when not in use, should be kept in a place where the wood will dry out. Loose tripod shoes are a common source of

serious error and every precaution should be taken to keep them tight. They should also be kept somewhat sharp for a better hold on the ground. When the tripod becomes wet, wipe dry and renew the varnish whenever necessary.

It is particularly important to keep a tripod with extension legs in a dry place. If, when the wood is dry, the tripod is put away in a damp place, the legs will swell and this is likely to split the metal clamp.

PART IV: STANDARD SURVEYS

INTRODUCTION

762. Standard Surveys are those in which requirements and procedures have become more or less standardized. In the *United States Coast and Geodetic Surveys* (see Chapter 56 of this book) and in the *United States Public-Land Surveys* (Chapter 59) requirements and procedures are prescribed by the federal government. In *Horizontal Control Surveys* (Chapter 56) to establish state systems of plane coordinates, requirements and procedures are prescribed by each state in cooperation with the U.S. Coast and Geodetic Survey. In *Surveys of Land Boundaries* (Chapter 58) requirements and procedures are generally prescribed by the state in which the land is situated, and often, in addition, by town or city ordinances. In *Surveys for Construction* (Chapter 57) requirements and procedures have become more or less standardized to meet the needs of builders and contractors. Procedures in *Aerial Photographic Surveys* (Chapter 60) are rapidly becoming standardized.

A working knowledge of the requirements and procedures in various types of standard surveys is best acquired through practical experience. Nevertheless, much of the information can be given in book form for ready reference. To do this has been the aim in Part IV. It is assumed that the engineer who consults any portion of Part IV will have a knowledge of the fundamentals of field work and office work; consequently explanations of field and office procedures are avoided in Part IV except in special cases not treated in Parts I, II, and III. Thus the greater portion of Part IV is primarily for reference by the surveyor rather than for use as a text by the student.

CHAPTER 56

HORIZONTAL CONTROL SURVEYS

763. Introduction. Just as permanent bench marks, used as starting points and checking points in leveling, form systems of vertical control, so do permanently monumented points, used as starting points and checking points in making horizontal surveys, form systems of horizontal control.

Monumented points for horizontal control have been established by governmental horizontal control surveys, such as those made by engineers of the federal, state, county, and municipal governments, and by surveys made by engineers engaged in private practice.

This chapter is devoted mainly to a description of two horizontal control surveys and to an explanation of the various ways in which they may be used to advantage. These two control surveys are the *United States Coast and Geodetic Survey* and the *State Systems of Plane Coordinates.* The technique of triangulation and of traversing used in these surveys will not be discussed since the methods are similar to those already explained in Part I.

764. National Geodetic Control Survey. The fundamental control survey, established by the U.S. Coast and Geodetic Survey, is a triangulation network composed of arcs (or chains) of first-order accuracy spaced at intervals of about 50 miles, and of arcs of second-order accuracy spaced approximately 25 miles apart. A further breakdown of this fundamental control is designated as supplemental control, and consists of triangulation and traverse established at a smaller spacing than 25 miles. As this geodetic system of control is extended from year to year, the United States is being covered with tens of thousands of control points permanently monumented. The geographic position of each of these stations is determined by its spherical coordinates, i.e., by its latitude and longitude. The observations and computations are made on a datum which approximates the true curved surface of the earth. This datum, known as the "North American Datum of 1927," is based on an assumed mathematical shape known as the "Clarke Spheroid of 1866" and upon the adopted position of a certain initial station in central Kansas called "Meades Ranch." All

stations located on this datum by standard observational methods have a definite fixed relationship to each other. Should a station marker be lost, it can be re-established in its original position by field observations, starting from adjacent station monuments which have not been disturbed. Once the latitude and longitude of a station have been accurately determined, the position of that station is forever fixed since obviously no other point on the earth's surface can have the same latitude and longitude. Thus the basic net of first order and second order, reaching into all parts of the United States, provides a rigid framework by which to determine the correct horizontal position on the earth's surface of all other surveys which may be connected with it.

764(*a*). TRIANGULATION ARC. An *arc* of the triangulation net generally consists of large systems or chains of overlapping triangles usually laid out in joined quadrilaterals (Art. 403). The length of sight varies from 2 or 3 miles up to a maximum of over 150 miles in western United States. More than 65,000 miles of arcs of triangulation that cut the country up into a great checkerboard have been completed . Soon every point in the country will be within 25 miles of a control station. This distance will be reduced as soon as possible to a maximum of 12 or 15 miles.

765. Monuments for Geodetic Stations. It is the practice in the U.S. Coast and Geodetic Survey to mark bench marks and stations by inscribed standard metal disks or tablets. These station marks are set in concrete posts, outcropping bed rock, boulders, abutments of highway and railroad bridges, and the walls and foundation of buildings. The location, depth of soil, and the presence of rock ledges and structures generally control the manner in which the tablet is set. The greatest number of marks are set in concrete posts. When these posts are where traffic passes over them, or where the soil is cultivated, the tops of the marks are placed slightly below the surface; but when the posts are located where they are not likely to be disturbed, the tops of the marks are generally 2 to 4 in. above the surface.

At each triangulation and traverse station marked by a concrete post an underground mark is established. This mark consists of a disk or tablet set in a mass of concrete at least 10 in. in diameter and centered under the station mark. If the station mark is disturbed, the underground mark can be uncovered and used for the station in surveying, or it can be used to re-establish the station mark.

Two or more reference marks, consisting of metal disks similar to the station marks but bearing an arrow which points to the station, are generally set in concrete posts or in bed rock at short

distances from the station. No subsurface marks are set under these concrete posts.

765(*a*). Azimuth marks. An azimuth mark is a monument, generally of the same size and nature as a reference mark, usually set within one-fourth mile of the station. A rod held on the azimuth mark can, in all cases, be seen by a surveyor standing on the ground at the station. The true bearing, or the true azimuth of the line from the station to the azimuth mark, is accurately determined, generally within 10″ of arc, often within 5″. This accuracy is amply sufficient for most local surveys. The description of each triangulation station gives the true bearing of the line to the azimuth mark and also describes the location of the mark. A survey may, therefore, be started from the station without sighting to another triangulation station. If, however, an azimuth that is correct within one second is desired, the sight should be along a line of the main scheme of triangulation. This will, in some cases, involve building a signal or tower at the triangulation station used as a starting point for the local survey, and, perhaps, also a signal on which to observe a second station of the main scheme.

At many of the triangulation stations, objects such as church spires and cupolas can be seen from the ground. The data furnished for those objects include the azimuths to and from the triangulation stations from which they were determined. For the stations established in recent years, the descriptions indicate those objects which are visible from the ground. For the triangulation stations established before 1927, no monumented azimuth marks are available, but there are usually prominent objects which can be seen from the ground.

765(*b*). Note. The engineer using these data must understand that the azimuths at different stations are given with reference to *different* meridians except for stations that happen to be on the same meridian. This is because each azimuth is a *true* azimuth referred to the meridian *through the station from which the azimuth proceeds.* If the engineer desires to start a traverse at one triangulation station, using an azimuth at that station, and close on another station and azimuth, his closure will include the amount of convergence of the two meridians through these stations. If the azimuth of the line from the starting station to the terminal station and the azimuth of the line from the terminal station to the starting station are given in the data, the difference of these two azimuths, plus or minus 180°, is the convergence of the meridians. This convergence may be applied to any azimuth at the closing station, in order to put it on a plane basis with reference to the starting meridian. The adjustment of the angles of the traverse can then be made by the ordinary processes of plane surveying.

A closure in length may be obtained, if the length between the terminal stations is given in the triangulation data, by comparing this length with the length of the line determined by computation from the latitudes and departures of the plane survey. The lengths of the lines given in the triangulation data are reduced to sea level, and they probably should be computed at the elevation of the ground to furnish an accurate check in length.

If the area under survey is so extensive as to include a number of triangulation stations, then an origin of plane coordinates should be adopted, and the various triangulation stations should be reduced to the plane system. This is easily and effectively done through the use of the tables contained in Coast and Geodetic Survey Special Publication 71, *Relation between Plane Rectangular Coordinates and Geographic Positions.* When such conversion has once been made, all surveys and their computations within a limited area may proceed according to the methods of plane surveying, as will be explained later.

766. Data Available. In order to ascertain what triangulation stations have been established in any locality and to procure information pertaining to these stations, it may be necessary to write to the director of the U.S. Coast and Geodetic Survey, Washington, D. C. Special triangulation and traverse publications are published from time to time by the Survey giving complete information (to the date of printing) for certain areas. However, the latest information is not always found in these publications, and for many areas no such publications exist.

The principal data available are: *

1. *Progress Charts of Areas.* The U.S. Coast and Geodetic Survey maintains index maps of each state showing arcs of first-order and second-order triangulation and traverse for which field work has been accomplished. Photostats of these may be obtained at a nominal cost.
2. *Tables of Geographic Positions and Geographic Azimuths.* On numbered sheets called "Lists of Geographic Positions" are recorded the geographic positions (latitudes and longitudes) of the triangulation stations for which the computations are complete, together with the geographic azimuths from these stations to a number of other triangulation stations, and to special azimuth marks for the more recent stations.
3. *Descriptions of Stations.* Descriptions of stations are available in various forms. For completed field work, they are placed on cards, photostats of which are obtainable. These descriptions are very complete and include angular measurements at the station between adjacent stations on the arc, instructions for reaching the station, data for finding the station, a description of the station mark, and reference marks and measurements by which the station mark can be completely located and identified.

For many stations the following are available: elevations; tables of plane coordinates and plane azimuths; reconnaissance sketches, made in the course of reconnaissance for triangulation work, which show the main scheme; the supplementary stations of an arc and the ties to other triangulation; progress charts of arcs, similar to reconnaissance sketches but made after the arc has been completed.

767. Supplemental Geodetic Control Surveys. The effective utilization of the fundamental triangulation network of the United States for the control of plane surveys requires the establishment of supplemental control systems based on the national network and of a considerably higher degree of accuracy than the surveys to be controlled. Such supplementary control surveys are to be found throughout the country; some are geodetic surveys and some are plane surveys.

* A more complete description of these data is given in Manual 20, *Horizontal Control Survey,* published by the American Society of Civil Engineers.

Supplemental geodetic surveys have been established to control many public surveys, such as those of certain states, counties, and cities, and also to control surveys for extensive engineering projects, such as highways, railroads, canals, power projects, flood control, and reclamation. Every such supplemental geodetic control survey, if it is made with first-order or second-order accuracy and if it is properly tied in with stations of the U.S. Coast and Geodetic Survey, becomes, in effect, an extension of the national triangulation network. Moreover, any plane survey controlled by such a supplemental survey is connected indirectly with that network.

The most important supplemental plane control surveys are those made in establishing the state systems of plane coordinates. These systems, as explained later, make it easy to connect a survey in almost any locality with the national network.

768. Methods of Connecting a Traverse with a Triangulation Station. Obstacles to linear measurement or to angular measurement, or to both, may make impracticable the use of simple methods in connecting a traverse with a triangulation station. Special methods of connection must then be devised. Several such methods or cases are shown in Figs. 768(I) to 768(X). In some cases it may be desirable to set two monuments on an accessible line near the triangulation station; these monuments can then be used as control stations in connecting future traverses with the station. Data pertaining to a triangulation station may be obtained as explained in Art. 766.

768(*a*). TYPES OF CONNECTION.* The methods or cases shown in the figures are numbered from the simplest to the more complex. Consequently in planning a traverse tie to a triangulation station, the methods should be considered in the order given; every effort should be made to use the first case before trying the second, and so on. A method should be avoided if in the measurement of the horizontal angles one or more of the lines of sight are steeply inclined. When a line of sight is inclined more than 15°, the instrument should be leveled by a special method (Art. 247).

In Figs. 768(I) to 768(X), inclusive, the following symbols apply:

———————— Line both measured and observed

— — — — — — Line observed only

△ Triangulation station

⊡ Traverse monument

⊙ Traverse point

* Taken from Manual 20, entitled *Horizontal Control Surveys*, published by the American Society of Civil Engineers.



Angle measured with accuracy sufficient to carry azimuth



Angle measured with accuracy sufficient to compute proper distance

Case I. Connection by Superimposing Traverse on Azimuth Line. This method represents the best form of tie to a triangulation station. (See Fig. 768(I).) The traverse line is run through the azimuth mark, and the distance from the azimuth mark to the triangulation station is measured. Coordinates can be computed for the azimuth mark, and then the triangulation station and azimuth mark become companion monuments on the traverse line.

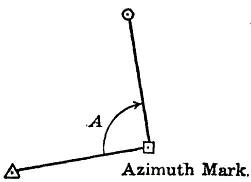


FIG. 768(I).

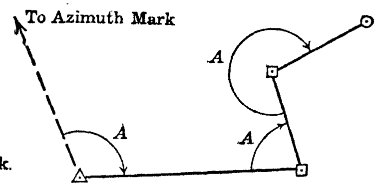


FIG. 768(II).

Case II. Standard Connection. Case II (see Fig. 768(II)) should be used if Case I cannot be adopted because the azimuth mark is inaccessible.

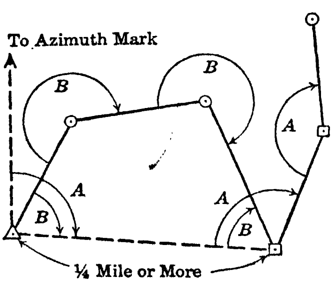


FIG. 768(III).

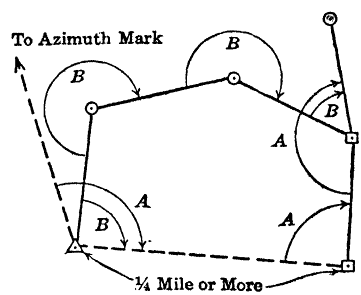


FIG. 768(IV).

Case III. Separate Azimuth and Position Tie for Ground Station. Case III (see Fig. 768(III)) is to be used when it is impossible to tape directly from the monument to the triangulation station.

Case IV. Special Case of Position Tie and Separate Azimuth Tie. Case IV (see Fig. 768(IV)) is a variation of Case III.

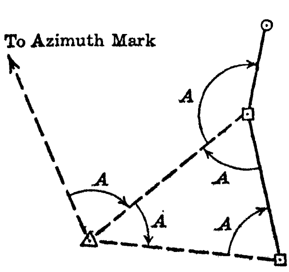


FIG. 768(V).

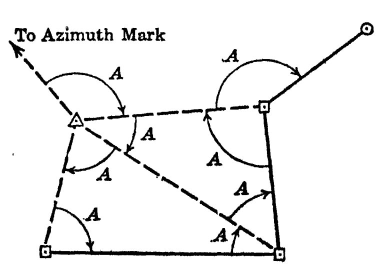


FIG. 768(VI).

Case V. Connection for a Triangulation Station on a Building, or on Ground, When Inaccessible. In this case (see Fig. 768(V)) monuments should be set so that an approximately equilateral triangle is formed. No angle should be less than 45°.

Case VI. Check Connection for a Triangulation Station on a Building, or on Ground, When Inaccessible. For Case VI (see Fig. 768(VI)) monuments should be set so that no angle is less than 30°. The other diagonal may be measured, and the minimum angle may then be reduced to 20°.

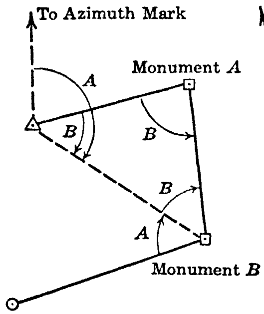


FIG. 768(VII).

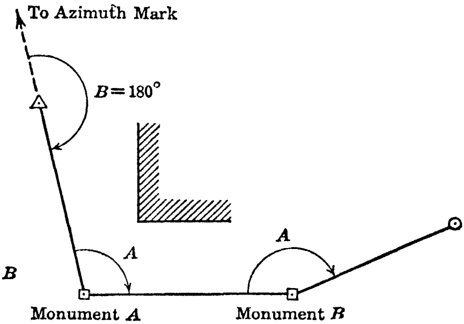


FIG. 768(VIII).

Case VII. For Ground Triangulation Station When Traverse Must be Close to Station and When It Is Impossible to Measure Directly from Monument B to Station. In Case VII (see Fig. 768(VII)) keep the angle at monument *A* close to a right angle and the distance from monument *A* to the triangulation station 100 ft. or less.

Case VIII. Superimposing Traverse on Azimuth, When Azimuth Mark Is on Distant Spire or Tank. In Case VIII (see Fig. 768(VIII)) it is important: (1) that no direct method be possible; (2) that monument *A* be located by three sets of double centering, and placed near triangulation station; and (3) that the angle at *A* be turned from the azimuth mark to monument *B*.

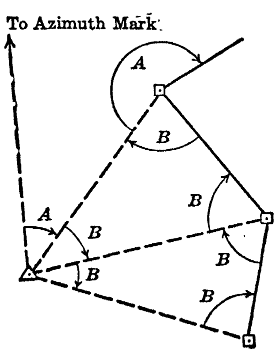


FIG. 768(IX).

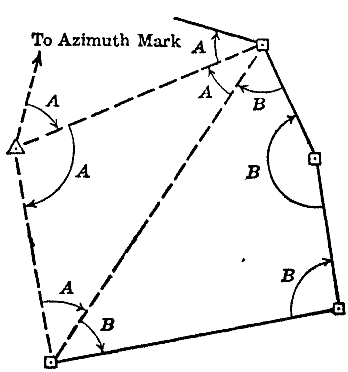


FIG. 768(X).

Case IX. Position Tie Separate From Azimuth Tie for Triangulation Stations on Buildings When Case V Is Impossible. For this case (see Fig. 768(IX)) use the special method of setting up the instrument for steeply inclined sights (Art. 247).

Case X. Broken Base. Case X (see Fig. 768(X)) should be used only when other methods cannot be applied.

769. State Systems of Plane Coordinates. The national geodetic triangulation net was designed to control relatively large areas. Its use for the control of the average plane survey has been extremely limited. For the survey of a relatively small tract of land it is obviously inconsistent if not absurd to use control points whose positions on the surface of the earth are fixed solely by latitudes and longitudes. What was needed was a system of control in which the positions of the control points are fixed not by spherical coordinates but by plane coordinates. In 1933 the U.S. Coast and Geodetic Survey devised and published tables for State Systems of Plane Coordinates for all the forty-eight states. This bureau then inaugurated state survey organizations throughout the United States to establish state systems of plane coordinates in accordance with general specifications issued by the bureau. Unlike the national geodetic triangulation net, there is not one system cutting across state boundaries but a separate system for each state. However, each of these state systems is based on and tied in with the U.S. Geodetic Survey. Conequently, all the state systems, taken together, constitute, in effect, a national plane system of control which is supplemental to the national geodetic triangulation net. The advantages inherent in state systems of plane coordinates and the benefits which accrue from their use are so great that the development of these systems is a most valuable contribution to progress in surveying. The advantages and benefits will be discussed later.

770. Characteristics of Plane-Coordinate Systems. Since the earth's surface is curved, no part of it can be represented on a plane without distortions and discrepancies entering into the computations. The area covered by a plane coordinate system should be sufficiently small to bring the errors due to distortion within the limits of error required in any local plane survey except one of considerable extent and of a high degree of precision. For a small state a single system covering the entire state meets this requirement, but it has been found necessary to divide each of the larger states into two or more zones or sections, each with its own plane-coordinate system. The boundary between zones follows county boundaries and is, therefore, generally a zigzag line.

770(a). ELEMENTS OF A PLANE-COORDINATE SYSTEM. (1) A coordinate or *central meridian* arbitrarily chosen near the center, east and west, of the state or the zone covered by the system. (2) A *Y axis* parallel to the central meridian and west of the most

westerly point in the state or the zone; an X *axis* perpendicular to the central meridian and south of the most southerly point in the state or the zone. The two axes are chosen in these positions in order that there may be no minus coordinates. The Y axis is generally assumed at such a distance west of the central meridian that the value of the x coordinate to the meridian is 2,000,000 ft., although in a small state, such as Connecticut, a smaller value such as 600,000 ft. may be used. (3) *Control points or stations* throughout the state or zone. The position of each point is determined by its x and y coordinates and it is preserved by a permanent monument.

770(*b*). PLANE AZIMUTH. The azimuths used in state systems of plane coordinates are commonly called "plane azimuths" or "grid azimuths." All such azimuths are measured with reference to one and only one meridian, namely, the central meridian. A line of zero azimuth from any station is parallel to the central meridian, *but it is not a true north-south line unless the station is on the central meridian.* It follows that the plane azimuth of a line from any station to any point is not a true azimuth, nor is the corresponding bearing of the line a true bearing unless the station is on the central meridian. If the station is west of the central meridian, the plane azimuth will be larger than the true azimuth. If the station is east of the central meridian, the plane azimuth will be smaller. The nearer a station is to the central meridian the less the divergence in azimuth. All plane azimuths are measured clockwise from the *south.* (If a true azimuth is desired at any point of a survey it may be determined by one of the methods explained in Chapters 34 and 35.)

770(*c*). COMPARISONS BETWEEN THE GEODETIC AND THE PLANE-COORDINATE SYSTEMS OF CONTROL. (1) Lines between geodetic control points are on a curved surface; those between control points of plane-coordinate systems are, for all practical purposes, on a plane surface. (2) Control points of the geodetic system are located by spherical coordinates; those of the state systems by plane coordinates. (3) In the geodetic system of control, the azimuth of a line from any station to any point is a true azimuth; in the plane-coordinate system, the plane azimuth is not a true azimuth unless the station is on the central meridian.

770(*d*). SIMPLICITY OF THE SYSTEM. The coordinates in a state system of plane coordinates are used in exactly the same manner as any system of plane coordinates. The use of the system requires no knowledge of geodosy. The local surveyor, in the use of the control points of the system, has no specialized computations to make, no new surveying procedures to learn, and nothing unusual to do but to start his survey from the nearest control points instead of at random. From there on, he may carry on his field work by the usual methods of plane surveying, compute the latitudes and departures in the customary manner, and adjust for the errors of closure as he has always done. Yet there is this important difference that his survey is connected with a state coordinate system and through that system with the national geodetic triangulation network, rather than with an arbitrary local control or with no control at all.

770(*e*). NOTE. In various explanations of state plane-coordinate systems considerable emphasis has often been placed on their mathematical development; consequently some surveyors get the impression that in order to use the control stations one must be familiar with geodetic methods of field work and computation. This, of course, is not true, for the average local surveyor needs to pay no attention to the computations that are involved in the transfer of spherical coordinates, in the form of latitudes and longitudes, to plane coordinates. That is done for him by the U.S. Coast and Geodetic Survey or the State Geodetic Survey. Although it is well for the surveyor to understand the mathematical development of a state plane-coordinate system (see Arts. 778–780), it is not essential that he should. Only simple mathematical principles, such as those used in the solution of a right plane triangle, are needed for the computations involved in tying in a local survey with a state control system.

771. State Control Stations. In establishing control stations in a state system of plane coordinates it has been necessary to extend the triangulation net of the U.S. Coast Survey by both first-order and second-order triangulation. The plane coordinates of each station of this extension as well as of each station in the national triangulation net within the state are known or may easily be computed; hence each of these stations becomes a control station in the state system of plane coordinates. The most useful control stations, however, are those which have been located (by second-order traverses) in streets and highways where they are easily accessible. In some states, such stations are established in *pairs*. Each station of a pair can be seen from the other station; hence the line between the two stations can be used as the starting line for a local survey. In other states, each station has its azimuth mark, and the line from the station to the mark may be used as a starting line.

Unless there are control stations in rather close proximity to the place of a local survey, it is inconvenient and expensive to tie the survey to the state system of control. Moreover, the greater the distance from the local survey to the control stations, the more difficult it may become to make an accurate connection. It is debatable just how long a traverse a local surveyor should be compelled to run in order to connect his survey with stations of the state system. There are other conditions, in addition to those just mentioned, which should also be considered in determining locations for control stations, and these conditions vary so greatly in different localities that no rigid rule can be followed. For example, in one state, the original plan was to set a pair of monuments every mile, where conditions permit, and to place them a minimum distance of 1200 ft. apart, but it was found that at this minimum distance many of the winding roads in the state would not have a pair of monuments for miles. The general practice now in that state is to set the monuments 600 ft. to 1800 ft. apart. For pur-

poses of land description, it is desirable that a state control station should be within one-half mile of some point in the boundary described (Art. 774(*a*)).

The nature of a state system of coordinates is such that future extension will be necessary. As the need for more control stations in any locality becomes evident and funds become available, more stations will be established until eventually every local survey will be within a relatively short distance of control stations.

771(*a*). MONUMENTS. A concrete monument with a metal disk set in its top is the most common type used for marking points in a state control system. The Connecticut Geodetic Survey uses four types of disks, namely: (1) the standard triangulation mark and (2) the standard bench mark used by the U.S. Coast and Geodetic Survey; (3) a monument with the number of the station and its elevation inscribed and (4) a town survey monument with the name of the town inscribed in addition to the station number and elevation.

772. Data Available. Data needed by the surveyor in order to connect his survey to the state system of plane coordinates are furnished in slightly different forms in different states. The form in which the Connecticut Geodetic Survey furnishes data for each control station is illustrated in Fig. 772(*a*). These data are: (1) The town and county in which the station is located. (2) The station number. (3) A description of the general location of the station and of the station itself. (4) A description of each reference mark and of its approximate location. (5) The plane coordinates of the station. (6) The elevation of the station with reference to mean sea level. (7) The plane (or grid) azimuth to adjacent control stations (often to only one station) measured from the south. When no adjacent station is visible from the control station, the azimuth from that station to its azimuth mark is given. (8) The distance to each adjacent station, often to only one station. (9) The distances and directions to station reference marks. (Directions taken from a backsight on an adjacent control station.)

No matter in what form the data may be presented or how clearly they may be stated, they are likely to be misinterpreted. For this reason it is advisable to include in the data a simple sketch such as that shown in Fig. 772(*a*). The arrangement of the printed data and the sketch is designed primarily to conserve space and at the same time to present the data, as far as practicable, in the order in which they will be needed.

In addition to the information given for each station, other data are available such as: descriptions and the plane coordinates of monuments established by the U.S. Coast and Geodetic Survey; descriptions and elevations of bench marks established by the U.S. Coast and Geodetic Survey and by the state survey organization; maps of different sections of the state and of some of the larger towns on which are shown the general locations of control stations and bench marks. In some states these data and the data for each control station are published in book form, properly indexed. Ordinarily, the surveyor should have no difficulty in obtaining all the information which he may need in order to connect his survey with the state system of plane coordinates.

772(*a*). Typical form for station data. The form in which the information concerning a station is given by the Connecticut Geodetic Survey is shown below.

Connecticut Geodetic Survey Monuments

Town of Somers *Tolland County*

#1133

On the east side of Conn. 83, about 3.00 miles north of its intersection in Somers with Conn. 20, or about 0.80 mile south of Conn.-Mass State Line. The station is between C. L. & P. poles 882–883–E–10, about 19 feet east of center line of highway, and about 33 feet south of 8-inch ash. Station is marked by a standard C. G. S. disk, stamped "1133," set in the top of a concrete post, the top set about 1 inch above ground.

R.M. #1: South of station; drill hole in boulder, about 18 feet east of highway center line.

R.M. #2: Southwest of station; top of northwest corner of stone foundation of old "1730" farm house.

R.M. #3: West of station; top of southeast corner of concrete block foundation of small house on west side of highway.

Plane Coordinates and Elevation Feet	Plane Azimuth To Station No. 1134 ° ′ ″
X—680,696.66	159 51 20.8
Y—434,573.92	
El—236.251	

Reference Marks

Object	Distance Feet	Direction °	′	″
Mon. #1134	2364.51	0	00	00
R.M. #1	30.35	195	40	00
R.M. #2	186.00	220	42	00
R.M. #3	57.55	291	22	00

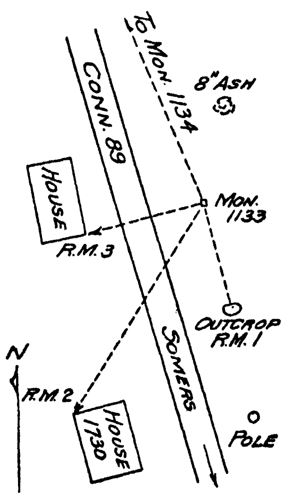


Fig. 772(*a*).

773. To Connect a Survey with a State System of Coordinates. In connecting a local survey with a state system of coordinates, the starting point is generally a monumented station of

that system. In some cases, however, it may be more practicable to start from a station of the U.S. Coast Survey. It is to be understood, therefore, that the term "control station," as used in this discussion, may refer to a station of either type.

The most desirable method of connecting a local survey with the state coordinate system is to start the traverse of the survey at one control station and to close on another, and thus obtain a check on the accuracy of the traverse as in Art. 773(*a*). When this cannot be done, the next best method is to start the traverse at a control station and complete it without closing on any other control station. (However, if it is not an open traverse, it should close on the single control station from which it started.) State coordinates can be carried through to the local survey by this method, but the advantage of a check on a second control station is lost. (See Art. 773(*b*).) A third case is that in which the local survey is so remote from any control station that it is necessary to run a special traverse which starts from the nearest control station and connects with the local survey. Here again there is no check on a second control station unless a second special traverse is run from the local survey to such a station, a procedure that often is not practicable (Art. 773(*b*)).

In each of the three methods just outlined, the first step in the field work is to set up over the monument at a control station and backsight on its azimuth mark or on another control station if one is visible. The latter method of orienting is generally possible where state control stations are established in pairs. Occasionally no azimuth mark or no other control station is visible from the starting station. In such a case the local survey can still be connected with the state system, provided that at some point the survey can be connected to a second control station, even if at that station also no other station or no azimuth mark is visible.

When it is necessary to start at a control station from which no other control station or no azimuth mark is visible, one of the procedures indicated in Art. 768 may be followed.

773(*a*). TO CONNECT AN OPEN-TRAVERSE SURVEY WITH A STATE SYSTEM OF PLANE COORDINATES. Let it be required to connect the open traverse of a survey along a highway with the state system of plane coordinates. A common procedure is this.

1. Ascertain the general location of *all* control stations in or near that portion of the highway to be covered by the survey, and obtain the data available for those stations. (See Art. 772.)

2. Set up at the control station nearest the beginning of the survey, and backsight to its azimuth mark, or to an adjacent control station if one is visible.

3. From the line of sight established in (2), measure the angle to the next point which the instrument is to occupy and compute the plane (grid) bearing of the line to this point, starting from the plane bearing to the azimuth mark or to an adjacent control station.

4. Proceed with the traverse, carrying calculated grid bearings of the traverse lines, in the usual manner, to the control station, nearest the end of the survey, and right on through that station to its azimuth mark or to another control station if one is visible. (This method of running the traverse is similar to that explained in Art. 305, except that the bearing of the backsight from the first station to its azimuth mark is obtained from the plane azimuth of that line as given in the data for the station.)

5. Compare the calculated azimuth of the line from the terminating control station to its azimuth mark (or to an adjacent station) with the grid azimuth of that line furnished in the official data. The discrepancy between the two is the angular error.

Note. It cannot be assumed that control stations were originally established and their coordinates computed with absolutely no error; consequently the discrepancy cannot be considered a true error. However, it may be assumed that the control points were established with a considerably greater degree of precision than that required in an ordinary survey; hence the discrepancy, for practical purposes, may be considered the angular error.

6. Distribute the angular error, provided it is within the limits specified for the survey.

7. Starting with the coordinates of the first control station (furnished in the official data), compute the plane coordinates (latitudes and departures) of the survey traverse. The calculated coordinates of the terminal control station should agree with those for that station furnished in the official data. Any discrepancy is the error of closure.

The procedure as outlined not only provides a check on the field work but also ties in the highway survey with the state system of coordinates.

If the surveyor prefers to run his traverse by the azimuth method, he first sets the vernier to read the official grid azimuth from the starting control station to its azimuth mark (or to an adjacent control station), and then backsights to the azimuth mark (or to the adjacent station). From there on, instead of measuring the traverse angles and calculating bearings, he simply reads the

azimuths of the various lines directly from the vernier plate, and closes on the known azimuth from the terminal control station to its azimuth mark, or to an adjacent control station. (For this method of running a traverse see Art. 303.) The principal disadvantage of this method is that angles at traverse stations cannot be measured by the method of repetition.

Without control stations it is difficult to check the field work of an open traverse. With control stations established along a highway, a traverse may be run for miles and still be checked as frequently as the spacing of the control stations will permit.

773(b). TO CONNECT A BOUNDARY SURVEY WITH A STATE SYSTEM OF PLANE COORDINATES it may be necessary to run a special traverse from the nearest control station to connect with the traverse of the boundary survey. The procedure in starting at the control station is the same as that given for the highway survey. The real object, however, is to determine the plane (grid) coordinates of at least one property corner. When the traverse from the control station has been properly connected to the traverse of the boundary survey, the grid coordinates of every station of that traverse can be computed; hence the coordinates of any property corner can also be computed, provided its position with respect to the traverse line is known. Frequently, however, it is possible to connect the special traverse from the control station directly with at least one property corner. From the computed plane coordinates of this corner and data obtained from the boundary survey the coordinates of the other boundary corners may be calculated. If there is a second control station sufficiently near the boundary survey to be connected to it, its coordinates can be used as a check both on the field work and on the computed coordinates of the boundary corners. When there is no such control station, there is no check on the accuracy of the connecting traverse from the first control station except such checks as may be applied to the field work as it progresses.

773(c). COMPUTATIONS. The minimum data required for the computation of the grid coordinates of the stations of an existing traverse are the grid coordinates of one station and the grid azimuth from that station to another station visible from it. In the computations of coordinates, it is well to keep in mind that a line between two stations is the hypotenuse of a right triangle. The length of one side of this triangle is equal to the difference in the values of the x coordinates of the two stations, and the length of the other side is equal to the difference in the values of the y coordinates. The length and the bearing (or azimuth) of the line

can easily be computed by the usual trigonometric or geometric methods of solving a right triangle, facilitated by the use of tables.

The large numbers used for grid coordinates in order to avoid minus values have been the cause of considerable criticism. Since the origin of the state system of coordinates was arbitrarily assumed, any other origin may be assumed, provided the relation of the grid azimuth is maintained. For example, in order to avoid carrying large numbers in his computations, the computer could substitute $x = 0$ and $y = 0$ for the official coordinate values of the starting control station. Generally, however, it is better to substitute some number sufficiently large to avoid minus values, such as 10,000, or as much greater as may be necessary. After computing the coordinates of any point on such a basis, the difference between the official values and the assumed values of the coordinates of the starting station should be added to the computed coordinates.

The coordinates of at least two boundary corners, each of which can be seen from the other, should be computed and recorded, and these corners should be marked with permanent monuments properly referenced. Thus a pair of private control points are established with respect to which all other corners can be located. Because of the connection with the state system of plane coordinates, every boundary corner, including the two used as control points, could then be obliterated and still be readily restored.

774. Uses of the Plane-Coordinate Control System. The state system of plane coordinates is particularly useful as a system of control for surveys and maps covering relatively large areas. Such surveys and maps are indispensable in planning extensive engineering projects such as highways, railroads, canals, power projects, reclamation projects, and flood control. The state system is equally important as a basis of control for state, county, city, and town surveys. It should be noted also that aerial navigation is greatly simplified by knowing just where prominent landmarks are and that accurate aerial surveying and mapping are dependent upon ground control points.

Although it may not be of great importance that the survey of a small area should be checked for accuracy by connecting it with a state control system, it may be of the highest importance to make this connection in order to establish permanently the locations of certain points in the survey, such as boundary corners. Moreover, if independent surveys of adjacent tracts of land are made on the same datum and are tied to the state system of control, these surveys and the corresponding maps can be joined at any time to cover

an extensive area. In fact, extensive surveys and maps are generally made in several sections with the control survey as a framework.

774(*a*). APPLICATION TO LAND BOUNDARIES. For a given pair of plane coordinates based on a state coordinate system there can be only one point on the earth's surface; consequently there is no more simple and certain method of defining the position of a point on a boundary line than by its plane coordinates. The use of a state coordinate system in the deed description of a tract of land is illustrated in Fig. 798(*f*), p. 1010. In this case, the plane coordinates of each of two intervisible corners are given. The coordinates for each of the remaining corners, if desired, can be easily computed. The field work of connecting the boundary survey with the state system of plane coordinates is explained in Art. 773(*b*). Another method used in deed descriptions is to give the plane coordinates of *each* boundary corner and of every intermediate monumented point on the boundary if there are such points. This method eliminates the necessity of giving the lengths and directions of boundary lines. By either method of description the boundary corners are so tied to the state coordinate system that should all monuments, reference marks, and other physical evidence of the positions of the corners be obliterated, the positions themselves cannot be lost so long as there is available a record of their plane coordinates. It should be noted also that if the spherical coordinates of any corner are desired, they may be readily computed from its plane coordinates. The advantages of connecting a boundary survey with the state system of coordinates are many, but it is obvious that unless state control stations are within a reasonable distance of the boundary survey the connection with the state control system is difficult if not impracticable. In order to insure a uniformly high grade of surveying in connecting a boundary survey with the state control stations, a half-mile limitation is specified in the laws of some states. For example, in the New York State law this limitation is stated as follows.

> No coordinates based on the New York Coordinate System purporting to define the position of a point of a land boundary shall be presented to be recorded in public land records or deed records unless a point in the survey is within one-half mile of a triangulation or traverse station established as prescribed in section four of this act, or unless the department of public works which is the agency hereby authorized to administer this act shall by rules and regulations increase or decrease such one-half mile distance for the whole state or any area or areas thereof.

An important step toward facilitating the use of a state coordinate system for the locations of property corners is to define and

name the system officially so that when reference is made to it there can be no doubt as to what is meant, and so as to preclude improper use of it, or ambiguous reference to it. This has been done by legislative enactment in a number of states.

In rerunning an old survey and in restoring lost corners, it is best, if practicable, to connect the boundary with the state control system. In at least one state, it has been the aim to restore and remonument all the government public land corners and to establish these corners permanently by determining the plane coordinates of each.

775. Advantages of the Plane-Coordinate Control System. (1) Standardization and unification of surveys. (2) Increased permanency of surveys. (3) Greater accuracy in surveys. (4) Reduction of cost and elimination of economic waste in surveys. (5) Better land surveys and land records.

775(*a*). STANDARDIZATION AND UNIFICATION OF SURVEYS. The fundamental reason for the importance of state plane coordinate systems is that they provide an easily usable yet uniform horizontal datum and control for the entire country. In the past, surveys, whether of the same piece of property or of different pieces of property, were generally entirely separate and had little or no relation to each other. Moreover, these uncontrolled surveys were based on an unlimited assortment of assumed, magnetic, or true meridians, and levels were based on every conceivable kind of datum plane.

Since state control systems are connected with the national level and triangulation nets, all surveys properly connected with a state control system become a part of one great national survey. This makes it possible to coordinate adjacent surveys and to combine data obtained from one survey with data obtained from another, regardless of whether the surveys were made concurrently or at different times. If, for example, the surveys of adjacent towns or cities are based on the same system of state control, including the same level datum, the surveys of one community can be coordinated with those of another, duplication of surveys can be avoided, and the maps and other data which belong to one community can be combined with those which belong to another. Because of mutual community interests, such coordination and combination are highly desirable. Similarly, it is often advantageous to coordinate and combine surveys of adjacent tracts of privately owned land.

In some states which have adopted the plane coordinate system, the aim is to be prepared for every type of survey requirement and to make it possible eventually to determine the relative position and elevation of every topographic feature or boundary line in the entire state with little expenditure of time and money. This means that eventually every survey in the state should be correlated with the whole. A controlled survey thus permanently connected with the state system can be drawn upon at any future time for many other purposes aside from that for which it was originally intended.

775(*b*). PERMANENCY OF SURVEYS. Monuments have long been regarded as the ultimate in marking boundary corners, yet the loss of innumerable corners shows the fallacy of relying on the corner monuments alone in delimiting land. Coordinates of a property corner will give permanent location to that corner as they will connect it to all the monuments of the control system as witness marks. "There is safety in numbers." All this is true not merely with respect to a boundary corner but also with respect to any other point whose location on the ground should be permanently fixed.

775(*c*). GREATER ACCURACY IN SURVEYS. When a survey is sufficiently extensive to be connected with two or more control stations of the state system, its degree of precision can be easily and conveniently checked by means of those stations. Moreover, when a surveyor knows that his survey is connected with the state system and that consequently it may be of future use

as a part of that system, he is likely to establish lines with greater accuracy and to be careful to monument important points as permanently as may be practicable. Too often in the past, surveys have been considered of minor or of temporary importance when, had they been made a little more accurately or monumented more permanently, they could have been used later for other than the purposes for which they were originally intended.

775(*d*). REDUCTION IN THE COST OF SURVEYS. There are numerous ways in which the use of state control stations may reduce the cost of a survey. For example, the cost of checking a survey, particularly of an open traverse, may be reduced or eliminated altogether. Moreover, by the use of a survey previously made and connected with the state system or by the combination of two or more surveys so connected, much duplication may be avoided with a corresponding reduction in cost. The economic loss of thousands of uncontrolled, unrelated surveys of all kinds and of transitory value has run into millions of dollars in a single state.

775(*e*). BETTER LAND SURVEYS AND LAND RECORDS. In no type of surveying is the use of the state coordinate system of greater benefit than in boundary surveying. With the adoption and development of the state systems most of the difficulties encountered in boundary surveys, particularly in resurveys, will gradually be reduced or eliminated. New boundaries will become more definitely and permanently located, and in the future there will be less necessity for rerunning boundary lines. Problems in the subdivision of land will gradually become less complex as mistakes and inconsistencies in the locations of existing boundaries are eliminated.

There is no more simple and definite method of describing the location of a boundary corner than by giving its plane coordinates; hence when property corners are described on the basis of a state coordinate system there is no doubt as to the meaning of the description. Moreover, it is possible for any surveyor to remark the property corners by recourse to the description alone. When once this is understood by property owners and by others concerned with real estate, confidence in the work of the surveyor will be more general. In the long run, the use of the state system of plane coordinates is bound to reduce the opportunity for disputes, confusion, litigation, and the resulting damage to title.

776. Difficulties in Securing the Adoption of the Plane-Coordinate System. Until closely spaced state control points of unimpeachable accuracy have been established throughout a state, there will be localities where it will be inconvenient or impracticable to connect a survey with the state system. Moreover, discontinuity may exist along state boundary lines and along lines separating the zones in a large state. Eventually, however, the state systems will provide overlaps so that no ordinary survey need be separated into parts because it lies across a state or a zone line.

The principal difficulties in securing a country-wide adoption of the state coordinate system are those generally inherent in the change from an old established practice to a new one. The plane coordinate system is little understood by the average person and may seem too mathematical and complex for common use. It is essential to win the confidence of lawyers, land owners, and real estate dealers in the simplicity as well as the value of the system by answering questions which are in their minds. Brief answers to some of the most common questions are suggested here.

776(*a*). Common questions.*

1. Does the adoption of the system of plane coordinates by a state make its use in that state obligatory? No, but the progressive engineer will promote its use wherever it is practicable and advantageous.

2. Would property lines be changed? No. The system may be regarded as simply a new method of expression. It does not change in any way the position of a boundary corner or other point whose location has been described by some other method of expression even though the original description resulted in an erroneous position of the point. There can, therefore, be no conflict with former descriptions and surveys.

3. Would the system conflict with other systems such as township surveys, army surveys, and public-land surveys? No. The use of the system would simply bring about a designation of the fixed points of the existing survey by a new, more accurate, and more permanent method of expression.

4. How should plane coordinates be applied to existing land descriptions? This would be done only in the event of the making of a new survey, at which time one or more corners or points would be fixed by their plane coordinates.

5. Does the use of the system involve a redescription of land in the states where the U.S. System of Surveying Public Land has been established? In the normal course of events the plane coordinate system would be used only in surveys yet to be made. It is recommended that the section, township, and range be stated in the deed; that supplementary survey data expressed in terms of state coordinates, wherever such data are in fact required, be included in the deed, either by statement or by reference to recorded plots or other official records. If a new survey is to be made—covering land now under a government rectangular survey, the points would be located by coordinates so that in time all the land may be so located. The lines would remain in the same positions and the same section and township designations would be used. There is no need of going back and redescribing all the land previously surveyed regardless of the pendency of any transaction relating to it.

6. Is it necessary that there should be agreement upon the location of boundaries before using the system? No. If there should be a dispute concerning a boundary, it would be decided by one of the usual methods and the results of the decision could not be lost through physical changes in existing monuments. It would always be possible and easy to relocate the lost monuments and to re-establish the points and lines which had been adjucated.

7. What weight would a description by coordinates have in relation to other types of calls in a deed, such, for example, as a call by metes and bounds? With the use of the plane coordinate system there probably will be a tendency to consider points fixed by that system as assuredly more definite and certain than other means of location, and quite possibly as more material.

8. Will the state system involve lawyers in a responsibility beyond what may reasonably be expected of them? No, because under this system, attorneys are not taking any responsibility that they have not already assumed when reference is made to a map of any kind.

777. Supplemental Control Surveys. When an accurate survey is required for a project of considerable magnitude and it is desired to connect the survey with the national or state system of control, it is often necessary to make a supplemental control survey. In areas where property values are relatively high or where existing control stations are several miles apart, supplemental triangulation, based upon the national geodetic triangulation net, may be necessary as explained in Art. 767. In many cases, however, an adequate control system can be established by traverses controlled

* Condensed from the Second Progress Report of the Joint Committee of the Real Property Division of the American Bar Association and the Surveying and Mapping Division, American Society of Civil Engineers, *Proceedings of the A.S.C.E.*, June 1941, p. 1066.

by stations of the state system of plane coordinates. Frequently the two types of control are combined. For example, the horizontal control system established by a city may consist of a supplemental geodetic triangulation net of at least second-order accuracy based on the national triangulation net and a supplemental traverse control survey of second-order accuracy based on the state control system. Since the plane coordinates of each triangulation station may be easily computed from its spherical coordinates, the plane coordinates of every station, including triangulation stations, are known.

Any city, because of its limited area, can have a single plane-coordinate system for its area without involving objectionable errors. For example, in an area 37 miles wide, the use of such a system would introduce errors in boundary location not greater than 1 part in 100,000. For a farm survey where corners need not be located with such great accuracy, a single coordinate system could be used of almost any width. For example, the errors involved in a system covering an area 158 miles in width need not exceed 1 part in 10,000.

777(*a*). A GOOD EXAMPLE OF A SUPPLEMENTAL CONTROL SURVEY is that made for flood control projects in the Pittsburgh District of the Corps of Engineers, U.S. Army.* The principal features of this survey are: (1) A main triangulation arc of control tied to national arcs. (2) Local triangulation nets for several reservoir areas. Lines of the main triangulation arc formed starting bases and check points for these local triangulation nets. As soon as these local arcs were adjusted, geodetic positions were computed and then these positions were transferred to the state coordinate system. (3) Traverses controlled by local triangulation nets and computed on the state coordinate system.

An interesting and unusual feature of the survey is the control that was used along several river stretches where, because of the small horizontal clearances, the terrain was not suited to arc triangulation. On each bank of the river, a precise traverse was run to form a control system. These two traverses, separated by 200 to 400 ft., were connected by cross ties at intervals of not more than one-half mile by measuring across bridges or by triangulation. This system, used on each of several rivers and creeks, furnished rigid and accurate control at a relatively low cost.

Every supplemental survey of at least second-order accuracy, such as that just described, can be made, in effect, an extension of the state coordinate system by permanently monumenting each control point and by properly recording a description of the nature and location of each monument together with its plane coordinates.

777(*b*). SPECIAL FIELD METHODS AND COMPUTATIONS used in supplemental control surveys are those commonly used whenever first-order or second-order accuracy is required. The field work involves: (1) precise linear measurements (Chapter 8); (2) precise differential leveling, including a knowledge of the specifications for first-order and second-order leveling (Chapters 15 and 11); (3) precise measurement of angles (Chapter 27); (4) triangulation (Chapter 28); (5) precise traversing, including a knowledge of the specifications for first-order and second-order traversing (Chapter 23).

* This control survey is described in *Civil Engineering*, Vol. 9, No. 5, May 1939, published by the American Society of Civil Engineers.

Computations involve adjustments of angles and lines in triangulation nets and in traverses and the transfer in some cases from spherical coordinates to plane coordinates.

Planning a supplemental triangulation net or a supplemental traverse involves a suitable distribution of control points, i.e., an adequate number located where they will be most needed, and the designing of a triangulation net or a traverse with due regard to the strength of figure, favorable conditions for accurate field measurements, and low cost of field work.

To discuss in detail the establishment of supplemental control systems would require more space than can be allotted in this book. An excellent treatment of the subject may be found in Manual 20, entitled *Horizontal Control Surveys to Supplement the Fundamental Net,* published by the American Society of Civil Engineers.

778. Projections. The plane coordinate system of any state is based on the national system of geodetic triangulation. This relationship between a plane surface and the spheroidal surface is such that the plane coordinates of any triangulation station within the area covered by the state system can be computed from its spherical coordinates (latitude and longitude). The scheme which thus relates each point in the curved surface to a definite point on the plane surface is called a "projection." A conformal projection is one which conserves the angles and therefore the shape of the area projected.

The U.S. Coast and Geodetic Survey uses two types of projections in developing state systems of plane coordinates. For a state of greater extent east and west than north and south, the *Lambert Conformal Conic Projection* is used; for a state of greater extent north and south than east and west, the *Transverse Mercator Projection* (which is also conformal) is used. In the application of either projection there is a limit to the extent of the spheroidal surface which can be projected without involving errors in the corresponding plane coordinate system too great to be permissible. Some of the smaller states come within this limit, and consequently for each of these states a *single* plane coordinate system, covering the entire state, has been developed. For the larger states, however, it has been necessary to subdivide the state into two or more coordinate zones each with its own origin of coordinates, i.e., each is a complete and separate system in itself. However, for all the zones in any state, the same type of projection has been used except in Florida, where both types have been utilized.

It is not the intention to explain, in the articles that follow, the mathematics of the procedure by which the plane coordinates of any point in a state are computed from the latitude and longitude of that point, but rather to give such general information con-

cerning the Lambert and the Transverse Mercator projections as the average surveyor may desire to have.*

779. The Lambert Conformal Conic Projection. Let it be assumed that a state of greater extent east and west than north and south is sufficiently small to permit the use of a single system of plane coordinates. The procedure in the use of the Lambert conic projection for the development of the system is as follows. Two standard parallels of latitude are selected, one a certain distance south of the most northerly point in the state, the other an approximately equal distance north of the most southerly point. A conical surface, with its apex in the axis of the earth produced through the North Pole, is imagined as passing through both the selected parallels of latitude. This surface is used as a surface of projection. The band of conical surface between the two parallels is relatively so narrow that it has very nearly the size and the shape of that portion of the spheroidal surface of the earth which also lies between the two parallels. When, therefore, lines on the earth's surface within the two parallels are projected on the surface of the cone, there is very little distortion either in the relative positions of the lines or in their lengths. Moreover, this conformity extends for a considerable distance on either side of the band. The surface of the cone within the band is wholly *inside* the surface of the spheroid whereas the conical surface north and south of the band is wholly outside; consequently, points on the spheroid within the band are projected in one direction and points outside are projected in the opposite direction.

The relationship between positions on the spheroid and corresponding positions on the projection cannot be expressed graphically. However, by the use of the conical surface of projection, which can be developed (unrolled into a plane surface), mathematical methods are derived for determining the relationship between positions on the spheroidal surface and corresponding positions on a plane surface. Upon this relationship is based the method of computing the plane coordinates of any point in the state, such, for example, as a triangulation station, from the latitude and longitude of the point.

In order that the projection on the conical surface may be as conformal as possible, the two standard parallels of latitude should be selected so that the distance between them bears a certain relationship to the distance between the northern and southern limits of the area to be projected. In Massachusetts, for example, the parallels selected by the U.S. Coast and Geodetic Survey are the north latitudes 41° 43′ and 42° 41′. The distance between these two parallels is approximately two-thirds of the distance between the northern and southern limits of the state, and each parallel is therefore approximately one-sixth of this distance from the corresponding state limit.

A distance on a horizontal plane which corresponds to the distance between two points on that portion of the earth's surface within the two standard parallels is slightly shorter than the corresponding distance on the surface of the earth, but outside the parallels the reverse is true. Only a distance between points on a standard parallel is the same on the plane as on the earth's surface. However, these discrepancies in distances are negligible in ordinary plane surveying.

In order that a desired degree of precision may be maintained, the distance between the northern and southern limits of the area covered by a state system developed by the use of the Lambert projection should be limited. Since the maximum error of closure accepted in a second-order traverse is 1 : 10,000, it is natural to accept this ratio as a criterion. This ratio will not be exceeded on a Lambert conformal conic projection if the distance between the northern and southern limits of the area does not exceed 158 miles. In Massachusetts, for example, the north-south dimension is considerably less, with consequent reduction of error in scale distances.

The statements made in the discussion of the application of the Lambert projection to the area of a small state apply equally well in principle to the application of the projection to the area of a zone in a large state.

The origin of coordinates is chosen so that the Y axis will be west of the

* Information pertaining to the mathematical development of the state plane coordinate systems as well as tables for use in the computation of coordinates may be obtained by application to the Director of the U.S. Coast and Geodetic Survey, Washington, D.C.

788(*a*). Plats

The first step is the survey for and the preparation of an *original plat* for the architect's use. This work has already been outlined in Art. 391.

The *architect's plat,* furnished the engineer by the architect, shows the location of the building with respect to street lines and other features given on the *original plat.* A considerable period of time may elapse between the date when the survey for the original plat was completed and the date when construction is begun. During this period changes may have occurred in street lines or in other important features; hence the engineer should check by field measurements the more important distances and angles given on the architect's plat.

788(*b*). Offset lines

It is usually impracticable to work from transit lines that coincide with street lines, hence an offset transit line, parallel to the street line, is generally run along the sidewalk. The offset distance to this line should be such as to provide for a clear line of sight during construction, say a distance of 5 ft. This offset line should be referenced so completely that there is no possibility that it will be lost. When the building is on a street corner, an offset line is established on each street.

788(*c*). Reference lines

Reference lines are lines parallel to the main lines of the building, so chosen and established on the ground that they can readily be used in locating all other lines of the building. Since the main lines of a building are usually either parallel with or perpendicular to the street lines, these reference lines are also usually parallel with or perpendicular to the street lines. They form, therefore, a system of rectangular coordinate lines. Any point of the building can be located by laying off two coordinate distances, one from each of the two nearest or most convenient coordinate lines. In establishing this system of reference lines the offset line on the sidewalk is used as a base.

It is not essential that any reference line should coincide with an axis of the building, nor is it essential that parallel reference lines should be equally spaced. On the contrary, each reference line should be established (1) where it will be as free as possible

from obstruction during construction and (2) where it will best serve for the location of important points of the building. It will be necessary to study the floor plans carefully in order to lay out a system of reference lines to meet these two requirements. In general, reference lines should be so located that they: (1) Will pass through windows on all floors. (2) Will pass through corridors on all floors. (3) Will be at intervals of 100 ft. or less in order that the maximum distance to be measured will not greatly exceed 50 ft. (4) Will be about 6 ft. from partition walls so that scaffolding will not interfere. (5) Will avoid such obstacles on all floors as chimneys, fireplaces, stairways, and steel columns. It may not be practicable so to locate reference lines that all these requirements will be met, but all of them should be considered when planning a system of lines which will serve for the duration of the construction work. When the building is on a street corner and the angle at which the two street lines intersect is other than 90°, two systems of coordinate lines may be established, one system with the offset line on one street as a base, the other system with the offset line on the other street as a base. The number of coordinate lines in a system depends upon the size and nature of the building. Often not more than one or two lines perpendicular to the street offset line and one or two parallel to that offset line will suffice. Each end of each reference line should be carefully referenced by means of several points.

Suggestions. (1) Reference points *outside* the area of construction activity are preferable. (2) When several reference lines are laid out perpendicular to the same line, each should be set by several repetitions of the 90° angle, then at the far ends of the lines distances between the lines should be measured in order to check the transit work. (3) It is well worth while for the engineer to draw his own *reference plan*. This plan need not necessarily be to scale, but it should show clearly by dimensions: (*a*) the location of reference lines with respect to street lines and with respect to each other; (*b*) the location of steel columns with respect to the reference lines. (4) Street lines on the reference plan should be marked as such, and each reference line should be designated by letters or numbers. (5) Each column and its designation should appear on the reference plan. It should be indicated in such a way that it will be clear whether its web is parallel with or perpendicular to the street line; otherwise mistakes may occur in setting anchor bolts. Distances between the centers of columns are shown on the reference plan in feet and inches as copied from the architect's plan. For convenience in chaining, distances of reference lines from street lines and street corners and distances of centers of columns from reference lines are shown in feet and decimals of a foot. The location of the center of a column is usually shown by the cumulative distance from the nearest reference line, measured along the line through column centers. (6) The reference plan should be thoroughly checked against and corrected to conform with each revised architect's foundation plan immediately after that plan is received. (7) Because the engineer will refer to his reference plan frequently, it should be small enough to be carried conveniently, yet large enough for all figures and notations to be legible. On a project covering a large area, several reference-line plans showing adjoining portions of the area may be more convenient than one large plan.

788(*d*). Use of reference lines

After the main excavation has been completed, reference lines are transferred to batter boards. The strings that are stretched between these batter boards now become the tangible coordinate lines for locating: (1) the excavation for the footings, (2) the forms for the footings, (3) the wooden blocks inside the forms for holding the column anchor bolts in place, (4) the billet plates (column-bearing plates).

When the erection of the steel frame is completed, the centers of the steel stairways are located by measurements from reference lines, and these centers are marked by chisel cuts on the steel floor beams. After the concrete floors have been poured, reference lines are placed at convenient intervals on these floors to be used in laying out partition walls and special details such as fireplaces.

In addition to the main reference lines, it is often desirable to establish supplementary lines. For example when there are several columns on the same line, this line may be established on the ground. The transit, set up on this line, can then be used for lining in footings, anchor bolts, and billet plates. Later a small offset from this line may be used for setting the wall-line batter boards.

Suggestions. (1) Short, rust-resisting nails such as "PK" nails, capable of being driven into cinder concrete, are convenient for reference points in the floors. To protect these points, it is well to have made up pieces of plywood 6 in. × 4 in. × 5/8 in., with a 1-in. hole bored in the center through which the reference point can be seen. These reference-point guards should be placed with the larger dimension along the line they indicate, and fastened to the floor with six-penny, steel-cut nails. (2) Transfers of reference lines as the construction work progresses should be made well ahead of time if there is any possibility that the old lines will be lost. (3) In spite of all precautions, a portion of some reference line may become lost. The engineer may then have to devise some method of restoring the line. For example, it may be necessary to offset around an obstacle, or it may be possible to suspend a plumb bob through openings in the floor, and, by using equal offsets on each floor, to restore the line.

788(*e*). Bench marks

The same bench marks that were used in making the original plat should be used as starting points in establishing other bench marks to be used for purposes of construction. These new bench marks are generally on column footings and on the columns themselves. The elevations used in building construction are in feet and inches.

Suggestions. (1) It is well to run a set of levels to check the given elevations of the bench marks used in the original plat plan before establishing

new bench marks. (2) Care should be taken in converting feet and inches to feet and decimals or vice versa since this simple process is a source of many mistakes. (3) A bolt about 6 in. long and ½ in. in diameter, imbedded in the base of a column footing before the concrete is set, makes a satisfactory bench mark, and such bench marks are among the first to be established. The bolt should be set in the corner of the footing base so as to be away from the column. Bench marks should be located so that they will be visible from as many points as possible. For example, if a bench mark is placed in a footing base near the corner of the building it can be seen from at least two directions. (4) Enough bench marks should be established so that if piles of materials obscure one, some other bench mark will be near enough to be visible. (5) As construction progresses, bench marks in the footings should be used rather than bench marks outside the building in order that there may be the proper intervals between the floors even though there is some settlement of the building.

788(*f*). Four-foot level marks

After the steel columns have been erected, plumbed, and guyed, bench marks known as "4-ft. level marks" may be set. The practice is to use an elevation of 4 ft. above the finished floor elevation on each floor. A notch may be cut in the outside corner of a column flange, the top of the notch being at the exact elevation. In the basement, the 4-ft. level marks are established from backsights on footing bench marks. Starting from the notches in the basement, measurements with the tape are made up along each of a few selected columns in order to establish 4-ft. level marks on each of the various floors.

Suggestions. (1) After the level has been set up on any floor, a backsight reading may be taken on a carpenter's rule held with the zero end on a 4-ft. level mark previously established for that floor. This reading is the distance that the level line of sight is above or below the 4-ft. level mark. To establish other 4-ft. level marks, the rodman simply holds the rule in such a position that the line of sight strikes the same reading. The zero end of the rule will then be at the point where the new notch should be cut. The rodman may check every 4-ft. level mark by reversing the carpenter's rule and having the instrument man observe whether or not the zero end coincides with the horizontal cross hair. (2) As the brick walls go up, the outside columns will be enclosed, but by this time window frames will have been set in the walls and pencil marks on these window frames may be made at the 4-ft. level. Later, 4-ft. level marks may be placed on each of the door bucks, and still later, after the partition walls have been completed, 4-ft. level marks may be placed on the walls of the larger rooms in order that the sleepers or nailing strips in the floor may be set to the proper elevation.

788(*g*). Excavation

The excavation plan will show what lines should be staked out. Stakes should be set for all major corners, but minor offsets or jogs should be disregarded and the main lines carried through. Usually the excavation is made at least one foot larger all around than the neat lines. The excavation plan will also show the elevation of the bottom of the excavation and the location of ramps.

Elevations of the tops of the excavation stakes are determined and the cuts marked on the stakes in feet and inches to the nearest inch. Before excavation is begun, it may be necessary to take additional elevations on the ground surface for the purpose of earthwork computations, and as the excavation work proceeds cross sections may have to be taken for the monthly estimates.

788(*h*). Sheet piling

It is frequently necessary for the surveyor to give lines and elevations for *temporary* construction. An excellent example of such work is that done in the location of sheet piling and its braces. At first thought this would seem to be purely a matter of construction, but it will be evident from the following description of the work that, in order to insure the stability of the building as well as to protect the workmen, the sheet piling and its braces must be carefully located and constructed. Here again the surveyor must work in cooperation with the men who are responsible for the construction.

Timber sheet piling is commonly used when the excavation is so near the sidewalk that the slope of the soil is as steep as, say, 1½ horizontal to 1 vertical. In order that the braces and the "dead men" (anchors which take the thrust of the braces) may not interfere with the column footings, they should be located with due regard to the positions and elevations of the footings as shown on the foundation plan. Braces should make an angle with the horizontal of not more than 50° nor less than 40°; the positions of the dead men and the length of the braces will be governed largely by this requirement. When it is necessary to set a brace in such a position that the vertical plane through it is not perpendicular to the plane of the piling, the end of the brace in contact with the ranger (waler) should not be more than 3 ft. from a vertical line in the plane of the piling and opposite the center of the dead man.

It is common practice with 3-in. plank and 8 in. × 8 in. rangers to limit the distances between braces to a maximum of 8 ft. and the vertical distance between rangers to 5 ft. The top ranger is usually about 2 ft. below the ground level. Spikes alone will not hold the thrust at points where the tops of the braces are in contact with the rangers, and the contractor should place cleats to resist the thrust. If the sheet piling is to be pulled out later, the bottom of the piling should be kept at an elevation higher than the bottom of the excavations for the footings in order that the latter excavations may not be disturbed. A slope from the bottom of the piling to the bottom of the footing excavation of 1½ horizontal to 1 vertical may be sufficient.

788(*i*). Batter boards

Batter boards should be rigidly fixed (see Art. 782(*d*)). Points on the reference lines are set on the batter boards, and the carpenter foreman is given a copy of the engineer's reference-line plan so that he may locate the column centers and set the forms for the column footings.

Batter boards for wall lines may be set on the steel columns. A ⅞-in. board nailed to a short piece of 2 in. × 4 in., which has been wedged between the column flanges, will be found satisfactory.

788(*j*). Column footings

The excavation for a column footing is usually made one foot larger all around to provide space for setting and bracing the forms and for easy stripping after the concrete has set. A stake is set at the center of the excavation area by measurements from reference lines and the column number marked on it. Stakes to mark the four corners of the excavation are then set by measurements from the center stake. A grade stake is driven about one foot outside the excavation area, the elevation of the top of the stake is determined, and the amount of cut in feet and inches to the nearest inch is marked on the side of the stake toward the excavation area.

After the excavation has been completed and the forms have been set for the base of the footing, the location of the centers of the four sides with respect to the reference lines should be checked. Grade nails are then set to mark the elevation of the top of the base. The steel reinforcing mat is then placed to grade and, using lines and grade nails already established for the base, forms are set for the cap. The column anchor bolts are then secured in place, care being taken to make the line between the two anchor bolts parallel to the flanges of the H columns. (The direction of the flanges is shown in the foundation plan.) Finally the engineer should check the location of the centers of the four sides of the cap forms with respect to reference lines and, what is even more important, the location of the centers of the anchor bolts.

Suggestions. (1) When the stake at the center of the excavation area has been set, the positions of the four corner stakes may be determined quickly as follows. Set the target of the level rod a distance from the base of the rod equal to one-half the length of the side of the excavation. Hold the base of the rod against the center stake, sight the rod into line parallel to one of the reference lines, and set a temporary stake at the target, thus marking the middle of one side of the excavation. Set the rod against this stake, sight it in parallel to a reference line perpendicular to the one first used, and set the corner stake. Set the other three corner stakes in a similar manner and then check the lengths of the four sides to the nearest tenth of a foot. (2) Where there is a difference of elevation between adjacent footings, the foundation plan should be consulted in order to be sure that, in making excavations one foot larger all around, adjacent excavations have not been brought so near together that the slope from the bottom of one to the bottom of the other is too steep for stability. In ordinary soil, if a slope of 2 horizontal to 1 vertical is exceeded, there is danger of the ground yielding under the higher footing. In order to reduce the slope, the higher footing may be lowered and the cap made correspondingly higher by the addition of concrete.

788(*k*). Billet plates

Billet plates are small rolled steel slabs grouted in place on top of the footing cap to serve as bearing plates for the steel columns. Each plate has holes through which the anchor bolts pass. These holes are drilled somewhat larger than the bolts in order that the plate may be shifted into correct alignment before it is grouted in place. Billet plates are set to line by marking the center of each side and locating these centers from the reference lines. They should be centered under the columns, the positions of which are shown on the engineer's reference-line plan.

The center portion of the larger billet plates is usually milled to receive the milled end of the column. In setting such billet plates to grade, the base of the level rod should rest on the milled surface. Each billet plate rests on steel shims placed at each corner so that it may be raised or lowered to the correct elevation by inserting or removing shims. The surface of the plate may thus be made level.

After billet plates have been set to correct line and elevation, they should be firmly bolted in place so that they will not be disturbed when the grout is rammed underneath them.

Suggestions. (1) A list of the columns, billet plate sizes, and elevations, all obtained from the steel erection plans, should be entered in the engineer's notebook for quick and ready reference. (2) The billet plates which have been placed loosely on the footings should first of all be measured for size by the engineer to make sure that each plate is placed as called for on the steel plans. Some misplaced plates may have escaped the notice of the steel inspector. (3) A billet plate may appear to be level when tested by an instrument and yet the shim under one corner may not be supporting that corner. In this case, the plate will teeter when the corner is stepped on. All four corners should be thus tested.

788(*l*). Steel columns

The engineer should make sure that the columns, especially those at exterior corners, are plumb in both directions after the floor beams have been connected up. The transit should be set as nearly on line with the flanges as possible, and the flange followed from the base up with the vertical cross wire. The amount that the column is out of plumb at each floor may be estimated pretty closely and the results recorded. A slight bow may be observed in the column due to imperfections in the rolling, or, if a floor beam is not of the correct length, both columns may have this bowed appearance. In such a case, an attempt should be made to have the top of the column plumbed over the bottom. The steel specifications will indicate the tolerance (expressed by a frac-

tion) to be allowed in the length of each member. The columns are plumbed by the steel-erecting contractor, generally by means of guy cables which are tightened or loosened with turnbuckles.

788(*m*). Wall lines

The architect's plans usually do not show the dimensions between the wall lines and the individual column centers, but these plans generally will show dimensions between wall lines and the street lines. The engineer, then, must use both his own reference-line plan and the architect's floor plans to obtain the dimensions between the wall lines and the column centers. These latter dimensions may be recorded on sketches in his notebook.

Since there is usually some sort of water table or rollback course, the outside line of the basement wall will not be directly under the corresponding lines of the stories above. The engineer should note carefully which set of plans he is using, whether basement or first-floor plan, so that his measurements will not be in error by the amount of the rollback.

The basement wall lines will first be marked out on the column footings so that the carpenters may set the forms for the concrete wall footings. After these wall footings have been poured and have set, batter boards may be fastened to the steel columns for use in lining up basement wall forms. Batter boards may also be attached to steel columns on the upper floors, and points on the brick or stone wall lines are then set on these batter boards.

788(*n*). Floors

Before the cinder concrete floor arches have been poured, the engineer may be asked to find the elevations of the steel floor beams so that the forms for the floor arches may be set correctly. After the floor arches have been poured, the engineer should take final cross sections of the floor in order to determine whether the rough floor has been kept low enough to allow the wooden sleepers or nailing strips to be placed later at the proper elevation. In order to assist in laying the subfloors and sleepers, the engineer should place 4-ft. level marks at frequent intervals on the hollow tile partition walls of each room.

788(*o*). General suggestions

1. Work with the latest revised plans only.
2. Check the dimensions taken from the plans in as many ways and in as many places as possible. Use other lines of partial and overall dimensions at both sides of the building to verify results.
3. Detailed plans and detailed cross sections usually take precedence over general floor plans or general elevations.
4. Precautions should be taken to eliminate errors in measuring with the tape, particularly errors due to changes in temperature.
5. In setting all grades, the rodman should have the instrument man check calculations and rod settings before the grades are given. When a high rod is used, the rodman can hold the back vernier up so that the instrument man can check the setting through the telescope.

Caution. During the period of steel erection, and until the forms for the floor arches have been set, keep out in the open so as to be visible to the workers above. After the forms for the floor arches have been set, keep away from any open space or shaft; material may be thrown down without warning. Surveying instruments should not be set up in such spaces.

CHAPTER 58

SURVEYS OF LAND BOUNDARIES

THIS chapter is a résumé of the different kinds of boundaries and the different methods of marking boundary lines; the functions and responsibilities of a land surveyor; the more important laws pertaining to boundaries and the ownership of land; the different methods of describing land in deeds and the principal requirements to be met in preparing a deed description; the general methods of searching for the evidence which a surveyor may need in order to restore lost or obliterated boundary lines; and the general procedures followed in land surveys, particularly in resurveys. No explanation of the technique of running traverse lines and of locating boundaries with respect to such lines is given in this chapter since this whole subject is treated in Part I, and particularly in Art. 386.

789. Introduction. (*a*). LAND BOUNDARIES. A land boundary may be one that has been *artificially* established such as a straight, curved, or irregular line marked by monuments, or it may be a *natural* boundary such as a river or the shore of a lake.

789(*b*). BOUNDARY SURVEYS. (1) An *original survey*, i.e., the first survey made of the boundaries of a tract of land which have been previously described and defined. (2) A *resurvey*, generally made for the purpose of checking or of relocating boundaries previously surveyed and established. (3) A *subdivision survey* made for the purpose of subdividing land. (4) A *cadastral survey* is a survey of or pertaining to landed property, especially as to its extent, value, and ownership. The term, formerly applied to extensive land surveys, is becoming obsolete.

789(*c*). THE LOCATION OF BOUNDARIES. The ownership of a tract of land is transferred from one owner to another by means of a legal instrument called a *deed*. When, from the description in the deed and from physical evidence on the ground, the exact location of each boundary of the tract can be readily determined, a boundary survey may be made by one of the common methods explained in Arts. 386, 387. When, however, there is any uncertainty concerning the exact location of one or more of the boundaries, that uncertainty must be removed before proceeding with the survey. Obviously, the surveyor must "find the land" before he can survey it. This work is often more difficult and time consuming than the field survey itself.

789(*d*). DESCRIPTION OF LAND IN A DEED. The surveyor is most concerned with that portion of a land deed in which the land is

described. In an original survey, the surveyor must make the measurements and furnish such other data as may be needed for the land description. Frequently he must make one or more plats or maps to supplement the written description. Such a plat is often the most important part of the description. In many cases, the surveyor is best qualified to prepare the entire description. In a resurvey, the surveyor must interpret and use the description in the deed as the authoritative basis for the relocation of the boundaries.

789(*e*). METES AND BOUNDS. When in a deed description the bearings and lengths of the several boundary lines are given, the tract of land is said to be described by *metes and bounds*. Many of the original surveys were made with the surveyor's compass and chain; consequently the bearings given were generally *magnetic* and the lengths were recorded in rods or chains. In a modern survey, made with transit and tape, directions of boundary lines are sometimes described by their *true* bearings or by their *true* azimuths calculated from angles at the corners, and lengths are recorded in feet.

789(*f*). LAND RECORDS. All deeds should be copied or "recorded" in public records, generally called "land records," kept by a town clerk, a city clerk or registrar, a county recorder or registrar, or by some other legally constituted official, depending upon the statutes of the particular state in which the land is situated. Maps or plats which supplement the written description of the land should be filed with the deed.

The records just described are open for public inspection and are indexed for the convenient use of surveyors, professional searchers of records, realtors, and others interested in the transfer of land.

789(*g*). LAND COURTS. In some states a special court known as a "land court" has been established whose function is to adjudicate both land titles and land boundaries. The "Torrens" system of transferring title to land by means of a certificate, validated by the land court, has also been adopted in a number of states. This method of transfer closely resembles that commonly used in transferring ownership of shares of stock in a corporation.

790. Landmarks and Monuments. A landmark, used to designate the boundary of land, is any object, natural or artificial, by which the boundaries of land may be known and preserved.

790(*a*). CORNERS. A *boundary corner* is the point of intersection of two established boundary lines. When a boundary corner falls in a place where it is impracticable to mark the corner, a point may be established on each of

the lines leading to the corner at a known distance from the corner. Such a point is called a *witness corner.* (See Art. 822(*e*).) A *meander corner* is the point in which a boundary line intersects a meander line. (A term used principally in U.S. Public Land Surveys. (See Art. 821.))

790(*b*). MONUMENTS. Any object used to mark a corner or any intermediate point on a boundary line is known as a monument. Among such objects are a stone, concrete, or wooden post; an iron pipe or a drain tile filled with cement; a small iron pipe set in a block of concrete; a small iron pipe driven in the ground; a metal marker set in concrete below the ground; a bolt or a metal plug leaded into a hole drilled in outcropping rock; a cross or other identifying mark chiseled on bed rock or on a boulder; a stone with an identifying marked place below the surface; a mound of earth above a buried stone; a mound around a tree; a mound or a pit with a charred stake or a small deposit of charcoal as a subsurface mark; a pile of stones, sometimes with a stake in the center; a substantial wooden stake.

790(*c*). DETAILS OF MODERN MONUMENTS. (See also Art. 335.) To be permanent, a monument should be made of durable material and be as immovable as practicable. These requirements are best met by a stone or concrete post 4 to 10 in. square or by an iron pipe about 3 in. in diameter filled with concrete. A concrete monument is often cast in the form of a frustum of a pyramid in order that its sloping faces may increase its stability. An iron pipe monument may be made more stable by splitting and spreading its lower end to form a base. A monument should extend to a depth in the ground of not less than 3 ft.; in cold climates this should be increased to at least 4 ft. in order that the lower portion of the monument may extend well below frost line. In some states a minimum depth of 4 ft. is required.

Modern monuments used in federal, state, county, and municipal surveys are generally capped at the top with a cast bronze or metal disk on which the corner point is marked. This disk may bear such information as the name or initials of the survey organization, elevation above sea level, and other basic data. When no metal cap is used, the corner point may be marked by a hole drilled in the top of the monument or, better still, by a bolt or metal plug cemented or leaded in a drilled hole. The exact point may be indicated by a prick mark in the plug. Another method is to chisel a cross or other identifying mark in the top of the monument. Monuments are generally set with their tops nearly flush with the ground surface. When greater protection is desired, a subsurface monument with its top 6 in. or more below the surface may be used. Access to such a monument is provided by a hand hole or shaft with a cover.

790(*d*). THE LOSS OR DISPLACEMENT OF MONUMENTS. The common term "permanent monument" may be misleading since slight but measurable movement is likely to occur even when a monument is carefully designed, set, and protected. Slight movement during a period of years has been detected in monuments that seemed permanent. No artificial monument should be considered absolutely permanent. The loss or displacement of monuments through construction or natural causes is sufficiently frequent to justify skepticism in the lasting properties of physical markers. For this reason, important monuments should be referenced or tied in with relatively permanent points (preferably points established by a governmental control survey) in order that they may be easily relocated if lost or displaced.

Corner marks or monuments may be displaced or destroyed during new construction, particularly in the older sections of a city where buildings are being demolished and reconstructed and sidewalks are being torn up. For this reason, in some cities, monuments marking street lines or property lines are placed on lines offset a whole number of feet from the street lines instead of on the street lines themselves. It is well when practicable to have such offset lines at least 2 ft. inside the curb line and 5 ft. or more from the property line.

790(*e*). FENCES AND STONE WALLS are important landmarks particularly in rural localities. The surveyor must know, however, where in a fence or a wall the boundary line is supposed to run. Local usage with regard to this differs somewhat in different communities, and when the location of a line is doubtful, the surveyor should consult local authorities, including carpenters and masons who build fences and walls.

Although no hard and fast rules can be given for the location of boundary lines in fences and walls, the following general statements are in accord with

common practice. They should never be accepted as true, however, unless confirmed by a knowledge of local usage:

In a *rail fence,* such as a Virginia rail fence, the line runs through the middle of the fence. Under the ends of the lower rails of such a fence one or more fairly flat stones are often placed to keep the rails off the ground. These "stagger stones," spaced only a rail length apart, often serve as evidence of the location of a fence after the fence itself has disappeared.

In a *picket or board fence,* the line generally runs between the bottom stringer and the posts, so that the posts are entirely on one side of the line and the remainder of the fence on the other. When, however, the stringers are sunk their full depth into the posts, both the posts and the stringers are on one side of the line and the pickets or boarding on the other. The line should be taken not at the upper part of a post but where the post enters the ground. The posts at the ends of a fence are more likely to be in a correct position than those elsewhere. When posts are leaning badly, it may be advisable to take the average of the bottoms of the faces of the posts as the true line.

A *wire fence* is generally built with the posts wholly on one side of the boundary line and the wire on the line.

In a *stone wall* the line generally runs through the center of the wall unless it is a bank wall. The exposed face of a bank wall is generally a little back of the line.

The statements that have been made concerning the location of boundary lines in fences and walls do not always apply when a boundary line is along a street or highway. This is particularly true of a picket or board fence. The surveyor should, therefore, ascertain and follow the practices concerning highway and street boundaries which prevail in the locality in which he is working. In any case, a fence or a wall has no value as evidence of the location of a highway boundary or of a street line if the line is defined by permanent landmarks (such as monuments) or by municipal ordinances.

790(*f*). Ditches and balks. In rural districts *ditches* are sometimes used to mark boundary lines. Generally the line runs along the center of the ditch and is often marked by a stake at either end or by a line of stakes.

A *balk* is a low ridge of earth which marks a boundary line. Generally the line runs through the center of the balk. When a balk is formed by excavating a ditch and throwing up the earth along one side, it is often difficult to determine whether the line is marked by the ditch or by the balk. When a boundary line lies between two cultivated fields, the level of the cultivated fields is likely to become gradually lower while the strip of land between may become raised. The low broad balk thus formed is often an eradicable mark of an ancient boundary.

790(*g*). Line trees. A common method of marking a boundary line, especially in woodland, is by means of marks on trees. A tree at a boundary corner is often marked in some distinctive way as, for example, by three notches or blazes cut on one side of the tree where the boundary lines intersect; sometimes the marks are on all four sides. A tree on line between corners is often marked by a blaze where the line strikes one side of the tree and by a second blaze where the line strikes the opposite side. When a tree is slightly off line, it may be marked by a blaze on the side of the tree toward the line. Blazes are sometimes painted; red is the color commonly used, but a light yellow or orange is more easily seen in the woods. The age of a blaze can generally be determined by cutting into the blaze crosswise of the grain and counting the annual rings.

A tree made to grow with its trunk in a horizontal position along a boundary line is a common landmark in some localities. Such a tree, sometimes called a *lopped tree,* is formed as follows. A sapling standing on or near the line is cut about halfway through in such a way that it can be bent parallel to the ground and in the direction of the line. The tree continues to grow, but its main trunk (or at least part of it) remains horizontal and on line. In some instances, the trunks of such trees, grown to considerable size, are the sole remaining evidence of the location of a boundary.

790(*h*). Natural boundaries. Among common natural boundaries are streams, rivers, lakes, ponds, shores, beaches, and trees. Such indefinite boundaries as ridges, bases of hills, and swamps are sometimes to be found specified in the land descriptions in old deeds.

Well-built and enduring fences, walls, highways, streets, and sidewalks, although the works of man, are legally classed as *natural* because of their

visibility, permanence, stability, certainty of identification, and independence of measurements considered simultaneously.

790(*i*). HIGHWAYS AND STREETS. When a boundary is a highway or a street, the boundary line may be the center line or it may be a side line, depending upon whether the right of the public is an easement or is in consequence of the public ownership of the right of way (Art. 797(*e*)). On modern highways and urban streets, lines are often monumented by the state, county, or municipality. These monuments are generally better maintained than private monuments. On rural roads, however, it is often difficult if not impossible to identify the center line or a side line, particularly if the road is old. In such a case, the line may be established by one of the methods of "establishment" (Art. 797(*b*)).

791. Riparian Boundaries and Rights. When any portion of a tract of land is bounded by water, the location of the corresponding boundary line is, in general, prescribed by state laws. These laws differ, however, in different states, and the surveyor should, therefore, be familiar with the general principles relating to riparian rights and with the statutes and precedents which govern in the state in which the land is situated. The laws pertaining to riparian boundaries are so varied and extensive that only some of the most general principles can be given here. An excellent treatment of the subject is to be found in the chapter, "Riparian Boundaries" in the book, *Legal Elements of Boundaries and Adjacent Properties*, by Ray Hamilton Skelton. Many of the definitions and statements in this article are taken from that chapter.*

791(*a*). DEFINITIONS. Among the terms used in defining riparian rights and boundaries the following are the most important:

1. *Navigable.* A river is navigable if the tide ebbs and flows therein, or if it is in fact navigable during some season of the year for the flotage of boats, lighters, logs, etc., and affords a channel for useful commerce of practical utility to the public.

2. The *banks* of a river are the elevations of land which confine the waters in their natural channel when they rise to the highest and do not overflow the banks.

3. *High and Low Water.* In general, in non-tidal waters such as streams, lakes, or ponds, high- and low-water marks are those lines to which the water recedes and advances at ordinary stages and not that occurring under unusual conditions.

The usual or ordinary high-water mark constituting a boundary on tidal water is not the limit of the monthly spring tides, but that reached by the neap tides, that is those occurring one or two days after the first and third quarters of the moon. Low water is the margin of the sea when the tide is out at ordinary low tides and not the line to which the tide ebbs when, from natural causes, it is at its lowest.

4. *Shores.* The shores of a river are the lands between high- and low-water marks. Likewise, the term seashore is that space of land between high- and low-water marks. For all ordinary purposes of a boundary, when the ocean or a bay or other body of water affected by the flux and reflux of the tide is made a limit, the words "seashore" and "sea" in the absence of any showing to the contrary appear to be synonymous.

5. *Thread of a River.* The thread of a stream or a river when called for as a boundary is the middle line between the shores, irrespective of the depth

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of the channel, taking them in the natural and ordinary stage of the water at medium height, neither swollen by freshets nor shrunk by droughts.

791(*b*). Where the boundary is the ocean or the sea. At common law, the title to that part of the beach below high-water mark is in the state. Consequently where land is bounded by the ocean, sea, or bay, or along the shore line of tidal water, the line of ordinary high water marks the limit of the grant; but if the body of water is distinctly called for in the deed, the water proper and not the shore is the boundary of the land conveyed.

In Massachusetts, Maine, and New Hampshire, the common law has been modified by colonial ordinances so as to give title to the soil of the shore as far as low-water mark, provided the tide does not ebb more than one hundred rods, and to that extent where the tide ebbs a greater distance.

In the three states just mentioned and in other jurisdictions where, by grant from the state, the seaward boundary of upland has been extended to the low-water mark, the construction of terms ordinarily used in the conveyancing of shore property is important. In general, terms "by the beach" (which is the equivalent of the shore) and "to the shore" exclude the land between high and low water; the terms "to the sea," "by the sea or beach," "along the shore," "along the bay," "on a stream," "by a stream," or "to a stream" include to low water; and land described as bounded by a named harbor has been held to include the flats between high and low water.

791(*c*). Where the boundary is a lake or pond. The Great Lakes have been classed with the tide water, and the same rules regarding riparian boundaries apply thereto, except in Michigan where the title of the upland owner is limited to the meander line which is presumed to have followed the existing shore when it was run. The land beyond the meander line is held by the state in trust for the public.

In some jurisdictions the title of a riparian owner extends to the middle of a navigable inland lake. In others it is limited to a shore line, and the title to the bed remains in the state in trust for public purposes. It is generally held that a boundary on a natural non-navigable lake carries title to the center. In a few jurisdictions, however, it is held to the contrary, and the fee is limited to the shore or low-water mark. In Massachusetts and Maine the decisions are based upon a colonial ordinance making every lake of 10 acres extent public.

791(*d*). Where the boundary is a river. It is generally held in the United States that private ownership of lands bordering on tidal and navigable water extends only to high-water mark, and the same is true where the stream is tidal but not navigable. However, the law pertaining to a navigable river as a boundary differs in different states. In a number of states, in the absence of any call in the deed to the contrary, it is presumed that a grant on a navigable fresh-water river is carried to the thread of the stream. In certain states, however, a boundary by a navigable stream takes to high water; in others it takes to low water. In Kentucky as regards riparian boundaries, all streams are non-navigable.

A grant or conveyance bounded by a non-navigable stream carries with it the bed of the stream to the center or thread of the stream and title to all islands on the grantees side within the limits of the side lines produced, but the general rule may be overcome by expressed intention to the contrary.

791(*e*). Meander lines. In surveys of the public lands of the United States, where land borders on water a *meander line* is run roughly following the bank or shore. (See Art. 821.) A meander line is not a property line unless it is so stated in the deed. In general, the ownership of the land extends to the shore line or water's edge of a lake, or to the bank of a navigable river, or to the bank of a navigable stream, or to the thread of a non-navigable stream. (See Art. 791(*a*).)

791(*f*). Property lines that determine riparian rights. These lines begin at the extremities of the dividing property lines on shore and continue into the body of water, but not necessarily in the same directions. When the property borders a stream or river, each line is generally established at right angles to the thread of the stream if it is non-navigable, or perpendicular to the low-water line if navigable. When the property borders a lake, the riparian property lines are established perpendicular to the center line of the lake, or, in the case of a circular shore line, toward the center of the circle.

In surveys of the public lands of the United States, the lines that determine riparian rights may begin (1) at the shore line or (2) at the meander line, depending upon which of two opposing lines of decisions is followed, but

the directions of the lines are generally established in accordance with the rules just given.

Shore lines frequently change owing to natural or artificial causes, and this shifting may result in an addition to riparian land made gradually and imperceptibly by the action of water (known as *alluvion*) or it may result in an addition to land due to the gradual subsidence of water (known as *reliction*). The term *accretion* is often used interchangeably for an addition to land by either process.

The surveyor should be reasonably familiar with the statutes and court decisions by which he should be governed in the survey of riparian boundaries, but there are so many exceptions to the general principles pertaining to riparian rights that he should proceed with caution in applying those principles. This is particularly necessary in a survey of land where riparian rights are valuable, as they often are. In no case should the surveyor assume the responsibility of making decisions concerning the legality of claims to such rights.

792. Difficulties. The difficulties encountered in boundary surveying arise, for the most part, in resurveys of old boundaries. Among these difficulties the following are the most important.

Inaccuracies, omissions, and inconsistencies in the original survey. Lengths and directions of boundary lines, because they were inaccurately measured, are inconsistent with the deed description or with the positions of existing landmarks. When the latitudes and departures of the boundary lines are computed the error of closure may be so great as to indicate great inaccuracy or possibly a blunder in measurement.

The description in the deed may be inadequate, or may contain errors, or may be inconsistent. Important measurements may have been omitted or landmarks inadequately described. The description may be such that no starting point or line can be established for the resurvey. Errors may have been made in transcribing data from the original survey or in copying and recording. There may be a geometric inconsistency in the description as, for example, when all four angles at the corners of a four-sided lot are described as right angles yet the length given for two opposite sides are different. (An actual case.)

The description, or part of it, may be based entirely on reference to adjoining land such as a highway or a tract of privately owned land. In such a case it is necessary to resort to an earlier description of the abutting land, which in turn may be indefinite or faulty in other respects.

Ambiguity in measurements given in the deed, or in some other statement in the description, may make interpretation of the description difficult.

Among other difficulties likely to be encountered are these. *Lost or obliterated* landmarks, or reference points, or other physical means of identifying points on boundaries. *Difficulty of determining the magnetic declination* at the time and place of a survey when the directions of the boundary lines are given in the deed by their magnetic bearings. *Conflicting court decisions or lack of court decisions. Reluctance of owners of adjacent land to agree upon the location of a new boundary line* when such a compromise is the best and perhaps the only method of settling a boundary dispute without recourse to the courts.

792(*a*). NOTE. If, in the resurvey of old boundaries, the surveyor can locate one corner and the direction of one line from that corner, a large portion of his difficulties are overcome. If the corner is marked by a natural or artificial monument he is fortunate, provided check measurements made from other points which he believes to be correct prove that the monument is in its original position. Frequently, however, the point of beginning is not marked, but must be found from the deed description of its location. This may be difficult if not impossible. For example, the point may be described as being so many feet along a center line or boundary of a highway measured from some other public line. Often neither of these lines can be found. Even if they are found, the surveyor may not be sure that the present location of these lines will agree with the location assumed by the person who first described the point of beginning. More evidence must then be brought to bear besides that found in the deed itself. So many elements must be considered in such a case that often more judgment than skill is required to establish even the initial point and direction.

The surveyor must be governed by legal principles, statutes, and court decisions pertaining to the ownership of land, and these legal aspects often present much greater difficulties than do the technical aspects of land surveying.

Improvements for land surveys and for title transfers are gradually being adopted which eventually should eliminate to a great extent diffculties in boundary surveying.

793. Functions and Responsibilities of a Land Surveyor. The main function of a land surveyor is to make boundary surveys for the purpose (1) of establishing new boundaries or (2) of re-establishing old boundaries. In connection with such surveys, it may be necessary for the surveyor to look up deeds and to interpret deed requirements; to set monuments or other landmarks and to reference them; to tie in the survey with control surveys; to furnish data that may be used in describing or identifying boundaries and frequently to write the deed description himself; to compute areas and dimensions; to make maps or plats; and to prepare extracts from field notes for filing with the deed.

When the information obtained from the deed and from the existing landmarks does not enable the surveyor to establish boundaries with certainty, it is his duty to seek additional evidence. When all the evidence which he can obtain is insufficient, his next step is to seek an agreement among the owners of the tracts of land involved as to the proper location of the boundary lines.

Should it become necessary to carry a boundary dispute to the courts, the surveyor should be of material assistance to his client's attorney in the preparation of his case, and he may serve as an important expert witness.

The land surveyor must accept responsibility for errors, mistakes, omissions, ambiguities or misstatements in his field work, field notes, office computations, maps or plats, deed interpretations, and deed descriptions. He is responsible also for failure to conform to laws or statutes governing surveys. The surveyor should be careful, however, *not* to assume responsibility which properly belongs to others. He has, for example, no judicial functions in questions of law and no judicial authority in the event of a dispute concerning the location of boundaries.

794. The Surveyor and the Law. This article is composed largely of extracts from an excellent paper, "The Surveyor and the Law," by Professor A. H. Holt, published in the *Proceedings of the American Society of Civil Engineers,* Vol. 67, p. 753, May 1941, and in the *Transactions Number for 1942,* p. 1379. It has been necessary to omit from that concisely written paper much that is important. Moreover, some of the abbreviated statements of law and procedure given here may not be as general as they seem; there may be exceptions. For a more complete treatment of the subject, the surveyor is advised to read the paper itself and to consult one or more of the books recommended at the end of this chapter.

794(*a*). QUALIFICATIONS OF A SURVEYOR. In most states of the Union, statutes provide that any person who is to do boundary (or land) surveying in the sense that he is to be in responsible charge of a boundary survey must be registered or licensed by an examining board. Thus, the representatives of the state have, in effect, pronounced the surveyor to have the minimum personal and professional qualifications that should permit him to practice. But what degree of care, skill, and judgment may a prospective client properly expect of him? The courts answer this question by quoting the phrases that have long been used in law such as the surveyor "must exercise that degree of care which a skilled engineer of ordinary prudence would have exercised under similar circumstances." One may desire a more definite criterion, but he will seek in vain. This standard must be used and interpreted in the light of the circumstances of each situation. The question must be resolved if it finally comes to a court—by expert testimony of those who profess to know what should be done.

794(*b*). DUTIES AND LIABILITIES OF THE SURVEYOR. The surveyor should make as thorough an examination of all reasonably available sources of information as circumstances seem to permit—and err a little on the side of safety in diligence of search. This advice applies to matters of record, to evidences of boundary or of occupancy to be found on the ground, and to any other items of information which his eyes or ears can discern. He must then pursue his field work and computations by such methods as will secure a degree of precision appropriate to the job and will eliminate the possibility of a mistake.

What happens if the surveyor fails to measure up to the required standard of care and skill? It is generally held that he is answerable in damages to his client for the loss or injury that results from such failure—at least for

any loss coming from the use of the land in a manner known to the surveyor to be contemplated at the time of the survey. This responsibility applies to a surveyor who has been registered or licensed as well as to one who has not, and regardless of whether the work is done in an official or a private capacity.

Warning. The knowledge and judgment required in boundary surveying obviously are best acquired from long and varied experience in that particular kind of surveying. A surveyor without that experience should be cautious about undertaking any but the simplest land surveys, not only because he may be held liable for damages resulting from incompetent work, but also because a poor boundary survey is likely to be a public detriment. Too many communities are already cursed with chaotic conditions due to erroneous and inadequate surveys.

794(*c*). Surveyor's right of entry. The survey of a boundary line will almost inevitably require going on other lands than those of the client. Ordinarily, if care is used to cause no appreciable damage, no objection will be raised by the owner or occupant of such land. And if and when such objection is raised, a frank explanation of the surveyor's need of permission to enter, a sincere promise of due care, and an explanation of the disinterested and non-partisan character of his work will often secure the needed permission. Such entry, however, cannot properly be claimed as a right unless the surveyor is acting in an official capacity or for a *legally authorized undertaking;* or unless, as in some states, the law provides that surveyors may enter upon land where titles are in dispute for the purpose of running doubtful or disputed lines and locating or searching for monuments. The invasion should be as brief and unobjectionable as practicable, limited strictly to the purposes of the survey. No reasonably avoidable damage should be done, and any unavoidable damage such, for example, as that due to clearing lines through timber, should be explained, preferably beforehand, and, of course, compensation should be made.

794(*d*). Finality of a survey. In proceeding with his survey, the surveyor may reach very firm and, in his judgment, very well-founded opinions as to where the lines with which he is concerned are, or should be. It is generally expected by his client that such will be the case, and that monuments of some kind will be set to represent such conclusions. The surveyor may find that his conclusions disagree from those of some other survey or from evidences of occupancy and use. May he, if he is "sure" he is right, "establish the boundary"? The same question is sometimes put differently by a client: "Is your survey 'legal'? Will it hold?" The answer probably should be that no survey, unless it is original and official, such as were the original surveys of the United States Public Lands, is "legal" in the sense in which people use that term in this connection. Nor will it necessarily "hold" without the occurrence of one of two things: (1) the acceptance of the result by those whose interests are affected or (2) a decree of the court that the survey represents the proper solution of the boundary problem involved. No modification of this statement is to be made because the surveyor may possess some official status, such as that of a city or county surveyor, or because that which such a surveyor has concluded is sanctioned by his superiors (the county commissioners, for example).

794(*e*). Agreement and arbitration. When there is doubt or uncertainty or a dispute has arisen as to the location of a boundary line, or when it would be disproportionately expensive, or even practically impossible, to establish the line with certainty and precision, the surveyor's best service, in the interest not only of his client but also of all concerned, may lead him to suggest that the best results from the standpoint of property rights and also with reference to the amicable relations of all concerned will be reached by compromise or by arbitration. If interested persons will either come to an understanding of all the conflicting elements involved and then hit upon and accept a solution, or if they will let the surveyor propose a compromise solution, with the reasons therefor, and then accept it, the chances are good that all will be happier than at the reaching of a conclusion in a court of law. After all, the decision of a court would have to be based on evidence, and it is unlikely that the court would have more evidence, or have it presented with less bias, than that in the possession of the surveyor. However, no suggestion is here made that any substantial well-founded rights of a client, properly enforceable in a court, should be recommended for surrender.

Nothing further than the parol (word-of-mouth) agreement of the owners concerned is needed to make their agreement binding; a written agreement is not required. A location agreed upon will stand even though subsequent developments show that it was not the true line. An agreement to fix a boundary line is void, however, if the owners know or if one of them knows that the agreed line is not the true line.

It may be that the parties will prefer, rather than to agree upon a location themselves, to submit the question to arbitration. Still assuming that the arbitrator is an honest and competent surveyor, that may be the very best solution. It is not necessary that the agreement to do so should be in writing, although, to prevent misunderstanding, a writing may sometimes be desirable. The award, resulting from an arbitration conducted in accordance with the terms of the submission, is valid and conclusive. It cannot be impeached because of an honest mistake of judgment on the part of the arbitrator.

All that is absolutely required to be done to make a location by agreement or by arbitration effective is to describe the boundary with certainty. However, to prevent a recurrence of the trouble, the line should be adequately and "permanently" monumented and platted, and the plat should be recorded with written, signed, and acknowledged acceptance of that which is shown thereon, or accompanied by a written submission to arbitration, and a certificate that the plat shows the result of the award.

Although acquiescence or agreement in a boundary line is binding on the parties thereto and on those claiming under them (grantees, heirs, etc.), only such persons are bound by the acquiescence or agreement.

794(*f*). The surveyor as an expert witness. If it becomes necessary to settle in court questions of boundary, the surveyor should be able to be of material assistance to his client's attorney; and this he can do better if he has some knowledge of the law concerned. Attorneys, especially those in general practice, are not always entirely familiar with matters of boundaries, with the engineering terms involved, with the practicability or difficulty of accomplishing, in special situations, things which seem theoretically possible. It should not disturb the professional pride of a lawyer to review his trial brief with the surveyor; likewise, a surveyor should not hesitate to consult a lawyer as to the correct procedure under certain circumstances. There must be a certain amount of "give-and-take" among the two professions to solve problems to their mutual advantage.

Ordinarily the surveyor will be used as an expert witness. That this may be done effectively requires that the examining attorney and the prospective witness go over the matter carefully beforehand. The witness may well explain first (if it is not already known to the attorney), just what the witness's testimony should show. Questions can then be framed which will bring these facts before the court clearly and expeditiously. If the surveyor listens carefully during the trial, especially as other surveyors give their testimony, he can frequently be of material help in catching points or suggesting questions of a technical nature.

794(*g*). Admissibility of plats and field notes. As a general rule, maps or plats of surveys are admissible in evidence regardless of whether they are made by an official surveyor or by a qualified surveyor in private practice. "Likewise plats and maps attached to and made a part of deeds or grants or referred to therein, either expressly or by clear implication, maps which have been recognized as correct by a former owner of the land, and ancient maps and plats when duly authenticated, are admissible to prove a disputed boundary. A map or plat referred to in the testimony is admissible as explanatory of it."

Unless a map or plat has been made a part of the deed, it must be verified or authenticated, and in the case of a surveyor's plat proof offered that the surveyor making the same actually traced the lines on the ground as depicted.

In situations quite analogous to those affecting the introduction in evidence of plats, the field notes of either public or private surveyors, when duly authenticated, are generally admissible.

794(*h*). Definition and ownership of field notes. Field notes have been defined as: "Notes made by the surveyor in the field while making the survey, describing by course and distance and by natural or artificial marks found or made by him where he ran the lines and made the corners." However, some usages seem to refer to a description *subsequently* "written up"

from the notes which were actually put down at the time measurements were made or objects observed.

It is believed that there is nothing that would lead one to think that there has been, or is, any custom of turning the field notes of private surveyors over to their clients. Nor does there appear to be anything in statute or reported decision to cause one to think that it is expected—let alone required. Field notes, particularly those in tabulated form, are generally of little use to a client unless he is familiar with survey work and notes; they may even confuse him. All the essential information that has been accumulated at the expense of the client may be turned over in a more understandable form. The most useful form (in addition to whatever physical evidences of the work are left on the site) will probably be a plat or map, together with whatever written explanation, if any, may be needed.

794(*i*). THE SURVEYOR'S REPORT TO HIS CLIENT. One of the chief elements of the surveyor's report to his client is an adequate description of the land surveyed. The two purposes which this description must serve should be kept constantly in mind. (1) It must make convenient and certain the identification of the tract of land for the purpose of tracing title. (2) It must be sufficient to enable any competent engineer to restore conveniently and at any future time a bound (or all bounds) which may have been obliterated.

For the first purpose, a written description is generally required. Highly useful for the first purpose, but absolutely essential in many cases for the second, is a plat or map. The surveyor should prepare the written description and the plat with the same care regardless of whether they are to be incorporated in the deed in the form in which he submits them or whether they are to serve merely as a basis for the deed description which is to be put in its final form by some other person. In any case, the surveyor should be familiar with what, in general, are the essentials of a satisfactory deed. (See "Deed Descriptions," Art. 798.)

794(*j*). THE SURVEYOR'S FINAL OBLIGATION. The surveyor should so do his work that it is unlikely that particular job will ever have to be done again. This requirement can best be attained by the use of distinctive and enduring monuments (Art. 790(*c*)). To economize by setting wooden stakes or other cheap monuments to perpetuate a survey is indeed "penny wise and pound foolish." The best advice that can be given to a client in this respect (and all too often the client must be "shown") is to add to the cost of the survey enough to monument it well. Finally, the survey should be "tied in," if practicable, with a municipal, county, state, or federal control survey (Arts. 774(*a*), 775).

795. Laws Pertaining to Boundaries. Laws affecting the surveyor's work in land surveying differ to some extent in different states. Moreover, any law pertaining to boundaries is generally subject to exceptions and restrictions. The land surveyor should not be expected to qualify as an authority on law or even to be familiar with all the complexities and intricacies of the laws affecting his work. Above all he should not be expected to make legal decisions; on the contrary he should be careful not to do so. Nevertheless, the land surveyor constantly encounters perplexing problems in boundary surveying in the solution of which he must be governed by legal requirements. The more he knows about these requirements, the better. He should keep in mind, however, that when the location of a boundary is uncertain or in dispute and the matter cannot be settled by agreement or arbitration, it is for the court to make the decision. This decision may be based on evidence which is only indirectly connected with surveying, as,

for example, evidence of undisputed possession for a number of years.

A general knowledge of laws pertaining to boundaries is particularly useful to the surveyor in preparing deed descriptions, in resurveying old boundaries partly obliterated, and in securing an agreement of owners in the relocation of uncertain or disputed boundaries. When, however, the interpretation of the legal principles involved is beyond the province of the surveyor, it is generally best for the surveyor to proceed with his work in accordance with able and explicit legal advice. In such a case he can protect himself by notifying his client in writing that he will be responsible for that part of the work which clearly and properly belongs within the province of the surveyor, but not for the interpretation of the legal principles involved.

Some of the laws defining landmarks and boundaries, both natural and artificial, are given in Arts. 790, 791. Other laws pertaining to the surveyor's qualifications, duties, responsibilities, liabilities, and rights are given in Art. 794. Laws governing the order of deed calls are outlined in Art. 796. Special laws pertaining to the ownership of land are explained in Art. 797.

795(*a*). Legal terms

Among the legal terms of particular interest to the land surveyor are the following:

Adjoiner. An owner of land adjoining a tract, his name being given in the deed or instrument by which the tract is conveyed.

Adverse Possession. A possession which is hostile, under a claim or color of title, actual, open, notorious, exclusive, and continuous. Such a possession, when continued for the required period of time (generally twenty years) gives an indefeasible right of possession or ownership, to the possessor, by the operation of the limitation of actions. In general, title by adverse possession cannot be acquired by one who enters into possession of real property by permission of the true owner, no matter how long the period of occupancy may be.

Alluvion or *alluvium.* An accession to land by the gradual or moment-to-moment addition of matter by the action of water, or (as used by some) by the insensible reliction of the water from its bank. In either case it belongs to the owners of the land to which it is added. (See also Art. 824(*c*) for accretion and alluvion.)

Avulsion. A sudden removal of land from the estate of one man to that of another by an inundation or current, or by the sudden change in the course of a river by which a part of the estate of one man is cut off and joined to the estate of another. The property in the part thus separated continues in the original owner.

Call. A reference to, or statement of, an object, course, distance, or other matter of description in a survey or grant requiring or calling for a corresponding object, etc., on the land.

Easement. An acquired privilege or right of use or enjoyment, falling short of ownership, which an owner or possessor of land has, by virtue of his ownership or possession, in the land of another, or, loosely, any of several rights which one person may have in the land of another.

Estoppel. The stopping of a person by law from asserting a fact or claim, irrespective of its truth, by reason of a previous representation, act, or adjucation inconsistent therewith. *Estoppel by deed,* estoppel which precludes a party to a deed, or instrument under seal, from denying, to the prejudice of the other party, anything stated therein. *Estoppel by record* or *estoppel by judgment,* which precludes the denial of the truth of anything appearing in the record of a final judgment. *Estoppel in pais* or *estoppel by conduct,* which, when a party whose conduct or language has caused another reasonably to believe in a certain state of things and (having a legal right so to do) to act upon the belief, precludes him from averring or setting up to the prejudice of the latter that a different state of things existed at the time in question. An intent to defraud or deceive is not essential to cause the estoppel. For example, when a grantor has pointed out a road as the boundary, he is estopped from denying the same as such. Or where the claimant of property has stood by and allowed it to be sold as the property of another without objection, the law holds him estopped from reclaiming it from the buyer.

Intent. The true meaning or purport.

More or Less. An expression used to signify that the exact quantity is unknown. It is intended, or should be intended, to apply to a small difference between the recorded and the actual quantity.

Parol. Executed or done by word of mouth, or by a writing not under seal. *Parol contract.* Any contract not of record or under seal.

Reliction. A recession of the sea or other water leaving land uncovered; also, the land so left uncovered.

795(*b*). Controlling calls

Control. Calls in deeds include reference to landmarks, distances, direction, adjoining land, and area. When a boundary line cannot be located with certainty from the calls in the deed description, the surveyor must weigh the relative importance of the calls. This is particularly true when there are conflicting calls. He should be guided by legal principles in deciding which call should be given the greatest weight. The call that takes precedence over another call is said to "control" or to be in "control of" that call. The control of one call over another has led to the expression "order of calls." The order of calls is based on the principle of evidence that "the certain will prevail over the uncertain." Each call in the order is, in general, more certain than *all* calls rated after it.

Order of Calls. In weighing the calls of a deed for the purpose of ascertaining the intention, certainty, and definiteness, precedence is given, in this order, to (1) natural monuments and boundaries over all other calls; (2) artificial monuments rank second, (3) adjoining land third, (4) courses and distances fourth, and quantity (such as area) last. This general rule should not be followed when it would result in a decision contrary to the expressed intention of the parties to the deed, or when, of two conflicting calls, the one ranked lower furnishes better evidence than the one ranked higher.

796. The Relative Importance of Deed Calls—Guide Rules. The old saying that "All sweeping statements are false—this is" suggests the limitations of almost any law, including laws pertaining to land boundaries and the ownership of land. General rules for determining the relative importance of different types of deed calls (order of calls) are subject to so many modifications, exceptions, and restrictions that in applying any one of them it is often essential to know its limitations. The ten rules which follow may be regarded as "sweeping statements" to be applied with intelligence and caution, but it is impracticable to discuss their limitations here. A thorough and able treatment of these rules and their application is to be found in *Legal Elements of Boundaries and Adjacent Properties*, by R. H. Skelton. The following rules are quoted almost verbatim from that excellent book.*

796(I). CONTROL BY INTENTION. *The cardinal rule for the interpretation of deeds is to discover the expressed intention of the parties, gathered from all parts of the instrument, giving each word its due force read in the light of existing conditions and circumstances at the time of conveyance. It is the intention definitely expressed in the instrument that controls, not intention merely surmised. This cardinal rule controls all others. A construction which is consistent with all the terms of the description should be given, rather than one consistent with some of the terms.*

How is the intention to be ascertained? "The only rule of much value—one which is frequently shadowed forth but seldom, if ever, expressly stated in books—is to place ourselves as nearly as possible in the seats which were occupied by the parties at the time the instrument was executed; then, taking it by the four corners, read it."

796(II). CONTROL BY LINES MARKED AND SURVEYED. *The lines and corners actually located upon the ground by the parties to a transaction prevail, if known, over all calls. This rule has no application to any line unless the line is run and marked before the execution of the deed or contemporaneously therewith, and is clearly regarded by the parties as one of the lines conveyed. The application of the rule is further dependent upon reference in the deed to the marked line. It should so be connected with the deed, either by intrinsic or extrinsic evidence, as to create a presumption that the grantor intended to adopt it.*

796(III). CONTROL BY NATURAL MONUMENTS. *When monuments mentioned in a deed are identified, they control both course and distance given, whether they were seen by parties to the deed or not, for monuments erected upon the land are facts; the field notes and plat returned by the surveyor, indicating course and distance and quantity, are but descriptions which serve to assist in ascertaining these facts. Natural monuments give control over artificial monuments, adjoiners, courses and distances, and quantity.*

A natural monument is one which has (1) visibility, (2) permanence, (3) stability, (4) certainty of identity, and (5) independence of measurement, considered simultaneously. From this viewpoint, certain monuments erected and placed by man deserve a high rating. Thus a sidewalk placed on a street line, party walls well built, enduring fences, highways and streets, although the work of man, are classed as natural. Among the works of nature are streams, rivers, lakes, ponds, shores, beaches, trees, and boulders.

796(IV). CONTROL BY ARTIFICIAL MONUMENTS. *Artificial monuments give control over adjoiners, courses and distances, and quantity.*

Artificial monuments are objects of lesser stability and endurance than natural monuments, but they must be visible, of fixed location, capable of

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identification, and possess a reasonable degree of permanence. Stakes have been rejected as artificial monuments because they were considered unsubstantial, but it has also been held that this applies only where there are no data in the description from which the location of the stake can be determined. It is difficult to draw any distinct line between natural and artificial monuments, and there are many cases where a monument might be regarded for purposes of control as either natural or artificial, according to the judgment of those making the decision.

796(V). CONTROL BY MAPS AND PLATS. *Where a map or plan of a tract of land, with lines drawn on it marking its boundaries and with the natural objects on its surface laid down, is referred to in a deed as containing a description of the premises therein conveyed, this map or plan is to be regarded as a true description of the land conveyed as much as if it were expressly recited and marked down in the deed itself, and "where there is a description by courses and distances and one by reference to known monuments, the latter prevails; and a map or plan referred to (certainly if it be a public record) stands upon the same footing as a monument, and is of even higher authority," but the courses and distances as indicated on the plat do not enjoy this prestige. Furthermore, where parties take possession by deeds according to measurements stated in a public plat of a city addition, their rights are governed by the rule that the lines of the plat, as actually located on the ground, control over monuments and other surveys.*

A map is merely a record of work performed and hence a transcript which is liable to error. This is especially true of courses and distances shown thereon, and it is evident that these when found can have no greater weight than when found in the deed. In general maps and plats are accorded relatively great prestige because of the numerous natural and artificial objects they generally call for. In this connection courts often disregard dimensions shown on the plat so that it may be interpreted as a whole and in accord with the intention of the maker.

796(VI). CONTROL BY ADJOINERS. *"A well established line of an adjacent survey is a natural monument within the rule that course and distance give way to natural monuments, because the line is more certain than course and distance." If an unmarked line is well known and generally recognized, it is a monument within the meaning of the rule although it has never been run and located on the ground.*

It is primarily in the extension of the rule to unmarked lines that exceptions are found, for not having physical evidence they lack visibility and definition, and therefore cannot be regarded as the practical intention of the deed. For this reason it is most essential that the calls of conveyance indicate that such lines unquestionably represent the common understanding of the parties at the time of the transaction.

"A call for the line of another tract is a 'natural boundary' and controls course and distance on the ground that there can be no mistake in respect to the intention to go to the line of the other tract, whereas in respect to course and distance there may be a mistake in entering upon the field notes or in transferring the entry to the description set out in the grant."

796(VII). METES AND BOUNDS (Art. 789(*c*)). Lines of adjoiners and monumented lines are within the definition of "metes and bounds." It therefore appears that metes and bounds need not be considered as a distinctive class of calls in the consideration of conflicting elements.

796(VIII). CONTROL BY COURSE AND DISTANCE. *Courses and distances control only when: (a) There are no calls for natural or artificial monuments, lines marked and surveyed, or adjoiners. (b) The monuments or lines called for cannot be located or identified. (c) The monuments or lines called for were referred to erroneously or upon conjecture or are impossible calls, and therefore compliance with them would defeat the intention expressed in the deed.*

In regard to the application of this rule to cases involving lost monuments, it should be borne in mind that "if monuments called for have been lost or removed, the places where they were originally located may be shown by parol or other competent evidence, and if proved to the satisfaction of the jury by a fair preponderance of evidence, these original locations will prevail over courses and distances."

796(IX). CONTROL BETWEEN COURSE AND DISTANCE. *While there is authority for giving calls for course control over a conflicting one for distance,*

in many jurisdictions it is recognized that the law cannot satisfactorily determine in all cases whether course or distance shall control when they do not correspond; that this must be determined by concurring testimony and the circumstances of each particular case, and that the one that convinces the judgment the most must be selected.

An example of a survey where courses are less likely to error than distances is one in open country where distances are long, the topography rough, and land values low. In such a survey courses may be readily observed, whereas accurate chaining is difficult and expensive.

796(X). CONTROL BY QUANTITY. *In general, quantity will control when (1) the superior calls are not set forth in the deed; (2) the superior calls leave the boundary doubtful; (3) quantity is made the essence of the contract or there is an expressed or implied covenant to convey a definite quantity; (4) when quantity complies with the intention as established in the deed; (5) the courses and distances run the lines outside of the grantor's property and quantity limits the grant to his land.*

A description of land by giving the number of acres is considered the weakest method of identification, but when there is doubt as to the true description, it may have a marked effect where the question is one of boundary. In some instances it may have a controlling weight.

797. Special Laws Pertaining to Ownership of Land. The surveyor is interested in special laws pertaining to the ownership of land primarily because they may and often do affect the location of boundaries. These laws with their many exceptions and limitations cannot be treated here. The following is merely a summary of some of the more important special laws which may affect the work of the surveyor. For a study of the application of these laws, the books named at the end of this chapter are recommended.

797(*a*). ADVERSE POSSESSION. To acquire title by adverse possession the several conditions specified in the definition of adverse possession (Art. 795(*a*)) must be satisfied simultaneously. Whether or not these requirements are met depends on matters of fact. It is for the courts and not the surveyor to decide the delicate questions of law which may be involved. However, if the matter is settled by agreement between the owners or by some other method of establishment, the surveyor will be better able to advise as to the location of a boundary line if he is familiar with the general legal aspects of adverse possession.

797(*b*). ESTABLISHMENT. Lines may be established by several methods. (1) By *agreement.* (See Art. 794(*e*).) (2) By *estoppel.* (See Art. 795(*a*).) (3) By *practical location,* i.e., the actual designation by the parties on the ground of the monuments and bounds called for in their deeds. One adjoiner cannot make a practical location without the other. (4) By *recognition and acquiescence.* In general, acquiescence depends on words, declarations, or silence of the parties thereto, or inferences and presumptions of their conduct, and exists when a person who knows that he is entitled to impeach a transaction or enforce a right neglects to do so for such a length of time that, under the circumstances of the case, the other party may fairly infer that he has waived or abandoned his right. Acquiescence does not presume an agreement expressed or implied. (5) By *arbitration.* Boundaries under dispute may, under the common law, be determined by arbitration, and the decision of the arbitrators is binding upon the parties concerned, although there be no written agreement. Nor is it essential that a conveyance be made, for the award acts as an estoppel. The surveyor is often the logical arbitrator (Art. 794(*e*)). (6) By *private survey.* The owners of adjoining land, under parol agreement, share the expense of a survey for the purpose of establishing a boundary between them, and after such survey the line is marked on the ground by the surveyor employed by the owners concerned. (7) By *statutory proceedings.* The statutes in many jurisdictions provide for

the appointment of commissioners or processioners to establish lost boundaries, or for the establishment of the same by the county surveyor. The procedures followed by these officers in the determination of boundaries and the effect given to their findings vary widely in the different states. In a few jurisdictions their determinations are final, but in general these officers report to the court which takes judicial action and by confirmation establishes the boundaries. Statutory proceedings have been simplified and rendered less expensive in some states by the establishment of "land courts" (Art. 789(*g*)). The surveyor can do much to avoid statutory proceedings by advocating some other method of establishment, particularly establishment by agreement or by arbitration (Art. 794(*c*)).

797(*c*). DEDICATION. Dedication is the appropriation of land by its owner for any general or public use, the owner reserving to himself no other rights in the soil than such as are perfectly compatible with the full exercise and enjoyment of the public use to which the property has been devoted. The important elements are intention to dedicate and the acceptance of the dedication. A consideration is not necessary. Dedications may be either common law or statutory. A *common law dedication* confers an easement only, and no particular formality is necessary to such a dedication since the intention may be shown by laying out or describing the land and allowing the public to use it. *Statutory dedications* which conform to the statutes are in the nature of a grant, and in most jurisdictions differ from common law dedications in that they vest legal title to grounds set apart for public purposes in the municipal corporation whereas the common law dedications leave the title in the original owner. In some states, however, statutory dedications, like common-law dedications, confer merely an easement and the fee remains in the dedicator.

797(*d*). EXCESS AND DEFICIENCY. When a tract of land is subdivided and platted, any excess or deficiency is to be apportioned among the several parcels. It cannot be presumed that the variance arose from the defective survey of any part, but it must be concluded in the absence of evidence that it arose from imperfect measurement of the whole line. The excess or deficiency, determined by actual measurement between recognized monuments as compared with plat dimensions, in the absence of the marking on the ground of the division lines between the lots, must be apportioned between the subdivisions of the whole line in proportion to their respective lengths; and if there is an excess, the title does not remain in the name of the original owner of the land. (There are exceptions and limitations to this rule of apportionment.)

797(*e*). OWNERSHIP OF HIGHWAYS AND STREETS. Laws pertaining to the ownership of land abutting on highways and streets differ in different states. In some states, ownership of such land extends to the center of the highway or the street, subject to public easement. In other states, highways and streets are owned in fee by the state or municipality. In general, however, when land is described as "bounded on" or "running along" a highway or a street, the grant carries title to the center of the road (subject to public easement) unless the terms and circumstances of the grant indicate a limitation of its extent by exterior lines. This rule holds, even though the expressed dimensions exclude the highway, since intent takes precedence. An intent should never be presumed but must appear from the terms of the deed. When it is clearly expressed that the side line of a highway or a street restricts the conveyance the highway or street is excluded.

When land is bounded upon a highway or a street without reservation and thereafter the highway or the street is abandoned by the public, the land originally granted for public use reverts to the grantors; but where the conveyance limits the grant to the side lines of the highway or street, the ownership of such right-of-way will not revert to the grantors.

It is often so difficult to locate the center line or the side lines of an old road, either from records or from physical evidence, that it is necessary to establish the lines by one of the methods of establishment (Art. 797(*b*)). (For monumented highway lines see Art. 790(*i*).)

797(*f*). RIPARIAN RIGHTS. A riparian owner is one who owns land bordering on a body of water. As such he has certain rights pertaining to the use of the shore or of the water, called *riparian rights*, which may be exclusive and valuable. Among these rights may be: the right of access, including the right to build and use piers and wharves; the right to make and reclaim land

under tidal water subject to certain limitations and restrictions; the right to the ownership of land added by *accretion* (alluvion) (Art. 795(*a*)) or by *reliction* (Art. 795(*a*)). Riparian rights are defined and limited by the statutes and precedents of the state in which the land is situated. (See also "Riparian Boundaries and Rights," Art. 791.)

797(*g*). Adjoining landowners and the law. "Each of two adjoining landowners has definite rights to the use and enjoyment of his property, and each must concede something for the preservation of the other. Where conflict is irreconcilable, right to use one's own property must prevail, but the owner may do so without compensation only where the resulting damage is not avoidable at all, or where the expense of obviating it would be practically prohibitory."

Among the laws which an adjoining owner should observe are those pertaining to: (1) rights and duties; (2) encroachments; (3) right to lateral support of the land; (4) right to excavate; (5) liability for dangerous conditions; (6) right to light and air; (7) ownership of trees on the line, near the line; (8) fences; (9) party walls.

798. Deed Descriptions. The preparation of a deed description is logically and properly the function of the surveyor. No surveyor who has repeatedly encountered difficulties in boundary surveying, such as those described in Art. 792, is likely to take this function lightly. He knows that a highly professional judgment, often involving a considerable legal knowledge, may be required. He should be familiar with the different types of boundaries and landmarks (Art. 790), and with the laws or statutes governing boundaries in the locality in which the land is situated (Art. 795). In the preparation of a deed description he should be able to anticipate difficulties which might arise in the use of that description and to forestall these difficulties. This he can best do by imagining himself in the place of those who, perhaps years hence, may seek to identify the land from the description.

798(*a*). General purposes. Lawyers and engineers look at the problem of land description from slightly different points of view. The lawyer must have a survey and a description which identify the land for the purpose of tracing title. The engineer requires positive identification of the boundaries in order that any competent surveyor can readily re-establish at any future time bounds that may have been obliterated. These two objectives are not identical; both should be attained in a deed description.

798(*b*). Basic requirements. In order to attain the two objectives just defined, there are certain requirements to be met in a deed description which may be considered basic. The requirements are these.

For the purpose of identifying the land. (1) *The identity of the property should be clearly stated.* (See rule 4, Art. 390(*a*).) (2) *The names of the owners of all adjoining lands, private or public, should be given.*

For the purpose of re-establishing bounds.

1. *The length and direction of each boundary line should be given;* the length should be stated with a degree of precision consistent with the requirements of the survey. The use of "more-or-less" distances is quite common in deed descriptions, but ordinarily this practice should be avoided. The directions may be given by the internal angles at the boundary corners, or by the *true* bearings, or the *true* azimuths of the lines. The adoption of state systems of plane coordinates has led to a third method of fixing, in a deed description, the positions of boundary corners, namely, by giving the coordinates of each corner with reference to the axes of the system, as explained in Art. 774(*a*). The direction should be given with a degree of precision consistent with that required for lengths of boundary lines. All bearings or azimuths must be relative, computed from measured angles. It is good practice to indicate on a plat the directions of all boundary lines by two methods, i.e., by interior angles at the corners and by bearings or azimuths. (See Fig. 798(*f*).) The sum of the adjusted values of the interior angles of the polygon formed by the boundary lines should satisfy the geometric requirement (Art. 298). Lengths and directions of the boundary lines should be adjusted to eliminate error of closure (Arts. 571, 572). These adjusted values are then used in the deed description. Curved and irregular boundary lines are best described by showing them with necessary dimensions on a plat as explained later.

2. *At least one point (often called the "starting point") and the direction of one boundary line extending from that point must be definitely and permanently fixed.* One of the best methods of doing this is to set a permanent monument at each end of one of the longer boundary lines. When a boundary coincides with a highway boundary or a street line or with some other officially monumented line, the starting point may well be a boundary corner which can be established and monumented on that line. Frequently it is advisable to set a permanent monument at each boundary corner.

3. *Since no artificial monument, even of the most durable type, can be considered absolutely permanent, every important monument should be referenced.* A description of the monument and reference points together with the reference measurements may be given in the deed description, generally on a plat. (See the northwest corner, Fig. 798(*f*).) The surest method of perpetuating the positions of boundary corners, however, is to tie the boundary survey to an official system of horizontal control, such as a state

system of plane coordinates or a municipal system of horizontal control (see Art. 774(*a*)) and to include in the deed an adequate description of this tie. The coordinates of any obliterated corner can then be computed and used in restoring the corner.

In addition to the three basic requirements just given, there are other elements of a deed description which are more or less essential. Among them are: (1) the name of the person responsible for the description; (2) the date when the person whose name appears as responsible for the description was assured of its accuracy; (3) the source of the survey data upon which the data in the description are based (see rules 1, 2, and 3 in Art. 390); (4) easement lines (see Fig. 798(*f*)); (5) direction of the true north or of the "grid" north of a system of plane coordinates (shown on the plat in Fig. 798(*f*)); (6) the area of the tract; (7) certification by the surveyor (see Fig. 798(*f*)).

In some communities, the surveyor is governed to a large extent in preparing a deed description by laws or regulations which specify in considerable detail what should be included in the description (Arts. 798(d) and 390(*c*)).

798(*c*). MAPS, PLATS, PLANS. The advantages of a map or plat as a part of a deed description are becoming generally recognized. It provides a means (and often the only means) of adequately showing every detail that would aid in relocating lines. It is less subject to error, as the scale and shape will generally indicate what was the intention when bearings become interchanged, or figures are reversed, or similar mistakes are made. In general it presents all essential data and information in such a way that they can be understood as easily by the layman as by the land surveyor.

The plat may be an integral part of the deed description, or reference may be made in the deed to the plat and its place of filing in the public records. In either case it should bear the certification by the surveyor. (See Fig. 798(*f*).) In at least one state, the common procedure is to trace the plat on cloth, attach this original tracing to the deed, and then insert a copy on tracing cloth in the book of record. Inexpensive methods now used in copying or reproducing plats and maps make this procedure much easier to carry out than formerly and enables the surveyor readily to obtain a copy of the plat for himself whenever he desires to do so. A map or plat is admissible as evidence when referred to in the deed even though it is not a part of the deed. It has been held that "where a plan is referred to in a deed, the lines, courses, distances, references to monuments, and other particulars appearing

on the plan, are to be regarded as much as they would be if expressly recited in the deed."

As a matter of economy, it often is not advisable to attach a separate plat to each deed description. A large number of small contiguous parcels of land can be incorporated in a single plat. In such cases, reference to the plat by number in the descriptions of all tracts shown on it should suffice.

When a plat is the sole land description in a deed, it should furnish the answers to all questions suggested by the basic requirements already discussed and, in addition, it should give such other information as may possibly be of use in identifying the land or in restoring obliterated boundaries. For this purpose, brief explanatory notes may be an important part of the plat. (See Art. 682.) When a plat supplements a written description, it generally can be used to advantage for the greater part of the description. Important details and data that should be shown on the plat, as well as some of the minor details, are given in Arts. 390 (*c*) and 798(*d*). In any case the best method of locating each boundary should be shown, and usually the method used in the field is the one reproduced on the plat. For example, when part of a boundary line is the arc of a circle, monuments may be set at the P.C. and the P.T. and possibly at the midpoint of the curve. Not only should measurements for locating these monuments be shown, but also information, such as the degree of the curve, essential for running in the curve should be given on the plat. When the curve is longer than the boundary line, the length of the chord between boundary corners on the curve may be indicated and possibly the offset from the chord to the midpoint of the curve as well. Similarly, when a boundary line is irregular and has been located in the field by offsets to a traverse line (or meander line), the same method of locating the boundary may be shown on the plat.

798(*d*). STATE REGULATIONS OR LOCAL BY-LAWS may specify what should appear on a plat. An example of such a town by-law is given in Art. 390(*c*). The following example is from instructions, issued to engineers and surveyors, for making plans or plats for the Land Court in the Commonwealth of Massachusetts (*A.S.C.E. Transactions,* 1940, p. 558).

1. The plan must be made on tracing cloth.
2. The sheet may be 18 in. by 24 in., or 24 in. by 36 in., or any convenient larger size.
3. The title must state the location, scale, date, name, and address of the surveyor.
4. The plan must show the north point, town, or county lines, names of adjoining owners, passageways, street lines, boundary monuments, all fences,

walls, buildings on or adjacent to lines claimed (using proper conventional signs), and all other natural monuments pertinent to the purpose.

5. The scale must be such as to show clearly all data necessary.

6. The area may be by calculation or by scale; if by scale, the sign, ±, is affixed.

7. Such parts of buildings and other physical features on adjoining land as are within 10 ft. of boundary lines are shown.

8. Boundaries of the property, pertinent dimensions, and all physical features are shown in black; all traverse and surveying data, in red.

9. The plan must indicate plainly how each corner, property line, and reference point are marked on the ground. This information is essential, for the reason that respondents often object to a plan and survey because a stone bound, partition wall, retaining wall, or other object used to mark a boundary has been erroneously included or excluded. On this account careful inquiry should be made to determine the ownship of such boundary monuments. If necessary, a note is included describing what part of the boundary monument is used.

10. The parcel sought to be registered must be enclosed within blue tinting.

11. The plan should be oriented so that the top is in a general northerly direction.

798(*c*). Different types of deed descriptions may be classified in accordance with the method used in describing land. Among the most common methods are these. (1) By metes and bounds. (2) By rectangular coordinates referred to a system of axes which have been legalized by a state, district, or other authority in which the land is situated. (3) By reference to a recorded map on which the land is represented by a numbered lot or by some other method. (4) By "adjoiners," i.e., by describing the way in which the tract is bounded by adjoining land belonging to owners whose names are given in the deed. (5) By reference to a former deed of record. (This method is objectionable unless the description in the former deed is clear and definite.) (6) By natural objects, such as a boulder, a tree, a stone wall, or a stream, that either mark corners or extend along boundaries. (A method used in earlier surveys and deeds but no longer deemed satisfactory.) (7) By the United Public Land Survey system. (See Chapter 59.)

When writing or when interpreting a deed description, it is often necessary to weigh the relative importance in the eye of the law of different types of "calls" (defined in Art. 795(*b*)). To do this it is necessary to know when one call "controls" another and, in general, the "order of calls." This matter is discussed in detail in Art. 796.

The trend in the description of land is toward a brief written description supplemented by a plan or plat that may be the principal part of the description. In a progress report of the Joint Committee of the Real Property Division, American Bar Association and the Surveying and Mapping Division, American Society of Civil Engineers, published in the *A.S.C.E. Proceedings,* June 1941,

the following statement appears: "It is the opinion of the Joint Committee that, as a basic principle in land description for the purpose of deeds and other instruments, it is best that the description written into the deed or other instrument be supplemented by a plan or plat of the land which may be on record at the office in which the instrument is to be recorded, or which may be prepared and filed for record with the instrument or in a Land Court."

798(*f*). SAMPLES OF DESCRIPTIONS OF LAND.* These descriptions do not include rights reserved, appurtenant rights, nor encumbrances upon the premises described. They are written solely to illustrate methods of description of land areas for the dual purpose required in deeds; for example:

(1) Identification of the tract for purposes of title; and

(2) Location of the tract upon the ground, providing data for the re-establishment of the bounds if this becomes necessary because of their obliteration.

As noted in each case, some of these descriptions are regarded as satisfactory, and are submitted as recommended forms; others, not satisfactory, are included to illustrate, by way of contrast, forms of description which are sometimes found in use, but which are deemed unsuitable.

No attempt has been made to show all practicable types of description—either good or bad.

These descriptions, although made up from actual cases, do not, as here written, refer to any existing tracts.

It will be found in some or all of the sample descriptions suggested in this or subsequent reports that there are certain examples of the determination of points or lines by more than one accurately stated method without words of qualification. The Committee believes that, in the event of a conflict between two or more calls, set forth unqualifiedly in the descriptions, the courts will continue to apply well-settled legal principles in determining the dominant call, and that legal rights and liabilities will be ascertained by the same methods as heretofore. The Committee does not believe, however, that the possibility of such conflicts should be an obstacle to the use, by lawyers, of all data available and for the accuracy of which a competent civil engineer has taken the responsibility. The modern perfection of surveying methods, with the concomitant potential accuracy, makes it desirable to abandon or modify traditional reluctance to provide too much information in a deed of real estate, but rather to adopt a more progressive policy of perpetuating on the land records themselves all data which may in the future facilitate the correct location of boundary lines and the relocation of points which have been lost.

1. Sample Description, Including a Plat as Part of the Description.

> "* * * situated in the Town of ...., County of ...., and State of ...., shown (....) † upon a plat drawn by Fred L. Connor, Civil Engineer, dated May 1, 1929, and filed (state here the place of filing) to which plat reference is hereby made for more particular description."

It is to be noted that the use of this form of description requires in every instance the use of a plat (see Fig. 798(*f*)) as an essential part thereof, which plat must be made a matter of public record.

The use of this form is not recommended when the omission from the body of the deed of words identifying the tract causes extra labor in title search

* This discussion of different types of descriptions of land is taken verbatim from the progress report of the Joint Committee referred to in the preceding paragraph.

† If the parcel has a name, lot number or other designation such as "Blackacre," or "containing .... sq. ft." that should be here stated. A sample plat is shown in Fig. 798(*f*). The foregoing description is to be used where only the one parcel described is shown on the plat. If the plat includes more area than that to be included in the description, there should appear in the place marked (....) † such a qualification as "see Lot 5 of Block 7."

(for example, when the place of storage of the plats is separate from that of the deeds). In such cases, the form Sample 2 should be used.

2. Sample Description, Including a Plat as Part of the Description. Where further identification in the body of the deed is desired than is provided by Description 1, the following is suggested:

"* * * situated in the Town of ...., County of ...., State of ...., and bounded and described as follows:

Southerly by Farm Road,	123.39 ft.
Westerly by land of Arthur C. Hicks,	145.82 ft.
Northerly by land of Peter L. Prince,	62.04 ft.
Easterly by land of Peter L. Prince,	133.09 ft.

Being the same premises shown on a plat of the land hereby conveyed, drawn by Fred L. Connor, Civil Engineer, dated May 1, 1929, and filed (state here the place of filing) to which plat reference is hereby made for more particular description."

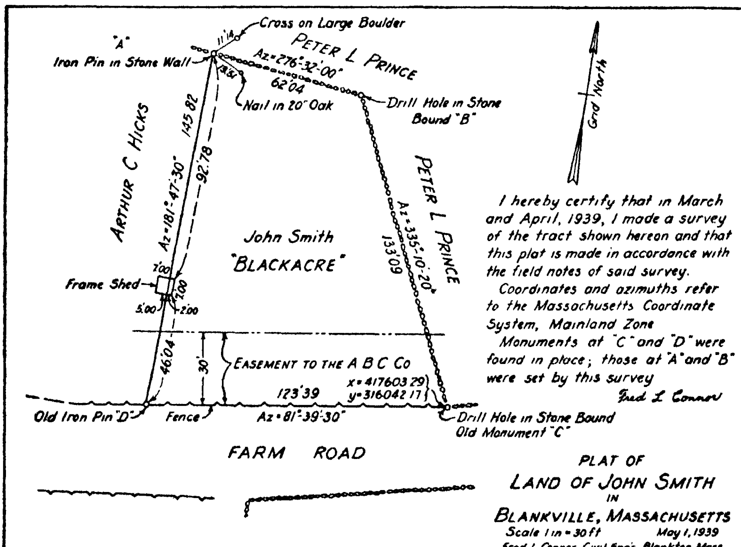


Fig. 798(*f*).

The verbal part of Description 2 furnishes positive identification of the tract for purposes of tracing title, especially if the brief description is used, verbatim, over and over in successive transfers, and, of course, much more conveniently when read with the plat at hand—as should always be the case. The plat (see Fig. 798(*f*), incorporated by reference, permits the clear showing of all data which are essential for recovering the boundaries on the ground. These data include, among other items, physical marks of boundary, a position and directions stated in terms of the state coordinate system, and names of adjoining owners. It will be noticed that the boundary on the highway is stated first, this being for the purpose of rapid identification.

3. Sample Description by Metes and Bounds, Where No Plat is Available.

"* * * situated in the Town of ...., County of ...., State of ...., and bounded as follows:

"Beginning at a drill hole in a stone bound which is set in the corner of a stone wall on the north line of Farm Road at the southwest corner of land of Peter L. Prince and at the southeast corner of land hereby conveyed, the coordinates of which monument referred to the Massachu-

setts Coordinate System, Mainland Zone, are: $x = 417{,}603.29$, $y = 316{,}042.17$.

"Thence, on an azimuth of 81° 39′ 30″, 123.39 feet along the northerly line of Farm Road to an iron pin at the southwest corner of the tract hereby conveyed;

"Thence, on an azimuth of 181° 47′ 30″, 145.82 feet along the easterly line of land of Arthur C. Hicks to an iron pin in a stone wall at the northwest corner of the tract hereby conveyed;

"Thence, on an azimuth of 276° 32′ 00″, 62.04 feet along a stone wall on the southerly line of land of Peter L. Prince to a drill hole in a stone bound in the wall at the northeast corner of land hereby conveyed;

"Thence, on an azimuth of 335° 10′ 20″, 133.09 feet along a stone wall on the westerly line of land of said Prince, to the point of beginning.

"Zero azimuth is grid south in the Massachusetts Coordinate System, Mainland Zone.

"This description was written June 1, 1939, from data secured by survey made by Fred L. Connor, Civil Engineer, in March and April, 1939.

"Together with all right, title, and interest in and to all roads and ways adjoining the above-described premises."

In Description 3 the location of the point of beginning is first definitely described, both as to physical monument and as to location with respect to adjoining land and in the Massachusetts Coordinate System. Each clause defining one course of the boundary contains the four essential elements of: (*a*) direction, (*b*) distance, (*c*) along a physical monument if one exists, such as a wall, and along the boundary of the adjoining tract, and (*d*) physical terminus. The manner of writing the directions and distances indicates the precision with which they are stated (number of decimal places, for example). Intentionally, there is no statement of "more or less." The bounds return the description to the point of beginning.

The meaning of the directions is definitely indicated by the sentence beginning "Zero azimuth is grid south * * *"; that is, there is no uncertainty as to whether directions are referred to the true meridian, to the magnetic needle, or to the direction of the central meridian of the state coordinate system.

However, this description, even for such a simple tract of land, is long and involved, and its use is not recommended. It is susceptible to error in transcription. Moreover, it is difficult to understand and visualize until a sketch showing the major features has been made up by the reader. For these reasons, the forms shown by Descriptions 1 and 2, involving the use of a plat, should always be required.

4. Sample Description by Metes and Bounds, Incomplete and Unsatisfactory.

"Beginning at a point on the north line of Farm Road at the southwest corner of land now or formerly of Peter L. Prince; thence north nine degrees and fifty minutes west (N 9° 50′ W) one hundred thirty-three and nine hundredths (133.09) feet by land of said Prince; thence north sixty-eight degrees and twenty-eight minutes west (N 68° 28′ W) sixty-two and four hundredths (62.04) feet by land of said Prince; thence south sixteen degrees and forty-seven minutes west (S 16° 47′ W) one hundred forty-five and eighty-two hundredths (145.82) feet by land now or formerly of Arthur C. Hicks to Farm Road; thence easterly along Farm Road to the place of beginning.

"Bearings stated are magnetic, as of May, 1939. Declination of the needle: 15° 00′ West."

This description, in a form quite commonly employed, is faulty in many respects. Among these are the following deficiencies:

The physical evidences of boundary on the ground (stone walls, stone bounds, iron pins, etc.) are not mentioned. The authenticity of these markers when they are found on the ground at some time in the future is thus left open to uncertainty and debate. Directions of lines stated in terms of bearings cannot be used directly with any considerable precision in the field. They must be translated into angles unless the very rough determinations by magnetic compass are to be used. Thus, the directions are really only rela-

tive; and the relations hold only within this one survey. This is true even though the declination is given. Usually the statement of declination is not reliable to the precision with which the bearings are stated. The statement that the third course runs "to Farm Road" raises the question, at least in some jurisdictions, whether the land conveyed runs to the center line of this road or whether it stops at the northerly boundary. Since the direction of the last course is omitted, the precision of the survey cannot be determined—the error of closure cannot be calculated. Hence no check can be applied to catch any mistake in length or direction of a course.

It would seem to be unnecessary to write numbers both in words and in numerals.

5. Description by Subdivision (Section, Quarter-Section, Etc.) of the Public Lands.

> "The northwest quarter of the southeast quarter (NW 1/4 SE 1/4) of Section Ten (10), Township Seventy-nine North (T79N), Range Six West (R6W) of the Fifth Principal Meridian." (See also Art. 819.)

This description, where practicable, is convenient, easy to express, easy to understand, and entirely satisfactory.

For the sake of perpetuating the positions of the corners that control these subdivisions, the statement of their positions in terms of the state coordinate system is very desirable.

Where descriptions in public land states must be written for parcels which cannot be described as fractional parts of sections or as whole government lots the method of Description 1 or 2 should be used. Any government corners near, on, or within, the boundaries of such parcels should appear on the plat.

Except with reference to fractions of subdivisions of the public lands, no reference to fractions of lots (such as "the west half of Lot 5") should be used in descriptions, because the interpretation of such a description is open to uncertainty and debate.

799. The Surveyor's Search for Evidence. The difficulties encountered in the resurvey of old boundaries have been described in Art. 792. When the correct location of a boundary line is uncertain the surveyor may seek evidence of its true location in order (1) to restore the line by a resurvey; or (2) to establish the line in accordance either with an agreement of the owners concerned or with a decision reached by arbitration; or (3) to testify in court as an expert witness. The search for evidence is often more difficult, requires more judgment based upon experience, and consumes more time than does the field work of the resurvey.

When the owner of land on one side of an uncertain boundary line and the owner on the other side disagree as to the true location of the line, it should not be assumed that the differences of opinion have reached a stage of hostility, or that a suit at law is the only recourse open to the disputants. Both parties are generally willing to accept the findings of the surveyor, provided they are convinced of his competence and are satisfied that all evidence of importance has been discovered and given due consideration. Where land is valuable and important interests are at stake, each owner may sometimes engage an independent surveyor to make a separate study of the conditions. This insures that no fact of importance to either owner is likely to be overlooked. A decision by

the two independent surveyors acting as arbiters will probably be fully as just and certainly less costly than a decision rendered by a court.

799(*a*). General suggestions. In the search for evidence so many elements must be considered in each case that no fixed rules of procedure can be given which will apply to all cases. It is suggested, however, that the following be kept in mind. (1) The surveyor has certain definite functions and responsibilities (Art. 793), duties and liabilities (Art. 794(*b*)), rights (Arts. 794(*c*), 794(*h*), and obligations (Arts. 794(*i*), (794(*j*)). (2) In the search for and the use of evidence, the surveyor should be governed by the general laws pertaining to boundaries (Arts. 795, 796) and to ownership (Art. 797). (3) The principal element to be looked for in *all* evidence is the intent of the grantor since this element controls all others (Art. 796(I)). (4) It is a cardinal principle that a boundary line having once been authoritively—even though inaccurately, or even erroneously—fixed in position on the ground, the true purpose of a resurvey is to reproduce that location, not to "correct," or to change it otherwise. (5) As much evidence as practicable should be obtained before starting the field work of a resurvey though it may be quite likely that additional evidence will be discovered during the field work. (6) Attention should be given to the broader considerations in the evidence, leaving the details until later. (7) Evidence should be weighed, the greatest weight being accorded to that which has the most important bearing on the problem of relocation; and doubtful evidence should be rejected if it conflicts with that which is obviously true. (8) If in the cumulative evidence there is a preponderance which points more or less clearly to one solution of the problem of relocation, that solution is generally the right one. If, however, the evidence for and against relocating an old line in a certain place is about evenly divided, it is generally best to establish the line, if possible, by acquiescence, or by agreement, or by arbitration (Art. 794(*c*)). (9) The surveyor should not assume the responsibility of deciding any question of evidence which properly belongs to the court. If he ventures to give a personal opinion, he should take precautions to prevent any person from holding him legally responsible for that opinion (Art. 795).

799(*b*). Evidence from public records. The surveyor's first search for evidence is in the land records kept by the legally constituted official (789(*f*)). In order to use the index to the records, the surveyor should know the name of the grantor, or the grantee of the property (or both), and the approximate date of the conveyance. He can then look up the deed and copy from the description and from the plat (if there is one) such data as he may require. It is often necessary to look up preceding transfers of the same tract and possibly the deeds by which adjoining lands have been conveyed. In such a case the search for evidence may become complicated and difficult. The legal intricacies connected with the search and guaranty of complicated titles are generally beyond the province of the surveyor though he may render intelligent help in the investigation of such titles. Most conveyances of valuable real estate are now handled for the parties to the transfer by specialists in tracing titles and interpreting deeds such as title insurance or title guarantee companies. The policies of these companies do not insure or guarantee certain features such, for example, as the location of the boundary lines as determined by the surveys. Conse-

quently it is good practice for both parties to place in the hands of a responsible company the entire transaction, including all questions pertaining to title or to the determination of boundaries, the filing of documents, and the payment of money. Even in such a case the surveyor is often called upon to advise as to the true location of boundaries.

In searching for evidence pertaining to the boundaries of a tract of land, it may be necessary for the surveyor to obtain data from three kinds of records: (1) records of the tract; (2) records of the private lands which bound the tract; (3) records of the public lands, such as streets and highways, which bound the tract. It may also be necessary to obtain special data from records of natural boundaries, particularly of riparian boundaries.

Unfortunately many deeds do not contain sufficient data to enable the surveyor to locate boundaries, and some contain so few as to be nearly worthless. Moreover, in deeds which are reasonably complete there are often mistakes, errors, and omissions in the description. Such difficulties are encountered most frequently in the records of rural lands, particularly in old records. The surveyor may find that the description in the deed for the last transfer of a tract did not apply at the time of the transfer but had been handed down from decade to decade, quite likely because of the desire to make sure at the time of each transfer that the tract conveyed was exactly the same piece of land that was conveyed by the next previous deed.

The surveyor should look for any or all of the following calls in a description, given here in the order of importance (Art. 796): (1) natural monuments and boundaries; (2) artificial monuments; (3) adjoining land; (4) courses and distances; (5) quantity. In addition he should obtain all other useful data and information which may be needed in the survey such as the data specified in Art. 798(*b*). It is particularly important to look for supplementary measurements which could be used in restoring an obliterated monument, such as those used in referencing the monument; also measurements for locating easement lines if there are any. It is important also to note carefully the description of each monument and the nature of each natural boundary such as a fence, a wall, or a hedge.

Any statement in a deed which is obviously at variance with the clear intention of the parties to the deed may be rejected. Likewise any mistake or error which obviously conflicts with such intention may be rejected; for example, a mistake of a whole chain or tape length in a linear measurement, or the wrong quadrant in a bear-

ing, or an obviously large mistake in area. Errors may occur in the field work or they may be clerical mistakes in transcription. When they are not evident from a careful examination of the deed, they may often be found by examining the recorded deeds of adjoining lands.

Maps and plats when recorded may be valuable sources of evidence, *provided they are authentic.* A plat to be valid as part of a conveyance, even when recorded, should be certified and either incorporated in the deed or be referred to in the deed as a part of the description. Surveyors have been held liable for damages due to the use of maps which were not authentic. The results of any boundary survey, including maps, to be legal in the sense that the survey will "hold" must either have been accepted by those whose interests are affected, or have been sanctioned by a decree of the court (Art. 794(*d*)). Nevertheless, a map or plat which is not authentic may furnish a clue to authentic evidence to be found elsewhere. Sometimes when there is no map of the tract, one plotted to scale from the deed description will be helpful in solving a problem in boundary location.

In interpreting a deed description, the following should be kept in mind. (1) A given distance is in a straight line unless otherwise specified. (2) A distance given as "along" a natural boundary, such as a street line, a fence, or a wall, is to be taken as measured along that boundary even though it is not straight. (3) Weight should be accorded various calls in accordance with the "order of calls" (Art. 796) unless there is evidence which would make such a procedure conflict with the intention of the conveyance. (4) A deed should be interpreted in such a way as to make it effective rather than void unless such a procedure would be contrary to that indicated by the preponderance of evidence.

A problem in boundary location due to insufficient or erroneous evidence in the deed may often be solved by the use of evidence obtained from landmarks, monuments, or other physical features on the ground, or from the field notes of the original survey, or from the testimony of persons familiar with the boundaries.

799(*c*). EVIDENCE OBTAINED ON THE GROUND consists largely of landmarks or monuments which mark the boundaries to be resurveyed. The surveyor should of course be familiar with all the different methods used in marking boundaries in the locality where he is making the survey. (See "Landmarks and Monuments," Art. 790.) Evidence may be found underground as well as above ground; for example, the remains of fence posts found wholly underground, or an iron pipe wholly below the surface.

Before beginning the field work of the resurvey, it is generally best for the surveyor to go around the boundary lines in order to ascertain what evidence is needed to locate doubtful lines. If he finds that the location of one or more of the bounds is doubtful, it is well to ascertain what the owner of the tract has to say concerning the proper location, and to seek such other evidence as may readily be obtainable on the ground. It is quite likely, however, that as the survey progresses evidence will be discovered which could not be obtained readily at the start. If during the survey it is discovered that evidence is needed which cannot be found in the deed, or on the plat, or on the ground, it may be possible to obtain it from the field notes of the original survey. Such notes are often not available, but when they are, they are likely to be a valuable source of evidence even though they may not have been officially filed with the deed.

Obliterated or lost corners are a common cause of difficulties in relocating boundary lines. If only one corner is missing it is generally possible to determine its position by running the boundary lines from the two nearest corners, one on either side, to the point at which they intersect. A search at that point may reveal traces of the monument that formerly marked it. Similarly, when two successive corners are missing, all the boundary lines except the one between the corners may be run and the points where the corners should be may thus be located. Sometimes the length or the direction of one or more of the boundary lines is not given in the deed description. It may be possible to calculate the missing quantities by the methods explained in Chapter 39. However, when a corner is located by any indirect method, a search should be made near the point thus located for material evidence of the original location of the corner.

When two calls in the deed description are found from observation on the ground to conflict, the controlling call should prevail unless the evidence indicates that the other call is clearly more in accord with the original intent (Art. 795(*b*)). For example, if each of the two monuments at the ends of a boundary line is found to be undisturbed and acceptable, the line is determined by these two monuments, even though its length or its direction as measured in the field is not the same as that called for in the deed (Art. 796(IV)).

When monuments or other landmarks have been partly or wholly obliterated, it sometimes is difficult even for a surveyor of long experience to discover evidence of their original position, for such evidence may escape the notice of the keenest observer. In order

to know what to look for, the surveyor should have a knowledge of local methods formerly used in marking corners and lines (Art. 790). For evidence of the original location of an obliterated corner, the surveyor may look for such objects as the following: the remains of a wooden stake or post under the surface of the ground, sometimes merely rotted wood or discolored soil; charcoal underground originally placed at the foot of a stake; two or three stones on or just under the surface, originally placed around a stake which no longer exists; a pile of stones or the remains of such a pile; a buried stone so far beneath the surface that it may be necessary to probe for it with a crowbar or a rod of smaller diameter; a pit or evidence of a pit, particularly in some of the western states; a tree stump or the stump of a "bearing tree"; an iron pipe wholly underground; a stone or concrete monument which was set below the surface or has become covered with soil.

For evidence of the original location of partly or wholly obliterated boundary lines, the surveyor may look for such objects as stumps of fence posts or underground remains of such posts; "stagger stones" upon which the lower rails of a fence once rested (Art. 790(*e*)); the remains of walls which have been removed—often scattered stones more or less underground; trees or stumps of trees on or quite near the line; blazed trees approximately on line—if the line has been blazed more than once, the oldest blazes prevail (Art. 790(*g*)); lopped trees or trees with a portion of their trunks horizontal and approximately on line (Art. 790(*g*)); ditches and balk lines (Art. 790(*f*)).

In evaluating evidence obtained on the ground, the surveyor should be guided by many of the considerations already given for evaluating evidence found in the deed. When there is a conflict between evidence obtained from two different monuments or landmarks for the same line, the greatest weight should be given to the one which, all things considered, appears to indicate with the greater certainty the original position of the line.

When no monuments or other landmarks have been obliterated and they are clearly in evidence, there may still be at least two questions concerning their positions. *Question 1. Is each monument or landmark in its original position?* Even a substantial monument at a corner may become displaced. A fence through years of rebuilding may gradually be moved from the line. Sometimes this is perfectly evident, as, for example, when there is a bend or an angle in a fence where obviously it should be straight. Even old stone walls may gradually become displaced. *Question 2. Exactly where is the boundary line in a fence, a wall, a hedge, a*

line of trees, or other somewhat indefinite natural boundary (Arts. 790(*e*) to 790(*h*)); *or where the land is bounded by a street or road* (Art. 790(*i*); *or by water* (Art. 791)?

799(*d*). EVIDENCE FROM TESTIMONY. When evidence in the deed and that obtained on the ground are not sufficient to enable the surveyor to restore an obliterated boundary line to its original position, he should seek the testimony of those who have had a good opportunity to become familiar with the obliterated monuments or landmarks. He is advised "to scour the neighborhood" and to "err a little on the side of safety in diligence of search." He should keep in mind the fact that in the opinion of arbiters or in the judgment of the court, the testimony of reliable and competent persons concerning lost landmarks sometimes outweighs the evidence obtained from maps and measurements furnished by surveyors. It should be noted, however, that the witness who is most positive in his opinion is sometimes most misleading in his testimony. It is important to obtain testimony from those who have no personal interest in the property, and to do this preferably before there is any serious controversy with respect to the location of the boundary line. The testimony of the owner is often helpful to the surveyor, but it is generally given little weight by the court unless it can be shown that his statements were made before controversy had arisen.

When the land is bounded on any side by public land, valuable evidence concerning the corresponding boundary line may often be obtained from the testimony of public authorities, particularly from state, county, or municipal engineers. Likewise, engineers employed by public utilities, such as railroads, telephone companies, and electric light and power companies, can frequently furnish valuable information concerning boundary lines.

It is sometimes desirable for the surveyor to obtain testimony concerning occupancy of land. It is not for him to judge the legality of a claim to ownership based on "adverse possession," but he may be able to use testimony concerning occupancy, as well as other evidence obtained on the ground, in persuading the parties interested to re-establish an obliterated boundary line by agreement or by arbitration.

799(*e*). EVIDENCE CONCERNING COMPASS BEARINGS. If the boundary lines of a tract of land were originally run by means of a surveyor's compass, it may be desirable to ascertain the magnetic declination at the time and place of the survey. The declination may sometimes be obtained from declination tables or from old isogonic charts, provided the date of the survey is known. However, if the date is uncertain or if tables or charts are not available, the magnetic declination may be determined by one of the methods explained

in Art. 801(*a*). Once the declination is known, it may be possible to use it in determining the date of the survey (Art. 801(*b*)).

800. Procedure for a Resurvey of Boundary Lines. Boundary surveys may be classified as: (1) original surveys; (2) resurveys; (3) subdivision surveys. Most of the perplexing problems in boundary surveying are encountered, not in original surveys but in resurveys, particularly resurveys of old boundaries in rural sections, many of which were originally made with the surveyor's compass and chain. This entire chapter has been written with such problems in mind.

When the description in the deed to a tract of land is accurate and adequate and all bounds are easily identified, the technique of the field work in a resurvey of the land boundaries is elementary, as explained in Art. 387. It is when some of the bounds have been obliterated that difficulties arise, and it is for a resurvey made under such conditions that the procedure outlined here is intended. No fixed procedure can be given that will apply to all cases, and even when a general procedure, such as the one outlined, is followed, difficulties may be so great as to call for extensive experience and great resourcefulness on the part of the surveyor. Resurveying boundary lines is in many cases more of an art than a science; it is one of the most difficult kinds of surveying.

The general method of procedure recommended is as follows:

1. *Search the records for evidence needed in the resurvey and, possibly, the restoration of lost boundary lines* (Art. 799). These records may include the deed, a map or plat, and field notes. It may be necessary also to search the records of adjoining lands. It is essential to discriminate between legally authentic records and those which are not authentic.

2. *Examine the records critically in order to discover errors and omissions.* If the directions and lengths of all boundary lines are given, check the sum of the interior angles of the boundary polygon (Art. 298), and compute the error of closure by the method of latitudes and departures (Art. 563). If not more than two bearings, or two lengths, or one bearing and one length are missing, compute the omitted measurements (Chapter 39). If the evidence obtainable from a search of the records, including those of adjoining lands, is insufficient for the location of the boundaries or if glaring errors are discovered, the search for evidence must be continued during the field work. Before going into the field it may be desirable to plot the boundary lines as completely as the evidence permits; the plat may be useful in the field work.

3. *Search for evidence on the ground* (Art. 799(*c*)). If from inspection of the readily identified boundary corners and lines it is evident that all boundary lines are clearly and definitely marked, only the technical field work remains. If, however, further search for evidence on the ground is necessary, this search should not be continued unduly. The additional physical evidence may often be discovered more easily from measurements and observations made as the field work progresses.

4. *Choose a starting point.* The point described in the records as the starting point of the original survey may not be the best point at which to begin the resurvey. The surveyor may find a point better marked or better in other respects as a starting point than the one described in the records. When the tract surveyed is bounded by public land, it may be possible to begin the resurvey on a publicly monumented line such as a street line or side line of a highway. Monuments on such lines are generally better maintained than private monuments.

Many times the surveyor has little or no choice. He may be unable to find more than two corners or even one corner which he can identify and use as a starting point. Sometimes he cannot find even one definite corner; in such a case he must use ingenuity and judgment in choosing at least a tentative starting point for a tentative resurvey.

5. *Survey the boundary lines.* If, from evidence in the records and from evidence on the ground the position of every boundary corner and line is known, the survey may be made by one of the methods explained in Art. 386.

If, after a reasonably thorough search, one or more of the boundary corners cannot be found, begin the boundary survey by running it from the starting point as far as possible around the tract in one direction, then as far as possible in the opposite direction. The portion of the survey thus completed may indicate where the evidence of the missing corners may be found. If further search is futile, try to obtain the desired evidence from the testimony of reliable persons who have been familiar with the bounds (Art. 799(*d*)). If evidence is still needed and cannot be obtained, locate the missing corners tentatively where it seems most probable that they should be and endeavor to have the doubtful boundaries thus located established by "practical location," or by agreement, or by arbitration, or by some other method of "establishment" (Art. 797(*b*)).

It may be necessary after locating corners tentatively by a preliminary survey to make such adjustments in the positions of the

corners as appear to be needed. When the corners thus located have been established by one of the methods of "establishment," the final resurvey can be completed.

6. *Set monuments* to mark corners and possibly other points which it is important to preserve. Any additional cost involved in the use of distinctive and enduring monuments is money well spent (Art. 794(*j*)). It is best to set the monuments while details of the survey are fresh in mind, and to reference each monument carefully to objects which are as permanent as possible.

7. *Tie in the survey, when it is practicable to do so, with a permanent system of horizontal control,* such as one established by the U.S. General Land Office, by a state, by a county, or by a municipality (Chapter 55).

8. *Prepare and submit an accurate and adequate report to the client* (Art. 794(*i*)). All directions and lengths given for boundary lines in this report should be those found by adjusting the field measurements until the error of closure is eliminated.

800(*a*). SUGGESTIONS.

1. The transit is generally used for measuring angles or directions even in resurveying lines originally run with the surveyor's compass and chain. Linear measurements are generally made with a steel tape or, in some cases, by the stadia method. The error in the length of the chain used in an old survey may be determined by comparing the old lengths of lines as measured with the chain with the corresponding lengths as measured with the tape. The correction for this error may then be applied in measuring with the tape.

2. It may be desirable to choose more than one staring point for a resurvey and then to complete as much of the survey from each point as the evidence will permit before spending too much time in a search for missing corners.

3. Lost or obliterated corners for which no physical evidence can be found may be restored in several ways. Three of the most common methods are: (1) By the use of reference measurements if they can be found in the deed or in the field notes. (2) By proportional measurements (Art. 824(*c*)). (3) By finding the intersection of two boundary lines, one run from the nearest corner on one side of the missing corner, the other run from the nearest corner on the other side. The position of the point thus found may often be checked, particularly if the tract is small, by running from a known corner of the boundary polygon the diagonal of the polygon which should pass through the required point. The direction and length of this diagonal may be computed by the method explained in Chapter 39. When only one boundary line can be run through the missing corner, the corner can often be located by finding the point in which the boundary line and one or more diagonals of the boundary polygon intersect. (For additional suggestions for the procedure in restoring lost corners, see Art. 824(*d*).)

4. When there is positive evidence of the original location of a corner, a monument may be set at that point even though the relocation measurements do not agree with the corresponding measurements given in the deed description. This is because a monument controls course and distance (Art. 796(III)). When, however, a monument is set without positive evidence of its correct position, that position should be established by one of the methods of "establishment" (Art. 797(*b*)).

5. A crooked line between two corners may be straightened or obvious encroachments on either side of the line may be corrected at the request or with the consent of the owners of the lands on both sides of the line. (A method of establishment, Art. 797(*b*).)

6. The surveyor should continually be on guard lest he assume a responsibility which does not properly belong to him. He should keep in mind the fact that the final decision as to the proper location of a boundary line is *not* within the jurisdiction of the surveyor. In general, the surveyor should assume responsibility for absolutely nothing but the honesty, accuracy, and adequacy of his survey within the limitations which exist because of the conditions under which the survey was made. (See Arts. 793, 794(*b*), and 795 for the duties, responsibilities, and liabilities of the land surveyor.)

801. Rerunning a Compass-and-Chain Survey. Special Problems. The general method of rerunning an old compass survey with a surveyor's compass or with a transit compass is explained in Art. 218. In rerunning a compass survey with a transit, magnetic bearings may be ignored and the angles at the corners measured or laid off instead, provided the corners can be found and identified. Sometimes, however, so many of the corners are missing that, in order to start the survey or to complete it, it becomes necessary to establish at least one boundary line by the use of its magnetic bearing. An extreme case of this kind is one in which only one corner can be found and identified. Assuming that the magnetic bearings of the two boundary lines meeting at this corner are given in the deed, the direction of either line can be established with the transit compass provided the magnetic declination at the place and the date of the original survey are known. When this declination is not given but the date of the survey is known, an approximate value of the declination may sometimes be obtained from old records or from declination tables. When neither the declination nor the date of the survey is known, the following procedure is recommended. (1) Compute the error of closure of the original survey by latitudes and departures in order to ascertain if there are any mistakes in the bearings and lengths given in the deed description. (2) Establish the true meridian through the known corner by one of the methods explained in Chapters 34 and 35. (3) With the transit at the corner and the true meridian as a backsight, establish the direction of at least one of the two boundary lines which meet at the corner. This is done by turning off the angle of magnetic bearing given for the line in the deed description. (4) The directions of the remaining boundary lines can then be established by laying off angles at the corners computed from the magnetic bearings. It is to be noted that since no comparison can be made between the length of the chain used in the original survey and that of the tape used in the resurvey, the corners established by the use of the tape may not be in their original positions even though the error of closure in each survey is small. Note also that, although the boundary polygons of the two surveys may have the same shape, the area within one may be greater than that within the other. Moreover, if there is any error in the direction of the first boundary line established in the resurvey, the orientation of the entire boundary polygon of that survey with respect to the true meridian will be different from that of the boundary polygon of the original survey.

The possibilities of error are so great that the surveyor may be compelled to locate the corners tentatively and then adjust their locations in accordance with his best judgment as the survey progresses. The positions of the relocated corners should then be legally established by one of the methods explained in Art. 797(*b*).

801(*a*). To DETERMINE THE MAGNETIC DECLINATION AT THE TIME AND PLACE OF AN OLD COMPASS SURVEY WHEN THE DATE OF THE SURVEY IS UNKNOWN. Where at least one line of the original survey and its magnetic bearing as determined in that survey are known, an approximate value of the required declination may be found by comparing the original bearing with the present magnetic bearing of the line, thus obtaining the change in declination. From this value and the value of the present declination the required declination may be computed. A more accurate value of the original declination may be obtained by comparing the true bearing of the known line, determined by observation on Polaris or on the sun, with its original magnetic bearing. When several successive original boundary lines can be identified on the ground and their magnetic bearings are known, a still more certain value of the required declination can generally be obtained as follows. (1) Determine the true bearing of any one of the identified lines. (2) From this bearing and the angles at the corners where the known boundary lines intersect, compute the true bearings of the known lines. (3) Compare the true bearings thus obtained with the corresponding magnetic bearings observed

during the original survey, and from this comparison determine the most probable value of the required magnetic declination.

801(*b*). THE APPROXIMATE DATE OF AN ORIGINAL SURVEY can be determined from the value of the magnetic declination at the date of that survey provided tables or isogonic charts are available which give the magnetic declinations for the place of the original survey and for various periods of time, including the period in which the survey was made.

802. Resurveys of Subdivided Lands. A complete resurvey of a tract of land which has been subdivided into smaller tracts may involve first of all a resurvey of the boundary lines of the original tract, and then a resurvey of the boundary lines of each of the smaller tracts. A boundary survey of one or more of the smaller tracts is, however, more often required. If the original subdivision survey was accurate and complete, the difficulties encountered in a resurvey may be few. If, however, it was inaccurate, a resurvey of even a single subdivision may be difficult. Moreover, the subdivisional survey may not have been done all at one time by one surveyor, but by stages and possibly by different surveyors. Sometimes owners themselves either establish erroneous boundary lines or gradually shift lines that were once correctly located. Thus the difficulties are multiplied and boundary disputes arise. In fact there is probably more strife and more litigation over the boundaries of small subdivisions than over the boundaries of large tracts which have not been subdivided. This is particularly true of small house lots whose owners will often defend to the last inch of land what they conceive to be their rightful ownership.

Some of the more common difficulties encountered in resurveys of subdivided lands are these. (1) The sum of the parts does not equal the whole. For example, the sum of the frontages of lots on a street line does not equal the length between monuments of the street line; the sum of the areas of the subdivisions does not equal the area of the subdivided tract. (2) The length of a common dividing line as given in the deed to the lot on one side is not the same as the length as given in the deed to the lot on the other side. (3) A dividing line described as making an angle of 90° with the street line is found to be not truly perpendicular to that line. (4) A fence is not exactly on the corresponding dividing line when the line is located in accordance with the deed description.

When the sum of the parts is not equal to the whole, the principle of "excess and deficiency" (Art. 797(*d*)) may, under proper conditions, be applied. The principle that monuments control courses and distances is also useful. In general, however, the laws pertaining to the subdivision of land are so numerous and complex

that unless the surveyor can persuade the parties to a boundary dispute to settle it by "practical location," or by agreement, or by arbitration (Art. 797(*b*)), he can seldom be of further service except, possibly, as an expert witness (Art. 794(*f*)).

It should be kept in mind that in making a resurvey of any part of a subdivided tract, such as a single subdivision or a group of several adjacent subdivisions, it may be necessary to examine not only the deeds to the land covered by the resurvey, but also the deeds to other subdivisions as well, particularly adjoining subdivisions.

References

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CHAPTER 59

UNITED STATES PUBLIC-LAND SURVEYS

THIS chapter is largely a résumé of the more important fundamentals of the United States system of public-land surveys as given in a manual issued by the United States Government.*

803. Introduction. *Original* public-land surveys are now made exclusively by surveyors under the direction of the General Land Office. However, once land has passed into private ownership, the General Land Office exercises no control over further surveys of that land. Such work as the subdivision of the land or the retracement and restoration of lost or obliterated boundaries is then carried on by surveyors employed by local governments, or by surveyors engaged in private practice. To do this work, the surveyor should have a thorough knowledge of the fundamentals of public-land surveying, including a knowledge of the legal requirements.

804. Legal Provisions Governing Subdivision of Public Lands. Authority for the rectangular system of surveying the public lands of the United States is based upon an act of Congress, passed May 20, 1785, providing for the subdivision of public lands into *townships* six miles square containing thirty-six *sections* each one mile square. The region first surveyed in accordance with this act is known as the "Seven Ranges." It forms the easterly part of the State of Ohio. A later act, passed May 18, 1796, provides that the sections should be numbered from 1 to 36, beginning at the northeast corner of the township, as shown in Fig. 817(*a*). This system of subdividing and numbering is still in constant use. Additional acts passed in 1800 and 1805 provide that the subdivision of public land into sections and into "half sections of three hundred and twenty acres each, as nearly as may be" shall be effected by means of parallel lines running "from *east* to *west* and from *south* to *north* at the distance of one mile from each other" and that in the subdivision of townships any "excess or deficiency shall be especially noted and added to or deducted from the WESTERN and NORTHERN ranges of sections or half sections in such townships." This act makes legal provision for unavoidable inaccuracies in field work and for the inconsistencies resulting from the attempt to lay out square plots bounded on the east and west sides by converging meridians. The act of 1805 also provides that the area of each half section and of each quarter section, although not specifically reported, shall be considered as containing the one-half or the one-fourth part, respectively, of the reported contents of the section of which they make a part. This provision is generally referred to as the "Law of Alequot Parts."

Subsequent acts of Congress have established a General Land Office and authorized the appointment of deputy surveyors to supervise the work of subdivision, marking corners, and the care of field notes, maps, and records. Quarter-section subdivisions of 160 acres were authorized in 1820 and quarter quarters of 40 acres in 1832.

* *Manual of Instructions for the Survey of the Public Lands of the United States,* prepared and published by the General Land Office. Copies may be obtained from the Superintendent of Documents, Washington, D. C.

In many states surveys have been completed and all field notes, maps, and other records pertaining to land subdivision have been transferred from the General Land Office in Washington to an appropriate office in each of the respective states.

In all, thirty-two *principal meridians* and *base lines* have been established in the United States. These control the surveys of public lands in every state except the original thirteen states, and Kentucky, Tennessee, and Texas. Three additional sets of rectangular control axes have also been established to govern surveys of public lands in Alaska. Each *principal meridian* is designated by a number, or a name, to facilitate the location of lands whose bounds they control. Thus, for example, all lands which are referred to the Huntsville meridian are situated in the state of Alabama. The complete list of principal meridians and base lines is given in the *Manual*. Some meridians and base lines control surveys in several adjacent states. The General Land Office publishes a map showing the extent of the surveys which are governed by a single set of coordinate axes. The name and address of the depository of maps and notes for each state are given in the *Manual*.

Previous to 1910 all surveying for public land subdivision was done under contract. Since that time the work has been carried on by a permanent corps of surveyors employed directly by the General Land Office. Subdivisional surveys and resurveys are still in progress. Under the old contract system it was but natural that all field work, including the setting and marking of monuments, should have been done hurriedly, and that many inaccuracies and omissions should have resulted. In spite of these inaccuracies, however, the statutes of the United States provide that "boundary lines actually run and marked in the surveys returned by the Surveyor General shall be established as the proper boundary lines of the sections or subdivisions of which they were intended, and the lengths of such lines as returned shall be held and considered as the true length thereof." Notwithstanding its defects, the system has been of great advantage to all owners of lands formerly a part of the public domain because it affords a method of locating land boundaries definitely and permanently and serves as a means of preventing overlapping titles.

804(*a*). NOTE. Many of the earlier surveys were carelessly and hurriedly made. Winter surveys in open country where there was neither stone nor timber for monuments made it impossible to mark the surveys at the time they were made. The surveyors were sometimes permitted to place the monuments later. Thus some corners were erroneously placed, and many were never marked at all. Some surveys were fraudulent. For example, in Arkansas, in 1849, it was found that in an area embracing twenty-six townships situated in prairie country, not an original corner could positively be identified. Instances of this kind unfortunately give uninformed persons the erroneous impression that many of the contract surveys were largely fraudulent.

805. The Manual of Instructions. Although the details of methods used to establish the rectangular system have been modified from time to time, its principal features remain unchanged. The *Manual of Instructions* contains specific directions for field work. The earliest directions were given to surveyors in manuscript or in printed circulars. Manuals, in book form, were issued in 1855, 1881, 1890, 1894, and 1902. Since then, revisions have been necessary to give effect to legislative changes in surveying procedure. A late edition of the *Manual* is practically indispensable to a surveyor in any locality where the United States system of subdividing public land has ever been established.* The chapters "Resurveys" and the "Restoration of Lost Corners" are of

* To facilitate computations, the surveyor should also have a copy of *Standard Field Tables*, published by the General Land Office.

particular value to surveyors engaged in retracing land boundaries. The local surveyor should also be familiar with court decisions which affect boundaries in his locality. The most important decisions have been summarized in convenient form for reference in legal digests to be found in law libraries.

805(a). NOTE. Every surveyor should be familiar with the regulations which were in force at the time when public lands in his state were subdivided. Not all the instructions, especially those issued in the early years, have been preserved, however. This condition adds to the surveyor's difficulties when monuments have been lost or obliterated and resurveys are necessary.

806. Standard Lines of the Rectangular System. In establishing a rectangular system in a particular section of the country, an *initial point* is selected as an origin.

Principal meridian, the true meridian through the initial point. (See Fig. 806.)

Base line, the true parallel of latitude through the initial point.

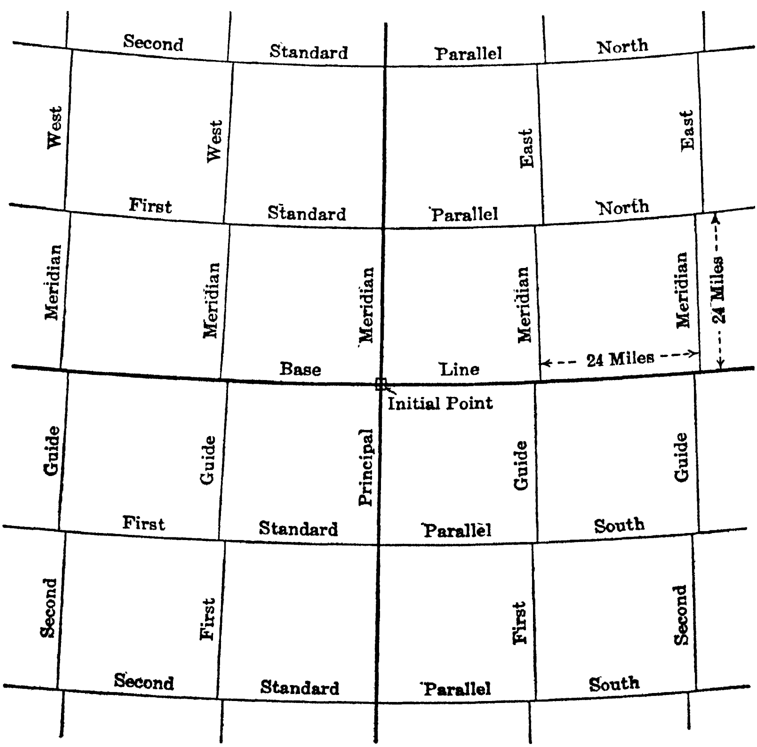


FIG. 806. Standard lines, namely, principal meridian; base line; parallels (or correction lines); guide meridians.

Standard parallels, also called *correction lines,* true parallels of latitude at intervals of 24 miles north or south of the base line.

Note. It should be kept in mind that a true meridian when established on the earth's surface is a straight line since it is part of a great circle, but that a true parallel of latitude when established on the earth's surface is a curve except at the equator.

Note. In some of the earlier surveys, *correction lines* were established at intervals of 30 miles and 36 miles north and south of the *base line.* Other *correction lines* at lesser distances apart than those mentioned are sometimes established in townships having defective exterior boundaries. The purpose of *sectional correction lines* is to prevent the further extension of surveys based upon defective control lines.

Guide meridians. Meridians, 24 miles long, extending north (and only north) from the base line and from each standard parallel at intervals of 24 miles east and west of the principal meridian. Each guide meridian may be conceived as starting from one parallel of latitude and terminating at another parallel of latitude 24 miles north.

807. Quadrangles. Standard parallels and guide meridians are *controlling lines* which subdivide an area into quadrangles (Fig. 806). The number of controlling lines and of the quadrangles formed by them is not fixed but varies in different sections of the country, depending upon the area covered. (In Fig. 806 there are sixteen quadrangles.) For purposes of explanation the number of quadrangles is immaterial. The essential characteristics of a quadrangle are these. (1) The eastern and western boundaries are true north-south lines; they are straight but not quite parallel. (2) The northern and southern boundaries are true east-west lines; hence they are parallel. (These lines, however, are slightly curved.) (3) The eastern, western, and southern boundaries are each 24 miles long, but the northern boundary is slightly less than 24 miles, owing to the convergence of the meridians. (4) The amount of convergence, i.e., the amount that the northern boundary is less than 24 miles, depends upon the latitude of the boundary; it is greater for a high latitude than for a low latitude.

The quadrangle is subdivided into sixteen *townships,* and each township is subdivided into thirty-six *sections.*

808. Townships. A quadrangle is subdivided into sixteen *townships* by true meridians and true parallels of latitudes. The true meridians extend *north* from points 6 miles apart measured on the parallel of latitude that forms the *southern* boundary of the quadrangle. The true parallels of latitude are also 6 miles apart. (In Fig. 815, p. 1034, there are four quadrangles, each subdivided into sixteen townships.) These are the characteristics of a township. (1) The eastern and western boundaries are true north-

south lines 6 miles long, but they are not quite parallel, owing to convergence of the meridians. (2) The northern and southern boundaries are true east-west lines, slightly curved, slightly different in length, and not quite 6 miles long, except southern boundaries on a *standard parallel*, each of which is 6 miles in length.

808(*a*). RANGE AND TIER. Any series of contiguous townships situated north and south of each other constitutes a *range;* such a series situated in an east and west direction constitutes a *tier*.

809. Sections. A township is subdivided into thirty-six *sections* by south-north lines parallel to the *east* boundary of the township and east-west lines parallel to the *south* boundary of the township (Fig. 817, p. 1039). The south-north lines extend from points set one mile apart on the south boundary of the township and terminate at points similarly set along the north boundary of the township. The east-west lines extend from points set one mile apart on the east boundary of the township and terminate at points similarly set on the west boundary of the township. A normal section is one mile square. Sections are subdivided into quarter sections and lots as will be explained later.

810. The Work of the Surveyor. In addition to running or rerunning the lines of the rectangular system, the work of the surveyor may include: setting monuments consisting of stone, metal, or wood; inscribing monuments with appropriate marks to identify the range, township, and section; blazing subdivisional lines through timber and marking trees or other natural objects in order to reference or to witness corners; running meander lines along bodies of navigable water and setting meander corners; establishing witness corners to witness or reference inaccessible corners; noting topographic and cultural features along subdivisional lines; preparing maps or plats to serve as permanent records of subdivision. This work will be discussed more in detail later.

811. Linear Measurements. The law prescribes the Gunter chain as the unit of linear measurement for the survey of the public lands because of its convenience in computing areas in acres, one acre being equal to 10 square chains. The link chain has been superseded in the field, however, by the more reliable steel tape. At present the most approved method of measurement employs steel ribbon tapes two to eight chains in length. Slope chaining is commonly used on later work, vertical angles being determined by means of clinometers for ordinary slopes whereas steep slopes are measured by means of the vertical arc of the transit. Reduction of slope distances to horizontal are commonly

made by means of diagrams or tables. The manual authorizes the use of the stadia method where natural obstacles are such as to render measurements by stadia more accurate than by the tape. In public-land surveying it is convenient to use fixed stadia wires having a ratio of 1 : 132 (instead of 1 : 100) but to take readings on a standard rod graduated into feet. The sum of two rod readings (intercepts) in feet will then represent the stadia distance in chains. Triangulation is also authorized for measurements across water or over precipitous slopes.

812. Methods of Establishing a Meridian. The true meridian through any point may be established by observation on a star, usually Polaris (Chapter 34), or by observation on the sun (Chapter 35). As explained in Chapter 35, the solar attachment, when skillfully used, affords a convenient instrumental solution of the "Pole-Zenith-Sun" triangle in such a manner as to bring the line of sight through the telescope into the plane of the true meridian. A meridian may be extended any desired distance by taking observations during the day when the sun is shining and when astronomical triangles are of suitable proportions. In this way, points may be established on a meridian at intervals sufficiently short to insure the determination of the meridian with the required degree of precision.

812(*a*). Testing the solar apparatus. The *Manual* specifies: "Surveyors using instruments with solar apparatus will be required to make azimuth observations on Polaris, or direct altitude observations upon the sun, at the beginning of every survey, to test the accuracy of the solar apparatus, and subsequent tests will be required at least at the beginning of the subdivision of every township. A test at the conclusion of a survey is necessary in order to prove the continued proper projection of transit lines or the continued satisfactory adjustment of the solar apparatus during the survey." (For the adjustments of solar apparatus see Arts. 750–752.)

"Should the successive independent tests of the alignment of any standard line, or the average tests of the solar attachment employed in the projection thereof, indicate that the line has deflected from the true cardinal course to exceed three minutes, the necessary corrections will be made to reduce the deviation in azimuth."

812(*b*). Note. The surveyor's compass was used in the earlier public-land surveys. "Beginning with the *Land Office Annual of 1890,* the use of the magnetic meridian was prohibited, except in subdividing and meandering, and then only in localities free from local attraction and with the use of suitably constructed needle instruments." Since 1894, the use of the magnetic needle has been prohibited altogether.

813. Convergence of Meridians. When two parallels of latitude are intercepted by two meridians the distance between the meridians measured on the northerly parallel will be somewhat shorter than the corresponding distance on the southerly parallel (Fig. 813(*a*)). The nearer the parallels of latitude are to the

north or to the south pole, the greater will be the convergence, i.e., the rate of convergence depends upon the latitude. The amount of convergence may be measured in either linear or angular units.

813(a). LINEAR CONVERGENCE. In Fig. 813(a), let l represent the distance in miles between two parallels and d the *mean* distance

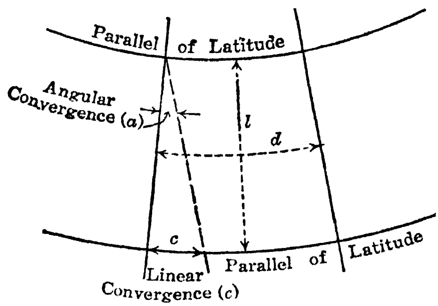


FIG. 813(a).

in miles between the two meridians measured on a parallel. Let c represent the linear convergence. An approximate equation for this linear convergence is:

$$\text{Linear convergence } c \text{ (in feet)} = \tfrac{4}{3} \times d \times l \times \tan \text{latitude}$$

For example, if the mean latitude of a township is 40°:

$$c = \tfrac{4}{3} \times 6 \times 6 \times 0.839 = 40.27 \text{ ft.}$$

813(b). ANGULAR CONVERGENCE. An approximate expression for angular convergence is:

$$\text{Angular convergence } a \text{ (in seconds)} = 52.13d \times \tan \text{latitude}$$

For example, if the mean latitude of a township is 40°:

$$a = 52.13 \times 6 \times 0.839 = 262.42'' = 4'\ 22''$$

A convenient rule for computing angular convergence is: *The convergence in seconds, per 100 ft. east or west, is equal, approximately, to 98.5 per cent of the tangent of the mean latitude.*

Using the same example as before:

$$a = 0.839 \times 0.985 \times 52.8 \times 6 = 261.80'' = 4'\ 22''$$

The approximate equations just given for computing linear and angular convergence are sufficiently accurate for ordinary land surveying. More accurate equations are derived in the *Manual.*

813(c). TABLE OF VALUES OF THE CONVERGENCE. The table on p. 1126 is taken from the *Standard Field Tables,* issued by the General

Land Office, and the following explanation of the use of the table is given in the *Manual*:

In Table 11, Standard Field Tables, are tabulated the linear amounts of the convergency of meridians, 6 miles long and 6 miles apart, for each degree of latitude from 25° to 70° N, together with the angle of convergency of the same meridians. These amounts of linear convergency are at once the proper corrections to apply to the north boundary of a regular township in the computation of the closing error around a township, or other computation by which a theoretical length of a north or south boundary of a township is compared with the length of the opposite boundary; the tabulated linear amounts of convergency are equal to double the amounts of the offsets from a tangent to the parallel at 6 miles for the same latitudes. Simple interpolation may be made for any intermediate latitude, and the amount of the convergency for a fractional township or other figure may be taken in proportion to the tabulated convergency as the fractional area is to 36 square miles.

The tabulated angle of convergency represents at once the deviation in azimuth of the tangent from the parallel at 6 miles; and ⅙, ⅓, ½, ⅔, and ⅚ of the tabulated angles of convergency represent at once the amounts of the correction in the bearing of meridional section lines to compensate for convergency within a township.

In the same table are given the differences of longitude for 6 miles in both angular and time measure, also the differences of latitude, for 1 or 6 miles, in angular measure, in the various tabulated latitudes.

814. Methods of Establishing a Parallel of Latitude. A parallel of latitude when established on the surface of the earth is not straight but curved. This is because the direction of a parallel of latitude at any point is perpendicular to the direction of the meridian through that point and meridians are not parallel but converge. Theoretically, a parallel of latitude is a straight east-west line at any point only for an infinitesimally short distance from the meridian through the point. Practically, however, the rate of curvature is so small that the direction of a parallel at a given point may generally be determined by the direction of a straight line from that point to another point on the parallel a quarter of a mile or more distant.

Since base lines, standard parallels, and latitudinal township lines are true parallels of latitude, the surveyor should know how to lay out such a parallel from any given starting point. There are three general methods.

814(*a*). Solar method. This method involves the determination of true meridians at successive intervals along the parallel of latitude, by means of a transit equipped with a solar attachment (Chapter 35). (1) At the starting point a true meridian is established. (2) An angle of 90° is turned from the meridian and a point set at a distance of 40 chains. (If more accurate results are desired the distance is 20 chains.) (3) The transit is set up at the point established in (2), a true meridian is established, an angle of 90° turned off from the meridian, and a second point on the parallel is determined. Thus the procedure is repeated at intervals of 40 chains (or 20 chains) until the desired points on the parallel have been set. In some cases where sighting is difficult because of ground irregularities, solar observations at much shorter intervals may be necessary.

814(*b*). Tangent method. (1) At the point of tangency from which the tangent to the parallel is to be established, an angle of 90° is turned either

east or west, as may be desired, from the true meridian through the point. (2) A straight line called the *tangent* is then run for the required distance—6 miles if it is desired to extend the parallel the entire distance between two corners of a township. (3) Points are set along the tangent at regular intervals of 40 chains from the point of beginning. (4) From these points, offsets are measured north from the tangent to the true parallel, and corner monuments are established. The lengths of the offsets for any given latitude can be taken from Table 13 in *Standard Field Tables*.

Note. Although the tangent is a straight line, it is not a true east-west line. The divergence at any point a given distance from the beginning of the tangent (point of tangency) is equal to the convergency of the meridians within that distance. The offset from the tangent to the parallel of latitude at a given distance from the point of tangency is approximately one-half the corresponding linear convergency. The values of the offsets vary as the square of the distance from the point of tangency.

Note. The principal objection to the tangent offset method, especially in timbered country, is the lengths of the offsets at considerable distances from the origin. For example, in latitude 45° 30′, an offset will have a length of about 25 ft. at the end of a 6-mile tangent, a distance that might require a large amount of clearing.

814(*c*). SECANT METHOD. This method, recommended by the General Land Office, is similar to the tangent method, except that offsets are taken from a straight line called a *secant*, 6 miles in length, as shown in Fig. 814(*c*). It will be noted that the secant intersects the true latitudinal curve

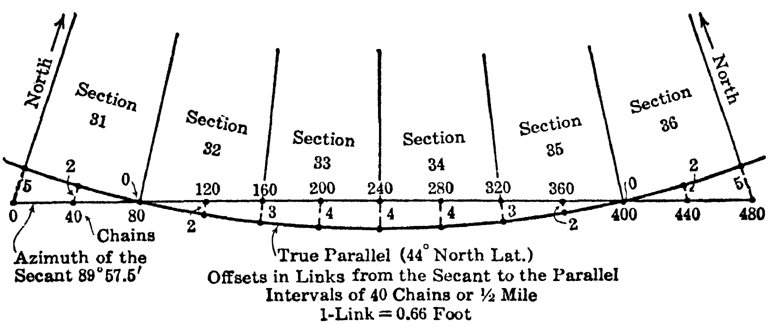


FIG. 814(*c*). Secant method of establishing a parallel of latitude.

at the ends of the first and fifth mile. Between these two points the true parallel lies to the south of the secant; throughout the first and sixth mile the true parallel lies to the north of the secant.

The steps in laying out a true parallel by the secant method are the following. (1) A true meridian is first established and the point of beginning determined as in the *tangent* method. The point of beginning will usually be a township corner already established on a standard line. (2) A point at the intersection of the secant with the meridian is found by measuring the required distance along the meridian to the south from the point of beginning. The required distance can be taken from the column headed "0 Miles" in Table XVIII, p. 1128, the value being that opposite the latitude of the place (5 links in Fig. 814(*c*)). (3) At the point of intersection determined in (2) an angle in the direction of the secant is turned either east or west of the true meridian. The angle to be turned off, i.e., the azimuth of the secant for any given latitude, can be taken from Table XVII, p. 1127 (89° 57′.5 for latitude 44°, Fig. 814(*c*)). (4) The secant is then extended as a straight line a distance of 6 miles in either direction, points being set at regular intervals of 40 chains (½ mile) from the point of intersection. (5) Offsets, from the points established in (4), are measured from the secant to the true parallel, and corner monuments are set. Values of the offsets in links at the half-mile intervals are taken from Table XVIII, p. 1128. (These offsets from the secant to the parallel are indicated in Fig. 814(*c*).)

The secant method is preferred to the tangent method because of its simplicity of execution and because of the close proximity of the secant to the required parallel (a maximum distance of about 3 feet in latitude 45°). All measurements are thus made approximately along the parallel itself, and both the secant and the parallel will be contained in a clearing of moderate width.

815. Establishing Standard Lines and Corners. (*a*). IN ESTABLISHING THE PRINCIPAL MERIDIAN of any system, permanent points are set on the meridian at intervals of 40 chains (½ mile)

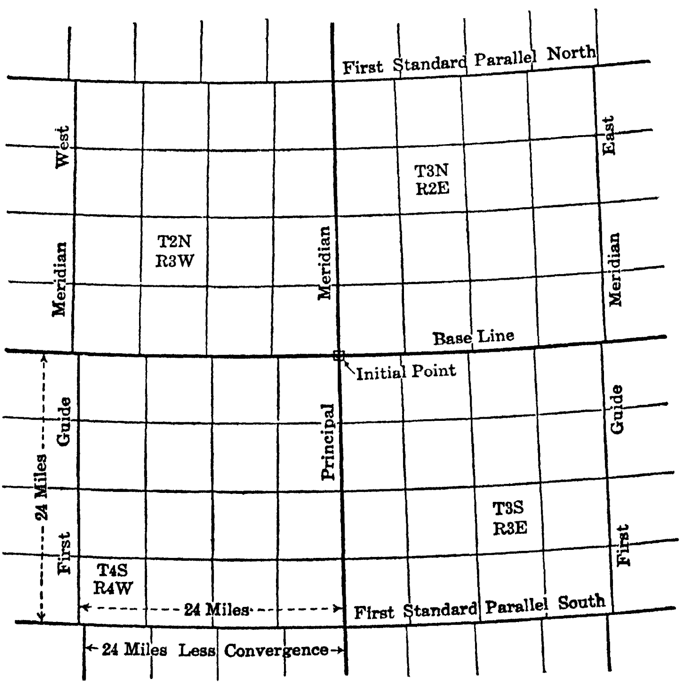


FIG. 815. Four quadrangles, each subdivided into sixteen townships.

beginning at the *initial point.* Each of these points is a *standard corner.* Points at intervals of 1920 chains (24 miles) are *standard quadrangle corners;* points at intervals of 480 chains (6 miles) are *standard township corners;* points at intervals of 80 chains (1 mile) are *standard section corners;* points at intervals of 40 chains are *standard quarter-section corners.*

815(*b*). IN ESTABLISHING THE BASE LINE on a true parallel of latitude through the initial point, permanent points are set at intervals of 40 chains (½ mile), each of which is a standard corner. The subdivisions on the base line correspond in length exactly to those on the principal meridian; hence the standard corners are the

same, namely, quadrangle corners, township corners, section corners, and quarter-section corners.

815(*c*). ESTABLISHING STANDARD PARALLELS. Through each standard quadrangle corner previously set on the principal meridian, a true parallel is established (Figs. 806 and 815). On each of these standard parallels, standard corners are set exactly as on the base line.

815(*d*). ESTABLISHING GUIDE MERIDIANS. From each of the standard quadrangle corners on each of the standard parallels, a guide meridian is run north (and only north) for a distance of 24 miles, thus ending at the standard parallel next north (Figs. 806 and 815). Each quadrangle will now be bounded on the east and west by true meridians 24 miles long, on the south by an arc of a standard parallel 24 miles long, and on the north by an arc of a standard parallel 24 miles long minus the convergency of the meridians that form the eastern and western boundaries of the quadrangle. On each guide meridian standard township corners, standard section corners, and standard quarter-section corners are set as on the principal meridian.

815(*e*). NOTE. The principal meridian and the guide meridians are established as explained in Art. 812. The base line and the standard parallels are established by one of the methods of Art. 814.

815(*f*). MONUMENTED STANDARD CORNERS serve as starting points in the subdivision of quadrangles into townships and of townships into sections and quarter sections. Not all corners on standard lines, however, are standard corners. Other corners called *correction corners* or *closing corners* are established on the base line and on standard parallels, as will be explained later.

816. Subdivision of a Quadrangle into Townships. Quadrangles 24 miles square are subdivided into sixteen townships, each approximately 6 miles square, by means of true meridians called *range lines* and by latitudinal lines called *township lines* (Fig. 816).

Let *ABCD* represent a quadrangle, *AD* the principal meridian, *BC* the first guide meridian east, *AB* the base line, and *DC* the first standard parallel north. On each of these four standard lines monuments for standard township corners were set when the line was established. Each of the three internal latitudinal township lines must intersect *AD* and *BC* at township corners previously established. The three internal range lines start from standard corners 1, 9, and 14 on the *south* boundary of the quadrangle, but they intersect the north boundary not at standard corners but at

closing corners 8, 13, and 18. However, none of the six interior lines that subdivide the quadrangle is run continuously from end to end but only for a distance of 6 miles at a time, as will be evident later.

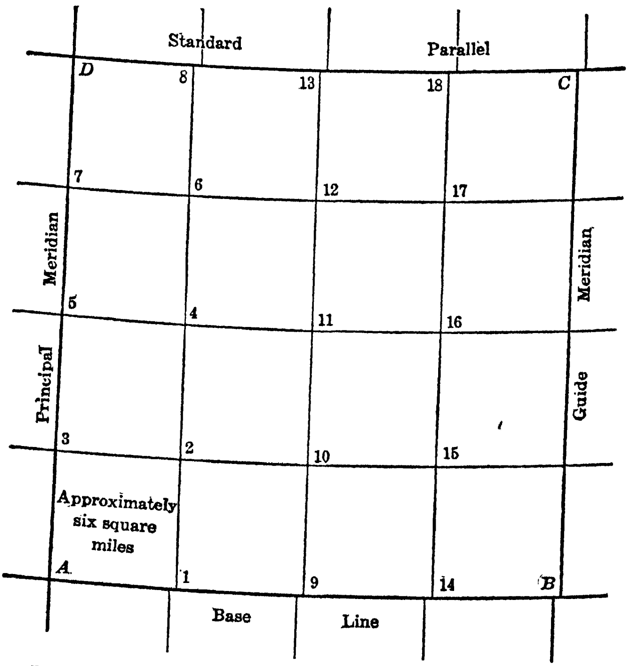


FIG. 816. A quadrangle subdivided into sixteen townships.

The usual procedure for subdividing the quadrangle is as follows. From the *standard* township corner 1, previously established 6 miles from *A*, on the base line (or on a standard parallel), a range line is run as a true meridian. On this line points are established at intervals of 40 chains (½ miles); these points are alternately quarter-section and section corners. At the end of 480 chains (6 miles) a *temporary* township corner 2 is set. From this corner a trial random line is run westward toward the *standard* township corner 3 previously established, temporary quarter-section and section corners being set alternately at intervals of 40 chains. The random line is likely to pass a short distance on one side or the other of corner 3, and from this distance the direction of the true parallel from 2 to 3 can be calculated. The true parallel is then run from corner 3 back to 2, and as the line progresses the

temporary quarter-section and section corners, including corner 2, are shifted to their proper positions on the parallel.

Starting at corner 2, the east and north boundaries 2–4 and 4–5 of the adjacent township on the north are established in a similar manner. Thus the procedure is repeated until the township corners 1, 2, 4, 6, and 8 have been established and all quarter-section and section corners on the boundaries of each township in the range have been set. (All such corners on the boundaries of the quadrangle were set when those boundaries were established.)

The range line terminates at the closing corner 8 on the standard parallel. Any excess or deficiency in the measured distance from 1 to 8 is placed in the most northerly half mile. Each of the latitudinal boundaries 1–*A*, 2–3, 4–5, 6–7, and 8–*D* terminates in a standard meridian. Any excess or deficiency in the length of any one of these lines is placed in the most westerly half mile. Thus the most northerly half mile of a range line and the most westerly half mile of a latitudinal township boundary may be more or less than 40 chains in length. This distribution of error is in accordance with the law. The error or discrepancy is due to errors in field work and, in the case of a latitudinal boundary, is due also to convergence of the meridians.

Corner 8 is a closing corner and its distance from the nearest standard corner should be measured and recorded.

Starting at the *standard* township corners 9 and 14, the township corners 9 to 13 and 14 to 18 are established in the order indicated by the numbers. From the township corners on the third range line, trial random lines are run east (instead of west) to connect with the standard township corners on the meridian *BC* which were set when that meridian was established. As these lines are corrected by running true parallels from east to west, quarter-section and section corners are established at intervals of 40 chains measured from the meridian *BC*. The errors are thus placed in the most westerly half mile of the townships in accordance with the law.

The township boundaries on the standard parallel *AB* are each 6 miles in length, but all other *latitudinal* boundaries of the townships in the quadrangle are slightly less than 6 miles long because of the convergence of the meridians. The discrepancy is least for the most southerly tier of townships and greatest for the most northerly tier.

The quadrangle chosen for the purpose of explanation corresponds to the northeast quadrangle in Fig. 815. The general

method of subdivision is practically the same, however, regardless of whether the quadrangle is east or west of the principal meridian, or north or south of the base line. In every case, a township boundary on a range line is run from the south toward the north. A random line used in the preliminary location of a latitudinal township boundary is run from the east toward the west except when the eastern boundary of the township coincides with a standard meridian, in which case it is run from the west toward the east. By this order of procedure a random line is always run from a corner that is being newly established toward a corner that has been previously set. However, *all* distances on a newly established latitudinal township boundary are measured from the east toward the west.

The procedure that has been outlined for establishing the exteriors of townships is an ideal procedure which cannot always be followed, particularly in irregular or partial surveys. The instructions for such surveys, as given in the *Manual*, are outlined in the following small-type sections.

816(*a*). IRREGULAR ORDER AND PARTIAL SURVEYS. As the remaining unsurveyed public lands are found to contain less and less extensive areas surveyable under the law, it becomes necessary to depart from the ideal procedure in order more directly to reach the areas authorized for survey. The many possible combinations are entirely too numerous to state in detail, but where an irregular order appears to be necessary, such departure from the ideal order of survey will be specifically outlined in the written special instructions. Such departure should always be based on the principle of accomplishing, by whatever plan, the same relation of one township boundary to another as would have resulted from regular establishment under ideal conditions.

In authorizing surveys to be executed it will not usually be provided that exteriors are to be carried forward until the township is to be subdivided; thus where causes operate to prevent the establishment of the boundaries in full it is not imperative that the survey of the exterior lines be completed; under such conditions it may be found necessary to run section lines as offsets to township exteriors and such section lines will be run either on cardinal courses or parallel to the governing boundaries of such townships, or even established when subdividing, as existing conditions may require.

816(*b*). GENERAL EXCEPTIONS. The above rules accord with former practice, except that in certain instances the random latitudinal boundaries will be run from west to east, instead of invariably from east to west, as heretofore required. It is also deemed advisable to incorporate other exceptions which will lessen the difficulties of subdivisional surveys frequently experienced in the past.

It is especially desirable that the alinement of a new latitudinal boundary (which becomes the governing south boundary of the township to the north) shall not depart more than 14′ from the true cardinal course; therefore the random line, run upon the cardinal course, may be made the true line where the falling would require a correction exceeding 14′ of arc.

Other exceptions and special cases are treated in the *Manual*.

816(*c*). TOWNSHIP DESIGNATIONS. A township is designated by symbols which indicate the tier and the range in which it lies. For example, in Fig. 815 *T3N* means tier 3 north of the base line

and *R2E* means range 2 east of the principal meridian. In addition, the number of the principal meridian is given as, for example, the Fourth Principal Meridian. The complete designation is: *T3N, R2E,* of the Fourth Principal Meridian. See Fig. 815 for other examples.

817. Subdivision of a Township into Sections. A normal township is subdivided into thirty-six sections, each approximately 80 chains (1 mile) square, as illustrated in Fig. 817. Let *AB, BC,*

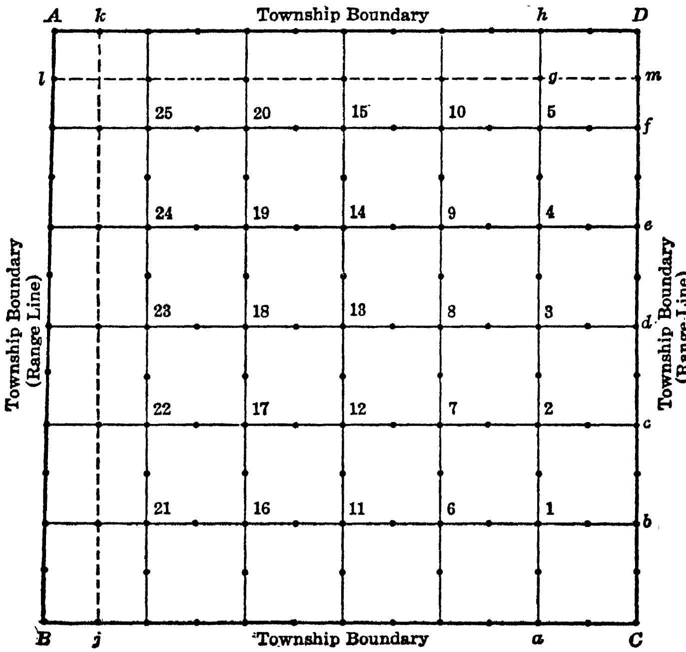


FIG. 817. A township subdivided into thirty-six sections. Numerals indicate the order in which the interior section corners are established.

CD, and *DA* represent the boundaries of a township. When these boundaries were originally established, section and quarter-section corners were set on each of them. Consequently, twenty-five section corners remain to be established, which is done in the order indicated by the numbers.

The general method of subdivision is as follows. Let *a, b, c, d, e, f,* and *h* represent some of the section corners which have already been established. Section corner 1 is set 80 chains (1 mile) from *a* and from *b*. When this has been done, corner 2 is set 80 chains from 1 and from *c;* corner 3, 80 chains from 2 and from

d; corner 4, 80 chains from 3 and from *e,* and finally corner 5, 80 chains from *f* and 80 chains, more or less, from *h.* Errors may accumulate in the measurements from *a* to 5 so that the distance from 5 to *h* may not be exactly 80 chains. Such an error, if it is within the prescribed limits of error, is thrown into the last or most northerly half mile, i.e., into the distance from the quarter-section corner *g* to the section corner *h.*

The most easterly range of sections having been completed, the procedure is repeated for the next range west, beginning with section corner 6, which is 80 chains from a section corner on *BC* (previously established), and from 1. Thus the ranges of sections are surveyed successively in the order from east to west, beginning at the easterly side of the township. Because of errors in field work and because of the convergence of the meridians *BA* and *CD,* section corners 21, 22, 23, 24, and 25 may not come exactly 80 chains from the boundary *BA.* In each case the error is thrown into the most westerly half mile, i.e., all errors in the east-west subdivision lines are thrown into the portions *west* of the line *jk,* which passes through the most westerly interior quarter-section corners. Similarly all errors in the south-north subdivision lines are thrown into the portions *north* of the line *lm,* which passes through the most northerly interior quarter-section corners. This distribution of the error is in accordance with legal requirements. As a result of this procedure, sections adjacent to *BA* or to *AD,* the west and north boundaries of the township, may not be exactly a mile square, thus leaving only twenty-five regular or complete sections in the township. Likewise, quarter sections adjacent to *BA* or *AD* may be irregular. However, the system of subdivision just outlined results in the maximum number of regular sections and quarter sections.

As each section boundary is established, a monument is set at its midpoint to mark a quarter-section corner, as indicated by dots in the figure; consequently every section line is marked by monuments at half-mile intervals. In addition, "meander corners will be established at the points of intersection of the several true lines with all meanderable bodies of water."

The south and east boundaries of every township may be considered as coordinate axes with respect to which all section corners of the township are located. This is true whether the township is east or west of the principal meridian, north or south of the base line. Since the subdivision of the entire township is thus controlled from its south and east boundaries, it is well, before beginning the work of subdivision, to test these two boundaries in order to make

sure that they are within the prescribed limits of 0° 14′ from the true cardinal directions.

From the method of subdivision it is evident that the interior south-north subdivision lines are *not* true meridians, although all of them were laid out parallel to the east boundary *CD* of the township, which was established as a true meridian. It is evident also that the east-west subdivision lines are not, theoretically, straight lines since they were made parallel to the southern boundary of the township and this boundary was established as a latitudinal arc.

817(*a*). Field work. The field work of subdividing a township into sections is explained in the following instructions taken from the *Manual:*

Thus, to recapitulate, the subdivisional survey will be commenced at the corner of sections 35 and 36 (Fig. 817(*a*)) on the south boundary of the township, and the line between sections 35 and 36 will be run parallel to the east boundary of the township, or to the mean course thereof, if it is imperfect in alinement, but within limits, establishing the quarter-section corner at 40 chains, and, at 80 chains, the corner common to sections 25, 26, 35, and 36. From the last-named corner, a random line will be run eastward,

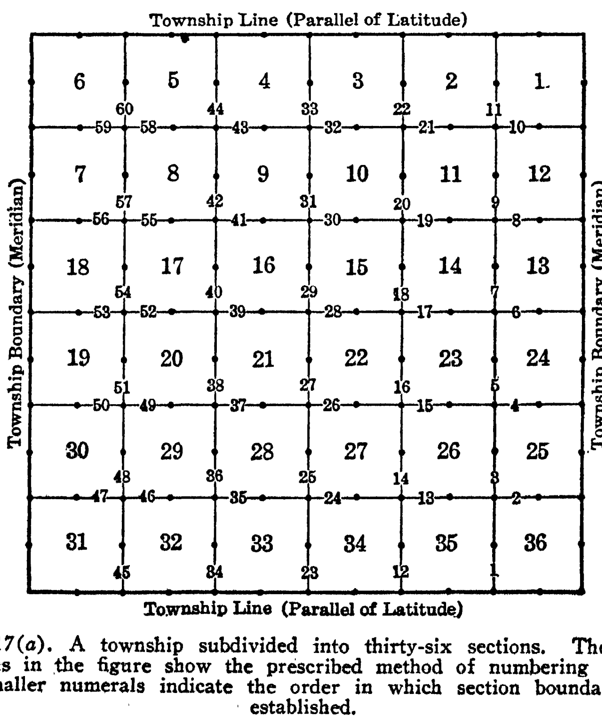


Fig. 817(*a*). A township subdivided into thirty-six sections. The larger numerals in the figure show the prescribed method of numbering sections. The smaller numerals indicate the order in which section boundaries are established.

without blazing, parallel to the south boundary of section 36, to its intersection with the east boundary of the township, placing at 40 chains from the point of beginning, a post for temporary quarter-section corner. If the random line intersects said township boundary exactly at the corner of sections 25 and 36, it will be blazed back and established as the true line, the permanent quarter-section corner being established thereon, midway between the initial and terminal section corners. If the random intersects said township boundary to the north or south of said corner, the falling will be carefully measured, and from the data thus obtained, the true return course will be calculated, and the true line blazed and established, and the position of the quarter-section corner determined, as directed above. The meridianal section line will be continued on the same plan, likewise the successive latitudinal section lines except that each random will be run parallel to the true south boundary of the section to which it belongs. After having established the west and north boundaries of section 12, the line between sections 1 and 2 will be projected northward, on a random line, parallel to the east boundary of the township, or to its mean course, as the case may be, setting a post for temporary quarter-section corner at 40 chains, to its intersection with the north boundary of the township. If the random intersects said north boundary exactly at the corner of sections 1 and 2, it will be blazed back and established as the true line, the quarter-section corner being established permanently in its original temporary position, and the fractional measurement thrown into that portion of the line between the permanent quarter-section corner and the north boundary of the township. If, however, said random intersects the north boundary of the township, to the east or west of the corner of sections 1 and 2, the falling will be carefully measured, and from the data thus obtained the true return course will be calculated and the true line established, the permanent quarter-section corner being placed upon the same at 40 chains from the initial corner of the random line, thereby throwing the fractional measurement in that portion lying between the quarter-section corner and the north boundary of the township. When the north boundary of a township is a base line or standard parallel, the line between sections 1 and 2 will be run as a true line parallel to the east boundary of the township, or to its mean course, as the case may be; the quarter-section corner will be placed at 40 chains, and a closing corner will be established at the point of intersection with such base or standard line; and in such case, the distance from said closing corner, to the nearest standard corner on such base or standard line, will be carefully measured and noted. The successive ranges of sections proceeding from east to west will be surveyed in the same manner; then after having established the west and north boundaries of section 32, a random line will be initiated at the corner common to sections 29, 30, 31, and 32, which will be projected westward parallel to the south boundary of the township, setting a temporary quarter-section corner at 40 chains, to an intersection with the west boundary of the township, where the falling will be measured and the bearing of the true line calculated, whereupon the line between sections 30 and 31 will be permanently marked between the section corners, and the quarter-section corner thereon will be established at 40 chains from the east, thereby placing the fractional measurement in the west half mile as required by law. The survey of the west two ranges of sections will be continued on the same plan, and the random line between sections 6 and 7 will be run westward parallel to the true line between sections 7 and 18; the random will be corrected to a true line and the fractional measurement placed in the west half mile; finally the random line between sections 5 and 6 will be run northward parallel to the true line between sections 4 and 5; the random will be corrected to a true line and the fractional measurement placed in the north half mile.

Note: Owing to changes from time to time in the details of subdivision and to many conditions which cause irregular boundaries of sections it is often impracticable to follow the ideal procedure for subdividing a township. Special cases of subdivision are treated in the *Manual.*

818. Subdivision of a Section. Acts of Congress contain the fundamental provisions for the subdivision of sections into quarter sections and quarter-quarter sections. The United States surveyors

set the monuments that mark the four corners of a section and the quarter-section corners that are on the boundaries of the section, but they set no monuments within those boundaries unless provision therefor is specifically mentioned in the written special instructions. (Such, for example, as provision for monuments at corners of meander lines.) However, certain subdivision-of-section lines are always protracted (plotted) upon the official plats of sections, and the local surveyor employed to run such lines in the field is required to correlate conditions as found upon the ground with those shown upon the approved plat. The function of the local surveyor begins when he is employed as an expert to identify the lands which have passed into private ownership; this may involve the restoration of lost or obliterated corners, a subject which will be treated later.

The subdivision of a township results in twenty-five complete or regular sections and eleven fractional sections. The method of subdividing a regular section will be explained first. Upon the official plat all regular sections are shown subdivided by protraction into quarter sections by two straight, broken lines. One of these lines (called the east-west quarter line) joins the midpoints of the east and west boundaries of the section, and the other line (called the north-south quarter line) joins the midpoints of the north and south boundaries of the section (Fig. 818). The point of intersection of these two quarter lines is the legal center of the section and the common corner to all four quarter sections. It is called the *interior quarter-section corner.* When a local surveyor has found all the original section and quarter-section monuments on the four boundaries of a regular section or has restored any missing ones, his next step is to establish this interior quarter-section corner. The two quarter lines passing through this corner may then be located on the ground, thus subdividing the section into quarter sections of 160 acres each.

A section adjacent to either the north or the west boundary of a township may not contain exactly 160 acres; consequently the method of subdivision differs slightly from that just explained. Such a section is subdivided by protraction on the official plat as shown in Fig. 818. In section 5, for example, the east-west quarter line is 40 chains from the south boundary of the section, but not from the north boundary, because in subdividing the township, the accumulated error in laying out a range of sections was thrown into the most northerly half mile of the most northerly section. The area north of the east-west quarter line is subdivided into two

regular half-quarter sections and the four lots 1, 2, 3, and 4 as indicated. Fractional areas are thus thrown into the four lots. Similarly, section 7 is subdivided into two regular quarter sections, two regular half-quarter sections, and four lots adjacent to the west boundary of the township, these lots containing the fractional areas which result from the method of subdividing a normal township.

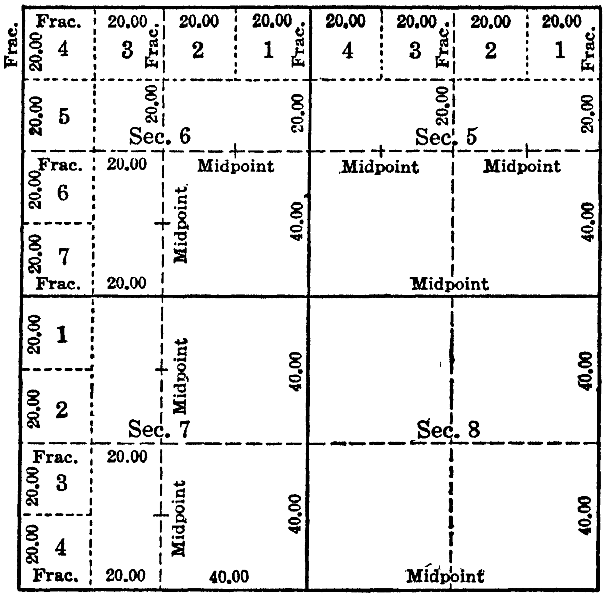


FIG. 818. Four sections, each subdivided into quarter sections. Fractional quarter-quarter sections also shown.

Section 6 is a special case since lots containing fractional areas are adjacent to both the north and west boundaries of the section; consequently the section is subdivided into one regular quarter section, two regular half-quarter sections, one regular quarter-quarter section, and seven lots.

Lots adjacent to the north boundary of a section are numbered 1, 2, 3, 4, from east to west as shown in section 5; lots adjacent to a west boundary are numbered 1, 2, 3, 4, from north to south as shown in section 7. The lots in the northwest section of a township are numbered from 1 to 7 as shown in section 6.

818(*a*). "SUBDIVISION OF QUARTER SECTIONS INTO QUARTER-QUARTER SECTIONS. Preliminary to the subdivision of quarter sections, the quarter-quarter- or sixteenth-section corners will be established at points midway between the section and quarter-

section corners, and between the quarter-section corners and the center of the section, except on the last half mile of the lines closing on irregular boundaries, where they should be placed at 20 chains, proportionate measurement, counting from the regular quarter-section corner. (Fig. 818.)

"The quarter-quarter- or sixteenth-section corners having been established as directed above, the center lines of the quarter section will be run straight between opposite corresponding quarter-quarter- or sixteenth-section corners on the quarter-section boundaries. The intersection of the lines thus run will determine the legal center of a quarter section."

818(*b*). FRACTIONAL AREAS. Fractional townships, fractional sections, and fractional quarter sections result from the existence within the tract of meanderable water courses or lakes and other segregated areas. The subdivision of such fractional areas involves special problems which are treated in the *Manual*.

819. Legal Description of a Parcel of Land. An important advantage of the rectangular system of subdivision is that the location of any subdivision can be described simply but definitely, and that this location is permanently fixed with reference to a principal meridian and the corresponding base line. For example, a certain parcel of land that was laid out in accordance with the United States land system is completely and definitely described as follows: "The NE ¼, SE ¼, NW ¼, Sec. 26, T45N, R3W of the Fourth Principal Meridian." In words this reads: "The northeast quarter of the southeast quarter of the northwest quarter of Section 26, Township 45 North, Range 3 West of the Fourth Principal Meridian" [State of Wisconsin]. The land thus described will be found in northern Wisconsin a few miles from the La Pointe Indian Reservation. References to *base lines* are usually omitted in descriptions of land.

819(*a*). FOR THE USE OF A STATE PLANE-COORDINATE SYSTEM in the description of land in a state where the U.S. System of Surveying Public Land has been established, see Art. 776(*a*), 5.

820. Limits of Error. Limits of error have been adopted for laying out the rectangular land system. To a considerable extent, these limits have been established in an arbitrary manner to fit prevailing conditions. When land was of little value, the compensation for surveying was so low and the instruments used were so inaccurate that even the skilled surveyor could not do his best work; consequently large limits of error were allowed. Increase

in the value of land, better instruments, and higher skills have made it possible and desirable to adopt smaller limits of error.

820(*a*). AZIMUTHS AND DISTANCES. Limits of error are smaller for important control lines, such as the standard meridians and parallels, than for lines of lesser importance, such as township lines and section lines. A standard line should not depart more than 3′ in azimuth from the cardinal line, and the maximum discrepancy between two measurements of the same distance should not exceed 20 links per mile of length. Boundary lines of townships and of sections are permitted to vary in azimuth from true north-south and true east-west lines as much as 14′ and, in some instances, as much as 21′. *Newly* established boundaries of townships should not depart in azimuth more than 14′, but "a previously established boundary, every part of which is within 21′ of cardinal will not be considered defective in alignment."

Note: A defective boundary may or may not be subject to rectification. The laws governing the rectification of defective boundaries and the legal methods of subdividing a township or section having a defective boundary are explained in the *Manual.*

The angular limits of 14′ and of 21′ correspond respectively to transverse linear displacements of 33 links and of 50 links in a distance of one mile. The limits of 14′ in azimuth and 33 links per mile in displacement which are required to preserve the rectangularity of township exteriors have been established as the maximum permissible limits of error for lines which are to serve as controls for the subdivision of a township into sections.

820(*b*). ERROR OF CLOSURE. The maximum permissible *errors of closure* prescribed for United States rectangular surveys are: for a closing line 1/452; for either latitude or departure 1/640. If the requirement of 1/640 is met in a survey, the requirement of 1/452 is bound to be satisfied also. In determining the ratio of error in any given case, the computed closing error is divided by the length of the *perimeter* of the rectangular area surveyed. The ratio of 1/640 is equivalent to a constant error of 12½ links per mile of perimeter. The following limits of error for either latitude or departure are found in each case by multiplying the length in miles of the perimeter of the rectangular area surveyed by 12½ links.

Township: 24 × 12½ = 300 links = 3 chains

Section: 4 × 12½ = 50 links = ½ chain

A range or a *tier* of sections = 14 × 12½ = 175 links = 1¾ chains

If the discrepancy between the calculated north and south departures or between the calculated west and east departures of the

four boundary lines exceeds in any of the above cases the corresponding limit of error, the survey should be corrected.

820(*c*). Note. It is to be noted that a survey around the four boundaries of a township or of a section might close within the limits just specified and yet, at the same time, one or more of those boundaries might depart from a cardinal line to such an extent that the township, or the section, would not be *rectangular*. If, however, no one of the four boundaries departs from a cardinal line more than is permitted by the limits of error given in Art. 820(*a*), rectangularity will be preserved within those limits.

821. Meander Lines. A traverse run along the margin of a permanent natural body of water is called a meander line. Meander lines are *not* land boundaries but are run for the purpose of defining shore lines and the irregular banks of streams and to ascertain the area of the subdivision adjacent to a particular stream or body of water. The true bearings and the lengths of the traverse lines are measured.

The ownership of lands which border navigable bodies of water or important rivers and lakes ceases at mean high-water line. In many cases a close determination of mean high-water is very difficult. This is particularly true of large rivers in the central and western states. An important court decision (47 Iowa, 370) contains the following definition: "High-water mark in the Mississippi River is to be determined from the river bed, and that only is river bed which the river occupies long enough to wrest it from vegetation." (See also Art. 791 for riparian boundaries and rights.)

Meander lines are not run along swamps or along the margins of overflowed lands but only along the edges of the lakes or rivers upon which they border. Monuments called meander corners are established at every point where township or section lines intersect the banks of a navigable stream, any river 3 chains or more in width, and any lake with an area of 25 acres or more. The meander line starts at one of these meander corners and runs along the bank to the next meander corner.

822. Corners and Monuments. "The term *corner* is employed to denote a point determined by the surveying process, whereas the *monument* is the physical structure erected for the purpose of marking the corner upon the earth's surface."

822(*a*). Corners to be marked by monuments. "Regulation monuments are employed to mark permanently the position of the quarter-section, section, township and meander corners; also to mark all angle points along an irregular boundary line, and at intermediate intervals of 40 and 80 chains along such limiting boundary."

The field notes are to record the establishment of a corner monument giving material, diameter of post, or tree, size of stone, depth set in the ground, markings upon the monument, and the nature of surrounding objects. When

the corner itself cannot be marked in the usual manner, an appropriate "witness corner" will be established as explained later.

822(*b*). DOUBLE CORNERS. It has already been explained that on base lines and on standard parallels or correction lines, there are *two* sets of corners, one set of *standard* corners to control boundaries of townships and of sections to the north of the line on which the corners are situated, and one set of *closing* corners to control boundaries of townships and of sections to the south. This results in *double corners,* i.e., two corners a short distance apart, one a standard corner, denoted by the letters SC, the other a closing corner, denoted by the letters CC. Standard corners always mark a *south* boundary; closing corners always mark a *north* boundary. Double corners occur only on a base line or on a correction line (standard parallel).

Note. In infrequent cases, double sets of corner monuments will be found along the margins of partial townships. They are the result of fragmentary surveys that are surrounded by lands which have previously been subdivided.

822(*c*). IRON POST MONUMENTS. Many corners are marked by square stone posts. Since about 1910, however, the General Land Office has used a standard type of post consisting of an iron pipe 1 to 3 in. in diameter, filled with concrete and capped with brass, to monument public-land surveys. Removal of the post is prevented through foot plates formed by splitting the lower end of the pipe and turning the two halves at right angles to the axis. Three-inch diameter posts are used to mark township corners, 2-inch diameter posts to mark sections corners, and 1-inch diameter posts to mark quarter-section and meander corners and to monument special surveys. The use of iron post monuments is at present almost universal. Monuments of stone or other material are used only under exceptional circumstances. Marks to identify the monument with respect to range, township, and section are stamped upon the proper quadrants of the cap by means of steel dies. Corners upon solid rock are marked with a cross cut at the exact point. A mound of stones is usually erected over the point.

Note. Mounds of earth, pits dug in the soil, and piles of stones, all of which have been in common use for many years to monument public lands, are omitted in current engineering practice.

822(*d*). AUTHORITY OF MONUMENTS. The law provides that "the original corners established during the process of the survey shall forever remain fixed in position." Technical errors in the execution of the survey which may have passed undetected prior to the acceptance of the survey and the opening of the lands to entry are to be disregarded. As an aid to permanency, Congress has stipulated that corner monuments of desirable material shall be erected, and has provided penalties for their defacement. "If it were possible to carry out the full intent of the surveying laws in regard to the aforementioned particulars, the most intricate of all technical and legal problems relating to surveys—the questions pertaining to the re-establishment of lost corners—would be avoided."

"The courts attach major importance to authentic evidence relating to the original position of an official corner monument, such evidence being given far greater weight than the technical record relating to bearings and lengths of lines, and it is assumed in the first instance that the original corners shall serve every necessary purpose for the identification of the survey delineated upon the official approved plat, and of the lands which have passed into private ownership. The legal significance of the original monuments, as thus briefly outlined, makes it mandatory upon the surveyor to exercise constant diligence in the workmanlike construction of lasting corners and alertness in skillfully connecting the same wlth natural objects or improvements, to the end that the greatest possible permanency may be secured for the public-land surveys."

822(*e*) WITNESS CORNERS are to be established wherever "the true point for a corner faces upon insecure ground, or in an inaccessible place, such as within an unmeandered stream, lake or pond, or in a marsh, or upon a precipitous slope or cliff." As the name implies, the witness corner is not the true corner but is used to reference the true corner. The *Manual* states: "Only one witness corner will be established in each instance (i.e., for each corner point) and the same will be placed upon any one of the surveyed lines leading to a corner, if a suitable place, within a distance of 10 chains, is available." If, in any direction within a distance of 5 chains no suitable place for a witness corner exists, one or more legal subdivisions will be

eliminated from the survey. The field notes will show every detail of the relation of a witness corner to the true corner, also "the direct connecting course and distance will be shown upon the plat of the survey." A witness corner will be constructed and marked similarly to a regular corner for which it stands, with the additional letters WC.

822(*f*). TRUE CORNER OF A TRAVELED ROADWAY. "Where the true point for a corner falls within a roadway in such a place as to interfere with travel, a stone marked with a cross will be deposited in the ground at the true corner point and a witness corner will be established at some suitable point, preferably on a surveyed line outside the roadway." Grading operations which are necessary for highway improvement have destroyed many corner monuments of this sort.

Note. Witness corners, mistaken for true corners, sometimes cause confusion in determining boundaries, particularly when a witness corner is in close proximity to a true corner.

823. Marking Monuments. For purposes of ready identification all classes of monuments have been marked in accordance with a system prescribed by the General Land Office. This is a list of some of the marks in most common use for monuments and for reference points of surveys.

Marks	Interpretations	Marks	Interpretations	Marks	Interpretations
BT	Bearing tree	NE	Northeast	SW	Southwest
CC	Closing corner	NW	Northwest	T	Township
E	East	R	Range	W	West
MC	Meander corner	S	Section	WC	Witness corner
N	North	SC	Standard corner	WP	Witness point
		SE	Southeast		

Markings are made on all corner monuments, meander corners, bearing trees, and on ledges situated at or near corners. Identifying marks consist of numbers, letters, grooves, and notches. A square township monument which is common to four townships is set cornerwise with the diagonals of the monuments on cardinal lines. On each of the four faces is chiseled the *range, township,* and *section* of the subdivision which lies directly opposite that face. For example, the northeast face of the monument might be inscribed R4E T6S 31. The inscription on the southeast face would be: R4E T7S 6. The latter marking would signify range four east (of the principal meridian), township seven south (of the base line), section six (of the township). On lines along the margins of unsurveyed land, posts are set as already described, but one or more faces are left vacant pending the completion of the survey. *Square* monuments on *township lines* are required to have as many notches, on opposite angles, as the monuments are distant in miles from the respective township corners. A quarter-section or half-mile monument is usually inscribed with the single mark ¼S. Sometimes the number of the section is added.

824. Restoration of Lost Corners. All corner monuments are subject to deteriorating influences which in time may result in their loss or obliteration. An *obliterated corner* is one where no visible evidence of the original remains. Its location may be preserved beyond question, however, by means of maps or notes, or through the acts of landowners, such as the erection of fences or buildings, or in the memory of those able to recall its initial position. A *lost corner* is one whose position cannot be determined, beyond reasonable doubt, either from original marks or from reliable external evidence. Many cases arise in which it is difficult for the surveyor to decide whether the monument is *lost* or merely *obliterated.*

824(*a*). General rules. Under the heading "General Rules" the *Manual* contains the following synopsis of congressional legislation pertaining to boundaries, to the subdivision of public lands, and to the restoration of lost or obliterated corners:

First. The boundaries and subdivisions of the public lands as surveyed under approved instructions by the duly appointed engineers, the physical evidence of which survey consists of monuments established upon the ground, and the record evidence of which consists of field notes and plats duly approved by the authorities constituted by law, are unchangeable after the passing of the title by the United States.

Second. The physical evidence of the original township, section, quarter-section, and other monuments must stand as the true corners of the subdivisions which they were intended to represent, and will be given controlling preference over the recorded directions and lengths of lines.

Third. Quarter-quarter-section corners not established in the process of the original survey shall be placed on the line connecting the section and quarter-section corners, and midway between them, except on the last half-mile of section lines closing on the north and west boundaries of the township, or on other lines between fractional or irregular sections.

[*Note.* It should be noted that in most of the older work, sections were legally subdivided by the method of "protraction," that is, by means of lines drawn on the map instead of by actual surveys run on the ground and marked by visible monuments.]

Fourth. The center lines of a regular section are to be straight, running from the quarter-section corner on one boundary of the section to the corresponding corner on the opposite section line.

Fifth. In a fractional section where no opposite corresponding quarter-section corner has been or can be established, the center line of such section must be run from the proper quarter-section corner as nearly in a cardinal direction to the meander line, reservation, or other boundary of such fractional section, as due parallelism with sectional lines will permit.

Sixth. Lost or obliterated corners of the approved surveys must be restored to their original locations whenever it is possible to do so. Actions or decisions by surveyors, Federal, State, or local, which may involve the possibility of changes in the established boundaries of patented lands, are subject to review by the state courts upon suit advancing that issue.

It should be kept in mind that the General Land Office assumes no control over the acts of local and county surveyors in the work of replacing lost or obliterated corners of original surveys where lands have passed into private ownership, nor will it issue instructions to surveyors in such cases. Legal requirements governing original subdivisions do not necessarily apply to the retracement

of boundaries of private property. To restore lost or obliterated boundaries correctly, the surveyor should be experienced in such work and must be familiar with the methods in force when lands in his locality were first subdivided.

824(*b*). ALTERATION OR RECONSTRUCTION OF ESTABLISHED MONUMENTS. "If a surveyor is called upon to alter the condition of a previously established point, the utmost regard should be shown for the evidence of the original location of the monument, and the corner will be carefully reconstructed by such additional means as may be appropriate, without destroying the evidence which served to identify its legal position. A complete record will be kept of the description of the old monument as identified, and all alterations and additions thereto."

824(*c*). ACCRETION AND ALLUVION. Accretion is the general term applied to land added to a shore by slow and imperceptible means. A riparian owner may gain title to such land and may also lose title through the effects of erosion or submergence. Alluvion is a closely synonymous term defined as "that increase of the earth on a shore or bank of a river, or to the shore of the sea, by the force of the water, as by a current or by the waves, which is so gradual that no one can judge how much is added at each moment of time." The doctrine applies to states and nations as well as to individuals. Except when making resurveys of lands along navigable waters, the surveyor has little interest in lands affected by accretion or erosion. In such cases the problem relating to boundaries is essentially legal rather than engineering in its nature. Consequently the surveyor should cooperate with a lawyer in making a survey of this sort. (See Arts. 794 to 797 for problems in land surveying which are essentially legal.)

824(*d*). PROCEDURE IN RESTORING LOST CORNERS. The surveyor's task of restoring obliterated or lost corners and of re-establishing property lines is often rendered increasingly difficult because authentic monuments may be found only at a considerable distance. It is the surveyor's duty to discover whatever evidence exists in the field, or in records, or in the memory of competent observers, which will enable him to replace corners in the exact position in which they were placed in the original survey, even though they were erroneously placed in the beginning. No definite procedure can be given for this work, but the following suggestions include, in general, the steps to be taken in nearly every case.

1. A field survey will usually be necessary. This should include a careful examination of existing fences, walls, hedges, posts, marks on trees or boulders, and similar evidences of property lines. Ditches, ridges, and furrows should be observed. Depressions in the soil may reveal the stumps of decayed fence posts or the roots of "bearing" or witness trees. Survey information can best be analyzed by means of an accurately drawn map, the scale of which will permit significant information, such as the alignment of roads and fences, to be shown clearly.
2. Original notes, plats, and other records will require close examination. Maps and field notes of surveys of adjacent properties may contain important measurements. Deeds of adjacent properties should also be examined. It is especially important to obtain the testimony of persons familiar with old fences or with structures on or near the line. Court decisions with respect to boundaries have, in many instances, been influenced more strongly by the testimony of former residents than by maps, surveys, or the opinions of surveyors.
3. In re-establishing lines or corners, the surveyor must keep constantly in mind the fact that he possesses no judicial powers. He cannot compel

adjacent owners to accept his relocation no matter how strongly it may be supported by his investigations. His relocation is "legal" only when it has been formally accepted by the owners of the land affected by the line or monument. If the owners refuse to accept the surveyor's judgment and prefer to have the court decide the issue, the surveyor has no power to prevent such action.

Note. Much of Chapter 58, "Surveys of Land Boundaries," is pertinent to the whole subject of the restoration of lost or obliterated boundaries that were originally established in the surveys of public land.

824(*e*). PROPORTIONATE MEASUREMENT. The principle of "*single* proportionate measurement" is used in restoring a lost corner on a boundary line by means of measurements from two authentic existing corners on that line, one on either side of the lost corner. These two existing corners may or may not be equidistant from the lost corner. The principle may be stated as follows. As the original recorded distance between the two existing corners is to the newly measured distance between those corners, so is the original recorded length of any portion of the line to the new measurement of that portion. For example, it is required to restore a lost corner E on a south-north boundary line. The nearest authentic existing corners to E on that line are corner B south of E and corner A north of E. The original recorded distances are: $AB = 120.22$ chains and $BE = 80.36$ chains. The newly measured distance for AB is 119.92 chains. The distance x to be measured from B to E in restoring E is found from the proportionate relation:

$$120.22 : 119.92 : : 80.36 : x, \text{ or } x = 80.16 \text{ chains from } B \text{ to } E$$

The method of "*double* proportionate measurement" is used in restoring a lost corner which is common to four townships or to four sections. (Such a corner is at the intersection of south-north with east-west boundaries.) The method may be illustrated as follows:

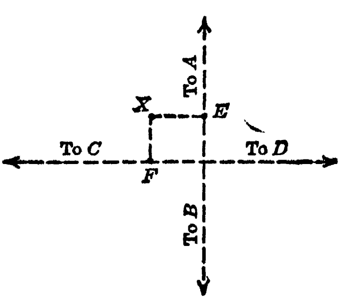


FIG. 824(*e*).

Let A and B represent two authentic existing corners on the south-north boundaries through the lost corner and C and D existing corners on the east-west boundaries through the corner. (1) By the method of *single* proportionate measurement, find E, the position of the lost corner with respect to B and A. (2) By the same method, find F, the position of the lost corner with respect to D and C. (3) Find X, at the intersection of

the east-west line through *E* with a south-north line through *F*. The distances from the restored corner when it is established at *X* to the existing corners, *A*, *B*, *C*, and *D*, will be consistent with the corresponding originally recorded distances, even though the point *X* may not be exactly in line with *B* and *A* or with *D* and *C*.

824(*f*). Referencing section corners. When township boundaries were established in the original subdivision of public lands, section corner monuments were set on the boundaries. Such monuments not only mark the township boundaries but in some states the county boundaries as well. They furnish the basis for the location and descriptions of all rural properties. It is important therefore that lost section corners should be restored *and carefully referenced.*

Many section corners are situated on the center lines of the public highways. Any corner so located should be accurately referenced by permanent reference monuments set as near the right-of-way lines as possible. At least three reference monuments are recommended for each section corner. The section corner itself should be accurately located by measurements, to the nearest hundredth of a foot, from each of the three monuments. Where two roads intersect at a section corner, there should be a reference monument on each of three of the road corners and possibly on the fourth. Where one road intersects another in a dead end, a reference monument may be set on each of the two corners and a third monument opposite the dead end, on or near its center line prolonged. In either case, a sketch should be made showing the road intersection, and the reference monuments with the distances to the section corner. There should also be a complete description of the sections which have the section corner in common.

It is well to tie section corners to the U.S. Coast and Geodetic Survey geodetic control or to a state control system when such control is available, as explained in Chapter 56.

825. Field Notes and Map. The field notes and map of a public-land survey constitute the only important records of the work and are designed to furnish not only a technical record of the subdivisional survey, but also a report upon the character of the soil and timber traversed, and a record of all topographic features encountered. The map is required to show the bearing and length of every line run together with its intersection with all rivers, ridges, Indian reservations, town sites, mine claims, or other private holdings. Field notes are to include a record of the material used for corner monuments and the marks placed thereon;

also a record of the kind, diameter, and position of all "bearing" or witness trees or other objects which serve to identify permanently the established corners. Springs, lakes, coal beds, ore bodies, quarries, roads, trails, mill sites, and United States mineral monuments are also to be noted, accompanied in each case by suitable measurements to facilitate mapping.

CHAPTER 60

AERIAL PHOTOGRAPHIC SURVEYS

AERIAL photographic surveying is a highly specialized type of surveying which is far too complex to be treated comprehensively in a single chapter. The aim in this chapter, therefore, is to give a general treatment of the subject that will be helpful to the average surveyor who may have occasion to use aerial photographs in connection with his work. A more comprehensive treatment of the subject may be found in numerous books and other publications on *photogrammetry* or *photogrammetric surveying.*

826. Photogrammetry. *Photogrammetry* is the science of measurement by means of photographs. *Photogrammetric surveying* by means of aerial photographs includes flying, photography, photogrammetry, ground control through ground measurements, and mapping. Obviously, flying and photography are specialties concerning which the surveyor needs only a general knowledge. Nor is it necessary for the surveyor to master the science of photogrammetry; all that he needs to know are a few fundamentals of the science which enable him intelligently to use aerial photographs in his work.

827. Use of Aerial Photographs in Surveying. Aerial photographs may serve as substitutes for maps or they may furnish data to be used in drawing maps. In either case, aerial photographs, with their wealth of detail, furnish a permanent record of the terrain which would be difficult to obtain in any other way. Moreover, aerial photography provides one of the most rapid and economical means of mapping a tract, particularly if the ground surface is rugged and more or less inaccessible. Although aerial photographic surveying was first developed as a means of obtaining small-scale maps of large land areas, it is also a valuable method of obtaining large-scale maps of relatively small areas. Since aerial photographs can be used to determine relative elevations of points on the earth's surface (Art. 838), *topographic* maps as well as planimetric maps may be obtained by means of aerial photography.

Surprisingly accurate results may be attained in aerial photographic surveying. However, the greatest value of aerial photographs to the average surveyor will be in the preparation of maps or the revision of old maps where wealth of detail is more important than exact scale and position. Aerial photographs are particularly useful in preliminary reconnaissance for route surveys,

hydrographic surveys, and other surveys of similar extent and detail; they are also particularly useful in the preparation of culture and zoning maps for real estate development or for various phases of city or town planning.

827(*a*). Mosaics. It is seldom that a single aerial photograph is sufficient for the purposes of surveying. Even when the area surveyed can be covered by a single photograph, distortion near the outer edges of the print leaves only the central portion approximately true to scale. However, if a number of *overlapped* photographs are taken which together cover the whole area, the outer portions of the photographs may be discarded and the central portions matched and assembled to form a composite photograph of the desired area. This assemblage of photographs, known as a *mosaic*, is generally rephotographed so that the final print appears on one sheet of paper. In a strict sense, a mosaic is not a map; nevertheless, it may often serve the same purpose.

827(*b*). How photographs, mosaics, and enlargements may be obtained. Most of the aerial photographic surveying is done by agencies of the federal government or by commercial companies that specialize in this type of surveying. Agencies of the federal government, such as the Department of Agriculture, have photographed most of the United States and are able to furnish either photographs made directly from negatives (contact prints) or enlarged photographs of almost any desired area. Commercial companies have contributed greatly to the advancement of aerial photographic surveying and can furnish contact prints, rectified prints, ratioed prints, mosaics, or enlargements for many portions of the country. Moreover, some of these companies can prepare topographic maps of a desired area that will compare favorably, both in cost and in accuracy, with plane-table surveys. When a special aerial survey is to be made by contract with a commercial company, the surveyor should be able to draw up specifications for the survey (Art. 844).

A visit to an office of a government agency or of a commercial company is often the most direct method of obtaining aerial photographs. By such a visit it is often possible to select prints which can be overlapped or made into mosaics to cover the desired area.

828. General Method of Aerial Photography. In photographing the area to be surveyed, the plane is flown along straight parallel courses called *flight strips*. These flight strips are planned to have a side overlap of 30 to 50 per cent. During the flight along a flight strip, exposures are made at such intervals as to provide a customary overlap of 60 per cent in the direction of flight. Owing

to this overlap and to the side overlap, the outer portions of the overlapped photographs may be discarded as explained in Art. 827(*a*). However, the most important reasons for overlap are these: (1) The overlapping portions are used in orienting and controlling a number of prints to form an assemblage of photographs, as explained later; (2) overlapping photographs make it possible to determine differences in ground elevations by the use of stereoscopic instruments.

The plane is generally flown at an altitude of at least 5000 ft. At this height, or lower, air conditions often make it difficult to maintain a straight course and a constant altitude; consequently higher altitudes, from 10,000 to 20,000 ft. or more, are favored. Only an experienced pilot can maneuver a plane satisfactorily for aerial photographic surveying.

The time interval between exposures in making a flight strip, generally a few seconds, varies according to the length of each negative in the direction of flight, the desired overlap, the altitude of the plane, and the ground speed corresponding to the speed of flight.

828(*a*). Aerial cameras. Aerial cameras may be of the single-lens or of the multiple-lens type. Cameras with more than one lens were developed in order to include more area in each exposure than can be covered with a single lens. In the 5-lens camera the center lens points vertically downward, but each of the other four is inclined at an angle of nearly 45° with the vertical, one to the right, one to the left, one forward, and one backward. In the best cameras, lenses operate in synchronism by means of electric shutters. Two-lens, 4-lens, 9-lens, and 10-lens cameras have been developed. The single-lens camera has the advantage of forming the images on a single negative, and owing to the improvement in wide-angle lenses, it can now cover a rather large area in a single exposure. For some purposes only a single-lens camera can be used.

829. Negatives and Prints. (*a*). Three types of aerial photographs. (1) *Vertical,* made with the axis of the camera vertical; (2) *inclined,* made with the axis of the camera intentionally inclined at a considerable known angle to the vertical; (3) *composite,* made with a multiple-lens camera in which at least one camera axis is vertical.

829(*b*). Film. Special aero film is generally used, with enough film in one roll to take one hundred to several hundred photographs. Common sizes of negatives are 7 in. by 9 in. and 9 in. by 9 in.

829(*c*). Fiducial marks. At the center of each of the four edges of the rectangular negative opening in the camera, a small mark is placed so that it records a corresponding mark at the center of the corresponding edge of every photograph taken. These marks are known as *fiducial marks* or *collimating marks.* A line

drawn between the marks on the two longer edges of a photograph will intersect a line drawn between the marks on the two shorter edges at the *principal point* of the photograph. The lines should be very nearly perpendicular to each other, and intersect very nearly at the center of the photograph.

829(*d*). THE SCALE OF THE NEGATIVE may be determined from the proportion:

$$\text{Scale} = \frac{AB}{ab} = \frac{H}{f}$$

AB = a distance on level ground and ab the corresponding distance on the negative or contact print; H = the altitude or height above the ground of the lens; f = the focal length of the lens. (In Fig. 421 (*a*), p. 436, imagine a camera in place of the telescope, the axis vertical instead of horizontal, S the altitude of the camera, and AB a distance on level ground. Then note that the ratio for determining scale is practically the same as that used in deriving the principle of the stadia.) When H is in feet and f in inches, the scale is feet per inch. For example, if H is 12,000 ft. and f is 12 in., the scale fraction is $12{,}000/12$ or 1000 ft. to the inch. When the numerator and denominator are both expressed in feet, the scale is generally expressed as a ratio, in this case $1/12{,}000$. It is to be remembered that unless the ground surface is level for the entire area covered by the aerial survey the altitude H above the ground will vary for different exposures; consequently the scale will differ for photographs of different portions of the area and for different portions of the same photograph even though the airplane is kept at a constant height above a datum plane.

For methods of checking or determining the scale of a photograph by measurement see Art. 832(*a*).

830. Causes of Distortion. A vertical aerial photograph is not an exact orthographic projection but is a perspective of the terrain photographed in which the scale varies with the distance to the object. Consequently such a photograph is not a map for it is not true to scale. The causes of distortion are (1) displacement due to ground relief and (2) displacement due to the tip and tilt of the camera.

830(*a*). DISPLACEMENT DUE TO GROUND RELIEF. In Fig. 830(*a*) the axis of the camera is assumed to be vertical, thus bringing the negative into a horizontal plane. The points marked "1" in the datum plane are the horizontal projections of the four points *A, B, C,* and *D*. The points marked "1" in the plan are the points on the negative where the images of *A, B, C,* and *D* would fall if all four

points were in the datum plane. The points marked "2" in the plan show the displaced positions on the negative of the four points. Note that the displacement of each point is along a radial line from the principal point *P* (Art. 836).

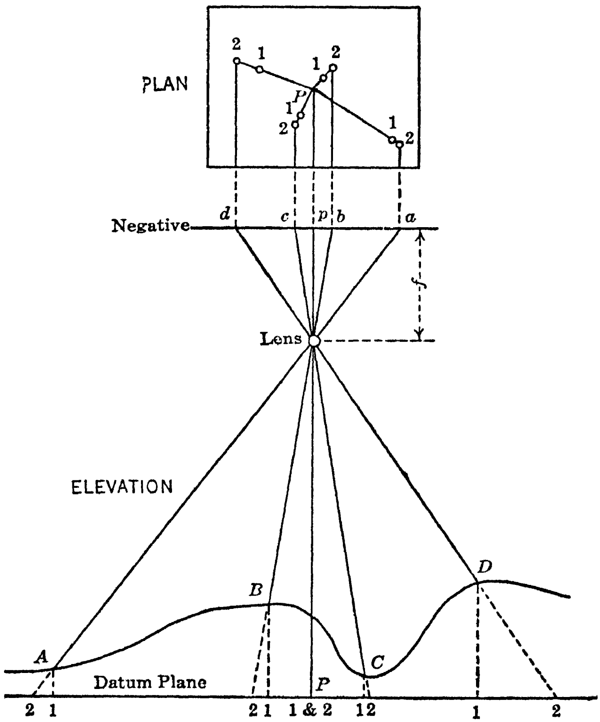


FIG. 830(*a*),1.

If two points are at the same elevation above or below the datum plane (adopted), the one nearer the center of the photograph will have the smaller displacement on the negative. For this reason the central portion of the photograph is likely to be the least distorted —one reason for using only the central portions of overlapped photographs. However, if two points are at equal distances from the center of the negative, the one nearer the datum plane will have the smaller displacement; consequently distortion is small when the surface of the terrain photographed is nearly level.

Note. From the formula $r = hd/H$, which is easily derived, the radial displacement r on a photograph, due to ground relief, varies directly with the altitude, h, of the mapped point from the map elevation or datum plane; and with the distance, d, of the image from the principal point of the photograph;

and inversely with the camera altitude, H, above the datum plane. (See Fig. 830(*a*),1.)

Displacement due to ground relief may cause such distortion in a photograph as to give an entirely erroneous impression of some portion of a terrain. For example, in Fig. 830(*a*),2 a diagram is shown of a straight road that passes over a hill. On a map this road would be represented by two straight lines, but on a photograph it would appear to curve as shown in the diagram. Any straight line with one end high and one end low may have its *direction* on the picture incorrect unless it lies near the principal point of the photograph.

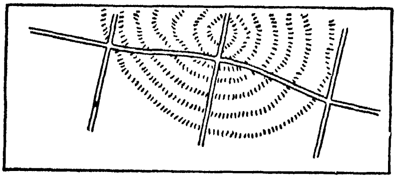


FIG. 830(*a*),2.

Displacement due to ground relief cannot be removed from an aerial photograph. It is inherent in photography and it should not be considered as an error.

830(*b*). DISPLACEMENT DUE TO TILT. The rotation of the camera about an axis normal to the line of flight is called *tip* and corresponds to the pitch of a ship at sea; the rotation of the camera about the line of flight is called *tilt* and corresponds to the roll of a ship. However, when considering single photographs, *tilt* is generally understood to mean the combination of the effects from both types of rotation.

When the camera is tilted, the film is not in a horizontal plane; consequently there is displacement in the negative. Equations may be derived for displacement due to tilt, but in ordinary surveying practice such equations are seldom used. If it is desired to eliminate displacement due to tilt, it can best be done by means of a rectifying camera specially constructed for the purpose. By this process, the original negative is rectified so that the resulting print is free from displacement due to tip and tilt. The process is known as *rectification*. Rectifying cameras have been developed which, in the hands of experts, produce very accurate rectification. Rectified prints may be produced either to an enlarged or to a reduced scale.

Displacements due to tilt are generally less than those due to

ground relief. Displacement due to tilt are regarded as errors. The error is zero at the center of the photograph and increases toward the elevated and depressed edges—another reason for using only the central portions of overlapped photographs.

831. Ground Control. Owing to displacements explained in the preceding article and to errors from other causes, most aerial photographs are distorted; consequently distances and areas cannot be scaled on such photographs except as an approximation. For some purposes, aerial photographs may be a valuable aid to the surveyor regardless of distortion. But if accuracy in scale and position is required, it is necessary to tie an aerial photographic survey to a ground control survey. Such a survey may include a *horizontal control survey* and a *vertical control survey*. The latter is important, particularly when contours are to be located.

The ground control stations should be well-defined points that can easily be identified in the photographs. At least one such point in each photograph is desirable but often impracticable; at least three points should appear in the first photograph, and a minimum of one point in at least every third photograph after the first. The number of ground control points required and their proper locations can be better judged after the uses of such points have been explained (Arts. 836(*a*)–836(*c*)). In general, the larger the scale of the map and the greater the accuracy desired, the more complete should be the ground control. For large-scale maps of relatively small areas the amount of ground control required may equal that required for maps made from conventional ground surveys.

A federal, state, municipal, or other control survey (described in Chapters 56 and 59) should be used for ground control whenever possible, provided it meets the requirements of the aerial survey. It is often necessary, however, for the surveyor either to extend an existing survey or to make an entirely new control survey. When a survey is to be made for the ground control of existing aerial photographs, control points on the photographs should be selected which are easily identified and advantageously located on the ground. When new photographs are required, it is desirable for the surveyor and the photographer to cooperate, prior to the aerial photography, in planning the ground control and the corresponding flight strips.

Ground control is particularly useful in orienting photographs when assembling them for a controlled mosaic or for an accurate map and in checking the positions and elevations of points shown on the mosaic or on the map. A control triangulation net or a traverse, for example, may constitute a framework upon which prints

can be correctly placed and oriented. It is obvious that ground control points should be so chosen that the lines between them can be used to the greatest advantage, both in orienting and in checking. Hence in planning the ground control survey, the surveyor should keep in mind the drafting-room methods used in assembling prints (Art. 836(*c*)) and in plotting maps. He should also keep in mind difficulties which might arise in the field measurement of angles, distances, and elevations, and plan his system of control to avoid, as far as possible, such difficulties.

832. Scales. The scale of a negative or contact print depends upon the altitude of the airplane and the focal length of the camera (Art. 829(*d*)). Most scales used in aerial photography fall between the limits of $1/2400$ and $1/24{,}000$ or, roughly, between limits of 1 in. = 200 ft. and 1 in. = 2000 ft. The scale has a most important bearing on the accuracy, and consequently the cost, of a survey. This may be illustrated as follows. Assume that a map is to be plotted to a scale of 1 in. = 1000 ft., a scale frequently used in aerial photographic mapping. Assume that the smallest distance that will be plotted accurately is $1/50$ in. To the scale chosen this represents $1/50$th of 1000 ft. or 20 ft. Displacements in the photograph as large as 20 ft. might not occur, particularly in the central portion of the print; therefore corrections of displacement could be omitted with a consequent saving in cost. However, if the scale chosen for the map had been 1 in. = 500 ft., an error in displacement of 10 ft. would be appreciable in plotting the map; consequently corrections of displacements might be necessary.

When photographs are desired to a larger scale than can be obtained in contact prints, enlargements can be made to almost any scale and at the same time the photographs can be rectified (Art. 830(*b*)). However, excessive enlargement may cause a loss in sharpness of essential detail so that the print may appear more or less blurred. Nevertheless, when sharpness of detail is not required, an enlargement ratio as great as 12 to 1 is often used. Thus if the original negative was to an approximate scale of 1 in. = 1200 ft., the corresponding enlargement would be to a scale of 1 in. = 100 ft. Despite their blurred appearance, large-scale enlargements are more useful for many purposes, such as tax appraisal and land subdivision, than small-scale enlargements would be. In making a large-scale enlargement, only a portion of a negative is generally used.

832(*a*). SCALE DETERMINATION. The scale of a contact print or an enlargement may be determined or checked by comparing the length of a line between two definite images on the photograph with

the corresponding horizontal distance on the ground. In checking the scale of a mosaic it is well to select several such "scale-determination lines" or "scale-check lines" located in different portions of the mosaic. When there is ground control, lines between ground control stations can be used as scale-check lines without the necessity for additional field work. When it is necessary to select and measure special check lines on the ground, each line should be carefully chosen (1) to be of sufficient length on the photograph to afford a good check and (2) to extend between two points that are sharply defined and easily identified on the photograph. (See selection of control points, Art. 835.)

Scales determined by the measurement of check lines in different portions of a mosaic may differ slightly from each other, particularly if the ground is not level. Consequently, the scale determination of the entire mosaic may be only an approximation, but it is generally sufficiently accurate for the purpose for which the mosaic is used. (i) Scale = f/H photo distance
(ii) Photo scale / map scale = map distance

832(*b*). NOTE. *The scale of a photographic negative* may be determined from *rectified* contact prints made preferably on non-shrinkable material. A *comparator* is an instrument for measuring accurately the rectangular coordinates of points on photographic prints. The scale of a negative is generally determined by the government agency or the commercial company that furnishes the photographs.

833. Use of Mosaics and Photogrammetric Maps. The purpose of this article is to give a general idea of the use of mosaics and photogrammetric maps in ordinary surveying. The use of aerial photographs in extensive surveys is treated in numerous books and other publications on photogrammetry or on photogrammetric surveying.

833(*a*). CLASSIFICATION OF MOSAICS. Mosaics are generally classified as (1) *uncontrolled* and (2) *controlled*. An *uncontrolled aerial mosaic* is made by assembling aerial photographs with little or no ground control (Art. 836(*b*)). A *controlled aerial mosaic* is made by assembling and orienting photographs by means of adequate ground control, as explained in Art. 836(*c*). Generally, such a mosaic is more nearly true to scale than an uncontrolled mosaic. Many mosaics used in ordinary surveying might well be regarded as roughly or partly controlled. *Since displacements due to ground relief cannot be eliminated even from a controlled mosaic, one should be cautious in scaling distances and areas on most mosaics.* However, the accumulation of error from displacement due to relief is largely prevented by ground control, and much of the distortion may be eliminated by using only central portions of the photographs and by using ratioed prints.

833(*b*). EXTENSIVE USE OF MOSAICS. Mosaics and maps made from them are used extensively by various federal and state agencies and by large business organizations such as oil companies and public utilities. Among the many purposes for which mosaics are used are these: preliminary location of railroads, highways, transmission lines, and pipe lines; preliminary surveys for dam sites and reservoirs; surveys of forests and the classification of growing timber; studies in soil conservation; petroleum geology; property ownership and tax maps. Although mosaics are particularly useful in such extensive surveys as those just listed, they may be useful also in surveys of relatively small areas, as will be apparent from the statement of advantages in the next article.

833(*c*). ADVANTAGES IN THE USE OF MOSAICS. Within limits dictated by common sense, mosaics and maps made from them can be used to great advantage in ordinary surveying. For some purposes mosaics are superior to maps made solely by ground methods. Even an uncontrolled mosaic is useful in many cases and it is of particular value for reconnaissance and for general planning purposes.

Among the advantages in the use of mosaics are: (1) A mosaic, as contrasted with a number of overlapping prints of the same area, shows on a single sheet the least distorted portions of all the photographs. These portions are cut and fitted to each other along irregular lines (much like a jigsaw puzzle) in such a way as to show the terrain to the best advantage as well as with the least distortion. (2) A mosaic can be made quickly and at a low cost, particularly when aerial photographs of the area can be readily obtained and ground control can be secured from ground surveys previously completed. (3) A mosaic shows better than any map a wealth of detail such as the variations in the character of walls, fences, roads, lanes, and trails; the character and extent of woodland, meadowland, swamps, and cultivated land; the size and distribution of individual trees; the meanders in streams and the irregularities in shore lines and channels; and the location of various structures such as bridges and buildings. (4) For purposes of reconnaissance, a mosaic enables the engineer to view the terrain as if he were passing over it in an airplane. (5) Contours can be transferred to a mosaic after having been located by stereoscopic observation of overlapped photographs, as explained later. The contours may be transferred to a sheet of paper to form a stereotopographic map or they may be drawn on the mosaic itself. In the latter case the mosaic becomes a substitute for a topographic map and for some purposes is better than a topographic map.

In comparing the relative advantages of an aerial photograph survey and a ground survey made by the customary methods, an important consideration is the character of the terrain. Obstacles to ground measurements such as swamps, lakes, thickly wooded areas, and rugged portions of the ground surface may make a ground survey too costly in both time and money.

833(*d*). CLASSIFICATION OF AERIAL PHOTOGRAMMETRIC MAPS. Although mosaics are often called maps, they are not maps and will not be included in this classification. Only those maps will be classified which are made directly from an assemblage of overlapping photographs that covers the map area. The first step in the preparation of such maps is to orient and to adjust the overlapped photographs, one by one, under a transparent celluloid control sheet, as explained in Art. 836. Details of the photographs can then be transferred to the map surface as explained in Art. 837.

An *aerial planimetric map* is a line map made by transferring details from aerial photographs to a map sheet, as explained in Art. 837. There are no contours on the map.

An *aerial topographic map* is similar to a planimetric map except that contours are added. These contours are located by field methods such as those explained in Chapter 31. One of the best methods is to transfer the planimetric map sheet to a plane table and then to locate the contours by the usual plane-table methods.

A *stereotopographic map* is similar to an aerial topographic map except that contours are located directly from pairs of photographs by means of a stereoscopic plotting instrument, as explained in Art. 839(*b*).

833(*e*). USE OF AERIAL PHOTOGRAMMETRIC MAPS. In the preparation of a photogrammetric map, detail may be transferred from the photographs either to the same scale as that of the photographs or to a different scale. Moreover, modern equipment and modern methods make possible a wide range of conversion from one scale to another so that a base map may be enlarged or reduced, within reasonable limits, to any desired scale. The map accuracy which can be attained by photogrammetric methods is surprisingly high, as indicated by the map accuracy specifications in Art. 844(*b*) at the end of this chapter. The wide range of scales which can be used and the high degree of accuracy attainable make practicable a wide variety of uses of photogrammetric maps, both planimetric and topographic.

In planning large projects, combinations of mosaics, photographs, and photogrammetric maps may be used. For example, in the design of highways, small-scale uncontrolled mosaics and topographic maps may be used for general study of the area; overlapping paired photographs and controlled mosaics for reconnaissance; overlapping paired photographs and larger-scale photogrammetric contour maps for preliminary location; and enlarged photographs and large-scale photogrammetric contour maps for the final survey and plans. (See Art. 843(*a*), Table I.)

In general, mosaics are used when a pictorial view of the terrain and its culture is desired and when only a moderate degree of accuracy is required. Photogrammetric maps are used when only details that can readily be shown on a map are desired and when a degree of accuracy not easily attainable in a mosaic is required.

It is sometimes advantageous to combine an aerial photographic survey with a ground survey to obtain a satisfactory map. For example, boundary lines of a tract may be surveyed with a high degree of precision by the customary field methods of ground sur-

veying, and the details within the boundaries may then be located by the use of aerial photographs.

Note. For a comprehensive discussion of the use of mosaics, aerial photographs, and photogrammetric maps see Art. 843(*a*).

833(*f*). ADVANTAGES IN THE USE OF AERIAL PHOTOGRAMMETRIC MAPS. When an area of land is sufficiently large to warrant an aerial photographic survey, a photogrammetric map can generally be produced at a lower cost than an ordinary map and in much less time than would be required for a conventional ground survey. Moreover, the photographs used in producing the map furnish a more or less permanent record of details which are more complete than can be shown on any map.

One of the greatest advantages in aerial photographic surveying is that accurate topographic maps can be made of large areas quickly and without ground surveys except those necessary in establishing horizontal and vertical ground control. Contours can be located directly from photographs with sufficient accuracy for most purposes. A common requirement in specifications for stereotopographic maps is "that eighty-five per cent (85%) of all elevations interpolated from the map's contours shall be correct within ½ the contour interval." (See Art. 844(*b*),IV.) When a contour interval is small, say 5 ft. or 2 ft., such a specification can be met only by means of large-scale precise photography, special ground surveys for the exact control of every photograph, and highly accurate photogrammetric mapping instruments operated by experts. The amount of ground control that is necessary may be equal to the amount required in a conventional ground survey.

834. Examples of the Use of Aerial Photographs in Ordinary Surveying. The following examples will illustrate the use of aerial photographs in making surveys of relatively small extent. All examples are of surveys actually made, and only simple surveys have been selected. For an example of the use of aerial photographs in more extensive and complex surveys, see Art. 843(*a*).

Example I. Reconnaissance for a Supplemental Control Survey. It was desired to make a supplemental control survey similar to that described in Art. 777(*a*). The preliminary reconnaissance of the area was made by inspection of aerial photographs. These were of value: (1) In showing the relative positions of the U.S. Coast and Geodetic Survey stations in the area; (2) in planning a supplemental control triangulation net tied to the U.S. stations and laid out with due regard to strength of figure (Art. 403(*f*)) and to low cost of towers and field work; (3) in planning supplemental, second-order control traverses, tied to the triangulation net.

Example II. Highway Location. Aerial photographs are extensively used in highway location. They are particularly valuable in the early planning of the more important highways, especially those of the limited access type, which, in many instances, are on new location and may extend over a considerable distance. By the use of mosaics in combination with the U.S. Geological Survey, contour maps, trial locations, with length, alignment, and rough profile, may be worked up in a relatively short time. Locations clearly impracticable from either an alignment or grade requirement may be discarded at this stage. Locations which appear feasible can then be given further study in the field. One size of mosaic used for the work of location covers an area of 5′ in longitude and 5′ in latitude. At the scale of 1 in. = 600 ft., with an overlap of 2 in. on all edges for matching purposes, the sheets measure about 3.5 ft. by 4.5 ft.

Some states have mosaics similar to those just described which together cover the entire state. In addition, stereotopographic maps (Art. 841(*b*)) on tracing cloth, made to a scale, say, of 1 in. = 200 ft., are used for particular sections of the state for which such maps are desirable. Both the mosaics and the stereotopographic maps may be made, under contract, by some company or corporation specializing in aerial surveying. The original photographic negatives should generally become the property of the state. Many of the particu-

lars which should be included in the specifications are suggested in Art. 844.

Note. For a more comprehensive explanation of the use of aerial photographs in highway engineering, see Art. 843(*a*).

Example III. Survey for Assessment Maps. Mosaics are used by some municipalities as assessment maps or tax maps. A set of such maps, covering a small town, consists of mosaics enlarged to a scale of 1 in. = 200 ft. Division lines marked by fences, hedges, and stone walls can readily be spotted on the mosaics. Ground control surveys are made with transit and tape, and boundaries, measured with the tape, are drawn, where necessary, directly on the mosaics. No tracings are made.

Example IV. Topographic Survey of a Tract of Land. A property owner desired a topographic survey of a tract of about 250 acres. Among the topographic features to be shown on the map were boundaries of the tract, buildings, walls, brooks, high ledges, ponds, important individual trees, outline of timber land, wood roads, and bridle paths.

The topography was very irregular with a maximum difference in elevation of about 150 ft. Brush and other conditions were such that the cost of a map by the usual ground methods would have been prohibitive.

Aerial photographs of the tract were taken from a plane flying at such an altitude that the scale of the negative with the particular lens used in the camera was approximately 1 in. = 450 ft. Seven photographs were selected which, overlapped, were sufficient to cover the entire area of the tract. Controlled enlargements were made to a scale of 1 in. = 150 ft., the scale desired for the finished map. The size of each enlargement was 15 in. by 15 in., a convenient size for use on a traverse board in the field.

In establishing horizontal ground control, transit and tape were used for the perimeter of the tract, transit and stadia for the interior. In establishing vertical control, the usual methods of spirit leveling were followed. A few of the more important points of ground control were marked on the ground by a white cross in order that the points could be more readily identified in the photographs. Each cross was made of two white cloth panels placed at right angles to each other.

In locating contours in the field, individual photographs were used as field sheets. Elevations of points on 5-ft. contours were determined by the use of altimeters (Art. 183(*a*)), and the contours were sketched directly on the aerial photographs. The final topographic map was traced from the photographs. Only the central portion of each photograph was used.

At a later time, surveys were made for the purpose of subdividing the entire area into acre squares. In running the lines to establish corner stakes, surveyors were impressed with the close agreement of the results of their field work with the topographic map made from the aerial photographs, even in the location of contours. It was evident that the topographic map was as accurate as the proposed use of it would warrant.

Note. In extensive surveys, the most economical method of locating contours is to trace them directly from photographs by the use of a stereoscopic instrument (Arts. 841(*b*), 842(*d*)).

Example V. Survey for a Map of a Water Distribution System. A water company desired a planimetric map of its distribution system covering a city and outlying country districts. The scale adopted for the map was 1 in. = 400 ft. Two mosaics were obtained, each about 39 in. by 44 in., made to a scale of 1 in. = 600 ft. The southern portion of one mosaic overlapped slightly the northern portion of the other. Thus overlapped, the two mosaics covered the area to be mapped.

Ground control was obtained from monuments which marked stations of the state system of plane coordinates. These monuments were numerous within the city limits and furnished abundant control. Positions of nearly all monuments could be readily spotted on the mosaics.

Two maps were made, one from each of the two mosaics, to a scale of 1 in. = 400 ft. The maps were drawn on tracing cloth from which subsequently black and white prints were made. When completed, each map was approximately 4 ft. by 6 ft. The northern portion of one map and the southern portion of the other map were identical for a short distance from the corresponding edges. The details of the distribution system shown on these maps included all distributing reservoirs, pumping stations, standpipes, mains or pipes owned by the water company, pipes not owned by the company, streets without pipe, gate valves, gate valves in manholes, hydrants, private hydrants, blowoffs, air traps, and meters. The small details were indicated by conven-

tional symbols. Details in addition to those of the distribution system included key buildings (such as municipal buildings, railroad station, and buildings owned by water company), streets and roads, lakes, rivers, brooks, bridges, and seashore lines.

The positions of the monuments which marked the ground-control stations were plotted first to the adopted scale of 1 in. = 400 ft., on the two sheets of tracing cloth. The rectangular coordinates used in plotting the stations were obtained from the State Geodetic Survey. In some of the outlying districts where monuments were farther apart than in the city, ground control from monuments alone was inadequate; consequently some field work was necessary to establish additional ground control.

Most of the details were transferred from the mosaics. Since the mosaic scale (1 in. = 600 ft.) was different from the map scale, details could not be traced directly from the mosaics; consequently they were transferred by means of a pantograph. Some of the minor details of the distribution system were located on the maps by means of data obtained from the water company records.

For the purposes for which the maps were made, discrepancies between the positions of plotted stations of the ground control and the positions of the corresponding points on the mosaic were found to be negligible.

835. Picture Control. Picture control is a type of control composed of points carefully selected on the photograph to be used in place of or in connection with ground control in orienting photographs, particularly by the radial-line method. Picture-control points have no corresponding ground-control points. A picture-control point should be some object or part of an object clearly defined and readily identified on each of at least two adjacent photographs, that is, within the area of overlap of the two photographs. Among points best suited for the purpose are corners of buildings free from shadows; intersections of walls, fences, and hedges; sharply defined intersections of streets, roads, and trails; isolated trees; and well-defined ends of structures such as bridges and culverts. In selecting control points, it is helpful to observe the photographs with a stereoscope. When aerial photographs are made to order for some particular purpose and it is important that certain points on the ground should appear in the photograph, white markers, sufficiently large to be clearly identified in the photograph, may be placed at the points. (See Art. 834, Example IV.)

Much depends upon the selection of the proper *positions* of picture-control points with respect to the edges of the photographs, as explained in Art. 836(*a*).

836. Radial-Line Method. The radial-line method is a graphic method of transferring control points from photographs to a transparent *mapping control sheet* in such a manner that all points are in their correct positions with respect to each other, free of relief and tilt distortion. When the picture-control points on all the photographs of a group have been thus transferred, any photograph, placed under the transparent sheet, can be moved to its proper position in the group by means of the picture-control points

on the photograph and the corresponding control points on the control sheet. If all the photographs of the group should be thus oriented under the sheet at one time, there would be an assemblage of overlapping photographs each of which would be in its correct position with respect to all the others. When accurate results are desired, picture control is used in combination with ground control.

Note. The radial-line method may also be used to add new features to an old map. Recognizable objects already mapped may be used as "ground-control" points for the orientation of the photographs and the location of the map positions of their principal points. Radial-line intersections from positioned photographs then serve to locate the new features.

A celluloid sheet (cellulose acetate) is generally used for a mapping control sheet. It is much more transparent than tracing cloth or tracing paper, changes in atmospheric conditions affect its dimensions very much less, and it is more satisfactory in other respects. The surface of celluloid can be improved for drafting purposes by rubbing it with some fine abrasive.

A *radial line* through any point on a photograph is the line that passes through that point and through the principal point of the photograph (Art. 829(*c*)). If there is a slight displacement of a point on a photograph due to tilt or to differences in ground elevation, the displacement will occur along the radial line through the point, and if the point is in the central portion of the photograph, the displacement will generally be negligible. The radial-line method is based partly on these two perspective properties of a photograph.

The radial-line method involves two principal steps: (1) the selection of picture-control points on the photograph and the marking of those points (Art. 836(*a*)); (2) the transfer of the picture-control points to the transparent sheet, a procedure known as *tracing* the points. There are two methods of tracing the points, one when there is little or no ground control (Art. 836(*b*)), and one when there is adequate ground control (Art. 836(*c*)).

836(*a*). The selection of picture-control points. In the selection of picture-control points the three principal requirements are these: (1) The points must be definite and easily identified (Art. 835). (2) Every control point on a photograph must appear also in an adjacent photograph, that is, within the overlap of the two photographs. (3) Control points must be in such positions that they can be used to the best advantage in the radial-line method.

The three rectangles in Fig. 836(*a*) represent the first three photographs of a flight strip before they have been placed in their

overlapped positions. (It should be kept in mind throughout the following explanation that the contact prints are much larger than the rectangles, generally 7 in. by 9 in. or 9 in. by 9 in.) The procedure for selecting and marking the control points, including one ground-control point, is as follows:

1. Number the photographs in the order in which they overlap in the flight strip.

2. Locate and mark the principal point on each photograph (Art. 829(*c*)). In Fig. 836(*a*) these points are marked 1*C*, 2*C*, and 3*C*.

3. At the principal point 1*C* of photograph 1, select and mark the image of a definite object which is also readily identified on photograph 2. When it is difficult to distinguish such an image exactly at the principal point, select an image as a *substitute center* near that point and on or near the line connecting adjacent principal points 1*C* and 2*C* (marked 1*S* in Fig. 836(*a*)).

4. On photograph 1 identify and mark the ground-control point (marked 1*G*). If there are several such points, each should be marked.

5. On photograph 1 select two picture-control points 1*L* and 1*R* nearly opposite 1*C*, one near the left edge, the other near the right edge of the photograph. Both points must also appear near the upper edge of photograph 2, as indicated in Fig. 836(*a*).

6. Near the lower corners of photograph 1 select two picture-control points 2*L* and 2*R* so chosen that they appear nearly opposite 2*C* in photograph 2 and near the upper corners of photograph 3, as indicated in Fig. 836(*a*).

7. At or near the center of photograph 2 select and mark an image 2*S* as a substitute center near 2*C* and on or near the line between 2*C* and 1*C*. It should be an image that can be easily identified on both photographs 1 and 2.

8. Near the lower corners of photograph 2 select two picture points, 3*L* and 3*R*, so chosen that they appear nearly opposite 3*C* in photograph 3.

9. Repeat the procedure outlined in (7) and (8) until all picture-control points and ground-control points have been selected and marked on the photographs throughout the strip. Each photograph, except the first and last, will then have eight picture-control points in addition to the principal point and possibly a substitute center for that point (marked *S*) near the center. (In photograph 3, a clearly defined image at its principal point made a substitute center unnecessary.) Each pair of photographs at each end of the strip should have at least one ground-control point in common if it is desired to determine the scale of the entire strip. This ground-

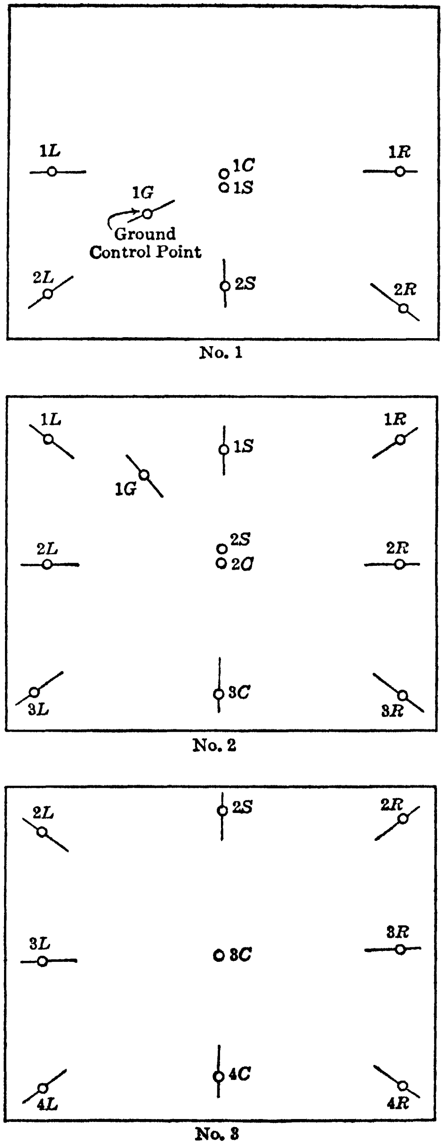


FIG. 836(*a*). The first three photographs in a flight strip as they are marked before placing them in their overlapped positions.

control point should lie within the area of overlap of the two photographs in order to permit the intersection of its radial line in one photograph by its radial line in the other. The scale of the strip will be the same as the mean scale of the photographs.

When there is an adjacent flight strip to the left of the points marked *L*, those points should be so chosen that they furnish satisfactory picture control in the adjacent strip as well as in the first strip. Similar precautions should be taken in choosing points marked *R* when there is an adjacent flight strip to the right of those points.

10. When the picture-control points and the ground-control points have been marked on all the photographs, draw radial lines on each photograph through all its control points. Only that portion of a radial line which extends a short distance each side of the corresponding control point is generally drawn. (See Fig. 836(*a*).)

In selecting control points, it may be helpful to examine the photograph under a magnifying stereoscope (Art. 838), particularly if it is desired to choose points near the average elevation of the terrain to be mapped.

A pin prick in the center of a very small circle drawn with a colored pencil is a common method of marking control points. Picture-control points and ground-control points should be marked in different colors.

836(*b*). USE OF THE RADIAL-LINE METHOD WHEN THERE IS LITTLE GROUND CONTROL AND WHEN IT IS DESIRED TO PLOT THE MAP TO THE SCALE OF THE PHOTOGRAPHS.

1. Place photograph 1 (Fig. 836(*a*)) under the celluloid sheet in such a position that when all the photographs of the area to be mapped are assembled in their proper overlapped positions, they will be within the limits of the sheet.

2. Trace the principal point 1*C*, the substitute center 1*S*, and the substitute center 2*S*. Trace also the radial lines to the other marked points, including the ground-control point. Remove photograph 1.

3. Place photograph 2 under the celluloid sheet in such a position that point 2*S* on that photograph and the corresponding point on the celluloid sheet (traced in (2)) coincide.

4. Turn the celluloid sheet about 2*S* as a center until the point 1*S* on the celluloid falls on its radial line on photograph 2. Fasten the sheet in this oriented position by thumb tacks or weights.

5. Trace radial lines from the principal point 2*C* to points on photograph 2 marked 1*L*, 1*R*, 2*L*, 2*R*, and 1*G*. The intersection of each of these radial lines from 2*C* with the corresponding radial

line from 1*C* (previously traced on the celluloid) should be the true location of the corresponding point on the map. However, owing to tilt and relief displacements, this point of intersection may not coincide exactly with the corresponding picture-control point, either on photograph 1 or on photograph 2. (See Fig. 836(*b*).) Trace the remaining radial lines from 2*C* to 3*L*, 3*R* and 3*C*. Remove photograph 2.

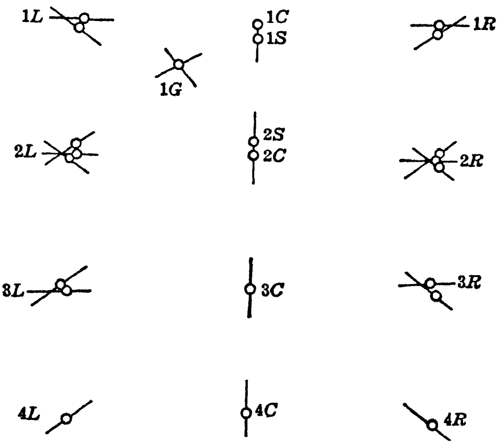


FIG. 836(*b*). The relative positions of control points after the three photographs represented in Fig. 836(*a*) have been overlapped and oriented.

Photograph 3 and all the remaining photographs of the strip can be oriented and their control points traced on the celluloid by the procedure just outlined for photograph 2. The same procedure is used for an adjacent strip except that adjacent strips must be "tied" to each other. This may be done by the three-point method applied to the picture-control points near the left-hand edge of one strip and near the right-hand edge of the other strip. The relative positions of the control points in the first three photographs after the photographs have been overlapped and oriented are shown in Fig. 836(*b*).

The scale of the tracing may be determined by comparing the distance between the two ground-control points on the tracing (one at each end of the strip) with the corresponding distance measured on the ground (Art. 832(*a*)). This scale should be the mean scale of the photographs. If the strip is long, it is well to establish intermediate ground-control points.

Since any photograph can be placed in its proper position under the celluloid sheet, details can be traced directly from a photograph.

Only the central or least distorted portion of each photograph should be used. The map may afterward be enlarged or reduced to a desired scale by one of the methods of Art. 663. If desired, photographs may be enlarged or reduced to produce approximately the scale desired in the finished map.

836(*c*). USE OF THE RADIAL-LINE METHOD WHEN GROUND CONTROL IS ADEQUATE AND WHEN IT IS DESIRED TO PLOT THE MAP TO A FIXED SCALE.

1. Plot the ground-control points on the celluloid to the scale chosen for the map. This scale should be nearly the same as the mean scale of the photographs. Some engineers prefer to plot the ground control on drawing paper of high quality. The plotted positions of the ground-control points are then transferred to the celluloid control sheet by pricking through each point with a fine needle.

2. Select two adjacent photographs in the flight strip which have at least three ground-control points in common—a prerequisite in beginning the orientation of the photographs.

3. Orient the first photograph by placing it under the celluloid sheet and moving the sheet until the three radial lines through the three *ground*-control points on the photograph pass through the corresponding ground-control points on the celluloid. Trace these radial lines on the celluloid.

4. Orient the second photograph and trace the radial lines in a similar manner. Both photographs are now in their correct positions with respect to the ground control, and the scale has been determined by that control.

5. Trace radial lines through all picture-control points on the two photographs. Each picture-control point on the celluloid will be at the intersection of two radial lines, one through the corresponding point on the first photograph and one through the corresponding point on the second photograph. Thus each picture-control point is located on the celluloid sheet by resection.

6. Assume that there are no ground-control points on the third photograph. Orient it by placing it in such a position that its principal point falls on the radial line previously drawn on the celluloid sheet and, simultaneously, the radial lines drawn through picture-control points near the upper corners of the photograph pass through the corresponding points previously located on the celluloid. This is similar to the method of orientation used throughout the procedure explained in Art. 836(*b*) for tracing control points when there is little or no ground control.

The procedure in (6) may be repeated for successive photographs until a photograph is reached on which there is at least one

ground-control point. The location of that point determined by resection may not coincide with its position as plotted in (1). In such a case slight graphic adjustment may be desirable.

The radial-line method of resection may be used in the determination of the true positions of important topographic features.

Remark. In transferring control points from photographs to the celluloid base, either with or without ground control, time, accuracy, and ease of adjustment may be gained by the use of templets (Arts. 836(*d*), 836(*e*), and 836(*f*)).

836(*d*). Use of transparent templets. Templets made of transparent celluloid sheets slightly larger than the photographs and numbered to correspond to the photographs are prepared as follows. All control points, including ground-control points, having been selected and marked on each photograph (Art. 836(*a*)), a celluloid sheet is laid over one of the photographs and secured in place. A point is pricked in the templet directly above the principal point of the photograph, and from that point radial lines are drawn on the template through all picture-control points and ground-control points (Art. 836(*a*),10). No radial lines are drawn on the photograph. When a templet for each photograph has been made in this manner, the templets may be used instead of photographs in locating picture-control points on the celluloid mapping sheet by either of the two procedures explained in Arts. 836(*b*), 836(*c*). If ink soluble in water is used in drawing radial lines on the templet, the celluloid sheets can be washed and used repeatedly.

836(*e*). Use of slotted templets.* The slotted templet is the same in principle as the transparent templet, but it is made of stiff Bristol board instead of celluloid. A templet for each photograph is prepared as follows. The photograph is laid on the templet and its principal point, picture-control points, and ground-control points are pricked through to the cardboard and marked by colored circles. Concentric with the principal point or substitute center, a small circular hole is cut through the Bristol board with a special punch. Instead of drawing radial lines on the templet, narrow slots 1½ in. to 2½ in. long are cut through the Bristol board with a special slot-cutting punch. Each slot on the templet serves in place of the corresponding radial line on the photograph; consequently radial lines are not drawn on the latter.

Templets are held in position on the celluloid mapping sheet by "posts." (Also known as "rivets" or "studs.") A post consists of a thin, flat disk which serves as the base of a short, cylindrical, vertical stem. Through the stem a small vertical hole is drilled so that a needle or pin may be pushed through to prick a point, or to hold the post in a fixed position, or to fix the position of a point by resection. The diameter of the vertical stem is equal to the width of a slot and to the diameter of a punched hole.

Like transparent templets, slotted templets may be used instead of photographs in locating picture-control points on the celluloid mapping sheet by either of the two procedures explained in Arts. 836(*b*), 836(*c*). For example, in the procedure outlined in Art. 836(*c*), the picture-control points on the first two photographs are transferred to the celluloid mapping sheet as follows:

1. Place the first templet on the mapping sheet with a post in each slot and each hole. The disks of the posts should be underneath the templet so that the stems project upward through the templet.
2. Orient the first templet by moving it to such a position that each of the three posts marking the three points of ground control falls exactly over its corresponding point of ground control previously plotted on the mapping sheet. Fasten the templet securely in place by pins pushed through the three posts. (See Art. 836(*c*),3.)
3. Place the second templet over the first in such an overlapped position that the slots to control points fit the corresponding posts that project through the first templet. (See Art. 836(*c*),4.)

* This method is covered by a U.S. Patent owned by the Fairchild Aerial Surveys, Inc.

As the procedure is repeated from templet to templet, a difference between the resected location of a ground-control point on the templet and its plotted location on the celluloid map sheet may require a slight adjustment. Since templets are held together by posts, they are free to move within the limits of their slots until fixed by ground-control points. In the final adjusted positions of the templates, the post for each ground-control point should fall exactly over the corresponding point as plotted on the celluloid mapping sheet. Points can now be pricked through the posts in all slots, thus marking on the mapping sheet the correct positions of all picture-control points.

836(*f*). "Lazy Daisy" mechanical triangulator.* This device consists of a set of slotted metal strips about one-half inch wide which are used as slotted templets in a procedure similar to that explained in Art. 836(*e*). These strips, called "radial arms," are of several different lengths, ranging from 3 to 24 in. At one end of each strip there is a slot of considerable length; at the other end there is a hole.

When in use, the radial arms, held together by pin guide shafts through the slots and holes, form a mechanical radial line control net work which accurately locates the control points. One advantage of this excellent device is that the radial-arm templets may be used again and again for different surveys.

837. Transfer of Detail from Photographs to Maps. The amount of detail to be transferred varies. Some mapping agencies compile only the most important lines, leaving minor details to be located in the field. Other agencies transfer directly from the photograph all the cultural detail that is to be shown on the map.

It is often difficult to distinguish the important from the unimportant detail since both are shown on the photograph with the same degree of density. A stereoscope, particularly a magnifying stereoscope, is helpful in observing photographs. For accurate compilation of culture, the compiler should be an expert in the interpretation of aerial photographs.

837(*a*). Transfer of detail by tracing. The detail may be traced directly on the celluloid control sheet. It is well to rub the tracing side of the sheet with a fine abrasive. Place any one of the photographs under the transparent sheet and carefully orient it by moving the sheet until the photograph is adjusted to the corresponding map control on the sheet. With pencil or pen, trace the detail in the central area of the photograph, that is, the detail within a distance from the center of about halfway to the nearest radial line points on the various radial lines. The area traced on each photograph is determined largely by the customary 60 per cent overlap in the direction of flight and by the customary side lap of 30 to 50 per cent.

Remove the photograph and replace it by an adjacent overlapping photograph, carefully oriented and adjusted, and trace the detail. Continue the procedure until the detail has been transferred from the central portions of all photographs, each of which has

* Manufactured by the Abrams Instrument Company, Lansing, Michigan. A handbook of instructions for the use of the triangulator is issued by the company.

been carefully adjusted to the map control. When, owing to relief displacement, there are small-scale differences between the control points on the sheet and the corresponding control points on the photographs, such discrepancies should be balanced between the points to prevent the accumulation of error at any one point. The accuracy of the transfer of detail depends very largely on the accuracy and density of the ground control.

Detail may be transferred from the celluloid sheet to the base map sheet by tracing over a carbon sheet. Detail may also be transferred, either to the same scale or to a different scale, by the use of a pantograph or by any one of the other methods of copying and transferring maps explained in Art. 663. The first step should be to transfer important points such as control points and to check the positions of the points, preferably by the coordinate method of plotting. The most important details should be transferred next. A temporary system of squares drawn lightly in pencil and to scale on both the celluloid sheet and the map sheet is useful in checking coordinates.

Contours may be added to a planimetric map by methods explained in Art. 841.

Note. The manual described in Art. 841(*c*) contains a more comprehensive treatment of the radial-line method and of the methods of transferring detail than it is possible to include within the limits of this chapter. The manual is highly recommended.

837(*b*). Transfer of detail by mechanical projection. Detail may be copied from a photograph to a map with considerable accuracy by a simple and relatively inexpensive projecting instrument known by the trade name Sketchmaster. (For a description, see Art. 842(*b*).) A more accurate projecting instrument is the overhead ratio projector, described in Art. 842(*c*). To obtain the highest degree of precision in mapping planimetric detail and contours from photographs, precision instruments are used, such as those described in Arts. 842(*e*) and 842(*f*).

838. The Stereoscope. An aerial photograph, viewed by the eye alone, shows the surface of a terrain as if it were in a horizontal plane. In order to observe differences in elevation, something must be done to make the photograph stand out in relief. For this purpose, some type of instrument must be used which is based on the principles of stereoscopic vision.

Stereoscopic human vision enables us to judge which of two objects is nearer to the eye. Human perception of distance is similar in effect to that obtained by two intersecting lines of sight from two transits, one at each end of a base line. Each eye records upon

its retina a separate image of the object observed; the base distance between the retinas is approximately 2½ in. Despite this short base, most persons are able to detect at distances up to about 2000 ft. which one of several objects is nearest the eye.

The old-fashioned stereoscope, with its pair of lenses, enables the observer to view a pair of nearly identical photographs, one with each eye, in such a manner that both photographs appear to fuse into a single three-dimensioned picture. The two photographs, though pictures of the same objects, are not exactly alike since they were taken from two different exposure points. The short distance between these points corresponds to a base line.

Because of a base distance of several hundred feet between successive exposure points in a flying strip, it is possible to view through a stereoscope any pair of adjacent photographs in the strip. Thus viewed, the ground surface stands out in relief in the single fused picture and differences in ground elevations become apparent. These differences in elevation become still more apparent when viewed through a *magnifying* stereoscope. The illusion produced may be so strong that the image of the ground, with all its variations in elevation, may appear solid as if one were looking at a small model of the ground surface. In photogrammetry, this apparent model is known as the *spatial model.*

In aerial surveys the stereoscope is used mainly: (1) for the study and interpretation of detail in photographs, particularly when transferring details from photographs to an aerial planimetric map; (2) for the location of contours on a stereotopographic map. For this second purpose, the stereoscope must be combined with a device for locating and sketching contours while the observer views the spatial model. In a simple form of such an instrument, a *floating mark,* observed through the stereoscope, is moved by the observer over the spatial model while at the same time a pencil is automatically moved to draw the contour. (See the description of the stereocomparagraph in Art. 842(*d*).)

One of the greatest advantages in aerial photographic surveying is that accurate topographic maps can be made of large areas quickly and at a relatively low cost. Savings as great as 50 per cent of the cost of mapping by conventional ground methods are not unusual.

839. Use of the Stereoscope.* A beginner in the use of a stereoscope should keep in mind some of the basic fundamentals of aerial photographs; it will make him more proficient in the use of

* This article contributed by Professor W. L. Willig.

the instrument and save him unnecessary eyestrain. A stereoscopic pair of photographs consists of two adjacent photographs having common coverage of a section of terrain. *Only that portion of terrain common to the two photographs may be viewed stereoscopically.* That is one reason for taking photographs with a 60 per cent overlap, which guarantees complete duplicate coverage and makes possible stereoscopic study of the whole area. For stereoscopic viewing of the second half of a photograph, the adjacent photograph on the other side is combined with it to form a pair. If there is less than 50 per cent overlap of the photographs, the entire area cannot be viewed in this manner. Two copies of the same photograph can never be used for stereoscopic study, although beginners sometimes interpret heavy shadows as giving some idea of relief and feel that this statement is not true.

Just as two transit angles from the end of a base line are essential in surveying for the location of an object by intersection, so are two photographs from ends of an "aerial base" essential for the reconstruction by the two eyes of the observer of an optical intersection and the creation in his mind of a three-dimensional or spatial image of the section of terrain being studied. It is exactly as if his two eyes were separated, one being put in the airplane where one picture was taken, and the other put in the airplane where the other picture was taken. The stereoscope is the device whereby each eye is forced to see its own picture, a miniature of the real terrain, with the *aerial base* reduced in like scale to equal his *eye base.*

It is absolutely necessary that the two pictures (images, not edges) be oriented to the same meridian on the table. In other words, all lines of one photograph within the overlap area will be parallel with the same lines as shown in the other photograph, except for such small distortions as may be occasioned by relief or tilt in the photographs. It follows that the two images of the aerial base as indicated separately on the two photographs lie along a common straight line. When photographs have been made by commercial aerial mapping companies, the oriented pictures usually have their edges nearly parallel with the mapped flight line of the plane, that is, the line connecting the photograph centers, or the aerial base. In order to make this possible, an observer in the plane must orient the camera by means of a view finder to make the camera plate axis parallel with the course of the plane's flight, since side winds usually prevent the plane's own axis from lying in the line of flight. However, war reconnaissance photographs may be taken by cameras which are rigidly fixed in the planes.

This results in the skewing or crabbing of the photographs with relation to the flight line or line of photograph centers. That is, the edges of the photographs may be entirely out of parallel with the aerial base. Where this is the case, it is especially necessary to orient the two photographs properly on the table if a spatial image is to be seen at all.

It is also essential that the two photographs be arranged on the same side of each other as when they were taken; otherwise an inverted image will be seen, in which hills appear as valleys and vice versa. If the suggested procedure is followed, the problems of orientation and the prevention of an inverted image will both be cared for automatically.

The beginner will do well to observe the following order of operation in detail:

1. Find the principal points of the two photographs by intersecting lines drawn between the collimating marks on the photographs. Using a drop compass and red or white ink, draw ¼-in. circles about these points on the two photographs. We shall call these points $1C$ and $2C$.

2. Find in each photograph, by observing minute details, the position of the principal point of the other photograph; and mark these conjugate principal points with circles of the same color and diameter as used in Step 1. We shall call these points $1C'$ and $2C'$, $1C'$ being the image of $1C$ on the other photograph, etc.

3. In both photographs draw in the lines connecting the two circles, using the same ink and weight of line as before. These are lines $1C$–$2C'$ and $2C$–$1C'$.

4. Place the photographs in overlapping position with matching details superimposed as closely as possible. Flipping the edges of the photographs up and down facilitates this matching. Lines $1C$–$2C'$ and $2C$–$1C'$ will now be found to be superimposed. In other words, the photographs could be oriented by placing the two ruled lines of the photographs under a common straight edge.

5. Draw the photographs apart in the *same direction* in which they are already partially separated, *with the ruled lines remaining along the straight edge,* as far as may be necessary for observation with the particular stereoscope in use. This amount should be a little less than the center-to-center of lens distance of a lens-type instrument, and about equal to the center-to-center distance of the outer mirrors of a mirror-type instrument.

6. Adjust the spacing of lenses of the stereoscope if that is possible so that it is the same as that of the eyes. This may be done as with binoculars. Look at some object at a distance and alter the

lens spacing until the two circular fields of vision merge into one circle. The object will be blurred but need only be identified to suffice for the setting. Or, if the object is viewed separately by one eye at a time while the stereoscope is held firmly against the brow, it should appear in the center of each lens.

7. Place the stereoscope with its axis of lenses or mirrors parallel with the lines ruled upon the photographs, which makes the eye base parallel with the air base. Rotation of the instrument in azimuth upon the table will make the two lines appear to swing past each other. The correct orientation of the stereoscope is obtained when the two lines appear to be one continuous line.

8. View the photographs simultaneously through both eyes. If the spacing of photographs is correct, a single, three-dimensional image will be seen and each principal point circle will be superimposed upon its conjugate circle. If the spacing is incorrect, each principal point will be seen distinct from its conjugate point. To correct for the latter condition, move one of the photographs along the same line as before until each principal point merges with its duplicate. Now the terrain should appear as a spatial image without further effort.

However, the eye will attempt to accommodate itself to improperly spaced pictures. If it is necessary to wait for the images to jump together or fuse, the photographs are not properly spaced and eyestrain will result from their study. Therefore, after the spatial image is seen, close and rest the eyes for a few moments. If upon reopening the eyes a double image is again seen, the spacing must be altered to bring the images together. When the spatial image appears immediately upon opening the eyes, the setting is satisfactory and may be studied for an indefinite time without unusual eyestrain. In this position the outer ends of the photographs may be fastened down to the table with masking tape. If the stereoscope is large enough to view the whole area of overlap, the two photographs may be fastened down to the table. The final spacing of the principal points of the two photographs may be noted and used immediately for setting up other stereoscopic pairs to be studied under the same stereoscope.

9. Where the stereoscope field is not large enough to cover the entire area of overlap of the setup photographs, the instrument should be moved *parallel with its first position* along the strip of overlap. Separation of the images of lines like roads is indicative of error of orientation of the stereoscope. Slight rotation of the instrument will bring the images into fusion again.

10. In order to view the whole area of overlap of large pictures with a pocket stereoscope, it will be necessary to reverse the overlap of the inner edges of the two pictures. That is, if photograph 1 is first lapping over photograph 2, then for the study of the under section of photograph 2, the inner edges of the two photographs are lifted and replaced with photograph 2 overlapping photograph 1. With wide (10-in.) photographs, it may be necessary to curl up the inner edge of the upper photograph (although the part being viewed is firmly on the table) so that middle parts of the overlap area may be studied.

After the above procedure is thoroughly understood, an experienced person can carry out the placing of the photographs and stereoscope without marking the principal points and base line. Conspicuous objects such as road intersections or large buildings may be used for perfecting the setup. However, for long study without eyestrain, it is recommended that the principal points be marked with needle holes or small crosses, and utilized as suggested above, although the inking of the circles and lines will no longer be necessary.

840. Measurement of Parallax. The difference in the elevations of two ground points whose images appear in each of two overlapped photographs may be determined by measuring the parallax difference between the two images. This is done by the use of a stereoscopic instrument designed for the purpose. The principle upon which the measurement of parallax is based will be explained here with particular reference to its application to the stereocomparagraph, an instrument described in Art. 842(*d*).

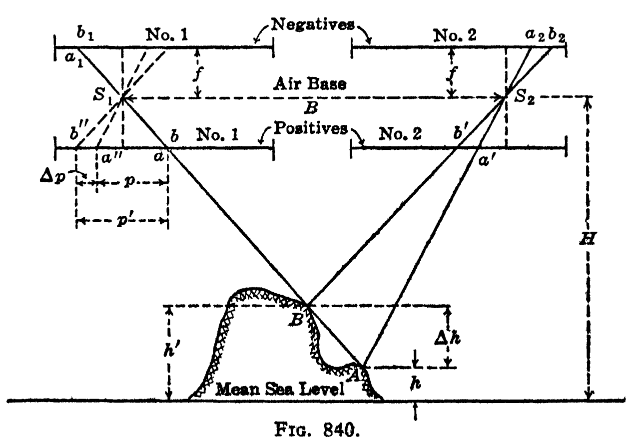


FIG. 840.

Let it be required to determine the difference in the elevations of two ground points, A and B, on the profile in Fig. 840. The images of these two points appear in each of two overlapped photographs, one taken with the camera lens at station S_1, the other with the lens at station S_2. It is assumed that the lens was at the same altitude at both stations.

The images of A and B will appear on negatives 1 and 2 in positions a_1, b_1 and a_2, b_2, respectively. The corresponding positions on the positives are a, b, and a', b'. If the lines S_1a'' and S_1b'' are drawn parallel to S_2a' and S_2b', respectively, the distance p is the absolute parallax of the image of A and p' is the absolute parallax of the image of B. Then Δp is the difference in absolute parallax due to the difference in the elevations of A and B.

From the relative positions of points shown in Fig. 840, the following two formulas may be derived:

$$\Delta h = \frac{\Delta p(H - h)}{B_m} \tag{1}$$

$$\Delta p = \frac{B_m(h' - h)}{H - h} = \frac{B_m \Delta h}{H - h} \tag{2}$$

In the use of the stereocomparagraph, the symbols have the following meanings:

$\Delta h = h' - h$ represents the difference in the elevations of points B and A expressed in feet. (See Fig. 840.)

$\Delta p = p' - p$, the difference in parallax expressed in millimeters, is equal to the difference in two micrometer readings, one taken when the floating mark is at a point B on the spatial model, the other taken when the floating mark is at a point A on the model.

H = the height of the camera lens (in feet) above the mean sea level, often called the *flight altitude.*

B_m = the length of the stereoscopic air base of the photographs expressed in millimeters. It is equal to the air base B (in feet) divided by the mean scale of the photograph; hence, $B_m = fB/(H - h)$. In this equation, the focal length of the camera, f, is expressed in millimeters.

The applications of formulas 1 and 2 in the use of the stereocomparagraph are explained in the latter part of Art. 842(d).

The solution of the formulas is simplified by the use of parallax tables or parallax diagrams.

Note that each of the formulas is independent of the focal length of the camera.

840(*a*). DERIVATION OF FORMULAS. From the similar triangles $S_1a''a$ and S_1S_2A in Fig. 840,

$$\frac{p}{f} = \frac{B}{H - h} \quad \text{or} \quad p = \frac{fB}{H - h}$$

From similar triangles $S_1b''a$ and S_1S_2B,

$$\frac{p'}{f} = \frac{B}{H - h'} \quad \text{or} \quad p' = \frac{fB}{H - h'}$$

$$\Delta p = p' - p = \frac{fB}{H - h'} - \frac{fB}{H - h} = \frac{fB(h' - h)}{(H - h')(H - h)}$$

From proportion,

$$\frac{B}{B_m} = \frac{H - h'}{f} \quad \text{or} \quad B = \frac{B_m(H - h')}{f}$$

Substituting this value of B in the equation for Δp,

$$\Delta p = \frac{B_m(h' - h)}{H - h}$$

Since $h' - h = \Delta p$,

$$\Delta h = \frac{\Delta p(H - h)}{B_m}$$

841. Topographic Maps. (*a*). AERIAL TOPOGRAPHIC MAPS are made in the same manner as aerial planimetric maps except that, in addition, contours are sketched on the map by field methods. Field sheets, prepared as explained in Art. 837, are taken into the field for contouring by the plane-table method or by one of the other field methods explained in Chapter 31. Enlargements of photographs are sometimes used instead of field sheets. Contours, located by field methods, are then sketched on the central portions of the enlarged photographs, and the final topographic map is traced from the photographs. (See Example IV, Art. 834.)

841(*b*). AERIAL STEREOTOPOGRAPHIC MAPS are made by adding contours to planimetric maps. Adequate ground control, vertical as well as horizontal, is essential (Art. 831). The contours are located not by field methods but from photographs by means of a stereoscopic mapping instrument. One of the simplest of these instruments is the stereocomparagraph. (See Art. 842(*d*).) Simple as this instrument is, however, considerable practical knowledge and experience are needed to use it successfully. An adequate explanation of the use of the stereocomparagraph, or of any other stereoscopic mapping instrument, cannot be given within the space available in this chapter. An inadequate explanation is likely to be misleading and of little value. Most of the printed explanations are designed for the college student, who can supplement his first basic ideas with hours of practice on the instruments under the guidance of an instructor.

If it is necessary to learn the use of stereoscopic mapping instruments from a book, a comprehensive manual which explains the various operations in considerable detail should be studied. For such study, the War Department Technical Manual (TM) 5-230, *Topographic Drafting*, is recommended. This manual may be purchased at a nominal price from the Superintendent of Documents, Government Printing Office, Washington, D. C.

841(*c*). NOTE. The War Department Technical Manual (TM) 5-230 is considered one of the most practical treatises on basic photographic mapping available. Some of the subjects treated are contours and form lines; selecting and marking control points on photographs; photogrammetry and photomechanical mapping methods; measurement of stereoscopic parallax; the stereocomparagraph; planimetric detail and contours from photographs; topography with the stereocomparagraph; compiling and finishing maps.
In addition to the operations which are described step by step in a very thorough manner, there is a great amount of detail about modern drafting methods in which aerial photographs are used.

841(*d*). NOTE. Aerial stereotopographic maps can generally be made most accurately and at the lowest cost by government or commercial organizations which specialize in photogrammetric mapping and have precision instruments operated by experts.

841(*c*). THE BROCK AND WEYMOUTH PROCESS OF TOPOGRAPHIC MAPPING * is an exceptionally precise photogrammetric method of obtaining a stereotopographic map. By the use of patented methods and equipment, true orthographic maps are obtained with planimetric detail and contours shown to a uniform scale. Moreover, the method is highly adaptable to the production of contoured mosaics. The Brock and Weymouth methods and equipment are used for commercial mapping by the Aero Service Corporation.

842. Mapping Instruments. There are many types of mapping instruments used in aerial photographic surveying, and many forms and sizes may be purchased from American manufacturers. Some of these instruments are so complex and expensive that they are used only by specialists. A practical working knowledge of any but the simplest instruments is best obtained by practice on the instrument under the guidance of an expert. The purpose here is merely to give a brief description of some of the more common instruments with brief comments on the use of each.

842(*a*). STEREOSCOPES. Many types and sizes of stereoscopes are available, ranging from a folding pocket stereoscope to a magnifying mirror stereoscope. A high-grade magnifying stereoscope differs from the usual type of stereoscope in several respects and is preferable when details of aerial photographs require close study. This instrument is compact, light, and comparatively inexpensive.

842(*b*). THE SKETCHMASTER. The vertical sketchmaster is a portable instrument by which the map draftsman can reflect a vertical aerial photograph on to his mapping surface for the purpose

* See "The Brock Process of Topographic Mapping," by E. A. Schuch, in *Photogrammetric Engineering*, Vol. VI, No. 2, p. 55, 1940.

of copying details. The instrument has a horizontal plate on which the photograph is clamped, and is supported on the mapping surface by three adjustable legs. The operator, looking vertically down upon his map, directs his vision through a half-silvered mirror (the reflecting surface is so thinly coated that about half the light entering his eye comes by reflection from the surface of the mirror and half comes straight through the mirror).

The light coming straight through the mirror shows the map and its control points. The light reflected from the half-silvered surface comes from the photograph by way of another inclined mirror above and behind the photograph. Thus the operator sees the photograph apparently superimposed directly upon the map. By lengthening or shortening all the legs of the sketchmaster, the scale of the reflection of the photograph upon the map may be altered to fit the scale of the map. By lengthening one or two of the legs, the scale may be altered in a variable manner to eliminate tilt in the original photograph. A set of lenses which may be interposed in the line of sight, together with the alteration of leg length, permits the photograph to be reflected with a scale change of about 50 per cent, either larger or smaller.

Relief displacements cannot be eliminated, and the reflection must be adjusted so that radial lines through the photograph images of the control points appear to pass through the corresponding mapped points. The best possible match of scale and control-point position is obtained by alteration of leg lengths. Then, slight movement of the eye is employed to keep the control matched in the area where detail is being copied.

842(*c*). THE OVERHEAD RATIO PROJECTOR. This instrument is used to project control points and detail on to a mapping sheet to any desired scale.

1. To project control points, the celluloid mapping-control sheet is placed in the instrument in a horizontal position. The control points, previously plotted on the control sheet by the methods of Art. 836(*b*) or Art. 836(*c*), are then projected downward to a mapping sheet held in place on a table and the projector is adjusted until the images of the control points are shown to the desired scale. These images are then carefully marked on the mapping sheet, and the celluloid control sheet is then removed.

2. To project detail, the first aerial photograph is placed in the instrument in a horizontal position. The control points, previously marked on the photograph (Art. 836(*a*)), will then be projected on the mapping sheet. The projector is adjusted until the image of each control point coincides with the corresponding control point projected and marked in (1). Detail projected from the photograph, now properly oriented, may then be drawn on the mapping sheet. The procedure is repeated for each of the succeeding photographs until detailing is completed.

The overhead ratio projector is not an extremely expensive instrument and it is likely to more than pay for itself when there is a considerable amount of mapping from aerial photographs.

842(d). THE TALLEY-FAIRCHILD STEREOCOMPARAGRAPH.* This is a simple and relatively inexpensive stereoscopic plotting instrument for locating and tracing contours directly from vertical aerial photographs. A pair of adjacent overlapping photographs are oriented and adjusted beneath the instrument so that stereoscopic fusion is obtained over the area common to the two photographs (Art. 839). The operator, looking through the stereoscope, sees a "floating mark" which can be kept to the elevation of any contour. The instrument is moved over the photographs in such a manner as to keep the floating mark in apparent contact with the ground surface as seen in the spatial model (Art. 838). This movement causes a special pencil at the end of a fixed drawing arm to trace a continuous contour, either on a photograph or upon a planimetric map previously drawn. During the movement the instrument is kept parallel to the stereoscopic base by a parallel motion device to which it is rigidly attached. The stereocomparagraph may be used for tracing planimetric detail as well as contours.

Lines traced by the pencil in the stereocomparagraph are exact reproductions of the corresponding lines of the left-hand photograph over which the pencil moves. Distortion or displacement in that photograph due to tilt, tip, or relief will be reproduced in the traced lines. Hence the parallactic displacement due to relief (Art. 840), which makes it possible to trace contour lines from a pair of overlapping photographs, makes it impossible for such lines to be true contours. Since the photograph is a perspective view and therefore not to a uniform scale, the contours will not be to a uniform scale, that is, each contour line will be to a slightly different scale from that of any other. Such lines are technically known as *form lines*. These form lines may be adjusted and redrawn by the use of proper map control to obtain contours to a uniform scale. One procedure is to compile the contoured photographs by the radial-line method (with adequate ground control), as explained in Art. 836(c), and then to transfer the contour lines to a map sheet by one of the methods of transferring detail (Art. 837). (See also the Brock and Weymouth process, Art. 841(e).)

Measurements by the stereocomparagraph are based on the principle of parallactic displacement explained in Art. 840. The measuring system consists of two index marks in the centers of two lenses which are carried in two mounting rings in the lower part

* Invented by B. B. Talley, Colonel, U.S.A., and manufactured by the Fairchild Aviation Corporation. The Abrams Contour Finder, a similar instrument, is manufactured by the Abrams Instrument Company.

of the instrument. During measurement one index mark is in contact with one photograph and the other mark is in contact with the other. Observed through the stereoscope these two index marks appear as one, that is, they are fused to form the floating mark. The lens carrying the right-hand mark can be moved either toward or away from the left-hand lens by means of a micrometer screw, and the extent of the movement is measured in millimeters by the micrometer. When the two index marks are moved closer together the floating mark is raised, and, vice versa, when the two marks are separated the floating mark is lowered. When the floating mark is brought into apparent contact with a point on the surface of a spatial model, the absolute parallax of that point is indicated by the corresponding micrometer reading. The difference in the elevations of two points A and B may be determined by measuring the parallax difference between the points. A micrometer reading is taken when the floating mark is on the fused image of A and a second reading is taken when the mark is on B. Since each reading is a measurement of parallax, the difference between the two readings is the parallax difference Δp expressed in millimeters. The difference in elevation Δh can be computed by substituting the value of Δp in formula 1, Art. 840.

When it is desired to establish a point B at a given vertical distance above or below a point A of known elevation, the difference in parallax Δp corresponding to the difference in the elevations of A and B is first computed from formula 2, Art. 840. This difference (in millimeters) is then added to or subtracted from the micrometer reading for A, according to whether B is above or below A. The result is the reading at which the micrometer should be set.

For tracing a contour, the parallax difference Δp between a point A of known elevation and a point B at the elevation of the contour is computed, and the reading at which the micrometer should be set is determined, as just explained. The micrometer is then clamped at that reading so that the floating mark will stay at the contour elevation as it is moved over the apparent surface of the spatial model. When the point A of known elevation is a point on the contour next above or next below the required contour, the parallax difference Δp is that for the contour interval.

842(*e*). THE AEROCARTOGRAPH. A complex and costly topographic mapping instrument used for drawing a complete topographic map from pairs of aerial photographs. It requires a minimum amount of ground control and extends that control, by aerial triangulation, from one photograph to the next throughout a flight strip. The instrument has the "floating mark" described in Art. 838. In the hands of an expert, the aerograph is an exceedingly accurate mapping instrument.

842(*f*). THE MULTIPLEX AEROPROJECTOR (commonly called the "multiplex"). A stereoscopic plotting instrument for producing topographic maps by means of the simultaneous spatial projection in complementary colors of diapositives (resembling lantern slides) of a series of overlapped photographs. The diapositives are inserted in the projectors and adjusted so that their images will fuse in space to form a spatial model. One diapositive of an adjacent pair is projected in red light and the other in green light. The observer views the spatial model thus projected through filters in the form of spectacles containing one red and one green lens.

The multiplex aeroprojector is extensively used, particularly by large mapping organizations.

Note. Stereoscopic plotting instruments are described in detail in books and various other publications on photogrammetry.

843. Planning for the Use of Aerial Photographs and Photogrammetric Maps. The production of aerial photographs and photogrammetric maps for a special project often requires combined planning by surveyor, photographer, and designer. Among the most important questions to be considered are those pertaining to: (1) scales of photographs or maps; (2) contour intervals of topographic maps; and (3) extent, accuracy, and general plan of ground control.

In designing a large project, uncontrolled mosaics, controlled mosaics, overlapping photographs, enlarged photographs, planimetric maps, and photogrammetric contour maps may all be required, each to be used at some stage of the design. Moreover, small-scale photographs or maps may be needed at one stage and large-scale photographs or maps at another stage, as illustrated in the next article.

843(*a*). USE OF AERIAL PHOTOGRAPHS AND PHOTOGRAMMETRIC MAPS IN HIGHWAY ENGINEERING. The use of aerial photographic surveys in the various stages of highway location, design, and construction affords an exceptionally suitable example of how different types of photographs and maps and different scales and contour intervals can be used most effectively, not only in highway projects but in various other engineering projects as well.

The remainder of this article is composed of portions of a paper, "The New Role of Aerial Photography," by Wilbur H. Simonson, Landscape Architect, Public Roads Administration, Washington, D. C. This paper was presented before the Surveying and Mapping Division of the American Society of Civil Engineers, and published by that society in *Civil Engineering,* Vol. 15, May 1945.

Three-dimensional aerial reconnaissance in wartime strategy focuses our attention on aerial photographic surveys and contour maps, and their use in peacetime plans. But how can we best use aerial survey methods in the preparation of plans for better highways? How can we re-gear outmoded ground survey methods to keep pace with faster aerial survey methods? How can aerial photographs and maps make it easier (1) to see the road, roadside, and related land as a single unit of development; (2) to work out an improved survey system for better design; and (3) to save time and

money in the preparation of project plans for large-scale construction programs?

The experience of highway departments in combining aerial and ground surveying and mapping methods points the way for their more effective use. Systematic use of photographic survey and map methods in every stage of plan preparation will simplify the work of administrative and technical staffs engaged in highway and related land development.

Four Stages of Plan Preparation

Clarity in discussing the use of aerial photographic surveys is gained by dividing highway design into four stages of plan preparation. Table I shows the trend toward the twofold use of photographs and maps in each stage. This table serves also to show how the scale of a photograph or map determines the plan stage in which it can be most effectively used.

TABLE I

Aerial Photographic Surveys for Highway Plans

Use	Type of Photograph or Map	Graphic Scale Feet to 1 In.	Contour Interval
I. General area study stage	Uncontrolled mosaic and topographic maps (U.S.G.S.)	2,000 or 1,667	20 ft.
II. Reconnaissance	Overlapping paired prints and controlled mosaics	1,000 or 660	10 ft.
III. Preliminary location	Overlapping paired prints and photogrammetric contour maps	500 400 200	5 ft.
IV. Final survey and plans	Enlarged photographs and photogrammetric contour maps	200 100 50	2 ft.

If we now think of the four progress stages as paired into two major groups, we find the top pair forming a rough study and reconnaissance group in which qualitative interpretation of small-scale photographs and maps is the primary function, and only general measurements by scaling are needed. We find that the bottom pair form a survey-control group in which quantitative measurements is an exacting requirement in map use, and more accurate large-scale photographs and maps are therefore essential. Although scales smaller than 500 ft. to the inch have proved very useful in the initial study stages, scales larger than 500 ft. to the inch have been found necessary in the later plan stages, as Table I shows.

Table II shows that the scale selected for highway design use is dependent upon (1) character of topography—whether level, rolling, or mountainous; (2) type of land-use—whether open or built-up; and (3) stage of survey—whether preliminary or final plan. All these factors must be considered together when preparing specifications for the making of aerial surveys.

Aerial Photographic Surveys

Complete aerial surveys comprise three operations: (1) the taking of the vertical photographs to provide overlapping paired prints; (2) the ground surveys required for photographic control, both horizontal and vertical; and (3) the combining of the photographs and the ground controls to produce either a controlled mosaic, or an accurate topographic map, or both.

As a survey method, aerial photography has three outstanding advantages—speed, economy, and the recording of details in a way that is impossible to obtain by any other method. Table I shows that the evolution of aerial

TABLE II

MAP SCALES FOR HIGHWAY DESIGN USE

Character of Topography	Type of Land Use	Survey and Plan Stages			
		Preliminary		Final Plans	
		Minimum *	Preferred *	Minimum *	Preferred *
Level to rolling	Open country Built-up sections	500 400	400 200	200 100	100 50
Rolling to hilly	Open country Built-up sections	400 200	200 100	100 100	50 50
Hilly to mountainous	Open country Built-up sections	200 100	100 50	100 50	50 20

* Scales in feet per inch.

photography in highway use has been through a series of steps—as aerial mosaics, as stereopaired prints, as photographic enlargements, and, finally, as photogrammetric contour maps.

AERIAL PHOTOGRAMMETRIC CONTOUR MAPS

Wartime developments in equipment, materials, and processing now make it possible to prepare complete topographic maps from aerial photographs in one continuous operation. Large-scale precise photography and considerable field surveying for exact control of every photograph are required in making special strip maps with contour intervals sufficiently close for highway use. *Such accurate contour maps cannot be made from the average existing aerial photographs which have not been closely tied in with sufficient ground-survey control.* The two types of aerial surveys are often confused, but this can easily be avoided by keeping in mind that one type is for small-scale general-purpose maps, and the other for large-scale special-purpose contour maps.

Discussion of these in detail is beyond the scope of this article. For large-scale topography, however, it suffices to point out that the selection of the contour interval controls the entire aerial survey operation. It fixes the flight altitude in relation to the stereo-plotting scale required by the kind of precision instrument used. The larger map-scales needed to provide 5-ft. and 2-ft. contour intervals within the specified limits of accuracy can be furnished by precise photographic survey methods in combination with special ground surveys. The art of "spotting" a strategic number of ground-control points directly from each of the photographs and marking them on the individual prints for the field survey crews is the key to aerial and ground surveying economy, and to accuracy in topographic mapping. Well-plotted ground-control points, easy of access to field survey crews for measurement of distances and elevations, simplify the correction of prints to a common or uniform scale (Art. 831). The scale-ratio of photographic enlargements can also be accurately controlled by such ground data.

Modern equipment with accurate ratio control makes possible a wide range of conversion from one scale to another. In Table II, for example, a 200-scale base map could be twice enlarged to 100, or four times to 50 if required, or the 200-scale map could be reduced to 400 scale if desired. A graphic scale and base-control grid should be placed on all photographs, maps, and plans selected for enlargement or reduction. Experience has proved the value of a rectangular grid with 5-in. by 5-in. spacing on large-scale maps of 400, 200 and 100 ft. to the inch. Such visible control references are an auto-

matic check on the accuracy of the converted scale, eliminate much confusion and loss of time in the drafting room, and simplify the interchange and transfer of map and plan data of different scales to any scale required.

Aerial photography can be geared to fit the degree of accuracy needed in each stage of plan preparation so that design and contract operations may be initiated with a minimum of ground-survey work, in three general types of highway location:

1. In mountainous terrain where ground elevation control is obviously difficult and expensive.
2. In extending mileages where the terrain is unexplored and mapping by ground survey methods is slow and time consuming.
3. In arterial highway problems, where the rural or urban importance of the improvement, land-use density of development, width and value of right-of-way, and cost of construction require most intensive reconnaissance and preliminary study of the whole area before routes are selected and preliminary location surveys begun.

Large-scale contour mapping in wartime marks the beginning of a new phase in highway survey and plan preparation. Teamwork of photographer, surveyor, draftsman, and designer in streamlining all aerial and ground-survey stages will speed up the whole job of preparing maps and plans for engineering work after the war. The problem is to select the map scales and survey and plan methods that will best fit a given job.

The use of maps of different scales in the preparation of plans for highway construction and land development is outlined in Table III. Planimetric or topographic maps at scales of one mile or more to the inch are too indefinite for project-plan use. These are classified as general and schematic groups for outlining interconnected systems of transportation, related facilities, and general relief. Such maps are also used as an index record to show progress stages in long-range construction programs in states and regions. But let us concentrate on the use of topographic and photographic maps in the larger scales needed for the preparation of plans for particular highway projects.

Table III gives a step-by-step breakdown of the whole range of working scales for highway use and shows how a complete series of scales can be grouped for ready conversion by photographic enlargement or reduction as desired. Take, for example, a single original aerial survey as made by the Agricultural Adjustment Agency with contact prints at a scale of 1667 ft. to the inch. Enlargement about 2½ times to 660 ft. to the inch (8 in. to the mile) makes the scale of the photograph useful for the study of alternate routes (reconnaissance stage). Or the original negative might be enlarged slightly more than four times to obtain the larger scale of 400 ft. to the inch suitable for preliminary location and design use. Under ideal conditions, with clear negatives, selected portions of the small, 1667-ft. photograph have been enlarged over 8 times to provide an even better working scale of 200 ft. to the inch. Base-control topographic maps or map sheets can be similarly converted to the working scales desired. A range of flexibility is provided in each of the basic scale-groups to fit available photographic and topographic maps where necessary. For dependable accuracy of control, however, enlargements are usually held within two to three times the original source-map data.

I. General Area Study Stage. Aerial mosaics and topographic maps of small scale are useful for general planning purposes where only area and type of work need be outlined, and scaled distances within about 100 ft. are considered sufficiently close to show the problem in skeleton or diagram form.

II. Reconnaissance Stage. Overlapping stereo-paired prints and controlled mosaics of intermediate scales ranging between 500 and 1000 ft. to the inch are of value in making preliminary reconnaissance studies of large areas. Such photographic surveys are economical and satisfactory for general study and comparison of alternate highway routes where scaled distances within 25 ft., or about the width of a two-lane pavement, are sufficiently close for making a route selection in advance of preliminary location and design plans.

III. Preliminary Location and Design Stage. Overlapping stereo-paired prints and contour maps larger than 500 ft. to the inch are ideal for preliminary highway location and overall control use where scaled distances within the close limits of 10 ft., or less than the width of one traffic lane, are required in the design computations.

IV. Final Survey and Plan Stage. Enlarged photographs and contour maps at a standard plan scale of 100 ft. to the inch, and larger, are useful in

TABLE III

ANALYSIS OF MAP SCALES FOR AERIAL PHOTOGRAPHS AND PHOTOGRAMMETRIC MAPS

<table>
<tr><th>Scale Type or Group</th><th>Scale Ratio</th><th>Graphic Scale (1 Inch =)</th><th>Accuracy of Scaled Distances on Prepared Plan Prints †</th><th>Map Type or Basic Data</th><th colspan="2">Map Use in Survey and Plan Stages</th></tr>
<tr><td>General</td><td>1 : 2,344,320
1 : 1,000,000
1 : 500,000</td><td>37 miles
* 16 miles
* 8 miles</td><td>Large areas, mileage base
Large areas, metric base
Interregional area, metric</td><td>Planimetric or topographic</td><td colspan="2">U.S. and federal-aid highways
World base map
Military strategic map of U.S.</td></tr>
<tr><td>Schematic</td><td>1 : 250,000
1 : 125,000
1 : 62,500</td><td>* 4 miles
* 2 miles
* 1 mile</td><td>Regional areas, metric base
State regions, metric base
Estimated 96% within 200′</td><td>Planimetric or topographic</td><td colspan="2">State-wide transport map
Selected state-area enlargement
Metric base control map (U.S.G.S.)</td></tr>
<tr><td>Small</td><td>1 : 31,680
1 : 24,000
1 : 20,000</td><td>½ mile
2,000′
1,667′</td><td>Estimated 96% within 100′
Estimated 96½% within 75′
Estimated 97% within 50′</td><td>Topographic
Topographic
Topographic or photographic</td><td colspan="2">Metropolitan area mileage base maps for general study of area and type only in key-map diagram stage of planning</td></tr>
<tr><td>Intermediate</td><td>1 : 12,000
1 : 10,000
1 : 8,000</td><td>1,000′
833′
666′</td><td>Estimated 97½% within 25
Estimated 97½% within 20′
Estimated 97½% within 16′</td><td>Topo-photo
Topo-photo
Topo-photo</td><td>Alternate route-study stage</td><td rowspan="2">Accurately ratioed photographic scale enlargements for progress studies</td></tr>
<tr><td>Medium</td><td>1 : 6,000
1 : 4,800
1 : 2,400</td><td>500′
400′
200′</td><td>Estimated 98% within 10′
Estimated 98% within 8′
Estimated 98% within 4′ †</td><td>Topo-photo
Topo-photo
Topo-photo</td><td>Overall control plan stage</td></tr>
<tr><td>Large</td><td>1 : 1,200
1 : 600
1 : 480</td><td>100′
50′
40′</td><td>Estimated 98½% within 1.5′
Estimated 98½% within 0.75′
Estimated 98½% within 0.6′</td><td>Base plan (1″ = 100′)
Detail plan (2.0 × base plan)
Detail plan (2.5 × base plan)</td><td colspan="2">Final survey and contract plan stage</td></tr>
<tr><td>Special</td><td>1 : 240</td><td>20′</td><td>Estimated 98½% within 0.3′</td><td colspan="3">Special details and plans for graphic accuracy and simplified procedure (5.0 × base-plan scale)</td></tr>
</table>

* Approximate.

† Example of limitations in scaling, including errors in transfer of base-map data to 1 in. = 200-ft. scale plan; 100% − 98% = 2% × 200 ft. = 4 ft. (coordinate control assumed for correction and compensation of scaled distances).

highway work where scaling of distances to the nearest foot or two is within sufficiently close limits for the practical needs of field construction. Photographic prints may be furnished to the designing engineers in advance so that rough studies may proceed without waiting for completion of the topographic maps. Construction work can be promptly initiated with a minimum of ground surveys.

Large-scale maps and plans, as suggested in Table II, are commonly used for the working plans, specifications, and estimates for grading, drainage, planting, and other construction and development work.

Where final location surveys and working plans require detail ground accuracy closer than obtainable graphically by scaled distances, the usual ground-survey procedure will be used as necessary to complete the contract plans. Material reduction in ground-survey work and computations in the early stages of reconnaissance and in the preliminary location and design stages of highway work by means of photographic processes saves a great deal of time. This of course speeds the final surveys and plans.

Aerial and Ground Surveys Interdependent

A reasonably uniform and economical scale-system as here outlined seems to be needed to bring all types of surveys and maps into closer relation. Aerial surveys need not be in competition with, nor tend to displace, essential ground surveys. On the contrary, air and ground surveys are complementary to each other. Combined aerial and ground-survey methods should result in more surveys, and more complete surveys, than ever. Detailed surveys of special development areas, as at bridges, grade separations, or parking layouts; final staking of locations for construction; the running of accurate base-line controls; precise surveys of property and right-of-way lines; and the measurement of construction quantities, will all tend to be done on the ground. Each survey method has a place and should be used to the best advantage for the job to be done. Continued improvement in equipment, materials, and methods has narrowed the one-time gap in accuracy between aerial and ground surveys. There is now no question of the advantages of large-scale aerial photographic surveys in combination with the old reliable ground methods.

Map-Making and Map-Using

A high degree of survey and map standards will only be attained when the individual scales are made generally uniform by map-making, map-servicing, and map-using groups. All must be in mesh. The reader should keep in mind the distinction between map-making and map-using. In this discussion, the use of reliable maps based on accurate horizontal and vertical ground survey controls is assumed. The relative accuracy of distances scaled from photo-enlargements, or other map sources, is of course dependent upon the initial accuracy of the control map used. Here we assume that the construction plans will be prepared from a coordinate control map so that the accuracy of scaled distances taken from prints in the field will safely fall within the percentage limits indicated for the respective working scales. The estimated accuracy of scaled distances is based on the assumption that maps made with the precision of typical aerial photogrammetric methods are equivalent to usual hand-drafted maps and meet War Department specifications and National Standard Map Accuracy requirements.

Since the use of a map determines the scale that should be selected, and the scale is dependent upon the relative degree of accuracy required of scaled distances on prepared prints, the use of a map should govern its making, and not the reverse—as too often happens—that map-making should unnecessarily limit the use. Since map accuracy and map cost are so closely related, sound engineering judgment needs to be exercised in the degree of accuracy required by the progress stage of the work. Table III may be helpful as a guide in selecting map scales appropriate for highway use.

844. Specifications. An organization which specializes in aerial surveying can often furnish a complete topographic map at a cost lower than would be incurred if the map were obtained in any other

way. In preparing specifications to be included in the contract for such a survey, it is well to proceed much as if the map were a structure to be built. The following digest of a set of specifications, prepared by a state department of public works, is suggestive of the procedure to be followed and the particulars to be specified in the preparation of specifications for a somewhat extensive aerial survey.

844(*a*). Digest of specifications for an aerial survey

1. *Area to Be Covered.* (Definite limits specified and, if practicable, indicated on an existing map.)

2. *Scale of Map.* Final copy to be inked on tracing cloth to a scale of 1 in. = 200 ft.

3. *Contour Interval.* Contour interval to be 5 ft. Elevations of specified important points, such as road intersections, to be shown to the nearest foot.

4. *Features to Be Shown on the Map.* (In correct horizontal projection.) Features such as buildings drawn by outline only; stone walls and fences which are visible on the original photograph and are not adjacent to roads; trails and roads; transmission or telephone lines; drainage lines. (The conventional signs to be used are specified.)

Relief to be shown by 5-ft. contours drawn in solid lines, clear and sharp, with each 25-ft. contour accentuated. Contour numbers to be shown at frequent intervals.

Political subdivision lines to be shown. "Grid lines" of the state coordinate system to be drawn at 1000-ft. intervals.

Lettering to be placed on the map. (Completely specified.)

5. *Drafting.* Tracings to be drawn in ink on high-grade tracing cloth. High-standard drafting required. Tracings to be not more than 5 ft. long, 2½ ft. wide to inside border lines. Match lines to be shown on each tracing. North point to be indicated. Information needed in drawing the title supplied by the state department.

6. *Control Surveys.* All field and office work required under the contract to be done by the contractor. Certain basic control already available may be used by the contractor if it is of first-order or second-order accuracy.

7. *Accuracy of Map.* The distance between any two well-defined cultural features as scaled on the map to check with the corresponding distance as measured on the ground within 10 ft. in 90 per cent of the distances tested, and no difference to be greater than 20 ft. Ninety per cent of all elevations interpolated from the map to be correct within 2½ ft. (one-half the contour interval) and the remaining 10 per cent not to be in error more than 5 ft. (See Art. 661(*a*).)

8. *Methods of Testing and Inspecting.* Sheets will be field-inspected and tested by the state department as rapidly as possible after they have been received from the contractor. Completeness of cultural and topographic detail to be checked by inspection in the field. Accuracy tests generally made by running "test profiles" over selected sections of the maps (Art. 661(*a*)). If a section is rejected, contractor to bring it to proper accuracy within 15 days and at his own expense. State department reserves the right to inspect all phases of the work at any time, either in the field or by viewing work in progress at contractor's plant. (See Art. 661(*a*).)

9. *Negatives and Photographs.* Original negatives to become the property of the state department. (Address to which the negatives are to be delivered after complete acceptance of map is specified.) Contractor to deliver one set of photographic prints, semi-matte, double-weight paper, scale 1 in. = 1000 ft., sheets to have sufficient overlap for stereoscopic study of the entire area. One copy required of a photographic index map at reduced scale. Photographs and index map to be delivered to specified address within ten days after completion of flying.

844(*b*). Map-accuracy specifications *

(*For planimetric and topographic map and charts prepared by stereophotogrammetric processes.*)

I. Control surveys

a. Horizontal: Each horizontal control station used in compiling the map shall be of sufficient accuracy in coordinate position (accurate in relation to the nearest first-order or second-order control stations, which are assumed to be absolutely correct) that its position on the map will not be in error by more than 1/200th in. (0.13 mm.), at publication scale.† (For example, if a map is to be published at a scale of 1 : 24,000, or 1 in. = 2000 ft., the maximum error in coordinates of any horizontal control point used should be 1/200th of 2000 ft. = 10 ft.)

b. Vertical: Each vertical control point used shall be correct in elevation (correct with respect to the nearest first-order or second-order level control, whose elevations are assumed to be absolutely correct) to within less than 1/10th of the contour interval.

(The horizontal and/or vertical control points should be of such number and so distributed throughout the map sheet's area that the specifications dealing with accuracy of planimetric detail, and accuracy of contours and elevations, can be met. Thus, the minimum number of horizontal and vertical control points required, and their distribution, will be determined in each case by the limitations of the mapping method to be used.)

II. Plotting projections and horizontal control

a. Projection: The projection lines of the map, at publication scale, shall scale correctly, both in overall dimensions and relative to each other, as follows:

1. The original manuscript map should be preferably on metal-mounted or other nondistortable base, and the projection lines should all scale correctly within 1/150th in. (0.17 mm.).

2. If the manuscript map is plotted on drawing material subject to expansion or shrinkage, the same accuracy in plotting the original map should be maintained. In the reproduction process, however, a method should be used which will as far as possible eliminate shrinkage or expansion distortions, and furnish inking plates, engraving plates, or press plates with projection lines scaling correctly in overall dimensions to within 1/100th in. (0.25 mm.).

b. Control Plotting: Each horizontal control point should be plotted in correct position on the original manuscript map, with respect to the nearest projection lines, to within 1/150th in. (0.17 mm.).

III. Horizontal accuracy of planimetric detail

Ninety-five per cent (95%) of all well-defined cultural and drainage features shall be plotted on the map in correct horizontal coordinate position within 1/50th in. (0.50 mm.), at the publication scale.

(If, for example, publication scale is 1 : 24,000, or 1 in. = 2000 ft., this specification insures that the scaled coordinates of 95% of the well-defined cultural features will be correct within 1/50th of 2000 ft. = 40 ft. No limitation is suggested for the remaining 5% of such points.)

IV. Vertical accuracy of topographic detail

a. Contours: Eighty-five per cent (85%) of all elevations interpolated from the map's contours shall be correct within ½ the contour interval. Not

* These specifications were prepared by the Committee on Map Specifications and Tests of the American Society of Photogrammetry and were adopted by that society in 1940. The specifications were originally printed in *Photogrammetric Engineering,* Vol. V, No. 1, p. 26, January, 1939. (See also Art. 661(*a*), *Tests and Specifications for Map Accuracy.*)

† If the compilation scale is the same, or nearly the same, as the publication scale, this accuracy should be required "at compilation scale."

more than 5% of all elevations interpolated from the map shall show errors in excess of the contour interval. Also, any contour which can be brought within the above-noted vertical tolerance by shifting its plotted location by 1/50th in. (0.50 mm.), in any direction, at the publication scale, shall be considered as correctly plotted.

b. "Spot" Elevations: All "spot" elevations shown on the map shall be correct to within 1/4th of the contour interval; except that no "spot" elevation need be shown closer than to the nearest foot.

V. Sheet edge matching

All cultural, drainage, and topographic features between adjoining sheets, along common margin lines, or in overlapping areas, shall be correctly transferred from one sheet to another, and shall match with each other, to within 1/100th in. (0.25 mm.) with respect to the nearest projection line; except that the map qualities herein specified shall not be sacrificed in order to join maps of lower order of accuracy.

Comments and suggestions

Among the comments and suggestions submitted by the Committee that prepared the specifications are the following:

Suggestions Regarding Map Testing. Sample testing of maps may often be found desirable and necessary to insure compliance with the specifications. If any map is indicated, from a relatively small sample test, to be substandard, it is suggested that further tests be made before rejection. More specifically, it is recommended, if the "test-profile" method is used on topographic maps, that the length of such test profile or profiles be equal at least to the width of the map sheet. If random "spot" elevations are used, they should be so distributed as to extend over at least 5% of the sheet's area before rejection; provided also that within this testing area there shall be an average of one "test shot" for each square inch of map area. Similarly, if horizontal coordinate positions of planimetric detail are believed to be substandard, the position tests should be extended over at least 5% of the map sheet's area before rejection.

Tests should be considered as being applied to the map at publication scale, regardless of compilation scale of manuscript map.

Tests by profile should be by transit-tape or transit-stadia traverse, originating and closing on coordinated positions, and of such accuracy that the plotted stations will be in error by not more than 1/200th in. on the map at publication scale. The elevations of the instrument stations, and plus stations on the line inserted for checking purposes, should be determined by methods that will assure the accuracy specified in I(*b*).

Tests by random spot elevations should be made in such manner that the elevations of test points will be of the accuracy specified in I(*b*).

Establishing Proper Contour Interval. It is suggested that if the specifications as to contour accuracy are adopted, they will, in considerable degree, govern the selection of the contour interval, or intervals, to be used on a map. When the publication scale and mapping methods are established, and the characteristics of the terrain known, the proper contour interval to use, in order to comply with the accuracy specifications, usually becomes obvious.

Comments on the Contour Specifications. The specifications on contour accuracy suggested herein are perhaps in the nature of a compromise. It is true that many topographic survey contracts have been let, and successfully executed, which require that 90% of the elevations interpolated from the map's contours shall be correct within one-half the interval. It is also believed, however, that a majority of the topographic maps executed in the past would prove by test to be less accurate than the specifications herein suggested. This Committee suggests that the time may come when it will perhaps be found desirable to raise the contour-accuracy specifications. The Committee does not believe (based on practical experiences from numerous current and previous mapping projects) that these recommended specifications are ever likely to prove too severe for practical map making and best map use.

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TABLES

TRIGONOMETRIC FUNCTIONS AND RIGHT TRIANGLES

Let A (Fig. 1) = angle BAC = arc BF, and let the radius $AF = AB = AH = 1$.

We then have

$\sin A$	$= BC$
$\cos A$	$= AC$
$\tan A$	$= DF$
$\cot A$	$= HG$
$\sec A$	$= AD$
$\operatorname{cosec} A$	$= AG$
$\operatorname{versin} A$	$= CF = BE$
$\operatorname{covers} A$	$= BK = HL$
$\operatorname{exsec} A$	$= BD$
$\operatorname{coexsec} A$	$= BG$
$\operatorname{chord} A$	$= BF$
$\operatorname{chord} 2A$	$= BI = 2BC$

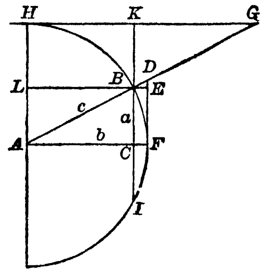


FIG. 1.

In the right-angled triangle ABC (Fig. 1)
Let $AB = c$, $AC = b$, and $BC = a$.
We then have:

1. $\sin A = \frac{a}{c} = \cos B$
2. $\cos A = \frac{b}{c} = \sin B$
3. $\tan A = \frac{a}{b} = \cot B$
4. $\cot A = \frac{b}{a} = \tan B$
5. $\sec A = \frac{c}{b} = \operatorname{cosec} B$
6. $\operatorname{cosec} A = \frac{c}{a} = \sec B$
7. $\operatorname{vers} A = \frac{c-b}{c} = \operatorname{covers} B$
8. $\operatorname{exsec} A = \frac{c-b}{b} = \operatorname{coexsec} B$
9. $\operatorname{covers} A = \frac{c-a}{c} = \operatorname{versin} B$
10. $\operatorname{coexsec} A = \frac{c-a}{a} = \operatorname{exsec} B$
11. $a = c \sin A = b \tan A$
12. $b = c \cos A = a \cot A$
13. $c = \frac{a}{\sin A} = \frac{b}{\cos A}$
14. $a = c \cos B = b \cot B$
15. $b = c \sin B = a \tan B$
16. $c = \frac{a}{\cos B} = \frac{b}{\sin B}$
17. $a = \sqrt{(c+b)(c-b)}$
18. $b = \sqrt{(c+a)(c-a)}$
19. $c = \sqrt{a^2+b^2}$
20. $C = 90° = A + B$
21. $\text{area} = \frac{ab}{2}$

Solution of Oblique Triangles

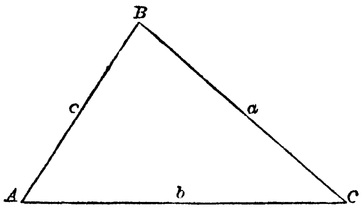


Fig. 2

	GIVEN	SOUGHT	FORMULÆ
22	A, B, a	C, b, c	$C = 180° - (A + B)$, $b = \frac{a}{\sin A} \cdot \sin B$, $c = \frac{a}{\sin A} \sin (A + B)$
23	A, a, b	B, C, c	$\sin B = \frac{\sin A}{a} \cdot b$, $C = 180° - (A + B)$, $c = \frac{a}{\sin A} \cdot \sin C$.
24	C, a, b	$\frac{1}{2}(A + B)$	$\frac{1}{2}(A + B) = 90° - \frac{1}{2} C$
25		$\frac{1}{2}(A - B)$	$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \frac{1}{2}(A + B)$
26		A, B	$A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B)$, $B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B)$
27		c	$c = (a + b) \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} = (a - b) \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)}$
28		area	$K = \frac{1}{2} a b \sin C$.
29	a, b, c	A	Let $s = \frac{1}{2}(a + b + c)$; $\sin \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{bc}}$
30			$\cos \frac{1}{2} A = \sqrt{\frac{s(s-a)}{bc}}$; $\tan \frac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$
31			$\sin A = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}$; $\text{vers } A = \frac{2(s-b)(s-c)}{bc}$
32		area	$K = \sqrt{s(s-a)(s-b)(s-c)}$
33	A, B, C, a	area	$K = \frac{a^2 \sin B \cdot \sin C}{2 \sin A}$

TABLE I

Longitude and Time

TO REDUCE DEGREES TO TIME.						TO REDUCE TIME TO DEGREES.					
°	H. M.	°	H. M.	Degrees.	Hours. Minutes.	Hours.	Degrees.	M.	° ′	M.	° ′
′	M. S.	′	M S.					S.	′ ″	S.	′ ″
″	S. T.	″	S. T.					T.	″ ‴	T.	″ ‴
1	0 4	51	3 24	101	6 44	1	15	1	0 15	51	12 45
2	0 8	52	3 28	102	6 48	1½	22½	2	0 30	52	13 0
3	0 12	53	3 32	103	6 52	2	30	3	0 45	53	13 15
4	0 16	54	3 36	104	6 56	2½	37½	4	1 0	54	13 30
5	0 20	55	3 40	105	7 0	3	45	5	1 15	55	13 45
6	0 24	56	3 44	106	7 4	3½	52½	6	1 30	56	14 0
7	0 28	57	3 48	107	7 8	4	60	7	1 45	57	14 15
8	0 32	58	3 52	108	7 12	4½	67½	8	2 0	58	14 30
9	0 36	59	3 56	109	7 16	5	75	9	2 15	59	14 45
10	0 40	60	4 0	110	7 20	5½	82½	10	2 30	60	15 0
11	0 44	61	4 4	115	7 40	6	90	11	2 45	61	15 15
12	0 48	62	4 8	120	8 0	6½	97½	12	3 0	62	15 30
13	0 52	63	4 12	125	8 20	7	105	13	3 15	63	15 45
14	0 56	64	4 16	130	8 40	7½	112½	14	3 30	64	16 0
15	1 0	65	4 20	135	9 0	8	120	15	3 45	65	16 15
16	1 4	66	4 24	140	9 20	8½	127½	16	4 0	66	16 30
17	1 8	67	4 28	145	9 40	9	135	17	4 15	67	16 45
18	1 12	68	4 32	150	10 0	9½	142½	18	4 30	68	17 0
19	1 16	69	4 36	155	10 20	10	150	19	4 45	69	17 15
20	1 20	70	4 40	160	10 40	10½	157½	20	5 0	70	17 30
21	1 24	71	4 44	165	11 0	11	165	21	5 15	71	17 45
22	1 28	72	4 48	170	11 20	11½	172½	22	5 30	72	18 0
23	1 32	73	4 52	175	11 40	12	180	23	5 45	73	18 15
24	1 36	74	4 56	180	12 0	12½	187½	24	6 0	74	18 30
25	1 40	75	5 0	185	12 20	13	195	25	6 15	75	18 45
26	1 44	76	5 4	190	12 40	13½	202½	26	6 30	76	19 0
27	1 48	77	5 8	195	13 0	14	210	27	6 45	77	19 15
28	1 52	78	5 12	200	13 20	14½	217½	28	7 0	78	19 30
29	1 56	79	5 16	205	13 40	15	225	29	7 15	79	19 45
30	2 0	80	5 20	210	14 0	15½	232½	30	7 30	80	20 0
31	2 4	81	5 24	215	14 20	16	240	31	7 45	81	20 15
32	2 8	82	5 28	220	14 40	16½	247½	32	8 0	82	20 30
33	2 12	83	5 32	225	15 0	17	255	33	8 15	83	20 45
34	2 16	84	5 36	230	15 20	17½	262½	34	8 30	84	21 0
35	2 20	85	5 40	235	15 40	18	270	35	8 45	85	21 15
36	2 24	86	5 44	240	16 0	18½	277½	36	9 0	86	21 30
37	2 28	87	5 48	245	16 20	19	285	37	9 15	87	21 45
38	2 32	88	5 52	250	16 40	19½	292½	38	9 30	88	22 0
39	2 36	89	5 56	255	17 0	20	300	39	9 45	89	22 15
40	2 40	90	6 0	260	17 20	20½	307½	40	10 0	90	22 30
41	2 44	91	6 4	270	18 0	21	315	41	10 15	91	22 45
42	2 48	92	6 8	280	18 40	21½	322½	42	10 30	92	23 0
43	2 52	93	6 12	290	19 20	22	330	43	10 45	93	23 15
44	2 56	94	6 16	300	20 0	22½	337½	44	11 0	94	23 30
45	3 0	95	6 20	310	20 40	23	345	45	11 15	95	23 45
46	3 4	96	6 24	320	21 20	23½	352½	46	11 30	96	24 0
47	3 8	97	6 28	330	22 0	24	360	47	11 45	97	24 15
48	3 12	98	6 32	340	22 40			48	12 0	98	24 30
49	3 16	99	6 36	350	23 20			49	12 15	99	24 45
50	3 20	100	6 40	360	24 0			50	12 30	100	25 0

TABLE II

Temperature Corrections per Foot for Steel Tapes

(Based on a Coefficient of Expansion of 0.00000645 per Degree Fahrenheit)

−F Degrees	Correction	+F Degrees	−F Degrees	Correction	+F Degrees	−F Degrees	Correction	+F Degrees	−F Degrees	Correction
68	.00000000	68	48	.00012900	88	28	.00025800	108	9	.00038055
67	.00000645	69	47	.00013545	89	27	.00026445	109	8	.00038700
66	.00001290	70	46	.00014190	90	26	.00027090	110	7	.00039345
65	.00001935	71	45	.00014835	91	25	.00027735	111	6	.00039990
64	.00002580	72	44	.00015480	92	24	.00028380	112	5	.00040635
63	.00003225	73	43	.00016125	93	23	.00029025	113	4	.00041280
62	.00003870	74	42	.00016770	94	22	.00029670	114	3	.00041925
61	.00004515	75	41	.00017415	95	21	.00030315	115	2	.00042570
60	.00005160	76	40	.00018060	96	20	.00030960	116	1	.00043215
59	.00005805	77	39	.00018705	97	19	.00031605	117	0	.00043860
58	.00006450	78	38	.00019350	98	18	.00032250	118	−1	.00044505
57	.00007095	79	37	.00019995	99	17	.00032895	119	−2	.00045150
56	.00007740	80	36	.00020640	100	16	.00033540	...	−3	.00045795
55	.00008385	81	35	.00021285	101	15	.00034185	...	−4	.00046440
54	.00009030	82	34	.00021930	102	14	.00034830	...	−5	.00047085
53	.00009675	83	33	.00022575	103	13	.00035475	...	−6	.00047730
52	.00010320	84	32	.00023220	104	12	.00036120	...	−7	.00048375
51	.00010965	85	31	.00023865	105	11	.00036765	...	−8	.00049020
50	.00011610	86	30	.00024510	106	10	.00037410	...	−9	.00049665
49	.00012255	87	29	.00025155	107	..	....	...	...	....

Use of Table III

Correction for a tape length less than 100 ft. may be obtained as follows:

1. Divide the difference of elevation h' for the short tape length by the short tape length and multiply by 100. The result is the corresponding difference of elevation h for a 100-ft. tape length on the same slope.

2. From Table III determine the correction C for a difference of elevation h.

3. Multiply the correction C by the short tape length and divide by 100. The result is the correction for the short tape length.

Example. Short tape length = 14.558 ft. Difference in elevation h' = 1.72 ft.

1. $(1.72 \div 14.558) \times 100 = 11.81$ ft. $= h$.
2. From Table III the correction for $h = 11.81$ ft. is 0.700 ft.
3. $(0.700 \times 14.558) \div 100 = 0.102$ ft. = correction for 14.56 ft.

The difference in elevation for 100 ft. is often known and step 1 is then unnecessary.

Example. To determine the correction for a distance of 614.558 ft. measured along a slope of 11.81 ft. vertical per 100-ft. tape length. From step 3 of the preceding example the correction for the short tape length is 0.102 ft. The total correction is:

$$6 \times 0.700 \text{ ft.} + 0.102 \text{ ft.} = 4.302 \text{ ft.}$$

Horizontal distance = 614.558 ft. − 4.302 ft. = 610.256 ft.

TABLE III

Inclination Corrections for 100-Foot Tape Lengths

h	0.00	0.01	0.02	0.03	0.04	0.05	0 06	0.07	0.08	0.09
ft.	ft.	ft.	ft.	ft.	ft.	ft.	ft.	ft.	ft.	ft.
0.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
.1	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.2	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.3	.000	.000	.001	.001	.001	.001	.001	.001	.001	.001
.4	.001	.001	.001	.001	.001	.001	.001	.001	.001	.001
0.5	0.001	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.002	0.002
.6	.002	.002	.002	.002	.002	.002	.002	.002	.002	.002
.7	.002	.003	.003	.003	.003	.003	.003	.003	.003	.003
.8	.003	.003	.003	.003	.004	.004	.004	.004	.004	.004
.9	.004	.004	.004	.004	.004	.005	.005	.005	.005	.005
1.0	0.005	0.005	0.005	0.005	0.005	0.006	0.006	0.006	0.006	0.006
.1	.006	.006	.006	.006	.006	.007	.007	.007	.007	.007
.2	.007	.007	.007	.008	.008	.008	.008	.008	.008	.008
.3	.008	.009	.009	.009	.009	.009	.009	.009	.010	.010
.4	.010	.010	.010	.010	.010	.011	.011	.011	.011	.011
1.5	0.011	0.011	0.012	0.012	0.012	0.012	0.012	0.012	0.012	0.013
.6	.013	.013	.013	.013	.013	.014	.014	.014	.014	.014
.7	.014	.015	.015	.015	.015	.015	.015	.016	.016	.016
.8	.016	.016	.017	.017	.017	.017	.017	.017	.018	.018
.9	.018	.018	.018	.019	.019	.019	.019	.019	.020	.020
2.0	0.020	0.020	0.020	0.021	0.021	0.021	0.021	0.021	0.022	0.022
.1	.022	.022	.022	.023	.023	.023	.023	.024	.024	.024
.2	.024	.024	.025	.025	.025	.025	.026	.026	.026	.026
.3	.026	.027	.027	.027	.027	.028	.028	.028	.028	.029
.4	.029	.029	.029	.030	.030	.030	.030	.031	.031	.031
2.5	0.031	0.032	0.032	0.032	0.032	0.033	0.033	0.033	0.033	0.034
.6	.034	.034	.034	.035	.035	.035	.035	.036	.036	.036
.7	.036	.037	.037	.037	.038	.038	.038	.038	.039	.039
.8	.039	.039	.040	.040	.040	.041	.041	.041	.041	.042
.9	.042	.042	.043	.043	.043	.044	.044	.044	.044	.045
3.0	0.045	0.045	0.046	0.046	0.046	0.047	0.047	0.047	0.047	0.048
.1	.048	.048	.049	.049	.049	.050	.050	.050	.051	.051
.2	.051	.052	.052	.052	.053	.053	.053	.053	.054	.054
.3	.054	.055	.055	.055	.056	.056	.056	.057	.057	.057
.4	.058	.058	.058	.059	.059	.060	.060	.060	.061	.061
3.5	0.061	0.062	0.062	0.062	0.063	0.063	0.063	0.064	0.064	0.064
.6	.065	.065	.066	.066	.066	.067	.067	.067	.068	.068
.7	.068	.069	.069	.070	.070	.070	.071	.071	.071	.072
.8	.072	.073	.073	.073	.074	.074	.075	.075	.075	.076
.9	.076	.076	.077	.077	.078	.078	.078	.079	.079	.080
4.0	0.080	0.080	0.081	0.081	0.082	0.082	0.082	0.083	0.083	0.084
.1	.084	.084	.085	.085	.086	.086	.087	.087	.087	.088
.2	.088	.089	.089	.090	.090	.090	.091	.091	.092	.092
.3	.092	.093	.093	.094	.094	.095	.095	.096	.096	.096
.4	.097	.097	.098	.098	.099	.099	.100	.100	.100	.101
4.5	0.101	0.102	0.102	0.103	0.103	0.104	0.104	0.104	0.105	0.105
.6	.106	.106	.107	.107	.108	.108	.109	.109	.110	.110
.7	.111	.111	.111	.112	.112	.113	.113	.114	.114	.115
.8	.115	.116	.116	.117	.117	.118	.118	.119	.119	.120
.9	.120	.121	.121	.122	.122	.123	.123	.124	.124	.125
5.0	0.125	0.126	0.126	0.127	0.127	0.128	0.128	0.129	0.129	0.180
.1	.130	.131	.131	.132	.132	.133	.133	.134	.134	.135
.2	.135	.136	.136	.137	.137	.138	.138	.139	.139	.140
.3	.141	.141	.142	.142	.143	.143	.144	.144	.145	.145
.4	.146	.146	.147	.148	.148	.149	.149	.150	.150	.151
5.5	0.151	0.152	0.152	0.153	0.154	0.154	0.155	0.155	0.156	0.156
.6	.157	.157	.158	.159	.159	.160	.160	.161	.161	.162
.7	.163	.163	.164	.164	.165	.165	.166	.167	.167	.168
.8	.168	.169	.170	.170	.171	.171	.172	.172	.173	.174
.9	.174	.175	.175	.176	.177	.177	.178	.178	.179	.180
6.0	0.180	0.181	0.181	0.182	0.183	0.183	0.184	0.184	0.185	0.186
.1	.186	.187	.187	.188	.189	.189	.190	.191	.191	.192
.2	.192	.193	.194	.194	.195	.196	.196	.197	.197	.198
.3	.199	.199	.200	.201	.201	.202	.202	.203	.204	.204
.4	.205	.206	.206	.207	.208	.208	.209	.210	.210	.211
6.5	0.211	0.212	0.213	0.213	0.214	0.215	0.215	0.216	0.217	0.217
.6	.218	.219	.219	.220	.221	.221	.222	.223	.223	.224
.7	.225	.225	.226	.227	.227	.228	.229	.229	.230	.231
.8	.231	.232	.233	.234	.234	.235	.236	.236	.237	.238
.9	.238	.239	.240	.240	.241	.242	.243	.243	.244	.245
7.0	0.245	0.246	0.247	0.247	0.248	0.249	0.250	0.250	0.251	0.252
.1	.252	.253	.254	.255	.255	.256	.257	.257	.258	.259
.2	.260	.260	.261	.262	.262	.263	.264	.265	.265	.266
.3	.267	.268	.268	.269	.270	.270	.271	.272	.273	.273
.4	.274	.275	.276	.276	.277	.278	.279	.279	.280	.281

TABLE III (*Continued*)

Inclination Corrections for 100-Foot Tape Lengths

h	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
ft.	ft.	ft.	ft.	ft.	ft.	ft.	ft.	ft.	ft.	ft.
7.5	0.282	0.282	0.283	0.284	0.285	0.285	0.286	0.287	0.288	0.288
.6	.289	.290	.291	.292	.292	.293	.294	.295	.295	.296
.7	.297	.298	.298	.299	.300	.301	.302	.302	.303	.304
.8	.305	.305	.306	.307	.308	.309	.309	.310	.311	.312
.9	.313	.313	.314	.315	.316	.317	.317	.318	.319	.320
8.0	0.321	0.321	0.322	0.323	0.324	0.325	0.325	0.326	0.327	0.328
.1	.329	.329	.330	.331	.332	.333	.333	.334	.335	.336
.2	.337	.338	.338	.339	.340	.341	.342	.343	.343	.344
.3	.345	.346	.347	.348	.348	.349	.350	.351	.352	.353
.4	.353	.354	.355	.356	.357	.358	.359	.359	.360	.361
8.5	0.362	0.363	0.364	0.364	0.365	0.366	0.367	0.368	0.369	0.370
.6	.370	.371	.372	.373	.374	.375	.376	.377	.377	.378
.7	.379	.380	.381	.382	.383	.384	.384	.385	.386	.387
.8	.388	.389	.390	.391	.391	.392	.393	.394	.395	.396
.9	.397	.398	.399	.400	.400	.401	.402	.403	.404	.405
9.0	0.406	0.407	0.408	0.409	0.409	0.410	0.411	0.412	0.413	0.414
.1	.415	.416	.417	.418	.419	.419	.420	.421	.422	.423
.2	.424	.425	.426	.427	.428	.429	.430	.431	.432	.432
.3	.433	.434	.435	.436	.437	.438	.439	.440	.441	.442
.4	.443	.444	.445	.446	.447	.448	.448	.449	.450	.451
9.5	0.452	0.453	0.454	0.455	0.456	0.457	0.458	0.459	0.460	0.461
.6	.462	.463	.464	.465	.466	.467	.468	.469	.470	.471
.7	.472	.473	.474	.474	.475	.476	.477	.478	.479	.480
.8	.481	.482	.483	.484	.485	.486	.487	.488	.489	.490
.9	.491	.492	.493	.494	.495	.496	.497	.498	.499	.500
10.0	0.501	0.502	0.503	0.504	0.505	0.506	0.507	0.508	0.509	0.510
.1	.511	.512	.513	.514	.515	.516	.517	.518	.520	.521
.2	.522	.523	.524	.525	.526	.527	.528	.529	.530	.531
.3	.532	.533	.534	.535	.536	.537	.538	.539	.540	.541
.4	.542	.543	.544	.545	.546	.548	.549	.550	.551	.552
10.5	0.553	0.554	0.555	0.556	0.557	0.558	0.559	0.560	0.561	0.562
.6	.563	.564	.566	.567	.568	.569	.570	.571	.572	.573
.7	.574	.575	.576	.577	.578	.579	.581	.582	.583	.584
.8	.585	.586	.587	.588	.589	.590	.591	.593	.594	.595
.9	.596	.597	.598	.599	.600	.601	.602	.604	.605	.606
11.0	0.607	0.608	0.609	0.610	0.611	0.612	0.613	0.615	0.616	0.617
.1	.618	.619	.620	.621	.622	.624	.625	.626	.627	.628
.2	.629	.630	.631	.633	.634	.635	.636	.637	.638	.639
.3	.641	.642	.643	.644	.645	.646	.647	.648	.650	.651
.4	.652	.653	.654	.655	.657	.658	.659	.660	.661	.662
11.5	0.663	0.665	0.666	0.667	0.668	0.669	0.670	0.672	0.673	0.674
.6	.675	.676	.677	.679	.680	.681	.682	.683	.684	.686
.7	.687	.688	.689	.690	.692	.693	.694	.695	.696	.697
.8	.699	.700	.701	.702	.703	.705	.706	.707	.708	.709
.9	.711	.712	.713	.714	.715	.717	.718	.719	.720	.721
12.0	0.723	0.724	0.725	0.726	0.727	0.729	0.730	0.731	0.732	0.734
.1	.735	.736	.737	.738	.740	.741	.742	.743	.745	.746
.2	.747	.748	.749	.751	.752	.753	.754	.756	.757	.758
.3	.759	.761	.762	.763	.764	.766	.767	.768	.769	.771
.4	.772	.773	.774	.776	.777	.778	.779	.781	.782	.783
12.5	0.784	0.786	0.787	0.788	0.789	0.791	0.792	0.793	0.794	0.796
.6	.797	.798	.800	.801	.802	.803	.805	.806	.807	.808
.7	.810	.811	.812	.814	.815	.816	.817	.819	.820	.821
.8	.823	.824	.825	.826	.828	.829	.830	.832	.833	.834
.9	.836	.837	.838	.839	.841	.842	.843	.845	.846	.847
13.0	0.849	0.850	0.851	0.853	0.854	0.855	0.856	0.858	0.859	0.860
.1	.862	.863	.864	.866	.867	.868	.870	.871	.872	.874
.2	.875	.876	.878	.879	.880	.882	.883	.884	.886	.887
.3	.888	.890	.891	.892	.894	.895	.896	.898	.899	.901
.4	.902	.903	.905	.906	.907	.909	.910	.911	.913	.914
13.5	0.915	0.917	0.918	0.920	0.921	0.922	0.924	0.925	0.926	0.928
.6	.929	.930	.932	.933	.935	.936	.937	.939	.940	.942
.7	.943	.944	.946	.947	.948	.950	.951	.953	.954	.955
.8	.957	.958	.960	.961	.962	.964	.965	.967	.968	.969
.9	.971	.972	.974	.975	.976	.978	.979	.981	.982	.983
14.0	0.985	0.986	0.988	0.989	0.991	0.992	0.993	0.995	0.996	0.998
.1	.999	1.000	1.002	1.003	1.005	1.006	1.008	1.009	1.010	1.012
.2	1.013	1.015	1.016	1.018	1.019	1.021	1.022	1.023	1.025	1.026
.3	1.028	1.029	1.031	1.032	1.034	1.035	1.036	1.038	1.039	1.041
.4	1.042	1.044	1.045	1.047	1.048	1.050	1.051	1.052	1.054	1.055
14.5	1.057	1.058	1.060	1.061	1.063	1.064	1.066	1.067	1.069	1.070
.6	1.072	1.073	1.074	1.076	1.077	1.079	1.080	1.082	1.083	1.085
.7	1.086	1.088	1.089	1.091	1.092	1.094	1.095	1.097	1.098	1.100
.8	1.101	1.103	1.104	1.106	1.107	1.109	1.110	1.112	1.113	1.115
.9	1.116	1.118	1.119	1.121	1.122	1.124	1.125	1.127	1.128	1.130

Table IV

MEAN TIMES OF UPPER CULMINATIONS OF POLARIS AT THE MERIDIAN OF GREENWICH IN THE YEAR 1941

Date	Civil Time of Upper Culmination	Variation per Day
	h. m.	m. s.
Jan. 1...	7 0.0 P.M.	−3 57.0
11...	6 20.4	3 57.0
21...	5 40.9	3 57.0
31...	5 1.4	3 57.0
Feb. 10...	4 21.9	3 57.0
20...	3 42.4	−3 56.9
Mar. 2...	3 2.9	3 56.8
12...	2 23.5	3 56.6
22...	1 44.0	3 56.4
Apr. 1...	1 4.7	3 56.2
11...	0 25.3 P.M.	3 56.0
21...	11 46.0 A.M.	−3 55.7
May 1...	11 6.8	3 55.5
11...	10 27.5	3 55.3
21...	9 48.3	3 55.1
31...	9 9.2	3 55.0
June 10...	8 30.0	3 54.9
20...	7 50.9	−3 54.8
30...	7 11.7	3 54.7
July 10...	6 32.6	3 54.6
20...	5 53.5	3 54.6
30...	5 14.4	3 54.7
Aug. 9...	4 35.3	3 54.7
19...	3 56.2	−3 54.8
29...	3 17.0	3 54.9
Sept. 8...	2 37.8	3 55.0
18...	1 58.6	3 55.2
28...	1 19.4	3 55.3
Oct. 8...	0 40.2	3 55.5
17...	0 4.9 A.M.	3 55.7
27...	11 21.6 P.M.	−3 55.9
Nov. 6...	10 42.3	3 56.1
16...	10 2.9	3 56.3
26...	9 23.5	3 56.5
Dec. 6...	8 44.1	3 56.7
16...	8 4.6	3 56.8
26...	7 25.2 P.M.	−3 56.9

Table IVa

CORRECTIONS IN MINUTES TO BE ADDED TO THE TIMES IN TABLE IV IN ORDER TO OBTAIN TIMES FOR YEARS OTHER THAN 1941

	min.
1942....................	add 1.5
1943....................	add 2.9
1944, before March 1.....	add 4.3
after March 1......	add 0.3
1945....................	add 1.7
1946....................	add 3.1
1947....................	add 4.5
1948, before March 1.....	add 6.1
after March 1	add 2.1
1949....................	add 3.7
1950....................	add 5.3

The data for the tables on this and the opposite page were furnished by the United States Naval Observatory.

TABLE IV*b*

Days Elapsed	Variation per Day					
	3 m. 57 s.		3 m. 56 s.		3 m. 55 s.	
	m.	s.	m.	s.	m.	s.
1	3	57	3	56	3	55
2	7	54	7	52	7	50
3	11	51	11	48	11	45
4	15	48	15	44	15	40
5	19	45	19	40	19	35
6	23	42	23	36	23	30
7	27	39	27	32	27	25
8	31	36	31	28	31	20
9	35	33	35	24	35	15

The Greenwich time of upper culmination for any day of the month not given in Table IV may be interpolated by the use of Table IV*b*.

The correction for the difference in longitude between the meridian of Greenwich and the meridian of the place of observation is approximately 10 seconds per hour (15°) of longitude.

To obtain standard time from the local civil time computed by means of the tables, determine the correction at the rate of four minutes for every degree of longitude that the place of observation is distant from the standard time meridian for that place (75°, 90°, 105°, or 120° in the United States). Add the correction when the place of observation is west of the standard time meridian, subtract when the place is east of the standard time meridian.

TABLE IV*c*

MEAN TIME INTERVAL BETWEEN UPPER CULMINATION AND ELONGATION

Latitude	Time Interval	Latitude	Time Interval
°	h. m.	°	h. m.
10	5 58.3	48	5 54.5
15	5 57.9	50	5 54.2
20	5 57.5	52	5 53.8
25	5 57.1	54	5 53.4
30	5 56.7	56	5 53.0
35	5 56.1	58	5 52.5
40	5 55.6	60	5 52.0
42	5 55.4	62	5 51.4
44	5 55.1	64	5 50.7
46	5 54.8	66	5 49.9

Eastern elongation precedes and western elongation follows upper culmination by the time interval given in Table IV*c*. Lower culmination precedes or follows upper culmination by $11^h\ 58^m$.

Table V

AZIMUTHS OF POLARIS AT ELONGATION, 1941 TO 1947

Latitude	1941	1942	1943	1944	1945	1946	1947
°	° ′	° ′	° ′	° ′	° ′	° ′	° ′
10	1 02.0	1 01.7	1 01.4	1 01.1	1 00.8	1 00.5	1 00.1
11	1 02.2	1 01.9	1 01.6	1 01.3	1 01.0	1 00.7	1 00.3
12	1 02.4	1 02.1	1 01.8	1 01.5	1 01.2	1 00.9	1 00.5
13	1 02.6	1 02.4	1 02.1	1 01.8	1 01.4	1 01.1	1 00.8
14	1 02.9	1 02.6	1 02.3	1 02.0	1 01.7	1 01.4	1 01.0
15	1 03.2	1 02.9	1 02.6	1 02.3	1 02.0	1 01.7	1 01.3
16	1 03.5	1 03.2	1 02.9	1 02.6	1 02.3	1 02.0	1 01.6
17	1 03.8	1 03.5	1 03.2	1 02.9	1 02.6	1 02.3	1 01.9
18	1 04.2	1 03.9	1 03.6	1 03.3	1 03.0	1 02.6	1 02.3
19	1 04.6	1 04.3	1 04.0	1 03.7	1 03.3	1 03.0	1 02.6
20	1 05.0	1 04.7	1 04.4	1 04.0	1 03.7	1 03.4	1 03.0
21	1 05.4	1 05.1	1 04.8	1 04.5	1 04.1	1 03.8	1 03.4
22	1 05.8	1 05.5	1 05.2	1 04.9	1 04.6	1 04.2	1 03.9
23	1 06.3	1 06.0	1 05.7	1 05.4	1 05.0	1 04.7	1 04.3
24	1 06.8	1 06.5	1 06.2	1 05.9	1 05.5	1 05.2	1 04.8
25	1 07.3	1 07.0	1 06.7	1 06.4	1 06.1	1 05.7	1 05.3
26	1 07.9	1 07.6	1 07.3	1 07.0	1 06.6	1 06.3	1 05.9
27	1 08.5	1 08.2	1 07.9	1 07.5	1 07.2	1 06.8	1 06.5
28	1 09.1	1 08.8	1 08.5	1 08.2	1 07.8	1 07.4	1 07.1
29	1 09.8	1 09.5	1 09.2	1 08.8	1 08.5	1 08.1	1 07.7
30	1 10.5	1 10.2	1 09.8	1 09.5	1 09.1	1 08.8	1 08.4
31	1 11.2	1 10.9	1 10.6	1 10.2	1 09.9	1 09.5	1 09.1
32	1 12.0	1 11.7	1 11.3	1 11.0	1 10.6	1 10.2	1 09.8
33	1 12.8	1 12.5	1 12.1	1 11.8	1 11.4	1 11.0	1 10.6
34	1 13.6	1 13.3	1 13.0	1 12.6	1 12.2	1 11.8	1 11.4
35	1 14.5	1 14.2	1 13.8	1 13.5	1 13.1	1 12.7	1 12.3
36	1 15.4	1 15.1	1 14.8	1 14.4	1 14.0	1 13.6	1 13.2
37	1 16.4	1 16.1	1 15.7	1 15.4	1 15.0	1 14.6	1 14.1
38	1 17.5	1 17.1	1 16.8	1 16.4	1 16.0	1 15.6	1 15.1
39	1 18.5	1 18.2	1 17.8	1 17.4	1 17.0	1 16.6	1 16.2
40	1 19.7	1 19.3	1 19.0	1 18.6	1 18.2	1 17.7	1 17.3
41	1 20.9	1 20.5	1 20.1	1 19.7	1 19.3	1 18.9	1 18.5
42	1 22.1	1 21.8	1 21.4	1 21.0	1 20.6	1 20.1	1 19.7
43	1 23.5	1 23.1	1 22.7	1 22.3	1 21.9	1 21.4	1 21.0
44	1 24.9	1 24.5	1 24.1	1 23.7	1 23.2	1 22.8	1 22.3
45	1 26.3	1 25.9	1 25.5	1 25.1	1 24.7	1 24.2	1 23.7
46	1 27.9	1 27.5	1 27.1	1 26.6	1 26.2	1 25.7	1 25.2
47	1 29.5	1 29.1	1 28.7	1 28.3	1 27.8	1 27.3	1 26.8
48	1 31.2	1 30.8	1 30.4	1 30.0	1 29.5	1 29.0	1 28.5
49	1 33.0	1 32.6	1 32.2	1 31.7	1 31.3	1 30.8	1 30.3
50	1 35.0	1 34.5	1 34.1	1 33.6	1 33.2	1 32.6	1 32.1

The azimuths in Table V are reckoned clockwise from the true north for the eastern elongation, and counterclockwise from the true north for western elongation. For intermediate latitudes, values may be obtained by interpolation without, in general, exceeding an error of 0′.3.

TABLE V (*Continued*)

AZIMUTHS 1948 to 1950

1948	1949	1950	Latitude
° ′	° ′	° ′	°
0 59.8	0 59.4	0 59.1	10
1 00.0	0 59.6	0 59.3	11
1 00.2	0 59.8	0 59.5	12
1 00.4	1 00.1	0 59.7	13
1 00.7	1 00.3	1 00.0	14
1 01.0	1 00.6	1 00.3	15
1 01.3	1 00.9	1 0.6	16
1 01.6	1 01.2	1 0.9	17
1 01.9	1 01.6	1 1.2	18
1 02.3	1 01.9	1 1.6	19
1 02.7	1 02.3	1 1.9	20
1 03.1	1 02.7	1 2.3	21
1 03.5	1 03.1	1 2.8	22
1 04.0	1 03.6	1 3.2	23
1 04.5	1 04.1	1 3.7	24
1 05.0	1 04.6	1 4.2	25
1 05.5	1 05.1	1 4.8	26
1 06.1	1 05.7	1 5.3	27
1 06.7	1 06.3	1 5.9	28
1 07.3	1 06.9	1 6.5	29
1 08.0	1 07.6	1 7.2	30
1 08.7	1.08.3	1 7.9	31
1 09.4	1 09.0	1 8.6	32
1 10.2	1 09.8	1 9.4	33
1 11.0	1 10.6	1 10.2	34
1 11.9	1 11.5	1 11.1	35
1 12.8	1 12.4	1 11.9	36
1 13.7	1 13.3	1 12.9	37
1 14.7	1 14.3	1 13.9	38
1 15.8	1 15.3	1 14.9	39
1 16.9	1 16.4	1 16.0	40
1 18.0	1 17.6	1 17.1	41
1 19.2	1 18.8	1 18.3	42
1 20.5	1 20.0	1 19.6	43
1 21.9	1 21.4	1 20.9	44
1 23.3	1 22.8	1 22.3	45
1 24.8	1 24.3	1 23.8	46
1 26.3	1 25.8	1.25 3	47
1 28.0	1 27.5	1 27.0	48
1 29.8	1 29.2	1 28.7	49
1 31.6	1 31.1	1 30.6	50

TABLE V*a*

Month	Correction
	′
January......	−0.5
February.....	−0.4
March.......	−0.3
April........	−0.1
May.........	+0.1
June.........	+0.2
July.........	+0.2
August.......	+0.1
September....	−0.1
October......	−0.3
November....	−0.6
December....	−0.8

Table V was computed from the mean declination of Polaris for the beginning of each year. A more accurate result will be obtained by applying to the tabular values the corrections in Table V*a* which depend on the difference between the mean and apparent place of the star.

Tables V and V*a* were furnished by the United States Naval Observatory.

TABLE VI

Decimals of a Foot for Each 32d of an Inch

In.	0	1	2	3	4	5	6	7	8	9	10	11	In.
0	Foot	.0833	.1667	.2500	.3333	.4167	.5000	.5833	.6667	.7500	.8333	.9167	0
1-32	.0026	.0859	.1693	.2526	.3359	.4193	.5026	.5859	.6693	.7526	.8359	.9193	1-32
1-16	.0052	.0885	.1719	.2552	.3385	.4219	.5052	.5885	.6719	.7552	.8385	.9219	1-16
3-32	.0078	.0911	.1745	.2578	.3411	.4245	.5078	.5911	.6745	.7578	.8411	.9245	3-32
1-8	.0104	.0938	.1771	.2604	.3438	.4271	.5104	.5938	.6771	.7604	.8438	.9271	1-8
5-32	.0130	.0964	.1797	.2630	.3464	.4297	.5130	.5964	.6797	.7630	.8464	.9297	5-32
3-16	.0156	.0990	.1823	.2656	.3490	.4323	.5156	.5990	.6823	.7656	.8490	.9323	3-16
7-32	.0182	.1016	.1849	.2682	.3516	.4349	.5182	.6016	.6849	.7682	.8516	.9349	7-32
1-4	.0208	.1042	.1875	.2708	.3542	.4375	.5208	.6042	.6875	.7708	.8542	.9375	1-4
9-32	.0234	.1068	.1901	.2734	.3568	.4401	.5234	.6068	.6901	.7734	.8568	.9401	9-32
5-16	.0260	.1094	.1927	.2760	.3594	.4427	.5260	.6094	.6927	.7760	.8594	.9427	5-16
11-32	.0286	.1120	.1953	.2786	.3620	.4453	.5286	.6120	.6953	.7786	.8620	.9453	11-32
3-8	.0313	.1146	.1979	.2813	.3646	.4479	.5313	.6146	.6979	.7813	.8646	.9479	3-8
13-32	.0339	.1172	.2005	.2839	.3672	.4505	.5339	.6172	.7005	.7839	.8672	.9505	13-32
7-16	.0365	.1198	.2031	.2865	.3698	.4531	.5365	.6198	.7031	.7865	.8698	.9531	7-16
15-32	.0391	.1224	.2057	.2891	.3724	.4557	.5391	.6224	.7057	.7891	.8724	.9557	15-32
1-2	.0417	.1250	.2083	.2917	.3750	.4583	.5417	.6250	.7083	.7917	.8750	.9583	1-2
17-32	.0443	.1276	.2109	.2943	.3776	.4609	.5443	.6276	.7109	.7943	.8776	.9609	17-32
9-16	.0469	.1302	.2135	.2969	.3802	.4635	.5469	.6302	.7135	.7969	.8802	.9635	9-16
19-32	.0495	.1328	.2161	.2995	.3828	.4661	.5495	.6328	.7161	.7995	.8828	.9661	19-32
5-8	.0521	.1354	.2188	.3021	.3854	.4688	.5521	.6354	.7188	.8021	.8854	.9688	5-8
21-32	.0547	.1380	.2214	.3047	.3880	.4714	.5547	.6380	.7214	.8047	.8880	.9714	21-32
11-16	.0573	.1406	.2240	.3073	.3906	.4740	.5573	.6406	.7240	.8073	.8906	.9740	11-16
23-32	.0599	.1432	.2266	.3099	.3932	.4766	.5599	.6432	.7266	.8099	.8932	.9766	23-32
3-4	.0625	.1458	.2292	.3125	.3958	.4792	.5625	.6458	.7292	.8125	.8958	.9792	3-4
25-32	.0651	.1484	.2318	.3151	.3984	.4818	.5651	.6484	.7318	.8151	.8984	.9818	25-32
13-16	.0677	.1510	.2344	.3177	.4010	.4844	.5677	.6510	.7344	.8177	.9010	.9844	13-16
27-32	.0703	.1536	.2370	.3203	.4036	.4870	.5703	.6536	.7370	.8203	.9036	.9870	27-32
7-8	.0729	.1563	.2396	.3229	.4063	.4896	.5729	.6563	.7396	.8229	.9063	.9896	7-8
29-32	.0755	.1589	.2422	.3255	.4089	.4922	.5755	.6589	.7422	.8255	.9089	.9922	29-32
15-16	.0781	.1615	.2448	.3281	.4115	.4948	.5781	.6615	.7448	.8281	.9115	.9948	15-16
31-32	.0807	.1641	.2474	.3307	.4141	.4974	.5807	.6641	.7474	.8307	.9141	.9974	31-32
	0	1	2	3	4	5	6	7	8	9	10	11	

TABLE VII

Horizontal Distances for Different Slope Angles

Inclined Distance = 100 feet

Slope	Correction	Horizontal Distance	Slope	Correction	Horizontal Distance
0° 00′	....	100.000	8° 00′	0.973	99.027
30	0.004	99.996	30	1.098	98.902
1 00	0.015	99.985	9 00	1.231	98.769
30	0.034	99.966	30	1.371	98.629
2 00	0.061	99.939	10 00	1.519	98.481
30	0.095	99.905	30	1.675	98.325
3 00	0.137	99.863	11 00	1.837	98.163
30	0.187	99.813	30	2.008	97.992
4 00	0.244	99.756	12 00	2.185	97.814
30	0.308	99.692	30	2.370	97.630
5 00	0.381	99.619	13 00	2.563	97.437
30	0.460	99.540	30	2.763	97.237
6 00	0.548	99.452	14 00	2.970	97.030
30	0.643	99.357	30	3.185	96.815
7 00	0.745	99.255	15 00	3.407	96.593
30	0.856	99.144	30	3.637	96.363

TABLE VIII

Minutes in Decimals of a Degree

′	0″	10″	15″	20″	30″	40″	45″	50″	′
0	.00000	.00278	.00417	.00556	.00833	.01111	.01250	.01389	0
1	.01667	.01944	.02083	.02222	.02500	.02778	.02917	.03055	1
2	.03333	.03611	.03750	.03889	.04167	.04444	.04583	.04722	2
3	.05000	.05278	.05417	.05556	.05833	.06111	.06250	.06389	3
4	.06667	.06944	.07083	.07222	.07500	.07778	.07917	.08056	4
5	.08333	.08611	.08750	.08889	.09167	.09444	.09583	.09722	5
6	.10000	.10278	.10417	.10556	.10833	.11111	.11250	.11389	6
7	.11667	.11944	.12083	.12222	.12500	.12778	.12917	.13056	7
8	.13333	.13611	.13750	.13889	.14167	.14444	.14583	.14722	8
9	.15000	.15278	.15417	.15556	.15833	.16111	.16250	.16389	9
10	.16667	.16944	.17083	.17222	.17500	.17778	.17917	.18056	10
11	.18333	.18611	.18750	.18889	.19167	.19444	.19583	.19722	11
12	.20000	.20278	.20417	.20556	.20833	.21111	.21250	.21389	12
13	.21667	.21944	.22083	.22222	.22500	.22778	.22917	.23056	13
14	.23333	.23611	.23750	.23889	.24167	.24444	.24583	.24722	14
15	.25000	.25278	.25417	.25556	.25833	.26111	.26250	.26389	15
16	.26667	.26944	.27083	.27222	.27500	.27778	.27917	.28056	16
17	.28333	.28611	.28750	.28889	.29167	.29444	.29583	.29722	17
18	.30000	.30278	.30417	.30556	.30833	.31111	.31250	.31389	18
19	.31667	.31944	.32083	.32222	.32500	.32778	.32917	.33056	19
20	.33333	.33611	.33750	.33889	.34167	.34444	.34583	.34722	20
21	.35000	.35278	.35417	.35556	.35833	.36111	.36250	.36389	21
22	.36667	.36944	.37083	.37222	.37500	.37778	.37917	.38056	22
23	.38333	.38611	.38750	.38889	.39167	.39444	.39583	.39722	23
24	.40000	.40278	.40417	.40556	.40833	.41111	.41250	.41389	24
25	.41667	.41944	.42083	.42222	.42500	.42778	.42917	.43056	25
26	.43333	.43611	.43750	.43889	.44167	.44444	.44583	.44722	26
27	.45000	.45278	.45417	.45556	.45833	.46111	.46250	.46389	27
28	.46667	.46944	.47083	.47222	.47500	.47778	.47917	.48056	28
29	.48333	.48611	.48750	.48889	.49167	.49444	.49583	.49722	29
30	.50000	.50278	.50417	.50556	.50833	.51111	.51250	.51389	30
31	.51667	.51944	.52083	.52222	.52500	.52778	.52917	.53056	31
32	.53333	.53611	.53750	.53889	.54167	.54444	.54583	.54722	32
33	.55000	.55278	.55417	.55556	.55833	.56111	.56250	.56389	33
34	.56667	.56944	.57083	.57222	.57500	.57778	.57917	.58056	34
35	.58333	.58611	.58750	.58889	.59167	.59444	.59583	.59722	35
36	.60000	.60278	.60417	.60556	.60833	.61111	.61250	.61389	36
37	.61667	.61944	.62083	.62222	.62500	.62778	.62917	.63056	37
38	.63333	.63611	.63750	.63889	.64167	.64444	.64583	.64722	38
39	.65000	.65278	.65417	.65556	.65833	.66111	.66250	.66389	39
40	.66667	.66944	.67083	.67222	.67500	.67778	.67917	.68056	40
41	.68333	.68611	.68750	.68889	.69167	.69444	.69583	.69722	41
42	.70000	.70278	.70417	.70556	.70833	.71111	.71250	.71389	42
43	.71667	.71944	.72083	.72222	.72500	.72778	.72917	.73056	43
44	.73333	.73611	.73750	.73889	.74167	.74444	.74583	.74722	44
45	.75000	.75278	.75417	.75556	.75833	.76111	.76250	.76389	45
46	.76667	.76944	.77083	.77222	.77500	.77778	.77917	.78056	46
47	.78333	.78611	.78750	.78889	.79167	.79444	.79583	.79722	47
48	.80000	.80278	.80417	.80556	.80833	.81111	.81250	.81389	48
49	.81667	.81944	.82083	.82222	.82500	.82778	.82917	.83056	49
50	.83333	.83611	.83750	.83889	.84167	.84444	.84583	.84722	50
51	.85000	.85278	.85417	.85556	.85833	86111	.86250	.86389	51
52	.86667	.86944	.87083	.87222	.87500	.87778	.87917	.88056	52
53	.88333	.88611	.88750	.88889	.89167	.89444	.89583	.89722	53
54	.90000	.90278	.90417	.90556	.90833	.91111	.91250	.91389	54
55	.91667	.91944	.92083	.92222	.92500	.92778	.92917	.93056	55
56	.93333	.93611	.93750	.93889	.94167	.94444	.94583	.94722	56
57	.95000	.95278	.95417	.95556	.95833	.96111	.96250	.96389	57
58	.96667	.96944	.97083	.97222	.97500	.97778	.97917	.98056	58
59	.98333	.98611	.98750	.98889	.99167	.99444	.99583	.99722	59
′	0″	10″	15″	20″	30″	40″	45″	50″	′

TABLE IX

Mean Refractions in Declination

(For use with the solar attachment)

Latitude.	Hour Angle.	DECLINATIONS.								
		+20°	+15°	+10°	+5°	0°	−5°	−10°	−15°	−20°
2° 30′	0 h	−18″	−12″	−07″	−02″	02″	07″	12″	18″	23″
	2	−18	−12	−07	−02	02	07	12	18	23
	3	−17	−11	−06	−01	03	08	13	19	25
	4	−15	−10	−05	00	05	10	15	21	27
	5	−10	−05	00	+05	10	15	20	26	32
5°	0 h	−15″	−10″	−05″	00″	05″	10″	15″	20″	27″
	2	−15	−10	−05	00	05	10	15	20	27
	3	−13	−08	−03	+02	07	12	17	23	29
	4	−10	−05	00	+05	10	15	20	27	32
	5	−05	00	+05	+10	15	20	27	32	40
7° 30′	0 h	−13″	−08″	−02″	02″	08″	13″	18″	24″	29″
	2	−12	−07	−01	03	09	14	19	25	31
	3	−10	−05	00	05	10	15	20	26	32
	4	−05	00	+05	10	15	20	26	32	39
	5	+07	+12	+17	23	29	36	43	51	1′ 01
10°	0 h	−10″	−05″	00″	05″	10″	15″	20″	26″	32″
	2	−07	−03	02	07	12	17	22	28	34
	3	−05	00	03	08	13	19	25	31	38
	4	00	+05	10	15	20	26	32	39	46
	5	+15	+20	26	32	39	46	55	1′ 06	1′ 19
12° 30′	0 h	−08″	−02″	02″	08″	13″	18″	24″	30″	36″
	2	−06	00	05	10	15	20	26	32	39
	3	+02	+07	12	17	23	29	36	43	51
	4	+04	+09	14	20	25	31	40	48	55
	5	+21	+27	33	40	48	57	1′ 08	1′ 23	1′ 41
15°	0 h	−05″	00″	05″	10″	15″	21″	27″	33″	40″
	2	−03	02	07	12	18	23	29	36	43
	3	+01	05	11	16	22	28	34	41	49
	4	+08	12	19	24	30	37	44	53	1′ 04
	5	+29	34	41	49	59	1′ 10	1′ 24	1′ 43	2 08
17° 30′	0 h	−02″	02″	08″	13″	18″	24″	30″	36″	44″
	2	00	05	10	15	21	27	33	40	48
	3	+02	10	15	21	27	33	40	48	57
	4	+13	18	23	29	35	43	51	1′ 01	1′ 13
	5	+34	41	49	58	1′ 10	1′ 23	1′ 41	2 06	2 42
20°	0 h	00″	05″	10″	15″	21″	27″	33″	40″	48″
	2	03	07	13	18	24	30	36	44	52
	3	06	13	18	24	30	36	44	52	1′ 02
	4	17	22	28	35	42	50	1′ 00	1′ 11	1 26
	5	39	47	57	1′ 07	1′ 20	1′ 37	2 00	2 32	3 25
22° 30′	0 h	02″	08″	13″	18″	24″	30″	36″	44″	52″
	2	06	11	15	21	27	33	40	48	57
	3	11	15	21	27	33	40	48	57	1′ 08
	4	20	26	32	39	46	56	1′ 07	1′ 19	1 37
	5	45	53	1′ 03	1′ 16	1′ 31	1′ 52	2 21	3 07	4 28
25°	0 h	05″	10″	15″	21″	27″	33″	40″	48″	57″
	2	08	14	19	25	31	38	46	54	1′ 05
	3	12	18	24	30	37	44	58	04	1 18
	4	23	29	35	45	53	1′ 03	1′ 16	1 31	1 52
	5	49	59	1′ 10	1′ 24	1′ 52	2 07	2 44	3 46	5 43
		+20°	+15°	+10°	+5°	0°	−5°	−10°	−15°	−20°

TABLE IX (*Continued*)

MEAN REFRACTIONS IN DECLINATION

Latitude.	Hour Angle.	+20°	+15°	+10°	+5°	0°	−5°	−10°	−15°	−20°
		DECLINATIONS.								
27° 30′	0 h	08″	13″	18″	24″	30″	36″	44″	52″	1′ 02″
	2	11	16	22	28	34	41	49	1′ 00	1 10
	3	17	22	28	35	42	50	1′ 00	1 11	1 26
	4	28	35	42	50	1′ 00	1′ 11	1 26	1 43	2 09
	5	54	1′ 05	1′ 18	1′ 34	1 54	2 24	3 11	4 38	8 15
30°	0 h	10″	15″	21″	27″	33″	40″	48″	57″	1′ 08″
	2	14	19	25	31	38	46	54	1′ 05	1 18
	3	20	26	32	39	47	55	1′ 06	1 19	1 36
	4	32	39	46	52	1′ 06	1′ 19	1 35	1 57	2 29
	5	1′ 00	1′ 10	1′ 24	1′ 52	2 07	2 44	3 46	5 43	13 06
32° 30′	0 h	13″	18″	24″	30″	36″	44″	52″	1′ 02″	1′ 14″
	2	17	22	28	35	42	50	1′ 00	1 11	1 26
	3	23	29	35	43	51	1′ 01	1 13	1 28	1 47
	4	35	43	51	1′ 01	1′ 13	1 27	1 46	2 13	2 54
	5	1′ 03	1′ 15	1′ 31	1 53	2 20	3 05	4 25	7 36	
35°	0 h	15″	21″	27″	33″	40″	48″	57″	1′ 08″	1′ 21″
	2	20	25	32	38	46	55	1′ 05	1 18	1 35
	3	26	33	39	47	56	1′ 07	1 21	1 38	2 00
	4	39	47	56	1′ 07	1′ 20	1 36	1 59	2 32	3 25
	5	1′ 07	1′ 20	1′ 38	2 00	2 34	3 29	5 14	10 16	
37° 30′	0 h	18″	24″	30″	36″	44″	52″	1′ 02″	1′ 14″	1′ 29″
	2	22	28	35	42	50	1′ 00	1 12	1 26	1 45
	3	29	36	43	52	1′ 02	1 14	1 29	1 49	2 16
	4	43	51	1′ 01	1′ 13	1 27	1 49	2 14	2 54	4 05
	5	1′ 11	1′ 26	1 54	2 10	2 49	3 55	6 15	14 58	
40°	0 h	21″	27″	33″	40″	48″	57″	1′ 08″	1′ 21″	1′ 39″
	2	25	32	39	46	52	1′ 06	1 19	1 35	1 57
	3	33	40	48	57	1′ 08	1 21	1 38	2 02	2 36
	4	47	55	1′ 06	1′ 19	1 36	1 58	2 30	3 21	4 59
	5	1′ 15	1′ 31	1 51	2 20	3 05	4 25	7 34	25 18	
42° 30′	0 h	24″	30″	36″	44″	52″	1′ 02″	1′ 14″	1′ 29″	1′ 49″
	2	28	35	39	50	1′ 00	1 12	1 26	1 45	2 11
	3	36	43	52	1′ 02	1 13	1 29	1 49	2 17	2 59
	4	50	1′ 00	1′ 11	1 26	1 44	2 10	2 49	3 55	6 16
	5	1′ 19	1 36	1 58	2 30	3 22	5 00	9 24		
45°	0 h	27″	33″	40″	48″	57″	1′ 08″	1′ 21″	1′ 39″	2′ 02″
	2	32	39	46	52	1′ 06	1 19	1 35	1 57	2 29
	3	40	47	56	1′ 07	1 21	1 38	2 00	2 34	3 29
	4	54	1′ 04	1′ 16	1 33	1 54	2 24	3 11	4 38	8 15
	5	1′ 23	1 41	2 05	2 41	3 40	5 40	12 02		
47° 30′	0 h	30″	36″	44″	52″	1′ 02″	1′ 14″	1′ 29″	1′ 49″	2′ 18″
	2	35	42	50	1′ 00	1 12	1 26	1 45	2 01	2 51
	3	43	51	1′ 01	1 13	1 28	1 47	2 15	2 56	4 08
	4	56	1′ 09	1 23	1 40	2 05	2 40	3 39	5 37	11 18
	5	1′ 27	1 46	2 12	2 52	4 01	6 30	16 19		
		+20°	+15°	+10°	+5°	0°	−5°	−10°	−15°	−20°

TABLE IX (*Continued*)

Mean Refractions in Declination

Latitude.	Hour Angle.	DECLINATIONS.								
		+20°	+15°	+10°	+5°	0°	−5°	−10°	−15°	−20°
50°	0 h	33″	40″	48″	57″	1′ 08″	1′ 21″	1′ 39″	2′ 02″	2′ 36″
	2	38	46	55	1′ 06	1 18	1 35	1 57	2 28	3 19
	3	47	56	1′ 06	1 19	1 36	2 29	2 31	3 23	5 02
	4	1′ 02	1′ 14	1 29	1 48	2 16	2 58	4 18	6 59	19 47
	5	1 30	1 51	2 19	3 04	4 22	7 28	24 10		
52° 30′	0 h	36″	44″	52″	1′ 02″	1′ 14″	1′ 29″	1′ 49″	2′ 18″	3′ 05″
	2	43	50	59	1 11	1 26	1 42	2 23	2 49	3 55
	3	50	1′ 00	1′ 11	1 26	1 45	2 11	2 51	2 58	6 22
	4	1′ 05	1 18	1 35	2 10	2 28	3 19	4 53	8 42	
	5	1 34	1 56	2 27	3 16	4 47	8 52			
55°	0 h	40″	48″	57″	1′ 08″	1′ 21″	1′ 39″	2′ 02″	2′ 36″	3′ 33″
	2	46	55	1′ 05	1 18	1 34	1 56	2 30	3 15	4 47
	3	55	1′ 06	1 19	1 35	1 58	2 30	3 21	4 58	9 19
	4	1′ 10	1 23	1 42	2 06	2 43	3 44	5 49	12 41	
	5	1 37	2 01	2 34	3 28	5 15	10 18			
57° 30′	0 h	44″	52″	1′ 02″	1′ 14″	1′ 29″	1′ 49″	2′ 18″	3′ 05″	4′ 37″
	2	50	59	1 11	1 25	1 43	2 09	2 47	3 51	6 04
	3	58	1′ 10	1 24	1 42	2 07	2 43	3 45	5 50	12 47
	4	1′ 11	1 25	1 43	2 10	2 50	3 55	6 14	14 49	
	5	1 41	2 06	2 42	3 42	5 46	12 26			
60°	0 h	48″	57″	1′ 08″	1′ 21″	1′ 39″	2′ 02″	2′ 36″	3′ 33″	5′ 23″
	2	54	1′ 04	1 17	1 33	1 54	2 24	3 12	4 38	8 15
	3	1′ 03	1 15	1 30	1 51	2 20	3 04	4 24	7 31	24 44
	4	1 18	1 34	1 56	2 28	3 18	4 50	8 53		
	5	1 45	2 11	2 50	3 57	6 21	15 32			
62° 30′	0 h	52″	1′ 02″	1′ 14″	1′ 29″	1′ 50	2′ 18″	3′ 00″	4′ 17″	7′ 13″
	2	58	1 09	1 23	1 41	2 06	2 43	3 44	5 50	12 44
	3	1′ 07	1 23	1 38	2 01	2 35	3 30	5 16	10 24	
	4	1 23	1 40	2 05	2 40	3 40	5 37	11 50		
	5	1 48	2 17	2 59	4 14	7 03				
65°	0 h	57″	1′ 08″	1′ 21″	1′ 39″	2′ 02″	2′ 36″	3′ 33″	5′ 23″	10′ 51″
	2	1′ 03	1 16	1 31	1 52	2 21	3 07	4 28	7 44	
	3	1 12	1 27	1 46	2 12	2 52	4 02	6 33		
	4	1 27	1 47	2 13	2 54	4 05	6 40			
	5	1 52	2 22	3 08	4 30	7 52				
67° 30′	0 h	1′ 02″	1′ 14″	1′ 29″	1′ 50″	2′ 18″	3′ 00″	4′ 17″	7′ 13″	
	2	1 08	1 22	1 40	2 03	2 39	3 37	5 32	11 28	
	3	1 17	1 34	1 55	2 26	3 14	4 44	8 34		
	4	1 32	1 53	2 23	3 14	4 35	8 05			
	5	1 56	2 28	3 17	4 40	8 51				
70°	0 h	1′ 08″	1′ 21″	1′ 39″	2′ 02″	2′ 36″	3′ 33″	5′ 23″	10′ 51″	
	2	1 14	1 29	1 50	2 18	3 00	4 17	7 13		
	3	1 23	1 43	2 05	2 41	3 41	5 59	12 15		
	4	1 37	2 00	2 34	3 28	5 20	10 12			
	5	2 02	2 33	3 27	5 11	10 05				
		+20°	+15°	+10°	+5°	0°	−5°	−10°	−15°	−20°

TABLE X

HORIZONTAL AND VERTICAL DISTANCES FROM 100-FOOT STADIA READINGS

Minutes.	0°		1°		2°		3°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0′	100.00	.00	99.97	1.74	99.88	3.49	99.73	5.23
2	"	.06	"	1.80	99.87	3.55	99.72	5.28
4	"	.12	"	1.86	"	3.60	99.71	5.34
6	"	.17	99.96	1.92	"	3.66	"	5.40
8	"	.23	"	1.98	99.86	3.72	99.70	5.46
10	"	.29	"	2.04	"	3.78	99.69	5.52
12	"	.35	"	2.09	99.85	3.84	"	5.57
14	"	.41	99.95	2.15	"	3.90	99.68	5.63
16	"	.47	"	2.21	99.84	3.95	"	5.69
18	"	.52	"	2.27	"	4.01	99.67	5.75
20	"	.58	"	2.33	99.83	4.07	99.66	5.80
22	"	.64	99.94	2.38	"	4.13	"	5.86
24	"	.70	"	2.44	99.82	4.18	99.65	5.92
26	99.99	.76	"	2.50	"	4.24	99.64	5.98
28	"	.81	99.93	2.56	99.81	4.30	99.63	6.04
30	"	.87	"	2.62	"	4.36	"	6.09
32	"	.93	"	2.67	99.80	4.42	99.62	6.15
34	"	.99	"	2.73	"	4.48	"	6.21
36	"	1.05	99.92	2.79	99.79	4.53	99.61	6.27
38	"	1.11	"	2.85	"	4.59	99.60	6.33
40	"	1.16	"	2.91	99.78	4.65	99.59	6.38
42	"	1.22	99.91	2.97	"	4.71	"	6.44
44	99.98	1.28	"	3.02	99.77	4.76	99.58	6.50
46	"	1.34	99.90	3.08	"	4.82	99.57	6.56
48	"	1.40	"	3.14	99.76	4.88	99.56	6.61
50	"	1.45	"	3.20	"	4.94	"	6.67
52	"	1.51	99.89	3.26	99.75	4.99	99.55	6.73
54	"	1.57	"	3.31	99.74	5.05	99.54	6.78
56	99.97	1.63	"	3.37	"	5.11	99.53	6.84
58	"	1.69	99.88	3.43	99.73	5.17	99.52	6.90
60	"	1.74	"	3.49	"	5.23	99 51	6.96
$c+f=$ .75	.75	.01	.75	.02	.75	.03	.75	.05
$c+f=$ 1.00	1.00	.01	1.00	.03	1.00	.04	1.00	.06
$c+f=$ 1.25	1.25	.02	1.25	.03	1.25	.05	1.25	.08

TABLE X (*Continued*)

Horizontal and Vertical Distances from 100-Foot Stadia Readings

Minutes.	4°		5°		6°		7°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0′	99.51	6.96	99.24	8.68	98.91	10.40	98.51	12.10
2	"	7.02	99.23	8.74	98.90	10.45	98.50	12.15
4	99.50	7.07	99.22	8.80	98.88	10.51	98.48	12.21
6	99.49	7.13	99.21	8.85	98.87	10.57	98.47	12.26
8	99.48	7.19	99.20	8.91	98.86	10.62	98.46	12.32
10	99.47	7.25	99.19	8.97	98.85	10.68	98.44	12.38
12	99.46	7.30	99.18	9.03	98.83	10.74	98.43	12.43
14	"	7.36	99.17	9.08	98.82	10.79	98.41	12.49
16	99.45	7.42	99.16	9.14	98.81	10.85	98.40	12.55
18	99.44	7.48	99.15	9.20	98.80	10.91	98.39	12.60
20	99.43	7.53	99.14	9.25	98.78	10.96	98.37	12.66
22	99.42	7.59	99.13	9.31	98.77	11.02	98.36	12.72
24	99.41	7.65	99.11	9.37	98.76	11.08	98.34	12.77
26	99.40	7.71	99.10	9.43	98.74	11.13	98.33	12.83
28	99.39	7.76	99.09	9.48	98.73	11.19	98.31	12.88
30	99.38	7.82	99.08	9.54	98.72	11.25	98.29	12.94
32	99.38	7.88	99.07	9.60	98.71	11.30	98.28	13.00
34	99.37	7.94	99.06	9.65	98.69	11.36	98.27	13.05
36	99.36	7.99	99.05	9.71	98.68	11.42	98.25	13.11
38	99.35	8.05	99.04	9.77	98.67	11.47	98.24	13.17
40	99.34	8.11	99.03	9.83	98.65	11.53	98.22	13.22
42	99.33	8.17	99.01	9.88	98.64	11.59	98.20	13.28
44	99.32	8.22	99.00	9.94	98.63	11.64	98.19	13.33
46	99.31	8.28	98.99	10.00	98.61	11.70	98.17	13.39
48	99.30	8.34	98.98	10.05	98.60	11.76	98.16	13.45
50	99.29	8.40	98.97	10.11	98.58	11.81	98.14	13.50
52	99.28	8.45	98.96	10.17	98.57	11.87	98.13	13.56
54	99.27	8.51	98.94	10.22	98.56	11.93	98.11	13.61
56	99.26	8.57	98.93	10.28	98.54	11.98	98.10	13.67
58	99.25	8.63	98.92	10.34	98.53	12.04	98.08	13.73
60	99.24	8.68	98.91	10.40	98.51	12.10	98.06	13.78
$c+f=$ .75	.75	.06	.75	.07	.75	.08	.74	.10
$c+f=$ 1.00	1.00	.08	.99	.09	.99	.11	.99	.13
$c+f=$ 1.25	1.25	.10	1.24	.11	1.24	.14	1.24	.16

TABLE X (*Continued*)

Horizontal and Vertical Distances from 100-Foot Stadia Readings

Minutes.	8°		9°		10°		11°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0'	98.06	13.78	97.55	15.45	96.98	17.10	96.36	18.73
2	98.05	13.84	97.53	15.51	96.96	17.16	96.34	18.78
4	98.03	13.89	97.52	15.56	96.94	17.21	96.32	18.84
6	98.01	13.95	97.50	15.62	96.92	17.26	96.29	18.89
8	98.00	14.01	97.48	15.67	96.90	17.32	96.27	18.95
10	97.98	14.06	97.46	15.73	96.88	17.37	96.25	19.00
12	97.97	14.12	97.44	15.78	96.86	17.43	96.23	19.05
14	97.95	14.17	97.43	15.84	96.84	17.48	96.21	19.11
16	97.93	14.23	97.41	15.89	96.82	17.54	96.18	19.16
18	97.92	14.28	97.39	15.95	96.80	17.59	96.16	19.21
20	97.90	14.34	97.37	16.00	96.78	17.65	96.14	19.27
22	97.88	14.40	97.35	16.06	96.76	17.70	96.12	19.32
24	97.87	14.45	97.33	16.11	96.74	17.76	96.09	19.38
26	97.85	14.51	97.31	16.17	96.72	17.81	96.07	19.43
28	97.83	14.56	97.29	16.22	96.70	17.86	96.05	19.48
30	97.82	14.62	97.28	16.28	96.68	17.92	96.03	19.54
32	97.80	14.67	97.26	16.33	96.66	17.97	96.00	19.59
34	97.78	14.73	97.24	16.39	96.64	18.03	95.98	19.64
36	97.76	14.79	97.22	16.44	96.62	18.08	95.96	19.70
38	97.75	14.84	97.20	16.50	96.60	18.14	95.93	19.75
40	97.73	14.90	97.18	16.55	96.57	18.19	95.91	19.80
42	97.71	14.95	97.16	16.61	96.55	18.24	95.89	19.86
44	97.69	15.01	97.14	16.66	96.53	18.30	95.86	19.91
46	97.68	15.06	97.12	16.72	96.51	18.35	95.84	19.96
48	97.66	15.12	97.10	16.77	96.49	18.41	95.82	20.02
50	97.64	15.17	97.08	16.83	96.47	18.46	95.79	20.07
52	97.62	15.23	97.06	16.88	96.45	18.51	95.77	20.12
54	97.61	15.28	97.04	16.94	96.42	18.57	95.75	20.18
56	97.59	15.34	97.02	16.99	96.40	18.62	95.72	20.23
58	97.57	15.40	97.00	17.05	96.38	18.68	95.70	20.28
60	97.55	15.45	96.98	17.10	96.36	18.73	95.68	20.34
$c+f=.75$	.74	.11	.74	.12	.74	.14	.73	.15
$c+f=1.00$	.99	.15	.99	.16	.98	.18	.98	.20
$c+f=1.25$	1.23	.18	1.23	.21	1.23	.23	1.22	.25

TABLE X (*Continued*)

HORIZONTAL AND VERTICAL DISTANCES FROM 100-FOOT STADIA READINGS

Minutes.	12°		13°		14°		15°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0′	95.68	20.34	94.94	21.92	94.15	23.47	93.30	25.00
2	95.65	20.39	94.91	21.97	94.12	23.52	93.27	25.05
4	95.63	20.44	94.89	22.02	94.09	23.58	93.24	25.10
6	95.61	20.50	94.86	22.08	94.07	23.63	93.21	25.15
8	95.58	20.55	94.84	22.13	94.04	23.68	93.18	25.20
10	95.56	20.60	94.81	22.18	94.01	23.73	93.16	25.25
12	95.53	20.66	94.79	22.23	93.98	23.78	93.13	25.30
14	95.51	20.71	94.76	22.28	93.95	23.83	93.10	25.35
16	95.49	20.76	94.73	22.34	93.93	23.88	93.07	25.40
18	95.46	20.81	94.71	22.39	93.90	23.93	93.04	25.45
20	95.44	20.87	94.68	22.44	93.87	23.99	93.01	25.50
22	95.41	20.92	94.66	22.49	93.84	24.04	92.98	25.55
24	95.39	20.97	94.63	22.54	93.81	24.09	92.95	25.60
26	95.36	21.03	94.60	22.60	93.79	24.14	92.92	25.65
28	95.34	21.08	94.58	22.65	93.76	24.19	92.89	25.70
30	95.32	21.13	94.55	22.70	93.73	24.24	92.86	25.75
32	95.29	21.18	94.52	22.75	93.70	24.29	92.83	25.80
34	95.27	21.24	94.50	22.80	93.67	24.34	92.80	25.85
36	95.24	21.29	94.47	22.85	93.65	24.39	92.77	25.90
38	95.22	21.34	94.44	22.91	93.62	24.44	92.74	25.95
40	95.19	21.39	94.42	22.96	93.59	24.49	92.71	26.00
42	95.17	21.45	94.39	23.01	93.56	24.55	92.68	26.05
44	95.14	21.50	94.36	23.06	93.53	24.60	92.65	26.10
46	95.12	21.55	94.34	23.11	93.50	24.65	92.62	26.15
48	95.09	21.60	94.31	23.16	93.47	24.70	92.59	26.20
50	95.07	21.66	94.28	23.22	93.45	24.75	92.56	26.25
52	95.04	21.71	94.26	23.27	93.42	24.80	92.53	26.30
54	95.02	21.76	94.23	23.32	93.39	24.85	92.49	26.35
56	94.99	21.81	94.20	23.37	93.36	24.90	92.46	26.40
58	94.97	21.87	94.17	23.42	93.33	24.95	92.43	26.45
60	94.94	21.92	94.15	23.47	93.30	25.00	92.40	26.50
$c+f=$ .75	.73	.16	.73	.17	.73	.19	.72	.20
$c+f=$ 1.00	.98	.22	.97	.23	.97	.25	.96	.27
$c+f=$ 1.25	1.22	.27	1.21	.29	1.21	.31	1.20	.34

TABLE X (*Continued*)

Horizontal and Vertical Distances from 100-Foot Stadia Readings

Minutes	16°		17°		18°		19°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0′	92.40	26.50	91.45	27.96	90.45	29.39	89.40	30.78
2	92.37	26.55	91.42	28.01	90.42	29.44	89.36	30.83
4	92.34	26.59	91.39	28.06	90.38	29.48	89.33	30.87
6	92.31	26.64	91.35	28.10	90.35	29.53	89.29	30.92
8	92.28	26.69	91.32	28.15	90.31	29.58	89.26	30.97
10	92.25	26.74	91.29	28.20	90.28	29.62	89.22	31.01
12	92.22	26.79	91.26	28.25	90.24	29.67	89.18	31.06
14	92.19	26.84	91.22	28.30	90.21	29.72	89.15	31.10
16	92.15	26.89	91.19	28.34	90.18	29.76	89.11	31.15
18	92.12	26.94	91.16	28.39	90.14	29.81	89.08	31.19
20	92.09	26.99	91.12	28.44	90.11	29.86	89.04	31.24
22	92.06	27.04	91.09	28.49	90.07	29.90	89.00	31.28
24	92.03	27.09	91.06	28.54	90.04	29.95	88.96	31.33
26	92.00	27.13	91.02	28.58	90.00	30.00	88.93	31.38
28	91.97	27.18	90.99	28.63	89.97	30.04	88.89	31.42
30	91.93	27.23	90.96	28.68	89.93	30.09	88.86	31.47
32	91.90	27.28	90.92	28.73	89.90	30.14	88.82	31.51
34	91.87	27.33	90.89	28.77	89.86	30.19	88.78	31.56
36	91.84	27.38	90.86	28.82	89.83	30.23	88.75	31.60
38	91.81	27.43	90.82	28.87	89.79	30.28	88.71	31.65
40	91.77	27.48	90.79	28.92	89.76	30.32	88.67	31.69
42	91.74	27.52	90.76	28.96	89.72	30.37	88.64	31.74
44	91.71	27.57	90.72	29.01	89.69	30.41	88.60	31.78
46	91.68	27.62	90.69	29.06	89.65	30.46	88.56	31.83
48	91.65	27.67	90.66	29.11	89.61	30.51	88.53	31.87
50	91.61	27.72	90.62	29.15	89.58	30.55	88.49	31.92
52	91.58	27.77	90.59	29.20	89.54	30.60	88.45	31.96
54	91.55	27.81	90.55	29.25	89.51	30.65	88.41	32.01
56	91.52	27.86	90.52	29.30	89.47	30.69	88.38	32.05
58	91.48	27.91	90.48	29.34	89.44	30.74	88.34	32.09
60	91.45	27.96	90.45	29.39	89.40	30.78	88.30	32.14
$c+f=.75$	.72	.21	.72	.23	.71	.24	.71	.25
$c+f=1.00$	.96	.28	.95	.30	.95	.32	.94	.33
$c+f=1.25$	1.20	.36	1.19	.38	1.19	.40	1.18	.42

TABLE X (*Continued*)

Horizontal and Vertical Distances from 100-Foot Stadia Readings

Minutes.	20°		21°		22°		23°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0′	88.30	32.14	87.16	33.46	85.97	34.73	84.73	35.97
2	88.26	32.18	87.12	33.50	85.93	34.77	84.69	36.01
4	88.23	32.23	87.08	33.54	85.89	34.82	84.65	36.05
6	88.19	32.27	87.04	33.59	85.85	34.86	84.61	36.09
8	88.15	32.32	87.00	33.63	85.80	34.90	84.57	36.13
10	88.11	32.36	86.96	33.67	85.76	34.94	84.52	36.17
12	88.08	32.41	86.92	33.72	85.72	34.98	84.48	36.21
14	88.04	32.45	86.88	33.76	85.68	35.02	84.44	36.25
16	88.00	32.49	86.84	33.80	85.64	35.07	84.40	36.29
18	87.96	32.54	86.80	33.84	85.60	35.11	84.35	36.33
20	87.93	32.58	86.77	33.89	85.56	35.15	84.31	36.37
22	87.89	32.63	86.73	33.93	85.52	35.19	84.27	36.41
24	87.85	32.67	86.69	33.97	85.48	35.23	84.23	36.45
26	87.81	32.72	86.65	34.01	85.44	35.27	84.18	36.49
28	87.77	32.76	86.61	34.06	85.40	35.31	84.14	36.53
30	87.74	32.80	86.57	34.10	85.36	35.36	84.10	36.57
32	87.70	32.85	86.53	34.14	85.31	35.40	84.06	36.61
34	87.66	32.89	86.49	34.18	85.27	35.44	84.01	36.65
36	87.62	32.93	86.45	34.23	85.23	35.48	83.97	36.69
38	87.58	32.98	86.41	34.27	85.19	35.52	83.93	36.73
40	87.54	33.02	86.37	34.31	85.15	35.56	83.89	36.77
42	87.51	33.07	86.33	34.35	85.11	35.60	83.84	36.80
44	87.47	33.11	86.29	34.40	85.07	35.64	83.80	36.84
46	87.43	33.15	86.25	34.44	85.02	35.68	83.76	36.88
48	87.39	33.20	86.21	34.48	84.98	35.72	83.72	36.92
50	87.35	33.24	86.17	34.52	84.94	35.76	83.67	36.96
52	87.31	33.28	86.13	34.57	84.90	35.80	83.63	37.00
54	87.27	33.33	86.09	34.61	84.86	35.85	83.59	37.04
56	87.24	33.37	86.05	34.65	84.82	35.89	83.54	37.08
58	87.20	33.41	86.01	34.69	84.77	35.93	83.50	37.12
60	87.16	33.46	85.97	34.73	84.73	35.97	83.46	37.16
$c+f=.75$	.70	.26	.70	.27	.69	.29	.69	.30
$c+f=1.00$	.94	.35	.93	.37	.92	.38	.92	.40
$c+f=1.25$	1.17	.44	1.16	.46	1.15	.48	1.15	.50

TABLE X (*Continued*)

Horizontal and Vertical Distances from 100-Foot Stadia Readings

Minutes.	24°		25°		26°		27°	
	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.	Hor. Dist.	Diff. Elev.
0′	83.46	37.16	82.14	38.30	80.78	39.40	79.39	40.45
2	83.41	37.20	82.09	38.34	80.74	39.44	79.34	40.49
4	83.37	37.23	82.05	38.38	80.69	39.47	79.30	40.52
6	83.33	37.27	82.01	38.41	80.65	39.51	79.25	40.55
8	83.28	37.31	81.96	38.45	80.60	39.54	79.20	40.59
10	83.24	37.35	81.92	38.49	80.55	39.58	79.15	40.62
12	83.20	37.39	81.87	38.53	80.51	39.61	79.11	40.66
14	83.15	37.43	81.83	38.56	80.46	39.65	79.06	40.69
16	83.11	37.47	81.78	38.60	80.41	39.69	79.01	40.72
18	83.07	37.51	81.74	38.64	80.37	39.72	78.96	40.76
20	83.02	37.54	81.69	38.67	80.32	39.76	78.92	40.79
22	82.98	37.58	81.65	38.71	80.28	39.79	78.87	40.82
24	82.93	37.62	81.60	38.75	80.23	39.83	78.82	40.86
26	82.89	37.66	81.56	38.78	80.18	39.86	78.77	40.89
28	82.85	37.70	81.51	38.82	80.14	39.90	78.73	40.92
30	82.80	37.74	81.47	38.86	80.09	39.93	78.68	40.96
32	82.76	37.77	81.42	38.89	80.04	39.97	78.63	40.99
34	82.72	37.81	81.38	38.93	80.00	40.00	78.58	41.02
36	82.67	37.85	81.33	38.97	79.95	40.04	78.54	41.06
38	82.63	37.89	81.28	39.00	79.90	40.07	78.49	41.09
40	82.58	37.93	81.24	39.04	79.86	40.11	78.44	41.12
42	82.54	37.96	81.19	39.08	79.81	40.14	78.39	41.16
44	82.49	38.00	81.15	39.11	79.76	40.18	78.34	41.19
46	82.45	38.04	81.10	39.15	79.72	40.21	78.30	41.22
48	82.41	38.08	81.06	39.18	79.67	40.24	78.25	41.26
50	82.36	38.11	81.01	39.22	79.62	40.28	78.20	41.29
52	82.32	38.15	80.97	39.26	79.58	40.31	78.15	41.32
54	82.27	38.19	80.92	39.29	79.53	40.35	78.10	41.35
56	82.23	38.23	80.87	39.33	79.48	40.38	78.06	41.39
58	82.18	38.26	80.83	39.36	79.44	40.42	78.01	41.42
60	82.14	38.30	80.78	39.40	79.39	40.45	77.96	41.45
$c+f=.75$	.68	.31	.68	.32	.67	.33	.66	.35
$c+f=1.00$	.91	.41	.90	.43	.89	.45	.89	.46
$c+f=1.25$	1.14	.52	1.13	.54	1.12	.56	1.11	.58

TABLE XI

Elevations Corresponding to Barometric Readings

B.	A.	Diff. for .01	B.	A.	Diff. for .01.	B.	A.	Diff. for .01.
Inches	Feet	Feet	Inches	Feet	Feet	Inches	Feet	Feet
18.0	13,918	−15.1	22.4	7,960	−12.2	26.7	3,175	−10.2
18.1	13,767	15.0	22.5	7,838	12.1	26.8	3,073	10.1
18.2	13,617	14.9	22.6	7,717	12.0	26.9	2,972	10.1
18 3	13,468	14.9	22.7	7,597	12.0	27.0	2,871	10.1
18.4	13,319	14.7	22.8	7,477	11.9	27.1	2,770	10.0
18.5	13,172	14.7	22.9	7,358	11.9	27.2	2,670	10.0
18.6	13,025	14.6	23.0	7,239	11.8	27.3	2,570	10.0
18.7	12,879	14.6	23.1	7,121	11.7	27.4	2,470	9.9
18.8	12,733	14.4	23.2	7,004	11.7	27.5	2,371	9.9
18.9	12,589	14.4	23.3	6,887	11.7	27.6	2,272	9.9
19.0	12,445	14.3	23.4	6,770	11.6	27.7	2,173	9.8
19 1	12,302	14.2	23.5	6,654	11.6	27.8	2,075	9.8
19 2	12,160	14.2	23.6	6,538	11.5	27.9	1 977	9.7
19.3	12,018	14.1	23.7	6,423	11.5	28.0	1,880	9.7
19 4	11,877	14.0	23.8	6,308	11.4	28.1	1,783	9.7
19 5	11,737	13.9	23.9	6,194	11.4	28.2	1,686	9.7
19 6	11,598	13.9	24.0	6,080	11.3	28.3	1,589	9 6
19.7	11,459	13.8	24.1	5,967	11.3	28.4	1,493	9.6
19.8	11,321	13.7	24.2	5,854	11.3	28.5	1,397	9.5
19.9	11,184	13.7	24.3	5,741	11.2	28.6	1,302	9.5
20.0	11,047	13.6	24.4	5,629	11.1	28.7	1,207	9.5
20.1	10,911	13.5	24.5	5,518	11.1	28.8	1,112	9.4
20.2	10,776	13.4	24.6	5,407	11.1	28.9	1,018	9.4
20.3	10,642	13.4	24.7	5,296	11.0	29.0	924	9.4
20.4	10,508	13.3	24.8	5,186	10.9	29.1	830	9.4
20.5	10,375	13.3	24.9	5,077	10.9	29.2	736	9.3
20.6	10,242	13.2	25.0	4,968	10.9	29.3	643	9.3
20.7	10,110	13.1	25.1	4,859	10.8	29.4	550	9.2
20.8	9,979	13.1	25.2	4,751	10.8	29.5	458	9.2
20.9	9,848	13.0	25.3	4,643	10.8	29.6	366	9.2
21.0	9,718	12.9	25.4	4,535	10.7	29.7	274	9.2
21.1	9,589	12.9	25.5	4,428	10.7	29.8	182	9.1
21.2	9,460	12.8	25.6	4,321	10.6	29.9	91	9.1
21.3	9,332	12.8	25.7	4,215	10.6	30.0	00	9.1
21.4	9,204	12.7	25.8	4,109	10.5	30.1	−91	9.0
21.5	9,077	12.6	25.9	4,004	10.5	30.2	181	9.0
21.6	8,951	12.6	26.0	3,899	10.5	30.3	271	9.0
21.7	8,825	12.5	26.1	3,794	10.4	30.4	361	9.0
21.8	8,700	12.5	26.2	3,690	10.4	30.5	451	8.9
21.9	8,575	12.4	26.3	3,586	10.3	30.6	540	8.9
22.0	8,451	12.4	26.4	3,483	10.3	30.7	629	8.8
22.1	8,327	12.3	26.5	3,380	10.3	30.8	717	8.8
22.2	8,204	12.2	26.6	3,277	−10.2	30.9	805	−8.8
22.3	8,082	−12.2	26.7	3,175		31.0	−893	
22.4	7,960							

TABLE XII

CORRECTION COEFFICIENTS TO BAROMETRIC READINGS FOR TEMPERATURE AND HUMIDITY

t+t'.	C.	Diff. for 1°	t+t'.	C.	Diff. for 1°.	t+t'.	C.	Diff. for 1°.
0°	−0.1250		60°	−0.0380		120°	+0.0262	
		10.9			10.7			10.6
10°	− .0915		70°	− .0273		130°	+ .0368	
		10.9			10.7			10.4
20°	− .0806		80°	− .0166		140°	+ .0472	
		10.8			10.8			10.3
30°	− .0698		90°	− .0058		150	+ .0575	
		10.6			10.7			10.2
40	− .0592		100°	+ .0049		160°	+ .0677	
		10.6			10.7			10.2
50°	− .0486		110°	+ .0156		170°	+ .0779	
		10.6			10.6			10.0
60°	− .0380		120°	+ .0262		180°	+ .0879	

TABLE XIII

AMOUNT AND VARIATION OF THE MAGNETIC NEEDLE FROM ITS MEAN DAILY POSITION

The letters E and W indicate which side of the mean position the needle points.

Season and Position in Latitude.	Local Mean Time: Morning Hours.						
	6 h.	7 h.	8 h.	9 h.	10 h.	11 h.	12 h
December, January, February:	Min.	Min.	Min.	Min.	Min.	Min.	Min.
Latitude 37° to 49°.	0.7 E	1.1 E	1.9 E	2.2 E	1.5 E	0.1 W	1.8 W
Latitude 25° to 37°.	0.1 W	0.1 E	1.0 E	2.0 E	2.2 E	1.1 E	0.5 W
March, April, May:							
Latitude 37° to 49°.	2.6 E	3.8 E	4.4 E	3.5 E	1.2 E	1.6 E	3.8 W
Latitude 25° to 37°.	1.6 E	2.8 E	3.3 E	2.6 E	1.1 E	0.6 W	1.9 W
June, July, August:							
Latitude 37° to 49°.	4.0 E	5.6 E	5.7 E	4.5 E	1.7 E	1.6 E	4.1 W
Latitude 25° to 37°.	2.4 E	4.0 E	4.2 E	2.9 E	0.5 E	1.6 W	2.8 W
September, October, November.							
Latitude 37° to 49°.	1.8 E	2.6 E	3.1 E	2.5 E	1.0 E	1.5 E	3.3 W
Latitude 25° to 37°.	0.9 E	2.1 E	2.6 E	2.1 E	0.6 E	0.9 E	2.1 W

Season and Position in Latitude.	Local Mean Time: Afternoon Hours.						
	0 h.	1 h.	2 h.	3 h.	4 h.	5 h.	6 h.
December, January February:	Min.	Min.	Min.	Min.	Min.	Min.	Min.
Latitude 37° to 49°.	1.8 W	2.9 W	2.8 W	2.1 W	1.3 W	0.7 W	0.2 W
Latitude 25° to 37°.	0.5 W	1.5 W	1.8 W	1.6 W	1.0 W	0.4 W	0.1 W
March, April, May:							
Latitude 37° to 49°.	3.8 W	4.8 W	4.6 W	3.8 W	2.5 W	1.4 W	0.7 W
Latitude 25° to 37°.	1.9 W	2.6 W	2.8 W	2.4 W	1.6 W	0.9 W	0.5 W
June, July, August:							
Latitude 37° to 49°.	4.1 W	5.6 W	5.6 W	4.6 W	3.0 W	1.4 W	0.6 W
Latitude 25° to 37°.	2.8 W	3.2 W	3.1 W	2.4 W	1.5 W	0.8 W	0.4 W
September, October, November:							
Latitude 37° to 49°.	3.3 W	4.0 W	3.4 W	2.3 W	1.2 W	0.6 W	0.1 W
Latitude 25° to 37°.	2.1 W	2.3 W	1.9 W	1.2 W	0.7 W	0.4 W	0.2 W

TABLE XIV

CONVERSION OF ENGLISH INCHES INTO CENTIMETRES.

Ins.	0	1	2	3	4	5	6	7	8	9
	Cm.	Cm.	Cm.	Cm.	Cm.	Cm.	Cm.	Cm.	Cm.	Cm.
0	0.000	2.540	5.080	7.620	10.16	12.70	15.24	17.78	20.32	22.86
10	25.40	27.94	30.48	33.02	35.56	38.10	40.64	43.18	45.72	48.26
20	50.80	53.34	55.88	58.42	60.96	63.50	66.04	68.58	71.12	73.66
30	76.20	78.74	81.28	83.82	86.36	88.90	91.44	93.98	96.52	99.06
40	101.60	104.14	106.68	109.22	111.76	114.30	116.84	119.38	121.92	124.46
50	127.00	129.54	132.08	134.62	137.16	139.70	142.24	144.78	147.32	149.86
60	152.40	154.94	157.48	160.02	162.56	165.10	167.64	170.18	172.72	175 26
70	177.80	180.34	182.88	185.42	187.96	190.50	193.04	195.58	198.12	200.66
80	203.20	205.74	208.28	210.82	213.36	215.90	218.44	220.98	223.52	226.06
90	228.60	231.14	233.68	236.22	238.76	241.30	243.84	246.38	248.92	251.46
100	254.00	256.54	259 08	261.62	264 16	266.70	269 24	271.78	274.32	276.86

CONVERSION OF CENTIMETRES INTO ENGLISH INCHES.

Cm.	0	1	2	3	4	5	6	7	8	9
	Ins.	Ins.	Ins.	Ins.	Ins.	Ins.	Ins.	Ins.	Ins.	Ins.
0	0.000	0.394	0.787	1.181	1.575	1.969	2.362	2.756	3.150	3.543
10	3.937	4.331	4.742	5.118	5.512	5.906	6.299	6.693	7.087	7.480
20	7.874	8.268	8.662	9.055	9.449	9.843	10.236	10.630	11.024	11.418
30	11.811	12.205	12.599	12.992	13.386	13.780	14.173	14.567	14.961	15.355
40	15.748	16.142	16.536	16.929	17.323	17.717	18.111	18.504	18.898	19.292
50	19.685	20.079	20.473	20.867	21.260	21.654	22.048	22 441	22.835	23.229
60	23.622	24.016	24.410	24.804	25.197	25.591	25.985	26.378	26.772	27.166
70	27.560	27.953	28.347	28.741	29.134	29.528	29.922	30.316	30.709	31.103
80	31.497	31.890	32.284	32.678	33.071	33.465	33.859	34.253	34.646	35.040
90	35.434	35.827	36.221	36.615	37.009	37.402	37.796	38.190	38.583	38.977
100	39.370	39.764	40.158	40.552	40.945	41.339	41.733	42.126	42.520	42.914

CONVERSION OF ENGLISH FEET INTO METRES.

Feet.	0	1	2	3	4	5	6	7	8	9
	Met.	Met.	Met.	Met.	Met.	Met.	Met.	Met.	Met.	Met.
0	0.000	0.3048	0.6096	0.9144	1.2192	1.5239	1.8287	2.1335	2.4383	2.7431
10	3.0479	3.3527	3.6575	3.9623	4.2671	4.5719	4.8767	5.1815	5.4863	5.7911
20	6.0959	6.4006	6.7055	7.0102	7.3150	7.6198	7.9246	8.2294	8.5342	8.8390
30	9.1438	9.4486	9.7534	10.058	10.363	10.668	10.972	11.277	11.582	11.887
40	12.192	12.496	12.801	13.106	13.411	13.716	14.020	14.325	14.630	14.935
50	15.239	15.544	15.849	16.154	16.459	16.763	17.068	17.373	17.678	17.983
60	18.287	18.592	18.897	19.202	19.507	19.811	20.116	20.421	20.726	21.031
70	21.335	21.640	21.945	22.250	22.555	22.859	23.164	23.469	23.774	24.079
80	24.383	24.688	24.993	25.298	25.602	25.907	26.212	26.517	26.822	27.126
90	27.431	27.736	28.041	28.346	28.651	28.955	29.260	29.565	29.870	30.174
100	30.479	30.784	31.089	31.394	31.698	32.003	32.308	32.613	32.918	33.222

CONVERSION OF METRES INTO ENGLISH FEET

Met.	0	1	2	3	4	5	6	7	8	9
	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.	Feet.
0	0.000	3.2809	6.5618	9.8427	13.123	16.404	19.685	22.966	26.247	29.528
10	32.809	36.090	39.371	42.651	45.932	49.213	52.494	55.775	59.056	62.337
20	65.618	68.899	72.179	75.461	78.741	82.022	85.303	88.584	91.865	95.146
30	98.427	101.71	104.99	108.27	111.55	114.83	118.11	121.39	124.67	127.96
40	131.24	134.52	137.80	141.08	144.36	147.64	150.92	154.20	157.48	160.76
50	164.04	167.33	170.61	173.89	177.17	180.45	183.73	187.01	190.29	193.57
60	196.85	200.13	203.42	206.70	209.98	213.26	216.54	219.82	223.10	226.38
70	229.66	232.94	236.22	239.51	242.79	246.07	249.35	252.63	255.91	259.19
80	262.47	265.75	269.03	272.31	275.60	278.88	282.16	285.44	288.72	292.00
90	295.28	298.56	301.84	305.12	308.40	311.69	314.97	318.25	321.53	324.81
100	328.09	331.37	334.65	337.93	341.21	344.49	347.78	351.06	354.34	357.62

TABLE XIV (*Continued*)

CONVERSION OF ENGLISH STATUTE-MILES INTO KILOMETRES.

Miles.	0	1	2	3	4	5	6	7	8	9
	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.	Kilo.
0	0.0000	1.6093	3.2186	4.8279	6.4372	8.0465	9.6558	11.2652	12.8745	14.4848
10	16.093	17.702	19.312	20.921	22 530	24.139	25.749	27.358	28.967	30.577
20	32.186	33.795	35.405	37.014	38.623	40.232	41.842	43.451	45.060	46.670
30	48.279	49.888	51.498	53.107	54.716	56.325	57.935	59.544	61.153	62.763
40	64.372	65.981	67.591	69.200	70.809	72.418	74.028	75.637	77.246	78.856
50	80.465	82.074	83.684	85.293	86.902	88.511	90.121	91.730	93.339	94.949
60	96 558	98.167	99.777	101.39	102.99	104.60	106.21	107.82	109.43	111.04
70	112.65	114.26	115.87	117.48	119.08	120.69	122 30	123.91	125.52	127.13
80	128.74	130.35	131.96	133.57	135.17	136.78	138.39	140.00	141.61	143.22
90	144.85	146.44	148.05	149.66	151.26	152.87	154.48	156.09	157.70	159.31
100	160.93	162.53	164 14	165 75	167.35	168.96	170.57	172.18	173.79	175.40

CONVERSION OF KILOMETRES INTO ENGLISH STATUTE-MILES.

Kilom.	0	1	2	3	4	5	6	7	8	9
	Miles.	Miles.	Miles.	Miles.	Miles.	Miles.	Miles.	Miles.	Miles.	Miles.
0	0.0000	0.6214	1.2427	1.8641	2.4855	3.1069	3.7282	4.3497	4 9711	5.5924
10	6.2138	6.8352	7.4565	8.0780	8.6994	9.3208	9.9421	10.562	11.185	11.805
20	12.427	13.049	13.670	14.292	14.913	15.534	16.156	16.776	17.399	18.019
30	18 641	19.263	19.884	20.506	21.127	21.748	23 370	22.990	23 613	24.233
40	24.855	25.477	26.098	26.720	27.341	27.962	28.584	29.204	29.827	30.447
50	31.069	31.690	32.311	32.933	33.554	34.175	34.797	35.417	36.040	36.660
60	37.282	37.904	38.525	39.147	39.768	40.389	41.011	41.631	42.254	42.874
70	43.497	44.118	44.739	45.361	45.982	46.603	47.225	47.845	48.468	49.088
80	49.711	50.332	50.953	51.575	52.196	52.817	53.439	54.059	64.682	55.302
90	55.924	56.545	57.166	57.788	58.409	59.030	59.652	60.272	60.895	61.515
100	62.138	62.759	63.380	64.002	64.623	65.244	65.866	66.486	67.109	67.729

TABLE XV

LENGTH IN FEET OF 1′ ARCS OF LATITUDE AND LONGITUDE.

Lat.	1′ Lat.	1′ Long.	Lat.	1′ Lat.	1′ Long.
1°	6045	6085	31°	6061	5222
2°	6045	6083	32°	6062	5166
3°	6045	6078	33°	6063	5109
4°	6045	6071	34°	6064	5051
5°	6045	6063	35°	6065	4991
6°	6045	6053	36°	6066	4930
7°	6046	6041	37°	6067	4867
8°	6046	6027	38°	6068	4802
9°	6046	6012	39°	6070	4736
10°	6047	5994	40°	6071	4669
11°	6047	5975	41°	6072	4600
12°	6048	5954	42°	6073	4530
13°	6048	5931	43°	6074	4458
14°	6049	5907	44°	6075	4385
15°	6049	5880	45°	6076	4311
16°	6050	5852	46°	6077	4235
17°	6050	5822	47°	6078	4158
18°	6051	5790	48°	6079	4080
19°	6052	5757	49°	6080	4001
20°	6052	5721	50°	6081	3920
21°	6053	5684	51°	6082	3838
22°	6054	5646	52°	6084	3755
23°	6054	5605	53°	6085	3671
24°	6055	5563	54°	6086	3586
25°	6056	5519	55°	6087	3499
26°	6057	5474	56°	6088	3413
27°	6058	5427	57°	6089	3323
28°	6059	5378	58°	6090	3233
29°	6060	5327	59°	6091	3142
30°	6061	5275	60°	6092	3051

TABLE XVI

Convergency of Meridians, Six Miles Long and Six Miles Apart, and Differences of Latitude and Longitude

Lat.	Convergency			Difference of longitude per range			Difference of latitude for—	
	On the parallel	Angle		In arc		In time	1 mi.	1 Tp.
°	*Lks.*	′	″	′	″	*Seconds*		
25	33.9	2	25	5	44.34	22.96		
26	35.4	2	32	5	47.20	23.15		
27	37.0	2	39	5	50.22	23.35	0.871	5.229
28	38.6	2	46	5	53.40	23.56		
29	40.2	2	53	5	56.74	23.78		
30	41.9	3	0	6	0.26	24.02		
31	43.6	3	7	6	3.97	24.26		
32	45.4	3	15	6	7.87	24.52	0.871	5.225
33	47.2	3	23	6	11.96	24.80		
34	49.1	3	30	6	16.26	25.08		
35	50.9	3	38	6	20.78	25.39		
36	52.7	3	46	6	25.53	25.70		
37	54.7	3	55	6	30.52	26.03	0.870	5.221
38	56.8	4	4	6	35.76	26.38		
39	58.8	4	13	6	41.27	26.75		
40	60.9	4	22	6	47.06	27.14		
41	63.1	4	31	6	53.15	27.54		
42	65.4	4	41	6	59.56	27.97	0.869	5.216
43	67.7	4	51	7	6.29	28.42		
44	70.1	5	1	7	13.39	28.89		
45	72.6	5	12	7	20.86	29.39		
46	75.2	5	23	7	28.74	29.92		
47	77.8	5	34	7	37.04	30.47	0.869	5.211
48	80.6	5	46	7	45.80	31.05		
49	83.5	5	59	7	55.05	31.67		
50	86.4	6	12	8	4.83	32.32		
51	89.6	6	25	8	15.17	33.03		
52	92.8	6	39	8	26.13	33.74	0.868	5.207
53	96.2	6	54	8	37.75	34.52		
54	99.8	7	9	8	50.07	35.34		
55	103.5	7	25	9	3.18	36.22		
56	107.5	7	42	9	17.12	37.14		
57	111.6	8	0	9	31.97	38.13	0.867	5.202
58	116.0	8	19	9	47.83	39.19		
59	120.6	8	38	10	4.78	40.32		
60	125.5	8	59	10	22.94	41.52		
61	130.8	9	22	10	42.42	42.83		
62	136.3	9	46	11	3.38	44.22	0.866	5.198
63	142.2	10	11	11	25.97	45.73		
64	148.6	10	38	11	50.37	47.36		
65	155.0	11	8	12	16.82	49.12		
66	162.8	11	39	12	45.55	51.04		
67	170.7	12	13	13	16.88	53.12	0.866	5.195
68	179.3	12	51	13	51.15	55.41		
69	188.7	13	31	14	28.77	57.92		
70	199.1	14	15	15	10.26	60.68	0.866	5.193

TABLE XVII

Azimuths of the Secant

Lat.	0 mi.		1 mi.		2 mi.		3 mi.	Deflection angle 6 mi.	
°	°	′	°	′	°	′		′	″
25	89	58.8	89	59.2	89	59.6	90°	2	25
26		58.7		59.2		59.6	E or W	2	32
27		58.7		59.1		59.6	" " "	2	39
28		58.6		59.1		59.5	" " "	2	46
29		58.6		59.0		59.5	" " "	2	53
30		58.5		59.0		59.5	" " "	3	0
31		58.4		59.0		59.5	" " "	3	7
32		58.4		58.9		59.5	" " "	3	15
33		58.3		58.9		59.4	" " "	3	23
34		58.2		58.8		59.4	" " "	3	30
35		58.2		58.8		59.4	" " "	3	38
36		58.1		58.7		59.4	" " "	3	46
37		58.0		58.7		59.3	" " "	3	55
38		58.0		58.6		59.3	" " "	4	4
39		57.9		58.6		59.3	" " "	4	13
40		57.8		58.5		59.3	" " "	4	22
41		57.7		58.5		59.2	" " "	4	31
42		57.7		58.4		59.2	" " "	4	41
43		57.6		58.4		59.2	" " "	4	51
44		57.5		58.3		59.2	" " "	5	1
45		57.4		58.3		59.1	" " "	5	12
46		57.3		58.2		59.1	" " "	5	23
47		57.2		58.1		59.1	" " "	5	34
48		57.1		58.1		59.0	" " "	5	46
49		57.0		58.0		59.0	" " "	5	59
50		56.9		57.9		59.0	" " "	6	12
51		56.8		57.9		58.9	" " "	6	25
52		56.7		57.8		58.9	" " "	6	39
53		56.6		57.7		58.8	" " "	6	54
54		56.4		57.6		58.8	" " "	7	9
55		56.3		57.5		58.8	" " "	7	25
56		56.2		57.4		58.7	" " "	7	42
57		56.0		57.3		58.7	" " "	8	0
58		55.8		57.2		58.6	" " "	8	19
59		55.7		57.1		58.6	" " "	8	38
60		55.5		57.0		58.5	" " "	8	59
61		55.3		56.9		58.4	" " "	9	22
62		55.1		56.7		58.4	" " "	9	46
63		54.9		56.6		58.3	" " "	10	11
64		54.7		56.5		58.2	" " "	10	38
65		54.4		56.3		58.1	" " "	11	8
66		54.2		56.1		58.1	" " "	11	39
67		53.9		55.9		58.0	" " "	12	13
68		53.6		55.7		57.9	" " "	12	51
69		53.2		55.5		57.8	" " "	13	31
70	89	52.9	89	55.3	89	57.6	" " "	14	15
	σ mi.		5 mi.		4 mi.		3 mi.		

TABLE XVIII

Offsets, in Links, from the Secant to the Parallel

Lat.	0 mi.	½ mi.	1 mi.	1½ mi.	2 mi.	2½ mi.	3 mi.
°							
25	2 N	1 N	0	1 S	1 S	2 S	2 S
26	2	1	0	1	1	2	2
27	3	1	0	1	2	2	2
28	3	1	0	1	2	2	2
29	3	1	0	1	2	2	2
30	3	1	0	1	2	2	2
31	3	1	0	1	2	2	2
32	3	1	0	1	2	2	3
33	3	1	0	1	2	2	3
34	3	2	0	1	2	3	3
35	4	2	0	1	2	3	3
36	4	2	0	1	2	3	3
37	4	2	0	1	2	3	3
38	4	2	0	1	2	3	3
39	4	2	0	1	2	3	3
40	4	2	0	1	3	3	3
41	4	2	0	2	3	3	4
42	5	2	0	2	3	3	4
43	5	2	0	2	3	4	4
44	5	2	0	2	3	4	4
45	5	2	0	2	3	4	4
46	5	2	0	2	3	4	4
47	5	2	0	2	3	4	4
48	6	3	0	2	3	4	4
49	6	3	0	2	3	4	5
50	6	3	0	2	4	4	5
51	6	3	0	2	4	5	5
52	6	3	0	2	4	5	5
53	7	3	0	2	4	5	5
54	7	3	0	2	4	5	6
55	7	3	0	3	4	5	6
56	7	3	0	3	4	6	6
57	8	3	0	3	5	6	6
58	8	4	0	3	5	6	6
59	8	4	0	3	5	6	7
60	9	4	0	3	5	7	7
61	9	4	0	3	5	7	7
62	9	4	0	3	6	7	8
63	10	4	0	3	6	7	8
64	10	5	0	4	6	8	8
65	11	5	0	4	6	8	9
66	11	5	0	4	7	8	9
67	12	5	0	4	7	9	9
68	12	6	0	4	7	9	10
69	13	6	0	5	8	10	10
70	14 N	6 N	0	5 S	8 S	10 S	11 S
	6 mi.	5½ mi.	5 mi.	4½ mi.	4 mi.	3½ mi.	3 mi.

Suggestions for the Use of Tables XIX, XX, XXI, XXII, and XXIII

The values of natural trigonometric functions in Tables XIX and XX are expressed to seven decimal places for use in high precision surveys in which the office computations are often made by means of the computing machine. The values of logarithms in Table XXII and of the logarithms of trigonometric functions in Table XXIII are expressed to six decimal places. In ordinary surveying it generally will be sufficiently accurate to use not more than five decimal places in any of the four tables, XIX, XX, XXII, and XXIII. The number of decimal places to be used will depend upon the accuracy to be attained in the computations, and the highest accuracy that can be attained is determined by the *number* of certain figures in the data. (See Art. 546(*b*).) It is a waste of time to use more decimal places in a table than are necessary. (See Art. 546(*i*) for general rules for the number of places required.) For the elimination of worthless refinements in computations, study Art. 546, particularly Art. 546(*h*).

In using, say, five decimal places in a table, it may be desirable to glance at the sixth decimal place to see if it exceeds 5. If it does, 1 may then be added to the value of the fifth decimal. For example, in Table XIX, the value for the natural sine of 40° 00′ is 0.6427876. If it is desired to use five decimal places, the value is 0.64278. But, glancing at the sixth place, it is seen that it is 7, and since this is more than 5, the five-place value of 0.64279 may be used. A similar procedure may be followed in any one of the five tables and for any desired number of decimal places.

General suggestions for logarithmic computations and for the use of logarithmic tables are given in Art. 547. The more specific suggestions and explanations given here for the use of Tables XIX, XX, XXI, XXII, and XXIII as well as the tables themselves have been taken from the book entitled *Highway Curves*, by Howard Chapin Ives, published by John Wiley & Sons.

Table XIX. Contains natural sines and cosines for every minute of the quadrant. Corrections for fractions of a minute are made directly proportional to the difference of consecutive

values in the table; positive for sines, negative for cosines. *With degrees and functions at top of page use the minutes at the left, and use minutes at the right of the page with degrees and functions at bottom of page.*

TABLE XX. Contains natural tangents and cotangents. Corrections for fractions of a minute are made directly proportional to the difference of consecutive values in the table; positive for tangents, negative for cotangents. *For proper column for minutes see directions for Table XIX.*

TABLE XXI. Contains the squares, cubes, square roots, cube roots, and reciprocals of numbers 1 to 1054. Its use may be greatly extended by observing that if any number is multiplied by n its square is multiplied by n^2, its cube by n^3, and its reciprocal by $1/n$.

TABLE XXII. **The logarithm** of a number consists of two parts, a whole number called the *characteristic,* and a decimal called the *mantissa.* All numbers which consist of the same figures standing in the same order have the same mantissa, regardless of the position of the decimal point in the number, or of the number of ciphers which precede or follow the significant figures of the number. The value of the characteristic depends entirely on the position of the decimal point in the number. It is always one less than the number of figures in the number to the left of the decimal point. The value is therefore diminished by one every time the decimal point of the number is removed one place to the *left,* and *vice versa.* Thus

Number	*Logarithm*
13840.	4.141136
1384.0	3.141136
138.40	2.141136
13.84	1.141136
1.384	0.141136
.1384	−1.141136
.01384	−2.141136
.001384	−3.141136
etc.	etc.

The mantissa is always positive even when the characteristic is negative. We may avoid the use of a negative characteristic by arbitrarily adding 10, which may be neglected at the close of the calculation. By this rule we have

Number	*Logarithm*
1.384	0.141136
.1384	9.141136
.01384	8.141136
.001384	7.141136
etc.	etc.

No confusion need arise from this method in finding a number from its logarithm; for although the logarithm 6.141136 represents either the number 1,384,000, or the decimal .0001384, yet these are so diverse in their values that we can never be uncertain in a given problem which to adopt.

Table XXII contains the mantissas of logarithms, carried to six places of decimals, for numbers between 1 and 9999, inclusive. The first three figures of a number are given in the first column, the fourth at the top of the other columns. The first two figures of the mantissa are given only in the second column, but these are understood to apply to the remaining four figures in either column following, which are comprised between the same horizontal lines with the two.

If a number (after cutting off the ciphers at either end) consists of not more than four figures, the mantissa may be taken direct from the table; but by interpolation the logarithm of a number having six figures may be obtained. The last column contains the average difference of consecutive logarithms on the same line, but for a given case the difference needs to be verified by actual subtraction, at least so far as the last figure is concerned. The lower part of the page contains a complete list of differences, with their multiples divided by 10.

To find the logarithm of a number having six figures. Take out the mantissa for the four superior places directly from the table, and find the difference between this mantissa and the next greater in the table. Add to the mantissa taken out the quantity found in the table of proportional parts, opposite the difference, and in the column headed by the fifth figure of the number; also add one tenth the quantity in the

column headed by the sixth figure. The sum is the mantissa required, to which must be prefixed a decimal point and the proper characteristic.

Example.—Find the log of 23.4275.

For 2342 mantissa is	369587
For diff. 185 col. 7	129.5
For diff. 185 col. 5	9.2

Ans. For 23.4275 log is 1.369726

The decimals of the corrections are added together to determine the nearest value of the sixth figure of the mantissa.

To find the number corresponding to a given logarithm. If the given mantissa is not in the table, find the one next less, and take out the four figures corresponding to it; divide the difference between the two mantissas by the tabular difference in that part of the table, and annex the figures of the quotient to the four figures already taken out. Finally, place the decimal point according to the rule for characteristics, prefixing or annexing ciphers if necessary. The division required is facilitated by the table of proportional parts, which furnishes by inspection the figures of the quotient.

Example.—Find the number of which the logarithm is 8.263927

		8.263927
First 4 figures 1836 from		263873
	Diff.	54.0
Tabular diff. = 236	∴ 5th fig. = 2	47.2
		6.80
	6th fig. = 3	7.08

Ans. No. = .0183623 or 183,623,000.

The number derived from a six-place logarithm is not reliable beyond the sixth figure.

At the end of Table XXII is a small table of logarithms of numbers from 1 to 100, with the characteristic prefixed, for easy reference when the given number does not exceed two digits. But the same mantissas may be found in the larger table.

TABLE XXIII. **The logarithmic sine, tangent, etc.,** of an arc is the logarithm of the natural sine, tangent, etc., of the same arc, but with 10 added to the characteristic to avoid negatives. This table gives log sines, tangents, cosines, and cotangents for every minute of the quadrant. With the number of degrees at the left side of the page are to be read the minutes in the left-hand column; with the degrees on the right-hand side are to be read the minutes in the right-hand column. When the degrees appear at the top of the page the top headings must be observed, when at the bottom those at the bottom. Since the values found for arcs in the first quadrant are duplicated in the second, the degrees are given from 0° to 180°. The differences in the logarithms due to a change of 1″ in the arc are given in adjoining columns.

To find the log sin, cos, tan, or cot of a given arc. Take out from the proper column of the table the logarithm corresponding to the given number of degrees and minutes. If there be any seconds multiply them by the adjoining tabular difference, and apply their product as a correction to the logarithm already taken out. The correction is to be *added* if the logarithms of the table are increasing with the angle, or *subtracted* if they are decreasing as the angle increases. In the first quadrant the log sines and tangents increase, and the log cosines and cotangents decrease as the angle increases.

Example.—Find the log sin of 9° 28′ 20″.

Log sin of 9° 28′ is		9.216097
Add correction 20 × 12.62		252
	Ans.	9.216349

Example.—Find the log cot of 9° 28′ 20″.

Log cotan of 9° 28′ is		10.777948
Subtract correction 20 × 12.97		259
	Ans.	10.777689

To find the angle or arc corresponding to a given logarithmic sine, tangent, cosine, or cotangent. If the given logarithm is found in the proper column take out the degrees and minutes directly; if not, find the two consecutive logarithms between which the given logarithm would fall, and

adopt that one which corresponds to the least number of minutes; which minutes take out with the degrees, and divide the difference between this logarithm and the given one by the adjoining tabular difference for a quotient, which will be the required number of seconds.

With logarithms to six places of decimals the quotient is not reliable beyond the tenth of a second.

Example.—9.383731 is the log tan of what angle?

Next *less* 9.383682 gives		13° 36′
Diff.	49.00 ÷ 9.20 =	05″.3
		Ans. 13° 36′ 05″.3

Example.—9.249348 is the log cos of what angle?

Next *greater* 583 gives		79° 46′
Diff.	235 ÷ 11.67 =	20″.1
		Ans. 79° 46′ 20″.1

The above rules do not apply to the first two pages of this table (except for the column headed cosine at top) because here the differences vary so rapidly that interpolation made by them in the usual way will not give exact results.

On the first two pages, the *first* column contains the number of seconds for every minute from 1′ to 2°; the minutes are given in the *second*, the log sin in the *third*, and in the *fourth* are the last three figures of a logarithm which is the difference between the log sin and the logarithm of the number of seconds in the first column. The first three figures and the characteristic of this logarithm are placed, once for all, at the head of the column.

To find the log sin of an arc less than 2° given to seconds. Reduce the given arc to seconds, and take the logarithm of the number of seconds from the table of logarithms, and *add* to this the logarithm from the fourth column opposite the same number of seconds. The sum is the log sin required.

The logarithm in the fourth column may need a slight interpolation of the last figure, to make it correspond closely to the given number of seconds.

Example.—Find the log sin of $1° 39' 14''.4$.

$1° 39' 14''.4 = 5954''.4$	log	3.774838
	add $(q - l)$	4.685515
	Ans. log sin	8.460353

Log tangents of small arcs are found in the same way, only taking the last four figures of $(q - l)$ from the *fifth* column.

Example.—Find the log tan of $0° 52' 35''$.

$52' 35'' = (3120'' + 35'') = 3155''$	log	3.498999
	add $(q - l)$	4.685609
	Ans. log tan	8.184608

Theory. Approximately, the sines or the tangents of small angles are proportional to the angles themselves. As respects the second example above we have

$$\tan 52' 35'' : \tan 52' :: 52' 35'' : 52'$$

or

$$\tan 52' 35'' = 52' 35'' \frac{\tan 52'}{52'} = 3155 \frac{\tan 52'}{3120}.$$

In the above q denotes the log tan $0° 52'$, and l the log of 3120. The quantity $q - l$ is to be added to the log of 3155 to get the log tan $52' 35''$.

A similar equation might be written for the *sine* of a small angle.

To find the log cotangent of an angle less than 2° given to seconds. Take from the column headed $(q + l)$ the logarithm corresponding to the given angle, interpolating for the last figure if necessary, and from this *subtract* the logarithm of the number of seconds in the given angle.

Example.—Find the log cotan of $1° 44' 22''.5$.

	$q + l$	15.314292
$6240'' + 22''.5 = 6262.5$	log	3.796748
	Ans.	11.517544

These two pages may be used in the same way when the given angle lies between 88° and 92°, or between 178° and 180°; but

if the number of degrees be found at the *bottom* of the page, the title of each column will be found there also; and if the number of degrees be found on the *right-hand* side of the page, the number of minutes must be found in the right-hand column, and since here the minutes increase upward, the number of seconds on the same line in the first column must be *diminished* by the odd seconds in the given angle to obtain the number whose logarithm is to be used with $(q \pm l)$ taken from the table.

Example.—Find the log cos of 88° 41′ 12″.5.

	$(q - l)$	4.685537
4740″ − 12″.5 = 4727.5	log	3.674631
	Ans.	8.360168

Example.—Find the log tan of 90° 30′ 50″.

	$q + l$	15.314413
1800″ + 50″ = 1850″	log	3.267172
	Ans.	12.047241

To find the arc corresponding to a given log sin, cos, tan, or cotan which falls within the limits of the first two pages of Table 39. Find in the proper column two consecutive logarithms between which the given logarithm falls. If the title of the given function is found at the top of that column read the degrees from the top of the page; if at the bottom read from the bottom.

Find the value of $(q - l)$ or $(q + l)$, as the case may require, corresponding to the given log (interpolating for the last figure if necessary). Then if q = given log and l = log of number of seconds, n, in the required arc, we have at once $l = q - (q - l)$ or $l = (q + l) - q$, whence n is easily found.

Find in the first column two consecutive quantities between which the number n falls, and if the degrees are read from the *left-hand* side of the page, adopt the *less*, take out the minutes from the second column, and take for the seconds the difference between the quantity adopted and the number n. But if the degrees are read from the *right-hand* side of the page, adopt the *greater* quantity, take out the minutes on the same line from the

right-hand column, and for the seconds take the difference between the number adopted and the number n.

Example.—11.734268 is the log cot of what arc?

$q + l$		15.314376
q		11.734268
$\therefore\ n =$	3802″.8	3.580108
For 1° adopt	3780. giving 03′	
Difference	22″.8	

Ans. 1° 03′ 22″.8 or 178° 56′ 37″.2.

Example.—8.201795 is the log cos of what arc?

$q - l$		4.685556
q		8.201795
$\therefore\ n =$	3282″.8	3.516239
For 89° adopt	3300. giving 05′	
Difference	17″.2	

Ans. 89° 05′ 17″.2 or 90° 54′ 42″.8.

TABLE XIX

Natural Sines and Cosines

′	0° Sine	0° Cosine	1° Sine	1° Cosine	2° Sine	2° Cosine	′
0	0.0000000	1.0000000	0.0174524	0.9998477	0.0348995	0.9993908	60
1	.0002909	1.0000000	.0177432	.9998426	.0351902	.9993806	59
2	.0005818	.9999998	.0180341	.9998374	.0354809	.9993704	58
3	.0008727	.9999996	.0183249	.9998321	.0357716	.9993600	57
4	.0011636	.9999993	.0186158	.9998267	.0360623	.9993495	56
5	.0014544	.9999989	.0189066	.9998213	.0363530	.9993390	55
6	.0017453	.9999985	.0191974	.9998157	.0366437	.9993284	54
7	.0020362	.9999979	.0194883	.9998101	.0369344	.9993177	53
8	.0023271	.9999973	.0197791	.9998044	.0372251	.9993069	52
9	.0026180	.9999966	.0200699	.9997986	.0375158	.9992960	51
10	0.0029089	0.9999958	0.0203608	0.9997927	0.0378065	0.9992851	50
11	.0031998	.9999949	.0206516	.9997867	.0380971	.9992740	49
12	.0034907	.9999939	.0209424	.9997807	.0383878	.9992629	48
13	.0037815	.9999928	.0212332	.9997745	.0386785	.9992517	47
14	.0040724	.9999917	.0215241	.9997683	.0389692	.9992404	46
15	.0043633	.9999905	.0218149	.9997620	.0392598	.9992290	45
16	.0046542	.9999892	.0221057	.9997556	.0395505	.9992176	44
17	.0049451	.9999878	.0223965	.9997492	.0398411	.9992060	43
18	.0052360	.9999863	.0226873	.9997426	.0401318	.9991944	42
19	.0055268	.9999847	.0229781	.9997360	.0404224	.9991827	41
20	0.0058177	0.9999831	0.0232690	0.9997292	0.0407131	0.9991709	40
21	.0061086	.9999813	.0235598	.9997224	.0410037	.9991590	39
22	.0063995	.9999795	.0238506	.9997155	.0412944	.9991470	38
23	.0066904	.9999776	.0241414	.9997086	.0415850	.9991350	37
24	.0069813	.9999756	.0244322	.9997015	.0418757	.9991228	36
25	.0072721	.9999736	.0247230	.9996943	.0421663	.9991106	35
26	.0075630	.9999714	.0250138	.9996871	.0424569	.9990983	34
27	.0078539	.9999692	.0253046	.9996798	.0427475	.9990859	33
28	.0081448	.9999668	.0255954	.9996724	.0430382	.9990734	32
29	.0084357	.9999644	.0258862	.9996649	.0433288	.9990609	31
30	0.0087265	0.9999619	0.0261769	0.9996573	0.0436194	0.9990482	30
31	.0090174	.9999593	.0264677	.9996497	.0439100	.9990355	29
32	.0093083	.9999567	.0267585	.9996419	.0442006	.9990227	28
33	.0095992	.9999539	.0270493	.9996341	.0444912	.9990098	27
34	.0098900	.9999511	.0273401	.9996262	.0447818	.9989968	26
35	.0101809	.9999482	.0276309	.9996182	.0450724	.9989837	25
36	.0104718	.9999452	.0279216	.9996101	.0453630	.9989706	24
37	.0107627	.9999421	.0282124	.9996020	.0456536	.9989573	23
38	.0110535	.9999389	.0285032	.9995937	.0459442	.9989440	22
39	.0113444	.9999357	.0287940	.9995854	.0462347	.9989306	21
40	0.0116353	0.9999323	0.0290847	0.9995770	0.0465253	0.9989171	20
41	.0119261	.9999289	.0293755	.9995684	.0468159	.9989035	19
42	.0122170	.9999254	.0296662	.9995599	.0471065	.9988899	18
43	.0125079	.9999218	.0299570	.9995512	.0473970	.9988761	17
44	.0127987	.9999181	.0302478	.9995424	.0476876	.9988623	16
45	.0130896	.9999143	.0305385	.9995336	.0479781	.9988484	15
46	.0133805	.9999105	.0308293	.9995247	.0482687	.9988344	14
47	.0136713	.9999065	.0311200	.9995157	.0485592	.9988203	13
48	.0139622	.9999025	.0314108	.9995066	.0488498	.9988061	12
49	.0142530	.9998984	.0317015	.9994974	.0491403	.9987919	11
50	0.0145439	0.9998942	0.0319922	0.9994881	0.0494308	0.9987775	10
51	.0148348	.9998900	.0322830	.9994788	.0497214	.9987631	9
52	.0151256	.9998856	.0325737	.9994693	.0500119	.9987486	8
53	.0154165	.9998812	.0328644	.9994598	.0503024	.9987340	7
54	.0157073	.9998766	.0331552	.9994502	.0505929	.9987194	6
55	.0159982	.9998720	.0334459	.9994405	.0508835	.9987046	5
56	.0162890	.9998673	.0337366	.9994308	.0511740	.9986898	4
57	.0165799	.9998625	.0340274	.9994209	.0514645	.9986748	3
58	.0168707	.9998577	.0343181	.9994110	.0517550	.9986598	2
59	.0171616	.9998527	.0346088	.9994009	.0520455	.9986447	1
60	0.0174524	0.9998477	0.0348995	0.9993908	0.0523360	0.9986295	0
′	89° Cosine	89° Sine	88° Cosine	88° Sine	87° Cosine	87° Sine	′

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	3°		4°		5°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	0.0523360	0.9986295	0.0697565	0.9975641	0.0871557	0.9961947	60
1	.0526264	.9986143	.0700467	.9975437	.0874455	.9961693	59
2	.0529169	.9985989	.0703368	.9975233	.0877353	.9961438	58
3	.0532074	.9985835	.0706270	.9975028	.0880251	.9961183	57
4	.0534979	.9985680	.0709171	.9974822	.0883148	.9960926	56
5	.0537883	.9985524	.0712073	.9974615	.0886046	.9960669	55
6	.0540788	.9985367	.0714974	.9974408	.0888943	.9960411	54
7	.0543693	.9985209	.0717876	.9974199	.0891840	.9960152	53
8	.0546597	.9985050	.0720777	.9973990	.0894738	.9959892	52
9	.0549502	.9984891	.0723678	.9973780	.0897635	.9959631	51
10	0.0552406	0.9984731	0.0726580	0.9973569	0.0900532	0.9959370	50
11	.0555311	.9984570	.0729481	.9973357	.0903429	.9959107	49
12	.0558215	.9984408	.0732382	.9973145	.0906326	.9958844	48
13	.0561119	.9984245	.0735283	.9972931	.0909223	.9958580	47
14	.0564024	.9984081	.0738184	.9972717	.0912119	.9958315	46
15	.0566928	.9983917	.0741085	.9972502	.0915016	.9958049	45
16	.0569832	.9983751	.0743986	.9972286	.0917913	.9957783	44
17	.0572736	.9983585	.0746887	.9972069	.0920809	.9957515	43
18	.0575640	.9983418	.0749787	.9971851	.0923706	.9957247	42
19	.0578544	.9983250	.0752688	.9971633	.0926602	.9956978	41
20	0.0581448	0.9983082	0.0755589	0.9971413	0.0929499	0.9956708	40
21	.0584352	.9982912	.0758489	.9971193	.0932395	.9956437	39
22	.0587256	.9982742	.0761390	.9970972	.0935291	.9956165	38
23	.0590160	.9982570	.0764290	.9970750	.0938187	.9955893	37
24	.0593064	.9982398	.0767190	.9970528	.0941083	.9955620	36
25	.0595967	.9982225	.0770091	.9970304	.0943979	.9955345	35
26	.0598871	.9982052	.0772991	.9970080	.0946875	.9955070	34
27	.0601775	.9981877	.0775891	.9969854	.0949771	.9954795	33
28	.0604678	.9981701	.0778791	.9969628	.0952666	.9954518	32
29	.0607582	.9981525	.0781691	.9969401	.0955562	.9954240	31
30	0.0610485	0.9981348	0.0784591	0.9969173	0.0958458	0.9953962	30
31	.0613389	.9981170	.0787491	.9968945	.0961353	.9953683	29
32	.0616292	.9980991	.0790391	.9968715	.0964248	.9953403	28
33	.0619196	.9980811	.0793290	.9968485	.0967144	.9953122	27
34	.0622099	.9980631	.0796190	.9968254	.0970039	.9952840	26
35	.0625002	.9980450	.0799090	.9968022	.0972934	.9952557	25
36	.0627905	.9980267	.0801989	.9967789	.0975829	.9952274	24
37	.0630808	.9980084	.0804889	.9967555	.0978724	.9951990	23
38	.0633711	.9979900	.0807788	.9967321	.0981619	.9951705	22
39	.0636614	.9979716	.0810687	.9967085	.0984514	.9951419	21
40	0.0639517	0.9979530	0.0813587	0.9966849	0.0987408	0.9951132	20
41	.0642420	.9979343	.0816486	.9966612	.0990303	.9950844	19
42	.0645323	.9979156	.0819385	.9966374	.0993197	.9950556	18
43	.0648226	.9978968	.0822284	.9966135	.0996092	.9950266	17
44	.0651129	.9978779	.0825183	.9965895	.0998986	.9949976	16
45	.0654031	.9978589	.0828082	.9965655	.1001881	.9949685	15
46	.0656934	.9978399	.0830981	.9965414	.1004775	.9949393	14
47	.0659836	.9978207	.0833880	.9965172	.1007669	.9949101	13
48	.0662739	.9978015	.0836778	.9964929	.1010563	.9948807	12
49	.0665641	.9977821	.0839677	.9964685	.1013457	.9948513	11
50	0.0668544	0.9977627	0.0842576	0.9964440	0.1016351	0.9948217	10
51	.0671446	.9977433	.0845474	.9964195	.1019245	.9947921	9
52	.0674349	.9977237	.0848373	.9963948	.1022138	.9947625	8
53	.0677251	.9977040	.0851271	.9963701	.1025032	.9947327	7
54	.0680153	.9976843	.0854169	.9963453	.1027925	.9947028	6
55	.0683055	.9976645	.0857067	.9963204	.1030819	.9946729	5
56	.0685957	.9976445	.0859966	.9962954	.1033712	.9946428	4
57	.0688859	.9976245	.0862864	.9962704	.1036605	.9946127	3
58	.0691761	.9976045	.0865762	.9962452	.1039499	.9945825	2
59	.0694663	.9975843	.0868660	.9962200	.1042392	.9945523	1
60	0.0697565	0.9975641	0.0871557	0.9961947	0.1045285	0.9945219	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	86°		85°		84°		

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	6° Sine	6° Cosine	7° Sine	7° Cosine	8° Sine	8° Cosine	′
0	0.1045285	0.9945219	0.1218693	0.9925462	0.1391731	0.9902681	60
1	.1048178	.9944914	.1221581	.9925107	.1394612	.9902275	59
2	.1051070	.9944609	.1224468	.9924751	.1397492	.9901869	58
3	.1053963	.9944303	.1227355	.9924394	.1400372	.9901462	57
4	.1056856	.9943996	.1230241	.9924037	.1403252	.9901055	56
5	.1059748	.9943688	.1233128	.9923679	.1406132	.9900646	55
6	.1062641	.9943379	.1236015	.9923319	.1409012	.9900237	54
7	.1065533	.9943070	.1238901	.9922959	.1411892	.9899826	53
8	.1068425	.9942760	.1241788	.9922599	.1414772	.9899415	52
9	.1071318	.9942448	.1244674	.9922237	.1417651	.9899003	51
10	0.1074210	0.9942136	0.1247560	0.9921874	0.1420531	0.9898590	50
11	.1077102	.9941823	.1250446	.9921511	.1423410	.9898177	49
12	.1079994	.9941510	.1253332	.9921147	.1426289	.9897762	48
13	.1082885	.9941195	.1256218	.9920782	.1429168	.9897347	47
14	.1085777	.9940880	.1259104	.9920416	.1432047	.9896931	46
15	.1088669	.9940563	.1261990	.9920049	.1434926	.9896514	45
16	.1091560	.9940246	.1264875	.9919682	.1437805	.9896096	44
17	.1094452	.9939928	.1267761	.9919314	.1440684	.9895677	43
18	.1097343	.9939610	.1270646	.9918944	.1443562	.9895258	42
19	.1100234	.9939290	.1273531	.9918574	.1446440	.9894838	41
20	0.1103126	0.9938969	0.1276416	0.9918204	0.1449319	0.9894416	40
21	.1106017	.9938648	.1279302	.9917832	.1452197	.9893994	39
22	.1108908	.9938326	.1282186	.9917459	.1455075	.9893572	38
23	.1111799	.9938003	.1285071	.9917086	.1457953	.9893148	37
24	.1114689	.9937679	.1287956	.9916712	.1460830	.9892723	36
25	.1117580	.9937355	.1290841	.9916337	.1463708	.9892298	35
26	.1120471	.9937029	.1293725	.9915961	.1466585	.9891872	34
27	.1123361	.9936703	.1296609	.9915584	.1469463	.9891445	33
28	.1126252	.9936375	.1299494	.9915206	.1472340	.9891017	32
29	.1129142	.9936047	.1302378	.9914828	.1475217	.9890588	31
30	0.1132032	0.9935719	0.1305262	0.9914449	0.1478094	0.9890159	30
31	.1134922	.9935389	.1308146	.9914069	.1480971	.9889728	29
32	.1137812	.9935058	.1311030	.9913688	.1483848	.9889297	28
33	.1140702	.9934727	.1313913	.9913306	.1486724	.9888865	27
34	.1143592	.9934395	.1316797	.9912923	.1489601	.9888432	26
35	.1146482	.9934062	.1319681	.9912540	.1492477	.9887998	25
36	.1149372	.9933728	.1322564	.9912155	.1495353	.9887564	24
37	.1152261	.9933393	.1325447	.9911770	.1498230	.9887128	23
38	.1155151	.9933057	.1328330	.9911384	.1501106	.9886692	22
39	.1158040	.9932721	.1331213	.9910997	.1503981	.9886255	21
40	0.1160929	0.9932384	0.1334096	0.9910610	0.1506857	0.9885817	20
41	.1163818	.9932045	.1336979	.9910221	.1509733	.9885378	19
42	.1166707	.9931706	.1339862	.9909832	.1512608	.9884939	18
43	.1169596	.9931367	.1342744	.9909442	.1515484	.9884498	17
44	.1172485	.9931026	.1345627	.9909051	.1518359	.9884057	16
45	.1175374	.9930685	.1348509	.9908659	.1521234	.9883615	15
46	.1178263	.9930342	.1351392	.9908266	.1524109	.9883172	14
47	.1181151	.9929999	.1354274	.9907873	.1526984	.9882728	13
48	.1184040	.9929655	.1357156	.9907478	.1529858	.9882284	12
49	.1186928	.9929310	.1360038	.9907083	.1532733	.9881838	11
50	0.1189816	0.9928965	0.1362919	0.9906687	0.1535607	0.9881392	10
51	.1192704	.9928618	.1365801	.9906290	.1538482	.9880945	9
52	.1195593	.9928271	.1368683	.9905893	.1541356	.9880497	8
53	.1198481	.9927922	.1371564	.9905494	.1544230	.9880048	7
54	.1201368	.9927573	.1374445	.9905095	.1547104	.9879599	6
55	.1204256	.9927224	.1377327	.9904694	.1549978	.9879148	5
56	.1207144	.9926873	.1380208	.9904293	.1552851	.9878697	4
57	.1210031	.9926521	.1383089	.9903891	.1555725	.9878245	3
58	.1212919	.9926169	.1385970	.9903489	.1558598	.9877792	2
59	.1215806	.9925816	.1388850	.9903085	.1561472	.9877338	1
60	0.1218693	0.9925462	0.1391731	0.9902681	0.1564345	0.9876883	0
′	83° Cosine	83° Sine	82° Cosine	82° Sine	81° Cosine	81° Sine	′

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	9° Sine	9° Cosine	10° Sine	10° Cosine	11° Sine	11° Cosine	′
0	0.1564345	0.9876883	0.1736482	0.9848078	0.1908090	0.9816272	60
1	.1567218	.9876428	.1739346	.9847572	.1910945	.9815716	59
2	.1570091	.9875972	.1742211	.9847066	.1913801	.9815160	58
3	.1572963	.9875514	.1745075	.9846558	.1916656	.9814603	57
4	.1575836	.9875057	.1747939	.9846050	.1919510	.9814045	56
5	.1578708	.9874598	.1750803	.9845542	.1922365	.9813486	55
6	.1581581	.9874138	.1753667	.9845032	.1925220	.9812927	54
7	.1584453	.9873678	.1756531	.9844521	.1928074	.9812366	53
8	.1587325	.9873216	.1759395	.9844010	.1930928	.9811805	52
9	.1590197	.9872754	.1762258	.9843498	.1933782	.9811243	51
10	0.1593069	0.9872291	0.1765121	0.9842985	0.1936636	0.9810680	50
11	.1595940	.9871827	.1767984	.9842471	.1939490	.9810116	49
12	.1598812	.9871363	.1770847	.9841956	.1942344	.9809552	48
13	.1601683	.9870897	.1773710	.9841441	.1945197	.9808986	47
14	.1604555	.9870431	.1776573	.9840924	.1948050	.9808420	46
15	.1607426	.9869964	.1779435	.9840407	.1950903	.9807853	45
16	.1610297	.9869496	.1782298	.9839889	.1953756	.9807285	44
17	.1613167	.9869027	.1785160	.9839370	.1956609	.9806716	43
18	.1616038	.9868557	.1788022	.9838850	.1959461	.9806147	42
19	.1618909	.9868087	.1790884	.9838330	.1962314	.9805576	41
20	0.1621779	0.9867615	0.1793746	0.9837808	0.1965166	0.9805005	40
21	.1624650	.9867143	.1796607	.9837286	.1968018	.9804433	39
22	.1627520	.9866670	.1799469	.9836763	.1970870	.9803860	38
23	.1630390	.9866196	.1802330	.9836239	.1973722	.9803286	37
24	.1633260	.9865722	.1805191	.9835715	.1976573	.9802712	36
25	.1636129	.9865246	.1808052	.9835189	.1979425	.9802136	35
26	.1638999	.9864770	.1810913	.9834663	.1982276	.9801560	34
27	.1641868	.9864293	.1813774	.9834136	.1985127	.9800983	33
28	.1644738	.9863815	.1816635	.9833608	.1987978	.9800405	32
29	.1647607	.9863336	.1819495	.9833079	.1990829	.9799827	31
30	0.1650476	0.9862856	0.1822355	0.9832549	0.1993679	0.9799247	30
31	.1653345	.9862375	.1825215	.9832019	.1996530	.9798667	29
32	.1656214	.9861894	.1828075	.9831487	.1999380	.9798086	28
33	.1659082	.9861412	.1830935	.9830955	.2002230	.9797504	27
34	.1661951	.0860929	.1833795	.9830422	.2005080	.9796921	26
35	.1664819	.9860445	.1836654	.9829888	.2007930	.9796337	25
36	.1667687	.9859960	.1839514	.9829353	.2010779	.9795752	24
37	.1670556	.9859475	.1842373	.9828818	.2013629	.9795167	23
38	.1673423	.9858988	.1845232	.9828282	.2016478	.9794581	22
39	.1676291	.9858501	.1848091	.9827744	.2019327	.9793994	21
40	0.1679159	0.9858013	0.1850949	0.9827206	0.2022176	0.9793406	20
41	.1682026	.9857524	.1853808	.9826668	.2025024	.9792818	19
42	.1684894	.9857035	.1856666	.9826128	.2027873	.9792228	18
43	.1687761	.9856544	.1859524	.9825587	.2030721	.9791638	17
44	.1690628	.9856053	.1862382	.9825046	.2033569	.9791047	16
45	.1693495	.9855561	.1865240	.9824504	.2036418	.9790455	15
46	.1696362	.9855068	.1868098	.9823961	.2039265	.9789862	14
47	.1699228	.9854574	.1870956	.9823417	.2042113	.9789268	13
48	.1702095	.9854079	.1873813	.9822873	.2044961	.9788674	12
49	.1704961	.9853583	.1876670	.9822327	.2047808	.9788079	11
50	0.1707828	0.9853087	0.1879528	0.9821781	0.2050655	0.9787483	10
51	.1710694	.9852590	.1882385	.9821234	.2053502	.9786886	9
52	.1713560	.9852092	.1885241	.9820686	.2056349	.9786288	8
53	.1716425	.9851593	.1888098	.9820137	.2059195	.9785689	7
54	.1719291	.9851093	.1890954	.9819587	.2062042	.9785090	6
55	.1722156	.9850593	.1893811	.9819037	.2064888	.9784490	5
56	.1725022	.9850091	.1896667	.9818485	.2067734	.9783889	4
57	.1727887	.9849589	.1899523	.9817933	.2070580	.9783287	3
58	.1730752	.9849086	.1902379	.9817380	.2073426	.9782684	2
59	.1733617	.9848582	.1905234	.9816826	.2076272	.9782080	1
60	0.1736482	0.9848078	0.1908090	0.9816272	0.2079117	0.9781476	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	80°		79°		78°		

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	12° Sine	12° Cosine	13° Sine	13° Cosine	14° Sine	14° Cosine	′
0	0.2079117	0.9781476	0.2249511	0.9743701	0.2419219	0.9702957	60
1	.2081962	.9780871	.2252345	.9743046	.2422041	.9702253	59
2	.2084807	.9780265	.2255179	.9742390	.2424863	.9701548	58
3	.2087652	.9779658	.2258013	.9741734	.2427685	.9700842	57
4	.2090497	.9779050	.2260846	.9741077	.2430507	.9700136	56
5	.2093341	.9778442	.2263680	.9740419	.2433329	.9699428	55
6	.2096186	.9777832	.2266513	.9739760	.2436150	.9698720	54
7	.2099030	.9777222	.2269346	.9739100	.2438971	.9698011	53
8	.2101874	.9776611	.2272179	.9738439	.2441792	.9697301	52
9	.2104718	.9775999	.2275012	.9737778	.2444613	.9696591	51
10	0.2107561	0.9775387	0.2277844	0.9737116	0.2447433	0.9695879	50
11	.2110405	.9774773	.2280677	.9736453	.2450254	.9695167	49
12	2113248	.9774159	.2283509	.9735789	.2453074	.9694453	48
13	.2116091	.9773544	.2286341	.9735124	.2455894	.9693740	47
14	.2118934	.9772928	.2289172	.9734459	.2458713	.9693025	46
15	.2121777	.9772311	.2292004	.9733793	.2461533	.9692309	45
16	.2124619	.9771693	.2294835	.9733125	.2464352	.9691593	44
17	.2127462	.9771075	.2297666	.9732458	.2467171	.9690875	43
18	.2130304	.9770456	.2300497	.9731789	.2469990	.9690157	42
19	.2133146	.9769836	.2303328	.9731119	.2472809	.9689438	41
20	0.2135988	0.9769215	0.2306159	0.9730449	0.2475627	0.9688719	40
21	.2138829	.9768593	.2308989	.9729777	.2478445	.9687998	39
22	.2141671	.9767970	.2311819	.9729105	.2481263	.9687277	38
23	.2144512	.9767347	.2314649	.9728432	.2484081	.9686555	37
24	.2147353	.9766723	.2317479	.9727759	.2486899	.9685832	36
25	.2150194	.9766098	.2320309	.9727084	.2489716	.9685108	35
26	.2153035	.9765472	.2323138	.9726409	.2492533	.9684383	34
27	.2155876	.9764845	.2325967	.9725733	.2495350	.9683658	33
28	.2158716	.9764218	.2328796	.9725056	.2498167	.9682931	32
29	.2161556	.9763589	.2331625	.9724378	.2500984	.9682204	31
30	0.2164396	0.9762960	0.2334454	0.9723699	0.2503800	0.9681476	30
31	.2167236	.9762330	.2337282	.9723020	.2506616	.9680748	29
32	.2170076	.9761699	.2340110	.9722339	.2509432	.9680018	28
33	.2172915	.9761068	.2342938	.9721658	.2512248	.9679288	27
34	.2175754	.9760435	.2345766	.9720976	.2515063	.9678557	26
35	.2178593	.9759802	.2348594	.9720294	.2517879	.9677825	25
36	.2181432	.9759168	.2351421	.9719610	.2520694	.9677092	24
37	.2184271	.9758533	.2354248	.9718926	.2523508	.9676358	23
38	.2187110	.9757897	.2357075	.9718240	.2526323	.9675624	22
39	.2189948	.9757260	.2359902	.9717554	.2529137	.9674888	21
40	0.2192786	0.9756623	0.2362729	0.9716867	0.2531952	0.9674152	20
41	.2195624	.9755985	.2365555	.9716180	.2534766	.9673415	19
42	.2198462	.9755345	.2368381	.9715491	.2537579	.9672678	18
43	.2201300	.9754706	.2371207	.9714802	.2540393	.9671939	17
44	.2204137	.9754065	.2374033	.9714112	.2543206	.9671200	16
45	.2206974	.9753423	.2376859	.9713421	.2546019	.9670459	15
46	.2209811	.9752781	.2379684	.9712729	.2548832	.9669718	14
47	.2212648	.9752138	.2382510	.9712036	.2551645	.9668977	13
48	.2215485	.9751494	.2385335	.9711343	.2554458	.9668234	12
49	.2218321	.9750849	.2388159	.9710649	.2557270	.9667490	11
50	0.2221158	0.9750203	0.2390984	0.9709953	0.2560082	0.9666746	10
51	.2223994	.9749556	.2393808	.9709258	.2562894	.9666001	9
52	.2226830	.9748909	.2396633	.9708561	.2565705	.9665255	8
53	.2229666	.9748261	.2399457	.9707863	.2568517	.9664508	7
54	.2232501	.9747612	.2402280	.9707165	.2571328	.9663761	6
55	.2235337	.9746962	.2405104	.9706466	.2574139	.9663012	5
56	.2238172	.9746311	.2407927	.9705766	.2576950	.9662263	4
57	.2241007	.9745660	.2410751	.9705065	.2579760	.9661513	3
58	.2243842	.9745008	.2413574	.9704363	.2582570	.9660762	2
59	.2246676	.9744355	.2416396	.9703661	.2585381	.9660011	1
60	0.2249511	0.9743701	0.2419219	0.9702957	0.2588190	0.9659258	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	77°		76°		75°		

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	15°		16°		17°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	0.2588190	0.9659258	0.2756374	0.9612617	0.2923717	0.9563048	60
1	.2591000	.9658505	.2759170	.9611815	.2926499	.9562197	59
2	.2593810	.9657751	.2761965	.9611012	.2929280	.9561345	58
3	.2596619	.9656996	.2764761	.9610208	.2932061	.9560492	57
4	.2599428	.9656240	.2767556	.9609403	.2934842	.9559639	56
5	.2602237	.9655484	.2770352	.9608598	.2937623	.9558785	55
6	.2605045	.9654726	.2773147	.9607792	.2940403	.9557930	54
7	.2607853	.9653968	.2775941	.9606984	.2943183	.9557074	53
8	.2610662	.9653209	.2778736	.9606177	.2945963	.9556218	52
9	.2613469	.9652449	.2781530	.9605368	.2948743	.9555361	51
10	0.2616277	0.9651689	0.2784324	0.9604558	0.2951522	0.9554502	50
11	.2619085	.9650927	.2787118	.9603748	.2954302	.9553643	49
12	.2621892	.9650165	.2789911	.9602937	.2957081	.9552784	48
13	.2624699	.9649402	.2792704	.9602125	.2959859	.9551923	47
14	.2627506	.9648638	.2795497	.9601312	.2962638	.9551062	46
15	.2630312	.9647873	.2798290	.9600499	.2965416	.9550199	45
16	.2633118	.9647108	.2801083	.9599684	.2968194	.9549336	44
17	.2635925	.9646341	.2803875	.9598869	.2970971	.9548473	43
18	.2638730	.9645574	.2806667	.9598053	.2973749	.9547608	42
19	.2641536	.9644806	.2809459	.9597236	.2976526	.9546743	41
20	0.2644342	0.9644037	0.2812251	0.9596418	0.2979303	0.9545876	40
21	.2647147	.9643268	.2815042	.9595600	.2982079	.9545009	39
22	.2649952	.9642497	.2817833	.9594781	.2984856	.9544141	38
23	.2652757	.9641726	.2820624	.9593961	.2987632	.9543273	37
24	.2655561	.9640954	.2823415	.9593140	.2990408	.9542403	36
25	.2658366	.9640181	.2826205	.9592318	.2993184	.9541533	35
26	.2661170	.9639407	.2828995	.9591496	.2995959	.9540662	34
27	.2663973	.9638633	.2831785	.9590672	.2998734	.9539790	33
28	.2666777	.9637858	.2834575	.9589848	.3001509	.9538917	32
29	.2669581	.9637081	.2837364	.9589023	.3004284	.9538044	31
30	0.2672384	0.9636305	0.2840153	0.9588197	0.3007058	0.9537170	30
31	.2675187	.9635527	.2842942	.9587371	.3009832	.9536294	29
32	.2677989	.9634748	.2845731	.9586543	.3012606	.9535418	28
33	.2680792	.9633969	.2848520	.9585715	.3015380	.9534542	27
34	.2683594	.9633189	.2851308	.9584886	.3018153	.9533664	26
35	.2686396	.9632408	.2854096	.9584056	.3020926	.9532786	25
36	.2689198	.9631626	.2856884	.9583226	.3023699	.9531907	24
37	.2692000	.9630843	.2859671	.9582394	.3026471	.9531027	23
38	.2694801	.9630060	.2862458	.9581562	.3029244	.9530146	22
39	.2697602	.9629275	.2865246	.9580729	.3032016	.9529264	21
40	0.2700403	0.9628490	0.2868032	0.9579895	0.3034788	0.9528382	20
41	.2703204	.9627704	.2870819	.9579060	.3037559	.9527499	19
42	.2706004	.9626917	.2873605	.9578225	.3040331	.9526615	18
43	.2708805	.9626130	.2876391	.9577389	.3043102	.9525730	17
44	.2711605	.9625342	.2879177	.9576552	.3045872	.9524844	16
45	.2714404	.9624552	.2881963	.9575714	.3048643	.9523958	15
46	.2717204	.9623762	.2884748	.9574875	.3051413	.9523071	14
47	.2720003	.9622972	.2887533	.9574035	.3054183	.9522183	13
48	.2722802	.9622180	.2890318	.9573195	.3056953	.9521294	12
49	.2725601	.9621387	.2893103	.9572354	.3059723	.9520404	11
50	0.2728400	0.9620594	0.2895887	0.9571512	0.3062492	0.9519514	10
51	.2731198	.9619800	.2898671	.9570669	.3065261	.9518623	9
52	.2733997	.9619005	.2901455	.9569825	.3068030	.9517731	8
53	.2736794	.9618210	.2904239	.9568981	.3070798	.9516838	7
54	.2739592	.9617413	.2907022	.9568136	.3073566	.9515944	6
55	.2742390	.9616616	.2909805	.9567290	.3076334	.9515050	5
56	.2745187	.9615818	.2912588	.9566443	.3079102	.9514154	4
57	.2747984	.9615019	.2915371	.9565595	.3081869	.9513258	3
58	.2750781	.9614219	.2918153	.9564747	.3084636	.9512361	2
59	.2753577	.9613418	.2920935	.9563898	.3087403	.9511464	1
60	0.2756374	0.9612617	0.2923717	0.9563048	0.3090170	0.9510565	0
	Cosine	Sine	Cosine	Sine	Cosine	Sine	
′	74°		73°		72°		′

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	18° Sine	18° Cosine	19° Sine	19° Cosine	20° Sine	20° Cosine	′
0	0.3090170	0.9510565	0.3255682	0.9455186	0.3420201	0.9396926	60
1	.3092936	.9509666	.3258432	.9454238	.3422935	.9395931	59
2	.3095702	.9508766	.3261182	.9453290	.3425668	.9394935	58
3	.3098468	.9507865	.3263932	.9452341	.3428400	.9393938	57
4	.3101234	.9506963	.3266681	.9451391	.3431133	.9392940	56
5	.3103999	.9506061	.3269430	.9450441	.3433865	.9391942	55
6	.3106764	.9505157	.3272179	.9449489	.3436597	.9390943	54
7	.3109529	.9504253	.3274928	.9448537	.3439329	.9389942	53
8	.3112294	.9503348	.3277676	.9447584	.3442060	.9388942	52
9	.3115058	.9502443	.3280424	.9446630	.3444791	.9387940	51
10	0.3117822	0.9501536	0.3283172	0.9445675	0.3447521	0.9386938	50
11	.3120586	.9500629	.3285919	.9444720	.3450252	.9385934	49
12	.3123349	.9499721	.3288666	.9443764	.3452982	.9384930	48
13	.3126112	.9498812	.3291413	.9442807	.3455712	.9383925	47
14	.3128875	.9497902	.3294160	.9441849	.3458441	.9382920	46
15	.3131638	.9496991	.3296906	.9440890	.3461171	.9381913	45
16	.3134400	.9496080	.3299653	.9439931	.3463900	.9380906	44
17	.3137163	.9495168	.3302398	.9438971	.3466628	.9379898	43
18	.3139925	.9494255	.3305144	.9438010	.3469357	.9378889	42
19	.3142686	.9493341	.3307889	.9437048	.3472085	.9377880	41
20	0.3145448	0.9492426	0.3310634	0.9436085	0.3474812	0.9376869	40
21	.3148209	.9491511	.3313379	.9435122	.3477540	.9375858	39
22	.3150969	.9490595	.3316123	.9434157	.3480267	.9374846	38
23	.3153730	.9489678	.3318867	.9433192	.3482994	.9373833	37
24	.3156490	.9488760	.3321611	.9432227	.3485720	.9372820	36
25	.3159250	.9487842	.3324355	.9431260	.3488447	.9371806	35
26	.3162010	.9486922	.3327098	.9430293	.3491173	.9370790	34
27	.3164770	.9486002	.3329841	.9429324	.3493898	.9369774	33
28	.3167529	.9485081	.3332584	.9428355	.3496624	.9368758	32
29	.3170288	.9484159	.3335326	.9427386	.3499349	.9367740	31
30	0.3173047	0.9483237	0.3338069	0.9426415	0.3502074	0.9366722	30
31	.3175805	.9482313	.3340810	.9425444	.3504798	.9365703	29
32	.3178563	.9481389	.3343552	.9424471	.3507523	.9364683	28
33	.3181321	.9480464	.3346293	.9423498	.3510246	.9363662	27
34	.3184079	.9479538	.3349034	.9422525	.3512970	.9362641	26
35	.3186836	.9478612	.3351775	.9421550	.3515693	.9361618	25
36	.3189593	.9477684	.3354516	.9420575	.3518416	.9360595	24
37	.3192350	.9476756	.3357256	.9419598	.3521139	.9359571	23
38	.3195106	.9475827	.3359996	.9418621	.3523862	.9358547	22
39	.3197863	.9474897	.3362735	.9417644	.3526584	.9357521	21
40	0.3200619	0.9473966	0.3365475	0.9416665	0.3529306	0.9356495	20
41	.3203374	.9473035	.3368214	.9415686	.3532027	.9355468	19
42	.3206130	.9472103	.3370953	.9414705	.3534748	.9354440	18
43	.3208885	.9471170	.3373691	.9413724	.3537469	.9353412	17
44	.3211640	.9470236	.3376429	.9412743	.3540190	.9352382	16
45	.3214395	.9469301	.3379167	.9411760	.3542910	.9351352	15
46	.3217149	.9468366	.3381905	.9410777	.3545630	.9350321	14
47	.3219903	.9467430	.3384642	.9409793	.3548350	.9349289	13
48	.3222657	.9466493	.3387379	.9408808	.3551070	.9348257	12
49	.3225411	.9465555	.3390116	.9407822	.3553789	.9347223	11
50	0.3228164	0.9464616	0.3392852	0.9406835	0.3556508	0.9346189	10
51	.3230917	.9463677	.3395589	.9405848	.3559226	.9345154	9
52	.3233670	.9462736	.3398325	.9404860	.3561944	.9344119	8
53	.3236422	.9461795	.3401060	.9403871	.3564662	.9343082	7
54	.3239174	.9460854	.3403796	.9402881	.3567380	.9342045	6
55	.3241926	.9459911	.3406531	.9401891	.3570097	.9341007	5
56	.3244678	.9458968	.3409265	.9400899	.3572814	.9339968	4
57	.3247429	.9458023	.3412000	.9399907	.3575531	.9338928	3
58	.3250180	.9457078	.3414734	.9398914	.3578248	.9337888	2
59	.3252931	.9456132	.3417468	.9397921	.3580964	.9336846	1
60	0.3255682	0.9455186	0.3420201	0.9396926	0.3583679	0.9335804	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	71°		70°		69°		

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	21° Sine	21° Cosine	22° Sine	22° Cosine	23° Sine	23° Cosine	′
0	0.3583679	0.9335804	0.3746066	0.9271839	0.3907311	0.9205049	60
1	.3586395	.9334761	.3748763	.9270748	.3909989	.9203912	59
2	.3589110	.9333718	.3751459	.9269658	.3912666	.9202774	58
3	.3591825	.9332673	.3754156	.9268566	.3915343	.9201635	57
4	.3594540	.9331628	.3756852	.9267474	.3918019	.9200496	56
5	.3597254	.9330582	.3759547	.9266380	.3920695	.9199356	55
6	.3599968	.9329535	.3762243	.9265286	.3923371	.9198215	54
7	.3602682	.9328488	.3764938	.9264192	.3926047	.9197073	53
8	.3605395	.9327439	.3767632	.9263096	.3928722	.9195931	52
9	.3608108	.9326390	.3770327	.9262000	.3931397	.9194788	51
10	0.3610821	0.9325340	0.3773021	0.9260902	0.3934071	0.9193644	50
11	.3613534	.9324290	.3775714	.9259805	.3936745	.9192499	49
12	.3616246	.9323238	.3778408	.9258706	.3939419	.9191353	48
13	.3618958	.9322186	.3781101	.9257606	.3942093	.9190207	47
14	.3621669	.9321133	.3783794	.9256506	.3944766	.9189060	46
15	.3624380	.9320079	.3786486	.9255405	.3947439	.9187912	45
16	.3627091	.9319024	.3789178	.9254303	.3950111	.9186763	44
17	.3629802	.9317969	.3791870	.9253201	.3952783	.9185614	43
18	.3632512	.9316912	.3794562	.9252097	.3955455	.9184464	42
19	.3635222	.9315855	.3797253	.9250993	.3958127	.9183313	41
20	0.3637932	0.9314797	0.3799944	0.9249888	0.3960798	0.9182161	40
21	.3640641	.9313739	.3802634	.9248782	.3963468	.9181009	39
22	.3643351	.9312679	.3805324	.9247676	.3966139	.9179855	38
23	.3646059	.9311619	.3808014	.9246568	.3968809	.9178701	37
24	.3648768	.9310558	.3810704	.9245460	.3971479	.9177546	36
25	.3651476	.9309496	.3813393	.9244351	.3974148	.9176391	35
26	.3654184	.9308434	.3816082	.9243242	.3976818	.9175234	34
27	.3656891	.9307370	.3818770	.9242131	.3979486	.9174077	33
28	.3659599	.9306306	.3821459	.9241020	.3982155	.9172919	32
29	.3662306	.9305241	.3824147	.9239908	.3984823	.9171760	31
30	0.3665012	0.9304176	0.3826834	0.9238795	0.3987491	0.9170601	30
31	.3667719	.9303109	.3829522	.9237682	.3990158	.9169440	29
32	.3670425	.9302042	.3832209	.9236567	.3992825	.9168279	28
33	.3673130	.9300974	.3834895	.9235452	.3995492	.9167118	27
34	.3675836	.9299905	.3837582	.9234336	.3998158	.9165955	26
35	.3678541	.9298835	.3840268	.9233220	.4000825	.9164791	25
36	.3681246	.9297765	.3842953	.9232102	.4003490	.9163627	24
37	.3683950	.9296694	.3845639	.9230984	.4006156	.9162462	23
38	.3686654	.9295622	.3848324	.9229865	.4008821	.9161297	22
39	.3689358	.9204549	.3851008	.9228745	.4011486	.9160130	21
40	0.3692061	0.9293475	0.3853693	0.9227624	0.4014150	0.9158963	20
41	.3694765	.9292401	.3856377	.9226503	.4016814	.9157795	19
42	.3697468	.9291326	.3859060	.9225381	.4019478	.9156626	18
43	.3700170	.9290250	.3861744	.9224258	.4022141	.9155456	17
44	.3702872	.9289173	.3864427	.9223134	.4024804	.9154286	16
45	.3705574	.9288096	.3867110	.9222010	.4027467	.9153115	15
46	.3708276	.9287017	.3869792	.9220884	.4030129	.9151943	14
47	.3710977	.9285938	.3872474	.9219758	.4032791	.9150770	13
48	.3713678	.9284858	.3875156	.9218632	.4035453	.9149597	12
49	.3716379	.9283778	.3877837	.9217504	.4038114	.9148422	11
50	0.3719079	0.9282696	0.3880518	0.9216375	0.4040775	0.9147247	10
51	.3721780	.9281614	.3883199	.9215246	.4043436	.9146072	9
52	.3724479	.9280531	.3885880	.9214116	.4046096	.9144895	8
53	.3727179	.9279447	.3888560	.9212986	.4048756	.9143718	7
54	.3729878	.9278363	.3891240	.9211854	.4051416	.9142540	6
55	.3732577	.9277277	.3893919	.9210722	.4054075	.9141361	5
56	.3735275	.9276191	.3896598	.9209589	.4056734	.9140181	4
57	.3737973	.9275104	.3899277	.9208455	.4059393	.9139001	3
58	.3740671	.9274016	.3901955	.9207320	.4062051	.9137819	2
59	.3743369	.9272928	.3904633	.9206185	.4064709	.9136637	1
60	0.3746066	0.9271839	0.3907311	0.9205049	0.4067366	0.9135455	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	68°		67°		66°		

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	24°		25°		26°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	0.4067366	0.9135455	0.4226183	0.9063078	0.4383711	0.8987940	60
1	.4070024	.9134271	.4228819	.9061848	.4386326	.8986665	59
2	.4072681	.9133087	.4231455	.9060618	.4388940	.8985389	58
3	.4075337	.9131902	.4234090	.9059386	.4391553	.8984112	57
4	.4077993	.9130716	.4236725	.9058154	.4394166	.8982834	56
5	.4080649	.9129529	.4239360	.9056922	.4396779	.8981555	55
6	.4083305	.9128342	.4241994	.9055688	.4399392	.8980276	54
7	.4085960	.9127154	.4244628	.9054454	.4402004	.8978996	53
8	.4088615	.9125965	.4247262	.9053219	.4404615	.8977715	52
9	.4091269	.9124775	.4249895	.9051983	.4407227	.8976433	51
10	0.4093923	0.9123584	0.4252528	0.9050746	0.4409838	0.8975151	50
11	.4096577	.9122393	.4255161	.9049509	.4412448	.8973868	49
12	.4099230	.9121201	.4257793	.9048271	.4415059	.8972584	48
13	.4101883	.9120008	.4260425	.9047032	.4417668	.8971299	47
14	.4104536	.9118815	.4263056	.9045792	.4420278	.8970014	46
15	.4107189	.9117620	.4265687	.9044551	.4422887	.8968727	45
16	.4109841	.9116425	.4268318	.9043310	.4425496	.8967440	44
17	.4112492	.9115229	.4270949	.9042068	.4428104	.8966153	43
18	.4115144	.9114033	.4273579	.9040825	.4430712	.8964864	42
19	.4117795	.9112835	.4276208	.9039582	.4433319	.8963575	41
20	0.4120445	0.9111637	0.4278838	0.9038338	0.4435927	0.8962285	40
21	.4123096	.9110438	.4281467	.9037093	.4438534	.8960994	39
22	.4125745	.9109238	.4284095	.9035847	.4441140	.8959703	38
23	.4128395	.9108038	.4286723	.9034600	.4443746	.8958411	37
24	.4131044	.9106837	.4289351	.9033353	.4446352	.8957118	36
25	.4133693	.9105635	.4291979	.9032105	.4448957	.8955824	35
26	.4136342	.9104432	.4294606	.9030856	.4451562	.8954529	34
27	.4138990	.9103228	.4297233	.9029606	.4454167	.8953234	33
28	.4141638	.9102024	.4299859	.9028356	.4456771	.8951938	32
29	.4144285	.9100819	.4302485	.9027105	.4459375	.8950641	31
30	0.4146932	0.9099613	0.4305111	0.9025853	0.4461978	0.8949344	30
31	.4149579	.9098406	.4307736	.9024600	.4464581	.8948045	29
32	.4152226	.9097199	.4310361	.9023347	.4467184	.8946746	28
33	.4154872	.9095990	.4312986	.9022092	.4469786	.8945446	27
34	.4157517	.9094781	.4315610	.9020838	.4472388	.8944146	26
35	.4160163	.9093572	.4318234	.9019582	.4474990	.8942844	25
36	.4162808	.9092361	.4320857	.9018325	.4477591	.8941542	24
37	.4165453	.9091150	.4323481	.9017068	.4480192	.8940240	23
38	.4168097	.9089938	.4326103	.9015810	.4482792	.8938936	22
39	.4170741	.9088725	.4328726	.9014551	.4485392	.8937632	21
40	0.4173385	0.9087511	0.4331348	0.9013292	0.4487992	0.8936326	20
41	.4176028	.9086297	.4333970	.9012031	.4490591	.8935021	19
42	.4178671	.9085082	.4336591	.9010770	.4493190	.8933714	18
43	.4181313	.9083866	.4339212	.9009508	.4495789	.8932406	17
44	.4183956	.9082649	.4341832	.9008246	.4498387	.8931098	16
45	.4186597	.9081432	.4344453	.9006982	.4500984	.8929789	15
46	.4189239	.9080214	.4347072	.9005718	.4503582	.8928480	14
47	.4191880	.9078995	.4349692	.9004453	.4506179	.8927169	13
48	.4194521	.9077775	.4352311	.9003188	.4508775	.8925858	12
49	.4197161	.9076554	.4354930	.9001921	.4511372	.8924546	11
50	0.4199801	0.9075333	0.4357548	0.9000654	0.4513967	0.8923234	10
51	.4202441	.9074111	.4360166	.8999386	.4516563	.8921920	9
52	.4205080	.9072888	.4362784	.8998117	.4519158	.8920606	8
53	.4207719	.9071665	.4365401	.8996848	.4521753	.8919291	7
54	.4210358	.9070440	.4368018	.8995578	.4524347	.8917975	6
55	.4212996	.9069215	.4370634	.8994307	.4526941	.8916659	5
56	.4215634	.9067989	.4373251	.8993035	.4529535	.8915342	4
57	.4218272	.9066762	.4375866	.8991763	.4532128	.8914024	3
58	.4220909	.9065535	.4378482	.8990489	.4534721	.8912705	2
59	.4223546	.9064307	.4381097	.8989215	.4537313	.8911385	1
60	0.4226183	0.9063078	0.4383711	0.8987940	0.4539905	0.8910065	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	65°		64°		63°		

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	27°		28°		29°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	0.4539905	0.8910065	0.4694716	0.8829476	0.4848096	0.8746197	60
1	.4542497	.8908744	.4697284	.8828110	.4850640	.8744786	59
2	.4545088	.8907423	.4699852	.8826743	.4853184	.8743375	58
3	.4547679	.8906100	.4702419	.8825376	.4855727	.8741963	57
4	.4550269	.8904777	.4704986	.8824007	.4858270	.8740550	56
5	.4552859	.8903453	.4707553	.8822638	.4860812	.8739137	55
6	.4555449	.8902128	.4710119	.8821269	.4863354	.8737722	54
7	.4558038	.8900803	.4712685	.8819898	.4865895	.8736307	53
8	.4560627	.8899476	.4715250	.8818527	.4868436	.8734891	52
9	.4563216	.8898149	.4717815	.8817155	.4870977	.8733475	51
10	0.4565804	0.8896822	0.4720380	0.8815782	0.4873517	0.8732058	50
11	.4568392	.8895493	.4722944	.8814409	.4876057	.8730640	49
12	.4570979	.8894164	.4725508	.8813035	.4878597	.8729221	48
13	.4573566	.8892834	.4728071	.8811660	.4881136	.8727801	47
14	.4576153	.8891503	.4730634	.8810284	.4883674	.8726381	46
15	.4578739	.8890171	.4733197	.8808907	.4886212	.8724960	45
16	.4581325	.8888839	.4735759	.8807530	.4888750	.8723538	44
17	.4583910	.8887506	.4738321	.8806152	.4891288	.8722116	43
18	.4586496	.8886172	.4740882	.8804774	.4893825	.8720693	42
19	.4589080	.8884838	.4743443	.8803394	.4896361	.8719269	41
20	0.4591665	0.8883503	0.4746004	0.8802014	0.4898897	0.8717844	40
21	.4594248	.8882166	.4748564	.8800633	.4901433	.8716419	39
22	.4596832	.8880830	.4751124	.8799251	.4903968	.8714993	38
23	.4599415	.8879492	.4753683	.8797869	.4906503	.8713566	37
24	.4601998	.8878154	.4756242	.8796486	.4909038	.8712138	36
25	.4604580	.8876815	.4758801	.8795102	.4911572	.8710710	35
26	.4607162	.8875475	.4761359	.8793717	.4914105	.8709281	34
27	.4609744	.8874134	.4763917	.8792332	.4916638	.8707851	33
28	.4612325	.8872793	.4766474	.8790946	.4919171	.8706420	32
29	.4614906	.8871451	.4769031	.8789559	.4921704	.8704989	31
30	0.4617486	0.8870108	0.4771588	0.8788171	0.4924236	0.8703557	30
31	.4620066	.8868765	.4774144	.8786783	.4926767	.8702124	29
32	.4622646	.8867420	.4776700	.8785394	.4929298	.8700691	28
33	.4625225	.8866075	.4779255	.8784004	.4931829	.8699256	27
34	.4627804	.8864730	.4781810	.8782613	.4934359	.8697821	26
35	.4630382	.8863383	.4784364	.8781222	.4936889	.8696386	25
36	.4632960	.8862036	.4786919	.8779830	.4939419	.8694949	24
37	.4635538	.8860688	.4789472	.8778437	.4941948	.8693512	23
38	.4638115	.8859339	.4792026	.8777043	.4944476	.8692074	22
39	.4640692	.8857989	.4794579	.8775649	.4947005	.8690636	21
40	0.4643269	0.8856639	0.4797131	0.8774254	0.4949532	0.8689196	20
41	.4645845	.8855288	.4799683	.8772858	.4952060	.8687756	19
42	.4648420	.8853936	.4802235	.8771462	.4954587	.8686315	18
43	.4650996	.8852584	.4804786	.8770064	.4957113	.8684874	17
44	.4653571	.8851230	.4807337	.8768666	.4959639	.8683431	16
45	.4656145	.8849876	.4809888	.8767268	.4962165	.8681988	15
46	.4658719	.8848522	.4812438	.8765868	.4964690	.8680544	14
47	.4661293	.8847166	.4814987	.8764468	.4967215	.8679100	13
48	.4663866	.8845810	.4817537	.8763067	.4969740	.8677655	12
49	.4666439	.8844453	.4820086	.8761665	.4972264	.8676209	11
50	0.4669012	0.8843095	0.4822634	0.8760263	0.4974787	0.8674762	10
51	.4671584	.8841736	.4825182	.8758859	.4977310	.8673314	9
52	.4674156	.8840377	.4827730	.8757455	.4979833	.8671866	8
53	.4676727	.8839017	.4830277	.8756051	.4982355	.8670417	7
54	.4679298	.8837656	.4832824	.8754645	.4984877	.8668967	6
55	.4681869	.8836295	.4835370	.8753239	.4987399	.8667517	5
56	.4684439	.8834933	.4837916	.8751832	.4989920	.8666066	4
57	.4687009	.8833569	.4840462	.8750425	.4992441	.8664614	3
58	.4689578	.8832206	.4843007	.8749016	.4994961	.8663161	2
59	.4692147	.8830841	.4845552	.8747607	.4997481	.8661708	1
60	0.4694716	0.8829476	0.4848096	0.8746197	0.5000000	0.8660254	0
	Cosine	Sine	Cosine	Sine	Cosine	Sine	
′	62°		61°		60°		′

TABLE XIX (*Continued*)

Natural Sines and Cosines

'	30° Sine	30° Cosine	31° Sine	31° Cosine	32° Sine	32° Cosine	'
0	0.5000000	0.8660254	0.5150381	0.8571673	0.5299193	0.8480481	60
1	.5002519	.8658799	.5152874	.8570174	.5301659	.8478939	59
2	.5005037	.8657344	.5155367	.8568675	.5304125	.8477397	58
3	.5007556	.8655887	.5157859	.8567175	.5306591	.8475853	57
4	.5010073	.8654430	.5160351	.8565674	.5309057	.8474309	56
5	.5012591	.8652973	.5162842	.8564173	.5311521	.8472765	55
6	.5015107	.8651514	.5165333	.8562671	.5313986	.8471219	54
7	.5017624	.8650055	.5167824	.8561168	.5316450	.8469673	53
8	.5020140	.8648595	.5170314	.8559664	.5318913	.8468126	52
9	.5022655	.8647134	.5172804	.8558160	.5321376	.8466579	51
10	0.5025170	0.8645673	0.5175293	0.8556655	0.5323839	0.8465030	50
11	.5027685	.8644211	.5177782	.8555149	.5326301	.8463481	49
12	.5030199	.8642748	.5180270	.8553643	.5328763	.8461932	48
13	.5032713	.8641284	.5182758	.8552135	.5331224	.8460381	47
14	.5035227	.8639820	.5185246	.8550627	.5333685	.8458830	46
15	.5037740	.8638355	.5187733	.8549119	.5336145	.8457278	45
16	.5040252	.8636889	.5190219	.8547609	.5338605	.8455726	44
17	.5042765	.8635423	.5192705	.8546099	.5341065	.8454172	43
18	.5045276	.8633956	.5195191	.8544588	.5343523	.8452618	42
19	.5047788	.8632488	.5197676	.8543077	.5345982	.8451064	41
20	0.5050298	0.8631019	0.5200161	0.8541564	0.5348440	0.8449508	40
21	.5052809	.8629549	.5202646	.8540051	.5350898	.8447952	39
22	.5055319	.8628079	.5205130	.8538538	.5353355	.8446395	38
23	.5057828	.8626608	.5207613	.8537023	.5355812	.8444838	37
24	.5060338	.8625137	.5210096	.8535508	.5358268	.8443279	36
25	.5062846	.8623664	.5212579	.8533992	.5360724	.8441720	35
26	.5065355	.8622191	.5215061	.8532475	.5363179	.8440161	34
27	.5067863	.8620717	.5217543	.8530958	.5365634	.8438600	33
28	.5070370	.8619243	.5220024	.8529440	.5368089	.8437039	32
29	.5072877	.8617768	.5222505	.8527921	.5370543	.8435477	31
30	0.5075384	0.8616292	0.5224986	0.8526402	0.5372996	0.8433914	30
31	.5077890	.8614815	.5227466	.8524881	.5375449	.8432351	29
32	.5080396	.8613337	.5229945	.8523360	.5377902	.8430787	28
33	.5082901	.8611859	.5232424	.8521839	.5380354	.8429222	27
34	.5085406	.8610380	.5234903	.8520316	.5382806	.8427657	26
35	.5087910	.8608901	.5237381	.8518793	.5385257	.8426091	25
36	.5090414	.8607420	.5239859	.8517269	.5387708	.8424524	24
37	.5092918	.8605939	.5242336	.8515745	.5390158	.8422956	23
38	.5095421	.8604457	.5244813	.8514219	.5392608	.8421388	22
39	.5097924	.8602975	.5247290	.8512693	.5395058	.8419818	21
40	0.5100426	0.8601491	0.5249766	0.8511167	0.5397507	0.8418249	20
41	.5102928	.8600007	.5252241	.8509639	.5399955	.8416679	19
42	.5105429	.8598523	.5254717	.8508111	.5402403	.8415108	18
43	.5107930	.8597037	.5257191	.8506582	.5404851	.8413536	17
44	.5110431	.8595551	.5259665	.8505053	.5407298	.8411963	16
45	.5112931	.8594064	.5262139	.8503522	.5409745	.8410390	15
46	.5115431	.8592576	.5264613	.8501991	.5412191	.8408816	14
47	.5117930	.8591088	.5267085	.8500459	.5414637	.8407241	13
48	.5120429	.8589599	.5269558	.8498927	.5417082	.8405666	12
49	.5122927	.8588109	.5272030	.8497394	.5419527	.8404090	11
50	0.5125425	0.8586619	0.5274502	0.8495860	0.5421971	0.8402513	10
51	.5127923	.8585127	.5276973	.8494325	.5424415	.8400936	9
52	.5130420	.8583635	.5279443	.8492790	.5426859	.8399357	8
53	.5132916	.8582143	.5281914	.8491254	.5429302	.8397778	7
54	.5135413	.8580649	.5284383	.8489717	.5431744	.8396199	6
55	.5137908	.8579155	.5286853	.8488179	.5434187	.8394618	5
56	.5140404	.8577660	.5289322	.8486641	.5436628	.8393037	4
57	.5142899	.8576164	.5291790	.8485102	.5439069	.8391455	3
58	.5145393	.8574668	.5294258	.8483562	.5441510	.8389873	2
59	.5147887	.8573171	.5296726	.8482022	.5443951	.8388290	1
60	0.5150381	0.8571673	0.5299193	0.8480481	0.5446390	0.8386706	0
'	Cosine	Sine	Cosine	Sine	Cosine	Sine	'
	59°		58°		57°		

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	33° Sine	33° Cosine	34° Sine	34° Cosine	35° Sine	35° Cosine	′
0	0.5446390	0.8386706	0.5591929	0.8290376	0.5735764	0.8191520	60
1	.5448830	.8385121	.5594340	.8288749	.5738147	.8189852	59
2	.5451262	.8383536	.5596751	.8287121	.5740529	.8188182	58
3	.5453707	.8381950	.5599162	.8285493	.5742911	.8186512	57
4	.5456145	.8380363	.5601572	.8283864	.5745292	.8184841	56
5	.5458583	.8378775	.5603981	.8282234	.5747672	.8183169	55
6	.5461020	.8377187	.5606390	.8280603	.5750053	.8181497	54
7	.5463456	.8375598	.5608798	.8278972	.5752432	.8179824	53
8	.5465892	.8374009	.5611206	.8277340	.5754811	.8178151	52
9	.5468328	.8372418	.5613614	.8275708	.5757190	.8176476	51
10	0.5470763	0.8370827	0.5616021	0.8274074	0.5759568	0.8174801	50
11	.5473198	.8369236	.5618428	.8272440	.5761946	.8173125	49
12	.5475632	.8367643	.5620834	.8270806	.5764323	.8171449	48
13	.5478066	.8366050	.5623239	.8269170	.5766700	.8169772	47
14	.5480499	.8364456	.5625645	.8267534	.5769076	.8168094	46
15	.5482932	.8362862	.5628049	.8265897	.5771452	.8166416	45
16	.5485365	.8361266	.5630453	.8264260	.5773827	.8164736	44
17	.5487797	.8359670	.5632857	.8262622	.5776202	.8163056	43
18	.5490228	.8358074	.5635260	.8260983	.5778576	.8161376	42
19	.5492659	.8356476	.5637663	.8259343	.5780950	.8159695	41
20	0.5495090	0.8354878	0.5640066	0.8257703	0.5783323	0.8158013	40
21	.5497520	.8353279	.5642467	.8256062	.5785696	.8156330	39
22	.5499950	.8351680	.5644869	.8254420	.5788069	.8154647	38
23	.5502379	.8350080	.5647270	.8252778	.5790440	.8152963	37
24	.5504807	.8348479	.5649670	.8251135	.5792812	.8151278	36
25	.5507236	.8346877	.5652070	.8249491	.5795183	.8149593	35
26	.5509663	.8345275	.5654469	.8247847	.5797553	.8147906	34
27	.5512091	.8343672	.5656868	.8246202	.5799923	.8146220	33
28	.5514518	.8342068	.5659267	.8244556	.5802292	.8144532	32
29	.5516944	.8340463	.5661665	.8242909	.5804661	.8142844	31
30	0.5519370	0.8338858	0.5664062	0.8241262	0.5807030	0.8141155	30
31	.5521795	.8337252	.5666459	.8239614	.5809397	.8139466	29
32	.5524220	.8335646	.5668856	.8237965	.5811765	.8137775	28
33	.5526645	.8334038	.5671252	.8236316	.5814132	.8136084	27
34	.5529069	.8332430	.5673648	.8234666	.5816498	.8134393	26
35	.5531492	.8330822	.5676043	.8233015	.5818864	.8132701	25
36	.5533915	.8329212	.5678437	.8231364	.5821230	.8131008	24
37	.5536338	.8327602	.5680832	.8229712	.5823595	.8129314	23
38	.5538760	.8325991	.5683225	.8228059	.5825959	.8127620	22
39	.5541182	.8324380	.5685619	.8226405	.5828323	.8125925	21
40	0.5543603	0.8322768	0.5688011	0.8224751	0.5830687	0.8124229	20
41	.5546024	.8321155	.5690403	.8223096	.5833050	.8122532	19
42	.5548444	.8319541	.5692795	.8221440	.5835412	.8120835	18
43	.5550864	.8317927	.5695187	.8219784	.5837774	.8119137	17
44	.5553283	.8316312	.5697577	.8218127	.5840136	.8117439	16
45	.5555702	.8314696	.5699968	.8216469	.5842497	.8115740	15
46	.5558121	.8313080	.5702357	.8214811	.5844857	.8114040	14
47	.5560539	.8311463	.5704747	.8213152	.5847217	.8112339	13
48	.5562956	.8309845	.5707136	.8211492	.5849577	.8110638	12
49	.5565373	.8308226	.5709524	.8209832	.5851936	.8108936	11
50	0.5567790	0.8306607	0.5711912	0.8208170	0.5854294	0.8107234	10
51	.5570206	.8304987	.5714299	.8206509	.5856652	.8105530	9
52	.5572621	.8303366	.5716686	.8204846	.5859010	.8103826	8
53	.5575036	.8301745	.5719073	.8203183	.5861367	.8102122	7
54	.5577451	.8300123	.5721459	.8201519	.5863724	.8100416	6
55	.5579865	.8298500	.5723844	.8199854	.5866080	.8098710	5
56	.5582279	.8296877	.5726229	.8198189	.5868435	.8097004	4
57	.5584692	.8295252	.5728614	.8196523	.5870790	.8095296	3
58	.5587105	.8293628	.5730998	.8194856	.5873145	.8093588	2
59	.5589517	.8292002	.5733381	.8193189	.5875499	.8091879	1
60	0.5591929	0.8290376	0.5735764	0.8191520	0.5877853	0.8090170	0
′	Cosine 56°	Sine 56°	Cosine 55°	Sine 55°	Cosine 54°	Sine 54°	′

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	36° Sine	36° Cosine	37° Sine	37° Cosine	38° Sine	38° Cosine	′
0	0.5877853	0.8090170	0.6018150	0.7986355	0.6156615	0.7880108	60
1	.5880206	.8088460	.6020473	.7984604	.6158907	.7878316	59
2	.5882558	.8086749	.6022795	.7982853	.6161198	.7876524	58
3	.5884910	.8085037	.6025117	.7981100	.6163489	.7874732	57
4	.5887262	.8083325	.6027439	.7979347	.6165780	.7872939	56
5	.5889613	.8081612	.6029760	.7977594	.6168069	.7871145	55
6	.5891964	.8079899	.6032080	.7975839	.6170359	.7869350	54
7	.5894314	.8078185	.6034400	.7974084	.6172648	.7867555	53
8	.5896663	.8076470	.6036719	.7972329	.6174936	.7865759	52
9	.5899012	.8074754	.6039038	.7970572	.6177224	.7863963	51
10	0.5901361	0.8073038	0.6041356	0.7968815	0.6179511	0.7862165	50
11	.5903709	.8071321	.6043674	.7967058	.6181798	.7860367	49
12	.5906057	.8069603	.6045991	.7965299	.6184084	.7858569	48
13	.5908404	.8067885	.6048308	.7963540	.6186370	.7856770	47
14	.5910750	.8066166	.6050624	.7961780	.6188655	.7854970	46
15	.5913096	.8064446	.6052940	.7960020	.6190939	.7853169	45
16	.5915442	.8062726	.6055255	.7958259	.6193224	.7851368	44
17	.5917787	.8061005	.6057570	.7956497	.6195507	.7849566	43
18	.5920132	.8059283	.6059884	.7954735	.6197790	.7847764	42
19	.5922476	.8057560	.6062198	.7952972	.6200073	.7845961	41
20	0.5924819	0.8055837	0.6064511	0.7951208	0.6202355	0.7844157	40
21	.5927163	.8054113	.6066824	.7949444	.6204636	.7842352	39
22	.5929505	.8052389	.6069136	.7947678	.6206917	.7840547	38
23	.5931847	.8050664	.6071447	.7945913	.6209198	.7838741	37
24	.5934189	.8048938	.6073758	.7944146	.6211478	.7836935	36
25	.5936530	.8047211	.6076069	.7942379	.6213757	.7835127	35
26	.5938871	.8045484	.6078379	.7940611	.6216036	.7833320	34
27	.5941211	.8043756	.6080689	.7938843	.6218314	.7831511	33
28	.5943550	.8042028	.6082998	.7937074	.6220592	.7829702	32
29	.5945889	.8040299	.6085306	.7935304	.6222870	.7827892	31
30	0.5948228	0.8038569	0.6087614	0.7933533	0.6225146	0.7826082	30
31	.5950566	.8036838	.6089922	.7931762	.6227423	.7824270	29
32	.5952904	.8035107	.6092229	.7929990	.6229698	.7822459	28
33	.5955241	.8033375	.6094535	.7928218	.6231974	.7820646	27
34	.5957577	.8031642	.6096841	.7926445	.6234248	.7818833	26
35	.5959913	.8029909	.6099147	.7924671	.6236522	.7817019	25
36	.5962249	.8028175	.6101452	.7922896	.6238796	.7815205	24
37	.5964584	.8026440	.6103756	.7921121	.6241069	.7813390	23
38	.5966918	.8024705	.6106060	.7919345	.6243342	.7811574	22
39	.5969252	.8022969	.6108363	.7917569	.6245614	.7809757	21
40	0.5971586	0.8021232	0.6110666	0.7915792	0.6247885	0.7807940	20
41	.5973919	.8019495	.6112969	.7914014	.6250156	.7806123	19
42	.5976251	.8017756	.6115270	.7912235	.6252427	.7804304	18
43	.5978583	.8016018	.6117572	.7910456	.6254696	.7802485	17
44	.5980915	.8014278	.6119873	.7908676	.6256966	.7800665	16
45	.5983246	.8012538	.6122173	.7906896	.6259235	.7798845	15
46	.5985577	.8010797	.6124473	.7905115	.6261503	.7797024	14
47	.5987906	.8009056	.6126772	.7903333	.6263771	.7795202	13
48	.5990236	.8007314	.6129071	.7901550	.6266038	.7793380	12
49	.5992565	.8005571	.6131369	.7899767	.6268305	.7791557	11
50	0.5994893	0.8003827	0.6133666	0.7897983	0.6270571	0.7789733	10
51	.5997221	.8002083	.6135964	.7896198	.6272837	.7787909	9
52	.5999549	.8000338	.6138260	.7894413	.6275102	.7786084	8
53	.6001876	.7998593	.6140556	.7892627	.6277366	.7784258	7
54	.6004202	.7996847	.6142852	.7890841	.6279631	.7782431	6
55	.6006528	.7995100	.6145147	.7889054	.6281894	.7780604	5
56	.6008854	.7993352	.6147442	.7887266	.6284157	.7778777	4
57	.6011179	.7991604	.6149736	.7885477	.6286420	.7776949	3
58	.6013503	.7989855	.6152029	.7883688	.6288682	.7775120	2
59	.6015827	.7988105	.6154322	.7881898	.6290943	.7773290	1
60	0.6018150	0.7986355	0.6156615	0.7880108	0.6293204	0.7771460	0
′	Cosine 53°	Sine 53°	Cosine 52°	Sine 52°	Cosine 51°	Sine 51°	′

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	39°		40°		41°		′
	Sine	Cosine	Sine	Cosine	Sine	Cosine	
0	0.6293204	0.7771460	0.6427876	0.7660444	0.6560590	0.7547096	60
1	.6295464	.7769629	.6430104	.7658574	.6562785	.7545187	59
2	.6297724	.7767797	.6432332	.7656704	.6564980	.7543278	58
3	.6299983	.7765965	.6434559	.7654832	.6567174	.7541368	57
4	.6302242	.7764132	.6436785	.7652960	.6569367	.7539457	56
5	.6304500	.7762298	.6439011	.7651087	.6571560	.7537546	55
6	.6306758	.7760464	.6441236	.7649214	.6573752	.7535634	54
7	.6309015	.7758629	.6443461	.7647340	.6575944	.7533721	53
8	.6311272	.7756794	.6445685	.7645465	.6578135	.7531808	52
9	.6313528	.7754957	.6447909	.7643590	.6580326	.7529894	51
10	0.6315784	0.7753121	0.6450132	0.7641714	0.6582516	0.7527980	50
11	.6318039	.7751283	.6452355	.7639838	.6584706	.7526065	49
12	.6320293	.7749445	.6454577	.7637960	.6586895	.7524149	48
13	.6322547	.7747606	.6456798	.7636082	.6589083	.7522233	47
14	.6324800	.7745767	.6459019	.7634204	.6591271	.7520316	46
15	.6327053	.7743926	.6461240	.7632325	.6593458	.7518398	45
16	.6329306	.7742086	.6463460	.7630445	.6595645	.7516480	44
17	.6331557	.7740244	.6465679	.7628564	.6597831	.7514561	43
18	.6333809	.7738402	.6467898	.7626683	.6600017	.7512641	42
19	.6336059	.7736559	.6470116	.7624802	.6602202	.7510721	41
20	0.6338310	0.7734716	0.6472334	0.7622919	0.6604386	0.7508800	40
21	.6340559	.7732872	.6474551	.7621036	.6606570	.7506879	39
22	.6342808	.7731027	.6476767	.7619152	.6608754	.7504957	38
23	.6345057	.7729182	.6478984	.7617268	.6610936	.7503034	37
24	.6347305	.7727336	.6481199	.7615383	.6613119	.7501111	36
25	.6349553	.7725489	.6483414	.7613497	.6615300	.7499187	35
26	.6351800	.7723642	.6485628	.7611611	.6617482	.7497262	34
27	.6354046	.7721794	.6487842	.7609724	.6619662	.7495337	33
28	.6356292	.7719945	.6490056	.7607837	.6621842	.7493411	32
29	.6358537	.7718096	.6492268	.7605949	.6624022	.7491484	31
30	0.6360782	0.7716246	0.6494480	0.7604060	0.6626200	0.7489557	30
31	.6363026	.7714395	.6496692	.7602170	.6628379	.7487629	29
32	.6365270	.7712544	.6498903	.7600280	.6630557	.7485701	28
33	.6367513	.7710692	.6501114	.7598389	.6632734	.7483772	27
34	.6369756	.7708840	.6503324	.7596498	.6634910	.7481842	26
35	.6371998	.7706986	.6505533	.7594606	.6637087	.7479912	25
36	.6374240	.7705132	.6507742	.7592713	.6639262	.7477981	24
37	.6376481	.7703278	.6509951	.7590820	.6641437	.7476049	23
38	.6378721	.7701423	.6512158	.7588926	.6643612	.7474117	22
39	.6380961	.7699567	.6514366	.7587031	.6645785	.7472184	21
40	0.6383201	0.7697710	0.6516572	0.7585136	0.6647959	0.7470251	20
41	.6385440	.7695853	.6518778	.7583240	.6650131	.7468317	19
42	.6387678	.7693996	.6520984	.7581343	.6652304	.7466382	18
43	.6389916	.7692137	.6523189	.7579446	.6654475	.7464446	17
44	.6392153	.7690278	.6525394	.7577548	.6656646	.7462510	16
45	.6394390	.7688418	.6527598	.7575650	.6658817	.7460574	15
46	.6396626	.7686558	.6529801	.7573751	.6660987	.7458636	14
47	.6398862	.7684697	.6532004	.7571851	.6663156	.7456699	13
48	.6401097	.7682835	.6534206	.7569951	.6665325	.7454760	12
49	.6403332	.7680973	.6536408	.7568050	.6667493	.7452821	11
50	0.6405566	0.7679110	0.6538609	0.7566148	0.6669661	0.7450881	10
51	.6407799	.7677246	.6540810	.7564246	.6671828	.7448941	9
52	.6410032	.7675382	.6543010	.7562343	.6673994	.7446999	8
53	.6412264	.7673517	.6545209	.7560439	.6676160	.7445058	7
54	.6414496	.7671652	.6547408	.7558535	.6678326	.7443115	6
55	.6416728	.7669785	.6549607	.7556630	.6680490	.7441173	5
56	.6418958	.7667918	.6551804	.7554724	.6682655	.7439229	4
57	.6421189	.7666051	.6554002	.7552818	.6684818	.7437285	3
58	.6423418	.7664183	.6556198	.7550911	.6686981	.7435340	2
59	.6425647	.7662314	.6558395	.7549004	.6689144	.7433394	1
60	0.6427876	0.7660444	0.6560590	0.7547096	0.6691306	0.7431448	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	50°		49°		48°		

TABLE XIX (*Continued*)

Natural Sines and Cosines

′	42° Sine	42° Cosine	43° Sine	43° Cosine	44° Sine	44° Cosine	′
0	0.6691306	0.7431448	0.6819984	0.7313537	0.6946584	0.7193398	60
1	.6693468	.7429502	.6822111	.7311553	.6948676	.7191377	59
2	.6695628	.7427554	.6824237	.7309568	.6950767	.7189355	58
3	.6697789	.7425606	.6826363	.7307583	.6952858	.7187333	57
4	.6699948	.7423658	.6828489	.7305597	.6954949	.7185310	56
5	.6702108	.7421708	.6830613	.7303610	.6957039	.7183287	55
6	.6704266	.7419758	.6832738	.7301623	.6959128	.7181263	54
7	.6706424	.7417808	.6834861	.7299635	.6961217	.7179238	53
8	.6708582	.7415857	.6836984	.7297646	.6963305	.7177213	52
9	.6710739	.7413905	.6839107	.7295657	.6965392	.7175187	51
10	0.6712895	0.7411953	0.6841229	0.7293668	0.6967479	0.7173161	50
11	.6715051	.7410000	.6843350	.7291677	.6969565	.7171134	49
12	.6717206	.7408046	.6845471	.7289686	.6971651	.7169106	48
13	.6719361	.7406092	.6847591	.7287695	.6973736	.7167078	47
14	.6721515	.7404137	.6849711	.7285703	.6975821	.7165049	46
15	.6723668	.7402181	.6851830	.7283710	.6977905	.7163019	45
16	.6725821	.7400225	.6853948	.7281716	.6979988	.7160989	44
17	.6727973	.7398268	.6856066	.7279722	.6982071	.7158959	43
18	.6730125	.7396311	.6858184	.7277728	.6984153	.7156927	42
19	.6732276	.7394353	.6860300	.7275732	.6986234	.7154895	41
20	0.6734427	0.7392394	0.6862416	0.7273736	0.6988315	0.7152863	40
21	.6736577	.7390435	.6864532	.7271740	.6990396	.7150830	39
22	.6738727	.7388475	.6866647	.7269743	.6992476	.7148796	38
23	.6740876	.7386515	.6868761	.7267745	.6994555	.7146762	37
24	.6743024	.7384553	.6870875	.7265747	.6996633	.7144727	36
25	.6745172	.7382592	.6872988	.7263748	.6998711	.7142691	35
26	.6747319	.7380629	.6875101	.7261748	.7000789	.7140655	34
27	.6749466	.7378666	.6877213	.7259748	.7002866	.7138618	33
28	.6751612	.7376703	.6879325	.7257747	.7004942	.7136581	32
29	.6753757	.7374738	.6881435	.7255746	.7007018	.7134543	31
30	0.6755902	0.7372773	0.6883546	0.7253744	0.7009093	0.7132504	30
31	.6758046	.7370808	.6885655	.7251741	.7011167	.7130465	29
32	.6760190	.7368842	.6887765	.7249738	.7013241	.7128426	28
33	.6762333	.7366875	.6889873	.7247734	.7015314	.7126385	27
34	.6764476	.7364908	.6891981	.7245729	.7017387	.7124344	26
35	.6766618	.7362940	.6894089	.7243724	.7019459	.7122303	25
36	.6768760	.7360971	.6896195	.7241719	.7021531	.7120260	24
37	.6770901	.7359002	.6898302	.7239712	.7023601	.7118218	23
38	.6773041	.7357032	.6900407	.7237705	.7025672	.7116174	22
39	.6775181	.7355061	.6902512	.7235698	.7027741	.7114130	21
40	0.6777320	0.7353090	0.6904617	0.7233690	0.7029811	0.7112086	20
41	.6779450	.7351118	.6906721	.7231681	.7031879	.7110041	19
42	6781597	.7349146	.6908824	.7229671	.7033947	.7107995	18
43	.6783734	.7347173	.6910927	.7227661	.7036014	.7105948	17
44	.6785871	.7345199	.6913029	.7225651	.7038081	.7103901	16
45	.6788007	.7343225	.6915131	.7223640	.7040147	.7101854	15
46	.6790143	.7341250	.6917232	.7221628	.7042213	.7099806	14
47	.6792278	.7339275	.6919332	.7219615	.7044278	.7097757	13
48	.6794413	.7337299	.6921432	.7217602	.7046342	.7095707	12
49	.6796547	.7335322	.6923531	.7215589	.7048406	.7093657	11
50	0.6798681	0.7333345	0.6925630	0.7213574	0.7050469	0.7091607	10
51	.6800813	.7331367	.6927728	.7211559	.7052532	.7089556	9
52	.6802946	.7329388	.6929825	.7209544	.7054594	.7087504	8
53	.6805078	.7327409	.6931922	.7207528	.7056655	.7085451	7
54	.6807209	.7325429	.6934018	.7205511	.7058716	.7083398	6
55	.6809330	.7323449	.6936114	.7203494	.7060776	.7081345	5
56	.6811469	.7321467	.6938209	.7201476	.7062835	.7079291	4
57	.6813599	.7319486	.6940304	.7199457	.7064894	.7077236	3
58	.6815728	.7317503	.6942398	.7197438	.7066953	.7075180	2
59	.6817856	.7315521	.6944491	.7195418	.7069011	.7073124	1
60	0.6819984	0.7313537	0.6946584	0.7193398	0.7071068	0.7071068	0
′	Cosine	Sine	Cosine	Sine	Cosine	Sine	′
	47°		46°		45°		

TABLE XX

Natural Tangents and Cotangents

′	0° Tangent	0° Cotangent	1° Tangent	1° Cotangent	2° Tangent	2° Cotangent	′
0	0.0000000	Infinite	0.0174551	57.289962	0.0349208	28.636253	60
1	.0002909	3437.7467	.0177460	56.350590	.0352120	28.399397	59
2	.0005818	1718.8732	.0180370	55.441517	.0355033	28.166422	58
3	.0008727	1145.9153	.0183280	54.561300	.0357945	27.937233	57
4	.0011636	859.43630	.0186190	53.708583	.0360858	27.711740	56
5	.0014544	687.54887	.0189100	52.882109	.0363771	27.489853	55
6	.0017453	572.95721	.0192010	52.080673	.0366683	27.271486	54
7	.0020362	491.10600	.0194920	51.303157	.0369596	27.056557	53
8	.0023271	429.71757	.0197830	50.548506	.0372509	26.844984	52
9	.0026180	381.97099	.0200740	49.815726	.0375422	26.636690	51
10	0.0029089	343.77371	0.0203650	49.103881	0.0378335	26.431600	50
11	.0031998	312.52137	.0206560	48.412084	.0381248	26.229638	49
12	.0034907	286.47773	.0209470	47.739501	.0384161	26.030736	48
13	.0037816	264.44080	.0212380	47.085343	.0387074	25.834823	47
14	.0040725	245.55198	.0215291	46.448862	.0389988	25.641832	46
15	.0043634	229.18166	.0218201	45.829351	.0392901	25.451700	45
16	.0046542	214.85762	.0221111	45.226141	.0395814	25.264361	44
17	.0049451	202.21875	.0224021	44.638596	.0398728	25.079757	43
18	.0052360	190.98419	.0226932	44.066113	.0401641	24.897826	42
19	.0055269	180.93220	.0229842	43.508122	.0404555	24.718512	41
20	0.0058178	171.88540	0.0232753	42.964077	0.0407469	24.541758	40
21	.0061087	163.70019	.0235663	42.433464	.0410383	24.367509	39
22	.0063996	156.25908	.0238574	41.915790	.0413296	24.195714	38
23	.0066905	149.46502	.0241484	41.410588	.0416210	24.026320	37
24	.0069814	143.23712	.0244395	40.917412	.0419124	23.859277	36
25	.0072723	137.50745	.0247305	40.435837	.0422038	23.694537	35
26	.0075632	132.21851	.0250216	39.965461	.0424952	23.532052	34
27	.0078541	127.32134	.0253127	39.505895	.0427866	23.371777	33
28	.0081450	122.77396	.0256038	39.056771	.0430781	23.213666	32
29	.0084360	118.54018	.0258948	38.617738	.0433695	23.057677	31
30	0.0087269	114.58865	0.0261859	38.188459	0.0436609	22.903766	30
31	.0090178	110.89205	.0264770	37.768613	.0439524	22.751892	29
32	.0093087	107.42648	.0267681	37.357892	.0442438	22.602015	28
33	.0095996	104.17094	.0270592	36.956001	.0445353	22.454096	27
34	.0098905	101.10690	.0273503	36.562659	.0448268	22.308097	26
35	.0101814	98.217943	.0276414	36.177596	.0451183	22.163980	25
36	.0104724	95.489475	.0279325	35.800553	.0454097	22.021710	24
37	.0107633	92.908487	.0282236	35.431282	.0457012	21.881251	23
38	.0110542	90.463336	.0285148	35.069546	.0459927	21.742569	22
39	.0113451	88.143572	.0288059	34.715115	.0462842	21.605630	21
40	0.0116361	85.939791	0.0290970	34.367771	0.0465757	21.470401	20
41	.0119270	83.843507	.0293882	34.027303	.0468673	21.336851	19
42	.0122179	81.847041	.0296793	33.693509	.0471588	21.204949	18
43	.0125088	79.943430	.0299705	33.366194	.0474503	21.074664	17
44	.0127998	78.126342	.0302616	33.045173	.0477419	20.945966	16
45	.0130907	76.390009	.0305528	32.730264	.0480334	20.818828	15
46	.0133817	74.729165	.0308439	32.421295	.0483250	20.693220	14
47	.0136726	73.138991	.0311351	32.118099	.0486166	20.569115	13
48	.0139635	71.615070	.0314263	31.820516	.0489082	20.446486	12
49	.0142545	70.153346	.0317174	31.528392	.0491997	20.325308	11
50	0.0145454	68.750087	0.0320086	31.241577	0.0494913	20.205553	10
51	.0148364	67.401854	.0322998	30.959928	.0497829	20.087199	9
52	.0151273	66.105473	.0325910	30.683307	.0500746	19.970219	8
53	.0154183	64.858008	.0328822	30.411580	.0503662	19.854591	7
54	.0157093	63.656741	.0331734	30.144619	.0506578	19.740291	6
55	.0160002	62.499154	.0334646	29.882299	.0509495	19.627296	5
56	.0162912	61.382905	.0337558	29.624499	.0512411	19.515584	4
57	.0165821	60.305820	.0340471	29.371106	.0515328	19.405133	3
58	.0168731	59.265872	.0343383	29.122005	.0518244	19.295922	2
59	.0171641	58.261174	.0346295	28.877089	.0521161	19.187930	1
60	0.0174551	57.289962	0.0349208	28.636253	0.0524078	19.081137	0
′	89° Cotangent	89° Tangent	88° Cotangent	88° Tangent	87° Cotangent	87° Tangent	′

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	3° Tangent	3° Cotangent	4° Tangent	4° Cotangent	5° Tangent	5° Cotangent	′
0	0.0524078	19.081137	0.0699268	14.300666	0.0874887	11.430052	60
1	.0526995	18.975523	.0702191	14.241134	.0877818	11.391885	59
2	.0529912	18.871068	.0705115	14.182092	.0880749	11.353970	58
3	.0532829	18.767754	.0708038	14.123536	.0883681	11.316304	57
4	.0535746	18.665562	.0710961	14.065459	.0886612	11.278885	56
5	.0538663	18.564473	.0713885	14.007856	.0889544	11.241712	55
6	.0541581	18.464471	.0716809	13.950719	.0892476	11.204780	54
7	.0544498	18.365537	.0719733	13.894045	.0895408	11.168089	53
8	.0547416	18.267654	.0722657	13.837827	.0898341	11.131635	52
9	.0550333	18.170807	.0725581	13.782060	.0901273	11.095416	51
10	0.0553251	18.074977	0.0728505	13.726738	0.0904206	11.059431	50
11	.0556169	17.980150	.0731430	13.671856	.0907138	11.023676	49
12	.0559087	17.886310	.0734354	13.617409	.0910071	10.988150	48
13	.0562005	17.793442	.0737279	13.563391	.0913004	10.952850	47
14	.0564923	17.701529	.0740203	13.509799	.0915938	10.917775	46
15	.0567841	17.610559	.0743128	13.456625	.0918871	10.882921	45
16	.0570759	17.520516	.0746053	13.403867	.0921804	10.848288	44
17	.0573678	17.431385	.0748979	13.351518	.0924738	10.813872	43
18	.0576596	17.343155	.0751904	13.299574	.0927672	10.779673	42
19	.0579515	17.255810	.0754829	13.248031	.0930606	10.745687	41
20	0.0582434	17.169337	0.0757755	13.196883	0.0933540	10.711913	40
21	.0585352	17.083724	.0760680	13.146127	.0936474	10.678348	39
22	.0588271	16.998957	.0763606	13.095757	.0939409	10.644992	38
23	.0591190	16.915025	.0766532	13.045769	.0942344	10.611841	37
24	.0594109	16.831915	.0769458	12.996160	.0945278	10.578895	36
25	.0597029	16.749614	.0772384	12.946924	.0948213	10.546151	35
26	.0599948	16.668112	.0775311	12.898058	.0951148	10.513607	34
27	.0602867	16.587396	.0778237	12.849557	.0954084	10.481261	33
28	.0605787	16.507456	.0781164	12.801417	.0957019	10.449112	32
29	.0608706	16.428279	.0784090	12.753634	.0959955	10.417158	31
30	0.0611626	16.349855	0.0787017	12.706205	0.0962890	10.385397	30
31	.0614546	16.272174	.0789944	12.659125	.0965826	10.353827	29
32	.0617466	16.195225	.0792871	12.612390	.0968763	10.322447	28
33	.0620386	16.118998	.0795798	12.565997	.0971699	10.291255	27
34	.0623306	16.043482	.0798726	12.519942	.0974635	10.260249	26
35	.0626226	15.968667	.0801653	12.474221	.0977572	10.229428	25
36	.0629147	15.894545	.0804581	12.428831	.0980509	10.198789	24
37	.0632067	15.821105	.0807509	12.383768	.0983446	10.168332	23
38	.0634988	15.748337	.0810437	12.339028	.0986383	10.138054	22
39	.0637908	15.676233	.0813365	12.294609	.0989320	10.107954	21
40	0.0640829	15.604784	0.0816293	12.250505	0.0992257	10.078031	20
41	.0643750	15.533981	.0819221	12.206716	.0995195	10.048283	19
42	.0646671	15.463814	.0822150	12.163236	.0998133	10.018708	18
43	.0649592	15.394276	.0825078	12.120062	.1001071	9.9893050	17
44	.0652513	15.325358	.0828007	12.077192	.1004009	9.9600724	16
45	.0655435	15.257052	.0830936	12.034622	.1006947	9.9310088	15
46	.0658356	15.189349	.0833865	11.992349	.1009886	9.9021125	14
47	.0661278	15.122242	.0836794	11.950371	.1012824	9.8733823	13
48	.0664199	15.055723	.0839723	11.908682	.1015763	9.8448166	12
49	.0667121	14.989784	.0842653	11.867282	.1018702	9.8164140	11
50	0.0670043	14.924417	0.0845583	11.826167	0.1021641	9.7881732	10
51	.0672965	14.859616	.0848512	11.785333	.1024580	9.7600927	9
52	.0675887	14.795372	.0851442	11.744779	.1027520	9.7321713	8
53	.0678809	14.731679	.0854372	11.704500	.1030460	9.7044075	7
54	.0681732	14.668529	.0857302	11.664495	.1033400	9.6768000	6
55	.0684654	14.605916	.0860233	11.624761	.1036340	9.6493475	5
56	.0687577	14.543833	.0863163	11.585294	.1039280	9.6220486	4
57	.0690499	14.482273	.0866094	11.546093	.1042220	9.5949022	3
58	.0693422	14.421230	.0869025	11.507154	.1045161	9.5679068	2
59	.0696345	14.360696	.0871956	11.468474	.1048101	9.5410613	1
60	0.0699268	14.300666	0.0874887	11.430052	0.1051042	9.5143645	0
′	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	′
	86°		85°		84°		

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	6° Tangent	6° Cotangent	7° Tangent	7° Cotangent	8° Tangent	8° Cotangent	′
0	0.1051042	9.5143645	0.1227846	8.1443464	0.1405408	7.1153697	60
1	.1053983	9.4878149	.1230798	8.1248071	.1408375	7.1003826	59
2	.1056925	9.4614116	.1233752	8.1053599	.1411342	7.0854573	58
3	.1059866	9.4351531	.1236705	8.0860042	.1414308	7.0705934	57
4	.1062808	9.4090384	.1239658	8.0667394	.1417276	7.0557905	56
5	.1065750	9.3830663	.1242612	8.0475647	.1420243	7.0410482	55
6	.1068692	9.3572355	.1245566	8.0284796	.1423211	7.0263662	54
7	.1071634	9.3315450	.1248520	8.0094835	.1426179	7.0117441	53
8	.1074576	9.3059936	.1251474	7.9905756	.1429147	6.9971816	52
9	.1077519	9.2805802	.1254429	7.9717555	.1432115	6.9826781	51
10	0.1080462	9.2553035	0.1257384	7.9530224	0.1435084	6.9682335	50
11	.1083405	9.2301627	.1260339	7.9343758	.1438053	6.9538473	49
12	.1086348	9.2051564	.1263294	7.9158151	.1441022	6.9395192	48
13	.1089291	9.1802838	.1266249	7.8973396	.1443991	6.9252489	47
14	.1092234	9.1555436	.1269205	7.8789489	.1446961	6.9110359	46
15	.1095178	9.1309348	.1272161	7.8606423	.1449931	6.8968799	45
16	.1098122	9.1064564	.1275117	7.8424191	.1452901	6.8827807	44
17	.1101066	9.0821074	.1278073	7.8242790	.1455872	6.8687378	43
18	.1104010	9.0578867	.1281030	7.8062212	.1458842	6.8547508	42
19	.1106955	9.0337933	.1283986	7.7882453	.1461813	6.8408196	41
20	0.1109899	9.0098261	0.1286943	7.7703506	0.1464784	6.8269437	40
21	.1112844	8.9859843	.1289900	7.7525366	.1467756	6.8131227	39
22	.1115789	8.9622668	.1292858	7.7348028	.1470727	6.7993565	38
23	.1118734	8.9386726	.1295815	7.7171486	.1473699	6.7856446	37
24	.1121680	8.9152009	.1298773	7.6995735	.1476672	6.7719867	36
25	.1124625	8.8918505	.1301731	7.6820769	.1479644	6.7583826	35
26	.1127571	8.8686206	.1304690	7.6646584	.1482617	6.7448318	34
27	.1130517	8.8455103	.1307648	7.6473174	.1485590	6.7313341	33
28	.1133463	8.8225186	.1310607	7.6300533	.1488563	6.7178891	32
29	.1136410	8.7996446	.1313566	7.6128657	.1491536	6.7044966	31
30	0.1139356	8.7768874	0.1316525	7.5957541	0.1494510	6.6911562	30
31	.1142303	8.7542461	.1319484	7.5787179	.1497484	6.6778677	29
32	.1145250	8.7317198	.1322444	7.5617567	.1500458	6.6646307	28
33	.1148197	8.7093077	.1325404	7.5448699	.1503433	6.6514449	27
34	.1151144	8.6870088	.1328364	7.5280571	.1506408	6.6383100	26
35	.1154092	8.6648223	.1331324	7.5113178	.1509383	6.6252258	25
36	.1157039	8.6427475	.1334285	7.4946514	.1512358	6.6121919	24
37	.1159987	8.6207833	.1337246	7.4780576	.1515333	6.5992080	23
38	.1162936	8.5989290	.1340207	7.4615357	.1518309	6.5862739	22
39	.1165884	8.5771838	.1343168	7.4450855	.1521285	6.5733892	21
40	0.1168832	8.5555468	0.1346129	7.4287064	0.1524262	6.5605538	20
41	.1171781	8.5340172	.1349091	7.4123978	.1527238	6.5477672	19
42	.1174730	8.5125943	.1352053	7.3961595	.1530215	6.5350293	18
43	.1177679	8.4912772	.1355015	7.3799909	.1533192	6.5223396	17
44	.1180628	8.4700651	.1357978	7.3638916	.1536170	6.5096981	16
45	.1183578	8.4489573	.1360940	7.3478610	.1539147	6.4971043	15
46	.1186528	8.4279531	.1363903	7.3318989	.1542125	6.4845581	14
47	.1189478	8.4070511	.1366866	7.3160047	.1545103	6.4720591	13
48	.1192428	8.3862519	.1369830	7.3001780	.1548082	6.4596070	12
49	.1195378	8.3655536	.1372793	7.2844184	.1551061	6.4472017	11
50	0.1198329	8.3449558	0.1375757	7.2687255	0.1554040	6.4348428	10
51	.1201279	8.3244577	.1378721	7.2530987	.1557019	6.4225301	9
52	.1204230	8.3040586	.1381685	7.2375378	.1559998	6.4102633	8
53	.1207182	8.2837579	.1384650	7.2220422	.1562978	6.3980422	7
54	.1210133	8.2635547	.1387615	7.2066116	.1565958	6.3858665	6
55	.1213085	8.2434485	.1390580	7.1912456	.1568939	6.3737359	5
56	.1216036	8.2234384	.1393545	7.1759437	.1571919	6.3616502	4
57	.1218988	8.2035239	.1396510	7.1607056	.1574900	6.3496092	3
58	.1221941	8.1837041	.1399476	7.1455308	.1577881	6.3376126	2
59	.1224893	8.1639786	.1402442	7.1304190	.1580863	6.3256601	1
60	0.1227846	8.1443464	0.1405408	7.1153697	0.1583844	6.3137515	0
′	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	′
	83°		82°		81°		

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	9° Tangent	9° Cotangent	10° Tangent	10° Cotangent	11° Tangent	11° Cotangent	′
0	0.1583844	6.3137515	0.1763270	5.6712818	0.1943803	5.1445540	60
1	.1586826	6.3018866	.1766269	5.6616509	.1946822	5.1365763	59
2	.1589809	6.2900651	.1769269	5.6520516	.1949841	5.1286224	58
3	.1592791	6.2782868	.1772269	5.6424838	.1952861	5.1206921	57
4	.1595774	6.2665515	.1775270	5.6329474	.1955881	5.1127855	56
5	.1598757	6.2548588	.1778270	5.6234421	.1958901	5.1049024	55
6	.1601740	6.2432086	.1781271	5.6139680	.1961922	5.0970426	54
7	.1604724	6.2316007	.1784273	5.6045247	.1964943	5.0892061	53
8	.1607708	6.2200347	.1787274	5.5951121	.1967964	5.0813928	52
9	.1610692	6.2085106	.1790276	5.5857302	.1970986	5.0736025	51
10	0.1613677	6.1970279	0.1793279	5.5763786	0.1974008	5.0658352	50
11	.1616662	6.1855867	.1796281	5.5670574	.1977031	5.0580907	49
12	.1619647	6.1741865	.1799284	5.5577663	.1980053	5.0503690	48
13	.1622632	6.1628272	.1802287	5.5485052	.1983076	5.0426700	47
14	.1625618	6.1515085	.1805291	5.5392740	.1986100	5.0349935	46
15	.1628603	6.1402303	.1808295	5.5300724	.1989124	5.0273395	45
16	.1631590	6.1289923	.1811299	5.5209005	.1992148	5.0197078	44
17	.1634576	6.1177943	.1814303	5.5117579	.1995172	5.0120984	43
18	.1637563	6.1066360	.1817308	5.5026446	.1998197	5.0045111	42
19	.1640550	6.0955174	.1820313	5.4935604	.2001222	4.9969459	41
20	0.1643537	6.0844381	0.1823319	5.4845052	0.2004248	4.9894027	40
21	.1646525	6.0733979	.1826324	5.4754788	.2007274	4.9818813	39
22	.1649513	6.0623967	.1829330	5.4664812	.2010300	4.9743817	38
23	.1652501	6.0514343	.1832337	5.4575121	.2013327	4.9669037	37
24	.1655489	6.0405103	.1835343	5.4485715	.2016354	4.9594474	36
25	.1658478	6.0296247	.1838350	5.4396592	.2019381	4.9520125	35
26	.1661467	6.0187772	.1841358	5.4307750	.2022409	4.9445990	34
27	.1664456	6.0079676	.1844365	5.4219188	.2025437	4.9372068	33
28	.1667446	5.9971957	.1847373	5.4130906	.2028465	4.9298358	32
29	.1670436	5.9864614	.1850382	5.4042901	.2031494	4.9224859	31
30	0.1673426	5.9757644	0.1853390	5.3955172	0.2034523	4.9151570	30
31	.1676417	5.9651045	.1856399	5.3867718	.2037552	4.9078491	29
32	.1679407	5.9544815	.1859409	5.3780538	.2040582	4.9005620	28
33	.1682398	5.9438952	.1862418	5.3693630	.2043612	4.8932956	27
34	.1685390	5.9333455	.1865428	5.3606993	.2046643	4.8860499	26
35	.1688381	5.9228322	.1868439	5.3520626	.2049674	4.8788248	25
36	.1691373	5.9123550	.1871449	5.3434527	.2052705	4.8716201	24
37	.1694366	5.9019138	.1874460	5.3348696	.2055737	4.8644359	23
38	.1697358	5.8915084	.1877471	5.3263131	.2058769	4.8572719	22
39	.1700351	5.8811386	.1880483	5.3177830	.2061802	4.8501282	21
40	0.1703344	5.8708042	0.1883495	5.3092793	0.2064834	4.8430045	20
41	.1706338	5.8605051	.1886507	5.3008018	.2067867	4.8359010	19
42	.1709331	5.8502410	.1889520	5.2923505	.2070900	4.8288174	18
43	.1712325	5.8400117	.1892533	5.2839251	.2073934	4.8217536	17
44	.1715320	5.8298172	.1895546	5.2755255	.2076968	4.8147096	16
45	.1718314	5.8196572	.1898559	5.2671517	.2080003	4.8076854	15
46	.1721309	5.8095315	.1901573	5.2588035	.2083038	4.8006808	14
47	.1724304	5.7994400	.1904587	5.2504809	.2086073	4.7936957	13
48	.1727300	5.7893825	.1907602	5.2421836	.2089109	4.7867300	12
49	.1730296	5.7793588	.1910617	5.2339116	.2092145	4.7797837	11
50	0.1733292	5.7693688	0.1913632	5.2256647	0.2095181	4.7728568	10
51	.1736288	5.7594122	.1916648	5.2174428	.2098218	4.7659490	9
52	.1739285	5.7494889	.1919664	5.2092459	.2101255	4.7590603	8
53	.1742282	5.7395988	.1922680	5.2010738	.2104293	4.7521907	7
54	.1745279	5.7297416	.1925696	5.1929264	.2107331	4.7453401	6
55	.1748277	5.7199173	.1928713	5.1848035	.2110369	4.7385083	5
56	.1751275	5.7101256	.1931731	5.1767051	.2113407	4.7316954	4
57	.1754273	5.7003663	.1934748	5.1686311	.2116446	4.7249012	3
58	.1757272	5.6906394	.1937766	5.1605813	.2119486	4.7181256	2
59	.1760271	5.6809446	.1940784	5.1525557	.2122525	4.7113686	1
60	0.1763270	5.6712818	0.1943803	5.1445540	0.2125566	4.7046301	0
′	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	′
	80°		79°		78°		

TABLE XX (*Continued*)

NATURAL TANGENTS AND COTANGENTS

′	12° Tangent	12° Cotangent	13° Tangent	13° Cotangent	14° Tangent	14° Cotangent	′
0	0.2125566	4.7046301	0.2308682	4.3314759	0.2493280	4.0107809	60
1	.2128606	4.6979100	.2311746	4.3257347	.2496370	4.0058164	59
2	.2131647	4.6912083	.2314811	4.3200079	.2499460	4.0008636	58
3	.2134688	4.6845248	.2317876	4.3142955	.2502551	3.9959223	57
4	.2137730	4.6778595	.2320941	4.3085974	.2505642	3.9909924	56
5	.2140772	4.6712124	.2324007	4.3029136	.2508734	3.9860739	55
6	.2143814	4.6645832	.2327073	4.2972440	.2511826	3.9811669	54
7	.2146857	4.6579721	.2330140	4.2915885	.2514919	3.9762712	53
8	.2149900	4.6513788	.2333207	4.2859472	.2518012	3.9713868	52
9	.2152944	4.6448034	.2336274	4.2803199	.2521106	3.9665137	51
10	0.2155988	4.6382457	0.2339342	4.2747066	0.2524200	3.9616518	50
11	.2159032	4.6317056	.2342411	4.2691072	.2527294	3.9568011	49
12	.2162077	4.6251832	.2345479	4.2635218	.2530389	3.9519615	48
13	.2165122	4.6186783	.2348548	4.2579501	.2533484	3.9471331	47
14	.2168167	4.6121908	.2351617	4.2523923	.2536580	3.9423157	46
15	.2171213	4.6057207	.2354687	4.2468482	.2539676	3.9375094	45
16	.2174259	4.5992680	.2357758	4.2413177	.2542773	3.9327141	44
17	.2177306	4.5928325	.2360829	4.2358009	.2545870	3.9279297	43
18	.2180353	4.5864141	.2363900	4.2302977	.2548968	3.9231563	42
19	.2183400	4.5800129	.2366971	4.2248080	.2552066	3.9183937	41
20	0.2186448	4.5736287	0.2370044	4.2193318	0.2555165	3.9136420	40
21	.2189496	4.5672615	.2373116	4.2138690	.2558264	3.9089011	39
22	.2192544	4.5609111	.2376189	4.2084196	.2561363	3.9041710	38
23	.2195593	4.5545776	.2379262	4.2029835	.2564463	3.8994516	37
24	.2198643	4.5482608	.2382336	4.1975606	.2567564	3.8947429	36
25	.2201692	4.5419608	.2385410	4.1921510	.2570664	3.8900448	35
26	.2204742	4.5356773	.2388485	4.1867546	.2573766	3.8853574	34
27	.2207793	4.5294105	.2391560	4.1813713	.2576868	3.8806805	33
28	.2210844	4.5231601	.2394635	4.1760011	.2579970	3.8760142	32
29	.2213895	4.5169261	.2397711	4.1706440	.2583073	3.8713584	31
30	0.2216947	4.5107085	0.2400788	4.1652998	0.2586176	3.8667131	30
31	.2219999	4.5045072	.2403864	4.1599685	.2589280	3.8620782	29
32	.2223051	4.4983221	.2406942	4.1546501	.2592384	3.8574537	28
33	.2226104	4.4921532	.2410019	4.1493446	.2595488	3.8528396	27
34	.2229157	4.4860004	.2413097	4.1440519	.2598593	3.8482358	26
35	.2232211	4.4798636	.2416176	4.1387719	.2601699	3.8436424	25
36	.2235265	4.4737428	.2419255	4.1335046	.2604805	3.8390591	24
37	.2238319	4.4676379	.2422334	4.1282499	.2607911	3.8344861	23
38	.2241374	4.4615489	.2425414	4.1230079	.2611018	3.8299233	22
39	.2244429	4.4554756	.2428494	4.1177784	.2614126	3.8253707	21
40	0.2247485	4.4494181	0.2431575	4.1125614	0.2617234	3.8208281	20
41	.2250541	4.4433762	.2434656	4.1073569	.2620342	3.8162957	19
42	.2253597	4.4373500	.2437737	4.1021649	.2623451	3.8117733	18
43	.2256654	4.4313392	.2440819	4.0969852	.2626560	3.8072609	17
44	.2259711	4.4253439	.2443902	4.0918178	.2629670	3.8027585	16
45	.2262769	4.4193641	.2446984	4.0866627	.2632780	3.7982661	15
46	.2265827	4.4133996	.2450068	4.0815199	.2635891	3.7937835	14
47	.2268885	4.4074504	.2453151	4.0763892	.2639002	3.7893109	13
48	.2271944	4.4015164	.2456236	4.0712707	.2642114	3.7848481	12
49	.2275003	4.3955977	.2459320	4.0661643	.2645226	3.7803951	11
50	0.2278063	4.3896940	0.2462405	4.0610700	0.2648339	3.7759519	10
51	.2281123	4.3838054	.2465491	4.0559877	.2651452	3.7715185	9
52	.2284184	4.3779317	.2468577	4.0509174	.2654566	3.7670947	8
53	.2287244	4.3720731	.2471663	4.0458590	.2657680	3.7626807	7
54	.2290306	4.3662293	.2474750	4.0408125	.2660794	3.7582763	6
55	.2293367	4.3604003	.2477837	4.0357779	.2663909	3.7538815	5
56	.2296429	4.3545861	.2480925	4.0307550	.2667025	3.7494963	4
57	.2299492	4.3487866	.2484013	4.0257440	.2670141	3.7451207	3
58	.2302555	4.3430018	.2487102	4.0207446	.2673257	3.7407546	2
59	.2305618	4.3372316	.2490191	4.0157570	.2676374	3.7363980	1
60	0.2308682	4.3314759	0.2493280	4.0107809	0.2679492	3.7320508	0
′	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	′
	77°		76°		75°		

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	15°		16°		17°		′
	Tangent	Cotangent	Tangent	Cotangent	Tangent	Cotangent	
0	0.2679492	3.7320508	0.2867454	3.4874144	0.3057307	3.2708526	60
1	.2682610	3.7277131	.2870602	3.4835896	.3060488	3.2674529	59
2	.2685728	3.7233847	.2873751	3.4797726	.3063670	3.2640596	58
3	.2688847	3.7190658	.2876900	3.4759632	.3066852	3.2606728	57
4	.2691967	3.7147561	.2880050	3.4721616	.3070034	3.2572924	56
5	.2695087	3.7104558	.2883201	3.4683676	.3073218	3.2539184	55
6	.2698207	3.7061648	.2886352	3.4645813	.3076402	3.2505508	54
7	.2701328	3.7018830	.2889503	3.4608026	.3079586	3.2471895	53
8	.2704449	3.6976104	.2892655	3.4570315	.3082771	3.2438346	52
9	.2707571	3.6933469	.2895808	3.4532679	.3085957	3.2404861	51
10	0.2710694	3.6890927	0.2898961	3.4495120	0.3089143	3.2371438	50
11	.2713817	3.6848475	.2902114	3.4457635	.3092330	3.2338078	49
12	.2716940	3.6806115	.2905269	3.4420226	.3095517	3.2304780	48
13	.2720064	3.6763845	.2908423	3.4382891	.3098705	3.2271546	47
14	.2723188	3.6721665	.2911578	3.4345631	.3101893	3.2238373	46
15	.2726313	3.6679575	.2914734	3.4308446	.3105083	3.2205263	45
16	.2729438	3.6637575	.2917890	3.4271334	.3108272	3.2172215	44
17	.2732564	3.6595665	.2921047	3.4234297	.3111462	3.2139228	43
18	.2735690	3.6553844	.2924205	3.4197333	.3114653	3.2106304	42
19	.2738817	3.6512111	.2927363	3.4160443	.3117845	3.2073440	41
20	0.2741945	3.6470467	0.2930521	3.4123626	0.3121036	3.2040638	40
21	.2745072	3.6428911	.2933680	3.4086882	.3124229	3.2007897	39
22	.2748201	3.6387444	.2936839	3.4050210	.3127422	3.1975217	38
23	.2751330	3.6346064	.2939999	3.4013612	.3130616	3.1942597	37
24	.2754459	3.6304771	.2943160	3.3977085	.3133810	3.1910039	36
25	.2757589	3.6263566	.2946321	3.3940631	.3137005	3.1877540	35
26	.2760719	3.6222447	.2949483	3.3904249	.3140200	3.1845102	34
27	.2763850	3.6181415	.2952645	3.3867938	.3143396	3.1812724	33
28	.2766981	3.6140469	.2955808	3.3831699	.3146593	3.1780406	32
29	.2770113	3.6099609	.2958971	3.3795531	.3149790	3.1748147	31
30	0.2773245	3.6058835	0.2962135	3.3759434	0.3152988	3.1715948	30
31	.2776378	3.6018146	.2965299	3.3723408	.3156186	3.1683808	29
32	.2779512	3.5977543	.2968464	3.3687453	.3159385	3.1651728	28
33	.2782646	3.5937024	.2971630	3.3651568	.3162585	3.1619706	27
34	.2785780	3.5896590	.2974796	3.3615753	.3165785	3.1587744	26
35	.2788915	3.5856241	.2977962	3.3580008	.3168986	3.1555840	25
36	.2792050	3.5815975	.2981129	3.3544333	.3172187	3.1523994	24
37	.2795186	3.5775794	.2984297	3.3508728	.3175389	3.1492207	23
38	.2798322	3.5735696	.2987465	3.3473191	.3178591	3.1460478	22
39	.2801459	3.5695681	.2990634	3.3437724	.3181794	3.1428807	21
40	0.2804597	3.5655749	0.2993803	3.3402326	0.3184998	3.1397194	20
41	.2807735	3.5615900	.2996973	3.3366997	.3188202	3.1365639	19
42	.2810873	3.5576133	.3000144	3.3331736	.3191407	3.1334141	18
43	.2814012	3.5536449	.3003315	3.3296543	.3194613	3.1302701	17
44	.2817152	3.5496846	.3006486	3.3261419	.3197819	3.1271317	16
45	.2820292	3.5457325	.3009658	3.3226362	.3201025	3.1239991	15
46	.2823432	3.5417886	.3012831	3.3191373	.3204232	3.1208722	14
47	.2826573	3.5378528	.3016004	3.3156452	.3207440	3.1177509	13
48	.2829715	3.5339251	.3019178	3.3121598	.3210649	3.1146353	12
49	.2832857	3.5300054	.3022352	3.3086811	.3213858	3.1115254	11
50	0.2835999	3.5260938	0.3025527	3.3052091	0.3217067	3.1084210	10
51	.2839143	3.5221902	.3028703	3.3017438	.3220278	3.1053223	9
52	.2842286	3.5182946	.3031879	3.2982851	.3223489	3.1022291	8
53	.2845430	3.5144070	.3035055	3.2948330	.3226700	3.0991416	7
54	.2848575	3.5105273	.3038232	3.2913876	.3229912	3.0960596	6
55	.2851720	3.5066555	.3041410	3.2879487	.3233125	3.0929831	5
56	.2854866	3.5027916	.3044588	3.2845164	.3236338	3.0899122	4
57	.2858012	3.4989356	.3047767	3.2810907	.3239552	3.0868468	3
58	.2861159	3.4950874	.3050946	3.2776715	.3242766	3.0837869	2
59	.2864306	3.4912470	.3054126	3.2742588	.3245981	3.0807325	1
60	0.2867454	3.4874144	0.3057307	3.2708526	0.3249197	3.0776835	0
	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	
′	74°		73°		72°		′

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	18° Tangent	18° Cotangent	19° Tangent	19° Cotangent	20° Tangent	20° Cotangent	′
0	0.3249197	3.0776835	0.3443276	2.9042109	0.3639702	2.7474774	60
1	.3252413	3.0746400	.3446530	2.9014688	.3642997	2.7449927	59
2	.3255630	3.0716020	.3449785	2.8987314	.3646292	2.7425120	58
3	.3258848	3.0685694	.3453040	2.8959986	.3649588	2.7400352	57
4	.3262066	3.0655421	.3456296	2.8932704	.3652885	2.7375623	56
5	.3265284	3.0625203	.3459553	2.8905467	.3656182	2.7350934	55
6	.3268504	3.0595038	.3462810	2.8878277	.3659480	2.7326284	54
7	.3271724	3.0564928	.3466068	2.8851132	.3662779	2.7301674	53
8	.3274944	3.0534870	.3469327	2.8824033	.3666079	2.7277102	52
9	.3278165	3.0504866	.3472586	2.8796979	.3669379	2.7252569	51
10	0.3281387	3.0474915	0.3475846	2.8769970	0.3672680	2.7228076	50
11	.3284610	3.0445018	.3479107	2.8743007	.3675981	2.7203620	49
12	.3287833	3.0415173	.3482368	2.8716088	.3679284	2.7179204	48
13	.3291056	3.0385381	.3485630	2.8689215	.3682587	2.7154826	47
14	.3294281	3.0355641	.3488893	2.8662386	.3685890	2.7130487	46
15	.3297505	3.0325954	.3492156	2.8635602	.3689195	2.7106186	45
16	.3300731	3.0296320	.3495420	2.8608863	.3692500	2.7081923	44
17	.3303957	3.0266737	.3498685	2.8582168	.3695806	2.7057699	43
18	.3307184	3.0237207	.3501950	2.8555517	.3699112	2.7033513	42
19	.3310411	3.0207728	.3505216	2.8528911	.3702420	2.7009364	41
20	0.3313639	3.0178301	0.3508483	2.8502349	0.3705728	2.6985254	40
21	.3316868	3.0148926	.3511750	2.8475831	.3709036	2.6961181	39
22	.3320097	3.0119603	.3515018	2.8449356	.3712346	2.6937147	38
23	.3323327	3.0090330	.3518287	2.8422926	.3715656	2.6913149	37
24	.3326557	3.0061109	.3521556	2.8396539	.3718967	2.6889190	36
25	.3329788	3.0031939	.3524826	2.8370196	.3722278	2.6865267	35
26	.3333020	3.0002820	.3528096	2.8343896	.3725590	2.6841383	34
27	.3336252	2.9973751	.3531368	2.8317640	.3728903	2.6817535	33
28	.3339485	2.9944734	.3534640	2.8291426	.3732217	2.6793725	32
29	.3342719	2.9915766	.3537912	2.8265256	.3735532	2.6769951	31
30	0.3345953	2.9886850	0.3541186	2.8239129	0.3738847	2.6746215	30
31	.3349188	2.9857983	.3544460	2.8213045	.3742163	2.6722516	29
32	.3352424	2.9829167	.3547734	2.8187003	.3745479	2.6698853	28
33	.3355660	2.9800400	.3551010	2.8161004	.3748797	2.6675227	27
34	.3358896	2.9771683	.3554286	2.8135048	.3752115	2.6651638	26
35	.3362134	2.9743016	.3557562	2.8109134	.3755433	2.6628085	25
36	.3365372	2.9714399	.3560840	2.8083263	.3758753	2.6604569	24
37	.3368610	2.9685831	.3564118	2.8057433	.3762073	2.6581089	23
38	.3371850	2.9657312	.3567397	2.8031646	.3765394	2.6557645	22
39	.3375090	2.9628842	.3570676	2.8005901	.3768716	2.6534238	21
40	0.3378330	2.9600422	0.3573956	2.7980198	0.3772038	2.6510867	20
41	.3381571	2.9572050	.3577237	2.7954537	.3775361	2.6487531	19
42	.3384813	2.9543727	.3580518	2.7928917	.3778685	2.6464232	18
43	.3388056	2.9515453	.3583801	2.7903339	.3782010	2.6440969	17
44	.3391299	2.9487227	.3587083	2.7877802	.3785335	2.6417741	16
45	.3394543	2.9459050	.3590367	2.7852307	.3788661	2.6394549	15
46	.3397787	2.9430921	.3593651	2.7826853	.3791988	2.6371392	14
47	.3401032	2.9402840	.3596936	2.7801440	.3795315	2.6348271	13
48	.3404278	2.9374807	.3600222	2.7776069	.3798644	2.6325186	12
49	.3407524	2.9346822	.3603508	2.7750738	.3801973	2.6302136	11
50	0.3410771	2.9318885	0.3606795	2.7725448	0.3805302	2.6279121	10
51	.3414019	2.9290995	.3610082	2.7700199	.3808633	2.6256141	9
52	.3417267	2.9263152	.3613371	2.7674991	.3811964	2.6233196	8
53	.3420516	2.9235358	.3616660	2.7649822	.3815296	2.6210286	7
54	.3423765	2.9207610	.3619949	2.7624695	.3818629	2.6187411	6
55	.3427015	2.9179909	.3623240	2.7599608	.3821962	2.6164571	5
56	.3430266	2.9152256	.3626531	2.7574561	.3825296	2.6141766	4
57	.3433518	2.9124649	.3629823	2.7549554	.3828631	2.6118995	3
58	.3436770	2.9097089	.3633115	2.7524588	.3831967	2.6096259	2
59	.3440023	2.9069576	.3636408	2.7499661	.3835303	2.6073558	1
60	0.3443276	2.9042109	0.3639702	2.7474774	0.3838640	2.6050891	0
′	71° Cotangent	71° Tangent	70° Cotangent	70° Tangent	69° Cotangent	69° Tangent	′

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	21° Tangent	21° Cotangent	22° Tangent	22° Cotangent	23° Tangent	23° Cotangent	′
0	0.3838640	2.6050891	0.4040262	2.4750869	0.4244748	2.3558524	60
1	.3841978	2.6028258	.4043646	2.4730155	.4248182	2.3539483	59
2	.3845317	2.6005659	.4047031	2.4709470	.4251616	2.3520469	58
3	.3848656	2.5983095	.4050417	2.4688816	.4255051	2.3501481	57
4	.3851996	2.5960564	.4053804	2.4668191	.4258487	2.3482519	56
5	.3855337	2.5938068	.4057191	2.4647596	.4261924	2.3463582	55
6	.3858679	2.5915606	.4060579	2.4627030	.4265361	2.3444672	54
7	.3862021	2.5893177	.4063968	2.4606494	.4268800	2.3425787	53
8	.3865364	2.5870782	.4067358	2.4585987	.4272239	2.3406928	52
9	.3868708	2.5848421	.4070748	2.4565510	.4275680	2.3388095	51
10	0.3872053	2.5826094	0.4074139	2.4545061	0.4279121	2.3369287	50
11	.3875398	2 5803800	.4077531	2.4524642	.4282563	2.3350505	49
12	.3878744	2.5781539	.4080924	2.4504252	.4286005	2.3331748	48
13	.3882091	2.5759312	.4084318	2.4483891	.4289449	2.3313017	47
14	.3885439	2.5737118	.4087713	2.4463559	.4292894	2.3294311	46
15	.3888787	2.5714957	.4091108	2.4443256	.4296339	2.3275630	45
16	.3892136	2.5692830	.4094504	2.4422982	.4299785	2.3256975	44
17	.3895486	2.5670735	.4097901	2.4402736	.4303232	2.3238345	43
18	.3898837	2.5648674	.4101299	2.4382519	.4306680	2.3219740	42
19	.3902189	2.5626645	.4104697	2.4362331	.4310129	2.3201160	41
20	0.3905541	2.5604649	0.4108097	2.4342172	0.4313579	2.3182606	40
21	.3908894	2.5582686	.4111497	2.4322041	.4317030	2.3164076	39
22	.3912247	2.5560756	.4114898	2.4301938	.4320481	2.3145571	38
23	.3915602	2.5538858	.4118300	2.4281864	.4323933	2.3127092	37
24	.3918957	2.5516992	.4121703	2.4261819	.4327386	2.3108637	36
25	.3922313	2.5495160	.4125106	2.4241801	.4330840	2.3090206	35
26	.3925670	2.5473359	.4128510	2.4221812	.4334295	2.3071801	34
27	.3929027	2.5451591	.4131915	2.4201851	.4337751	2.3053420	33
28	.3932386	2.5429855	.4135321	2.4181918	.4341208	2.3035064	32
29	.3935745	2.5408151	.4138728	2.4162013	.4344665	2.3016732	31
30	0.3939105	2.5386479	0.4142136	2.4142136	0.4348124	2.2998425	30
31	.3942465	2.5364839	.4145544	2.4122286	.4351583	2.2980143	29
32	.3945827	2.5343231	.4148953	2.4102465	.4355043	2.2961885	28
33	.3949189	2.5321655	.4152363	2.4082672	.4358504	2.2943651	27
34	.3952552	2.5300111	.4155774	2.4062906	.4361966	2.2925442	26
35	.3955916	2.5278598	.4159186	2.4043168	.4365429	2.2907257	25
36	.3959280	2.5257117	.4162598	2.4023457	.4368893	2.2889096	24
37	.3962645	2.5235667	.4166012	2.4003774	.4372357	2.2870959	23
38	.3966011	2.5214249	.4169426	2.3984118	.4375823	2.2852846	22
39	.3969378	2.5192863	.4172841	2.3964490	.4379289	2.2834758	21
40	0.3972746	2.5171507	0.4176257	2.3944889	0.4382756	2.2816693	20
41	.3976114	2.5150183	.4179673	2.3925316	.4386224	2.2798653	19
42	.3979483	2.5128890	.4183091	2.3905769	.4389693	2.2780636	18
43	.3982853	2.5107629	.4186509	2.3886250	.4393163	2.2762643	17
44	.3986224	2.5086398	.4189928	2.3866758	.4396634	2.2744674	16
45	.3989595	2.5065198	.4193348	2.3847293	.4400105	2.2726729	15
46	.3992968	2.5044029	.4196769	2.3827855	.4403578	2.2708807	14
47	.3996341	2.5022891	.4200190	2.3808444	.4407051	2.2690909	13
48	.3999715	2.5001784	.4203613	2.3789060	.4410526	2.2673035	12
49	.4003089	2.4980707	.4207036	2.3769703	.4414001	2.2655184	11
50	0.4006465	2.4959661	0.4210460	2.3750372	0.4417477	2.2637357	10
51	.4009841	2.4938645	.4213885	2.3731068	.4420954	2.2619554	9
52	.4013218	2.4917660	.4217311	2.3711791	.4424432	2.2601773	8
53	.4016596	2.4896706	.4220738	2.3692540	.4427910	2.2584016	7
54	.4019974	2.4875781	.4224165	2.3673316	.4431390	2.2566283	6
55	.4023354	2.4854887	.4227594	2.3654118	.4434871	2.2548572	5
56	.4026734	2.4834023	.4231023	2.3634946	.4438352	2.2530885	4
57	.4030115	2.4813190	.4234453	2.3615801	.4441834	2.2513221	3
58	.4033496	2.4792386	.4237884	2.3596683	.4445318	2.2495580	2
59	.4036879	2.4771612	.4241316	2.3577590	.4448802	2.2477962	1
60	0.4040262	2.4750869	0.4244748	2.3558524	0.4452287	2.2460368	0
′	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	′
	68°		67°		66°		

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	24°		25°		26°		′
	Tangent	Cotangent	Tangent	Cotangent	Tangent	Cotangent	
0	0.4452287	2.2460368	0.4663077	2.1445069	0.4877326	2.0503038	60
1	.4455773	2.2442796	.4666618	2.1428793	.4880927	2.0487910	59
2	.4459260	2.2425247	.4670161	2.1412537	.4884530	2.0472800	58
3	.4462747	2.2407721	.4673705	2.1396301	.4888133	2.0457708	57
4	.4466236	2.2390218	.4677250	2.1380085	.4891737	2.0442634	56
5	.4469726	2.2372738	.4680796	2.1363890	.4895343	2.0427578	55
6	.4473216	2.2355280	.4684342	2.1347714	.4898949	2.0412540	54
7	.4476708	2.2337845	.4687890	2.1331559	.4902557	2.0397519	53
8	.4480200	2.2320433	.4691439	2.1315423	.4906166	2.0382517	52
9	.4483693	2.2303043	.4694988	2.1299308	.4909775	2.0367532	51
10	0.4487187	2.2285676	0.4698539	2.1283213	0.4913386	2.0352565	50
11	.4490682	2.2268331	.4702090	2.1267137	.4916997	2.0337615	49
12	.4494178	2.2251009	.4705643	2.1251082	.4920610	2.0322683	48
13	.4497675	2.2233709	.4709196	2.1235046	.4924224	2.0307769	47
14	.4501173	2.2216432	.4712751	2.1219030	.4927838	2.0292873	46
15	.4504672	2.2199177	.4716306	2.1203034	.4931454	2.0277994	45
16	.4508171	2.2181944	.4719863	2.1187057	.4935071	2.0263133	44
17	.4511672	2.2164733	.4723420	2.1171101	.4938689	2.0248289	43
18	.4515173	2.2147545	.4726978	2.1155164	.4942308	2.0233462	42
19	.4518676	2.2130379	.4730538	2.1139246	.4945928	2.0218654	41
20	0.4522179	2.2113234	0.4734098	2.1123348	0.4949549	2.0203862	40
21	.4525683	2.2096112	.4737659	2.1107470	.4953171	2.0189088	39
22	.4529188	2.2079012	.4741222	2.1091611	.4956794	2.0174331	38
23	.4532694	2.2061934	.4744785	2.1075771	.4960418	2.0159592	37
24	.4536201	2.2044878	.4748349	2.1059951	.4964043	2.0144869	36
25	.4539709	2.2027843	.4751914	2.1044150	.4967669	2.0130164	35
26	.4543218	2.2010831	.4755481	2.1028369	.4971297	2.0115477	34
27	.4546728	2.1993840	.4759048	2.1012607	.4974925	2.0100806	33
28	.4550238	2.1976871	.4762616	2.0996864	.4978554	2.0086153	32
29	.4553750	2.1959923	.4766185	2.0981140	.4982185	2.0071516	31
30	0.4557263	2.1942997	0.4769755	2.0965436	0.4985816	2.0056897	30
31	.4560776	2.1926093	.4773326	2.0949751	.4989449	2.0042295	29
32	.4564290	2.1909210	.4776899	2.0934085	.4993082	2.0027710	28
33	.4567806	2.1892349	.4780472	2.0918437	.4996717	2.0013142	27
34	.4571322	2.1875510	.4784046	2.0902809	.5000352	1.9998590	26
35	.4574839	2.1858691	.4787621	2.0887200	.5003989	1.9984056	25
36	.4578357	2.1841894	.4791197	2.0871610	.5007627	1.9969539	24
37	.4581877	2.1825119	.4794774	2.0856039	.5011266	1.9955038	23
38	.4585397	2.1808364	.4798352	2.0840487	.5014906	1.9940554	22
39	.4588918	2.1791631	.4801932	2.0824953	.5018547	1.9926087	21
40	0.4592439	2.1774920	0.4805512	2.0809438	0.5022189	1.9911637	20
41	.4595962	2.1758229	.4809093	2.0793942	.5025832	1.9897204	19
42	.4599486	2.1741559	.4812675	2.0778465	.5029476	1.9882787	18
43	.4603011	2.1724911	.4816258	2.0763007	.5033121	1.9868387	17
44	.4606537	2.1708283	.4819842	2.0747567	.5036768	1.9854003	16
45	.4610063	2.1691677	.4823427	2.0732146	.5040415	1.9839636	15
46	.4613591	2.1675091	.4827014	2.0716743	.5044063	1.9825286	14
47	.4617119	2.1658527	.4830601	2.0701359	.5047713	1.9810952	13
48	.4620649	2.1641983	.4834180	2.0685994	.5051363	1.9796635	12
49	.4624179	2.1625460	.4837778	2.0670646	.5055015	1.9782334	11
50	0.4627710	2.1608958	0.4841368	2.0655318	0.5058668	1.9768050	10
51	.4631243	2.1592476	.4844959	2.0640008	.5062322	1.9753782	9
52	.4634776	2.1576015	.4848552	2.0624716	.5065977	1.9739531	8
53	.4638310	2.1559575	.4852145	2.0609442	.5069633	1.9725296	7
54	.4641845	2.1543156	.4855739	2.0594187	.5073290	1.9711077	6
55	.4645382	2.1526757	.4859334	2.0578950	.5076948	1.9696874	5
56	.4648919	2.1510378	.4862931	2.0563732	.5080607	1.9682688	4
57	.4652457	2.1494021	.4866528	2.0548531	.5084267	1.9668518	3
58	.4655996	2.1477683	.4870126	2.0533349	.5087929	1.9654364	2
59	.4659536	2.1461366	.4873726	2.0518185	.5091591	1.9640227	1
60	0.4663077	2.1445069	0.4877326	2.0503038	0.5095254	1.9626105	0
	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	
′	65°		64°		63°		′

TABLE XX (*Continued*)

NATURAL TANGENTS AND COTANGENTS

′	27° Tangent	27° Cotangent	28° Tangent	28° Cotangent	29° Tangent	29° Cotangent	′
0	0.5095254	1.9626105	0.5317094	1.8807265	0.5543091	1.8040478	60
1	.5098919	1.9612000	.5320826	1.8794074	.5546894	1.8028108	59
2	.5102585	1.9597910	.5324559	1.8780898	.5550698	1.8015751	58
3	.5106252	1.9583837	.5328293	1.8767736	.5554504	1.8003408	57
4	.5109919	1.9569780	.5332029	1.8754588	.5558311	1.7991077	56
5	.5113588	1.9555739	.5335765	1.8741455	.5562119	1.7978759	55
6	.5117259	1.9541713	.5339503	1.8728336	.5565929	1.7966454	54
7	.5120930	1.9527704	.5343242	1.8715231	.5569739	1.7954162	53
8	.5124602	1.9513711	.5346981	1.8702141	.5573551	1.7941883	52
9	.5128275	1.9499733	.5350723	1.8689065	.5577364	1.7929616	51
10	0.5131950	1.9485772	0.5354465	1.8676003	0.5581179	1.7917362	50
11	.5135625	1.9471826	.5358208	1.8662955	.5584994	1.7905121	49
12	.5139302	1.9457896	.5361953	1.8649921	.5588811	1.7892893	48
13	.5142980	1.9443981	.5365699	1.8636902	.5592629	1.7880678	47
14	.5146658	1.9430083	.5369446	1.8623896	.5596449	1.7868475	46
15	.5150338	1.9416200	.5373194	1.8610905	.5600269	1.7856285	45
16	.5154019	1.9402333	.5376943	1.8597928	.5604091	1.7844107	44
17	.5157702	1.9388481	.5380694	1.8584965	.5607914	1.7831943	43
18	.5161385	1.9374645	.5384445	1.8572015	.5611738	1.7819790	42
19	.5165069	1.9360825	.5388198	1.8559080	.5615564	1.7807651	41
20	0.5168755	1.9347020	0.5391952	1.8546159	0.5619391	1.7795524	40
21	.5172441	1.9333231	.5395707	1.8533252	.5623219	1.7783409	39
22	.5176129	1.9319457	.5399464	1.8520358	.5627048	1.7771307	38
23	.5179818	1.9305699	.5403221	1.8507479	.5630879	1.7759218	37
24	.5183508	1.9291956	.5406980	1.8494613	.5634711	1.7747141	36
25	.5187199	1.9278228	.5410740	1.8481761	.5638543	1.7735076	35
26	.5190891	1.9264516	.5414501	1.8468923	.5642378	1.7723024	34
27	.5194584	1.9250819	.5418263	1.8456099	.5646213	1.7710985	33
28	.5198278	1.9237138	.5422027	1.8443289	.5650050	1.7698958	32
29	.5201974	1.9223472	.5425791	1.8430492	.5653888	1.7686943	31
30	0.5205671	1.9209821	0.5429557	1.8417709	0.5657728	1.7674940	30
31	.5209368	1.9196185	.5433324	1.8404940	.5661568	1.7662950	29
32	.5213067	1.9182565	.5437092	1.8392184	.5665410	1.7650972	28
33	.5216767	1.9168960	.5440862	1.8379442	.5669254	1.7639007	27
34	.5220468	1.9155370	.5444632	1.8366713	.5673098	1.7627053	26
35	.5224170	1.9141795	.5448404	1.8353999	.5676944	1.7615112	25
36	.5227874	1.9128236	.5452177	1.8341297	.5680791	1.7603183	24
37	.5231578	1.9114691	.5455951	1.8328610	.5684639	1.7591267	23
38	.5235284	1.9101162	.5459727	1.8315936	.5688488	1.7579362	22
39	.5238990	1.9087647	.5463503	1.8303275	.5692339	1.7567470	21
40	0.5242698	1.9074147	0.5467281	1.8290628	0.5696191	1.7555590	20
41	.5246407	1.9060663	.5471060	1.8277994	.5700045	1.7543722	19
42	.5250117	1.9047193	.5474840	1.8265374	.5703899	1.7531866	18
43	.5253829	1.9033738	.5478621	1.8252767	.5707755	1.7520023	17
44	.5257541	1.9020299	.5482404	1.8240173	.5711612	1.7508191	16
45	.5261255	1.9006874	.5486188	1.8227593	.5715471	1.7496371	15
46	.5264969	1.8993464	.5489973	1.8215026	.5719331	1.7484564	14
47	.5268685	1.8980068	.5493759	1.8202473	.5723192	1.7472768	13
48	.5272402	1.8966688	.5497547	1.8189932	.5727054	1.7460984	12
49	.5276120	1.8953322	.5501335	1.8177405	.5730918	1.7449213	11
50	0.5279839	1.8939971	0.5505125	1.8164892	0.5734783	1.7437453	10
51	.5283560	1.8926635	.5508916	1.8152391	.5738649	1.7425705	9
52	.5287281	1.8913313	.5512708	1.8139904	.5742516	1.7413969	8
53	.5291004	1.8900006	.5516502	1.8127430	.5746385	1.7402245	7
54	.5294727	1.8886713	.5520297	1.8114969	.5750255	1.7390533	6
55	.5298452	1.8873436	.5524093	1.8102521	.5754126	1.7378833	5
56	.5302178	1.8860172	.5527890	1.8090086	.5757999	1.7367144	4
57	.5305906	1.8846924	.5531688	1.8077664	.5761873	1.7355468	3
58	.5309634	1.8833690	.5535488	1.8065256	.5765748	1.7343803	2
59	.5313364	1.8820470	.5539288	1.8052860	.5769625	1.7332149	1
60	0.5317094	1.8807265	0.5543091	1.8040478	0.5773503	1.7320508	0
′	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	′
	62°		61°		60°		

TABLE XX (*Continued*)

NATURAL TANGENTS AND COTANGENTS

′	30°		31°		32°		′
	Tangent	Cotangent	Tangent	Cotangent	Tangent	Cotangent	
0	0.5773503	1.7320508	0.6008606	1.6642795	0.6248694	1.6003345	60
1	.5777382	1.7308878	.6012566	1.6631834	.6252739	1.5992991	59
2	.5781262	1.7297260	.6016527	1.6620884	.6256786	1.5982647	58
3	.5785144	1.7285654	.6020490	1.6609945	.6260834	1.5972312	57
4	.5789027	1.7274060	.6024454	1.6599016	.6264884	1.5961987	56
5	.5792912	1.7262477	.6028419	1.6588097	.6268935	1.5951672	55
6	.5796797	1.7250905	.6032386	1.6577189	.6272988	1.5941366	54
7	.5800684	1.7239346	.6036354	1.6566292	.6277042	1.5931070	53
8	.5804573	1.7227797	.6040323	1.6555405	.6281098	1.5920783	52
9	.5808462	1.7216261	.6044294	1.6544529	.6285155	1.5910505	51
10	0.5812353	1.7204736	0.6048266	1.6533663	0.6289215	1.5900238	50
11	.5816245	1.7193222	.6052240	1.6522808	.6293274	1.5889979	49
12	.5820139	1.7181720	.6056215	1.6511963	.6297336	1.5879731	48
13	.5824034	1.7170230	.6060192	1.6501128	.6301399	1.5869491	47
14	.5827930	1.7158751	.6064170	1.6490304	.6305464	1.5859261	46
15	.5831828	1.7147283	.6068149	1.6479490	.6309530	1.5849041	45
16	.5835726	1.7135827	.6072130	1.6468687	.6313598	1.5838830	44
17	.5839627	1.7124382	.6076112	1 6457893	.6317667	1.5828628	43
18	.5843528	1.7112949	.6080095	1.6447111	.6321738	1.5818436	42
19	.5847431	1.7101527	.6084080	1.6436338	.6325810	1.5808253	41
20	0.5851335	1.7090116	0.6088067	1.6425576	0.6329883	1.5798079	40
21	.5855241	1.7078717	.6092054	1.6414824	.6333959	1.5787915	39
22	.5859148	1.7067329	.6096043	1.6404082	.6338035	1.5777760	38
23	.5863056	1.7055953	.6100034	1.6393351	.6342113	1.5767615	37
24	.5866965	1.7044587	.6104026	1.6382630	.6346193	1.5757479	36
25	.5870876	1.7033233	.6108019	1.6371919	.6350274	1.5747352	35
26	.5874788	1.7021890	.6112014	1.6361218	.6354357	1.5737234	34
27	.5878702	1.7010599	.6116011	1.6350528	.6358441	1.5727126	33
28	.5882616	1.6999238	.6120008	1.6339847	.6362527	1.5717026	32
29	.5886533	1.6987929	.6124007	1.6329177	.6366614	1.5706936	31
30	0.5890450	1.6976631	0.6128008	1.6318517	0.6370703	1.5696856	30
31	.5894369	1.6965344	.6132010	1.6307867	.6374793	1.5686784	29
32	.5898289	1.6954069	.6136014	1.6297227	.6378885	1.5676722	28
33	.5902211	1.6942804	.6140018	1.6286597	.6382978	1.5666669	27
34	.5906134	1.6931550	.6144024	1.6275977	.6387073	1.5656625	26
35	.5910058	1.6920308	.6148032	1.6265368	.6391169	1.5646590	25
36	.5913984	1.6909077	.6152041	1.6254768	.6395267	1.5636564	24
37	.5917910	1.6897856	.6156052	1.6244178	.6399366	1.5626548	23
38	.5921839	1.6886647	.6160064	1.6233599	.6403467	1.5616540	22
39	.5925768	1.6875449	.6164077	1.6223029	.6407569	1.5606542	21
40	0.5929699	1.6864261	0.6168092	1.6212469	0.6411673	1.5596552	20
41	.5933632	1.6853085	.6172108	1.6201920	.6415779	1.5586572	19
42	.5937565	1.6841919	.6176126	1.6191380	.6419886	1.5576601	18
43	.5941501	1.6830765	.6180145	1.6180850	.6423994	1.5566639	17
44	.5945437	1.6819621	.6184166	1.6170330	.6428105	1.5556685	16
45	.5949375	1.6808489	.6188188	1.6159820	.6432216	1.5546741	15
46	.5953314	1.6797367	.6192211	1.6149320	.6436329	1.5536806	14
47	.5957255	1.6786256	.6196236	1.6138829	.6440444	1.5526880	13
48	.5961196	1.6775156	.6200263	1.6128349	.6444560	1.5516963	12
49	.5965140	1.6764067	.6204291	1.6117878	.6448678	1.5507054	11
50	0.5969084	1.6752988	0.6208320	1.6107417	0.6452797	1.5497155	10
51	.5973030	1.6741921	.6212351	1.6096966	.6456918	1.5487264	9
52	.5976978	1.6730864	.6216383	1.6086525	.6461041	1.5477383	8
53	.5980926	1.6719818	.6220417	1.6076094	.6465165	1.5467510	7
54	.5984877	1.6708782	.6224452	1.6065672	.6469290	1.5457647	6
55	.5988828	1.6697758	.6228488	1.6055260	.6473417	1.5447792	5
56	.5992781	1.6686744	.6232527	1.6044858	.6477546	1.5437946	4
57	.5996735	1.6675741	.6236566	1.6034465	.6481676	1.5428108	3
58	.6000691	1.6664748	.6240607	1.6024082	.6485808	1.5418280	2
59	.6004648	1.6653766	.6244650	1.6013709	.6489941	1.5408460	1
60	0.6008606	1.6642795	0.6248694	1.6003345	0.6494076	1.5398650	0
	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	
′	59°		58°		57°		′

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	33°		34°		35°		′
	Tangent	Cotangent	Tangent	Cotangent	Tangent	Cotangent	
0	0.6494076	1.5398650	0.6745085	1.4825610	0.7002075	1.4281480	60
1	.6498212	1.5388848	.6749318	1.4816311	.7006411	1.4272642	59
2	.6502350	1.5379054	.6753553	1.4807021	.7010749	1.4263811	58
3	.6506490	1.5369270	.6757790	1.4797738	.7015089	1.4254988	57
4	.6510631	1.5359494	.6762028	1.4788463	.7019430	1.4246171	56
5	.6514774	1.5349727	.6766268	1.4779197	.7023773	1.4237362	55
6	.6518918	1.5339969	.6770509	1.4769938	.7028118	1.4228561	54
7	.6523064	1.5330219	.6774752	1.4760688	.7032464	1.4219766	53
8	.6527211	1.5320479	.6778997	1.4751445	.7036813	1.4210979	52
9	.6531360	1.5310746	.6783243	1.4742210	.7041163	1.4202200	51
10	0.6535511	1.5301023	0.6787492	1.4732983	0.7045515	1.4193427	50
11	.6539663	1.5291308	.6791741	1.4723764	.7049869	1.4184662	49
12	.6543817	1.5281602	.6795993	1.4714553	.7054224	1.4175904	48
13	.6547972	1.5271904	.6800246	1.4705350	.7058581	1.4167153	47
14	.6552129	1.5262215	.6804501	1.4696155	.7062940	1.4158409	46
15	.6556287	1.5252535	.6808758	1.4686967	.7067301	1.4149673	45
16	.6560447	1.5242863	.6813016	1.4677788	.7071664	1.4140943	44
17	.6564609	1.5233200	.6817276	1.4668616	.7076028	1.4132221	43
18	.6568772	1.5223545	.6821537	1.4659452	.7080395	1.4123506	42
19	.6572937	1.5213899	.6825801	1.4650296	.7084763	1.4114799	41
20	0.6577103	1.5204261	0.6830066	1.4641147	0.7089133	1.4106098	40
21	.6581271	1.5194632	.6834333	1.4632007	.7093504	1.4097405	39
22	.6585441	1.5185012	.6838601	1.4622874	.7097878	1.4088718	38
23	.6589612	1.5175400	.6842871	1.4613749	.7102253	1.4080039	37
24	.6593785	1.5165796	.6847143	1.4604632	.7106630	1.4071367	36
25	.6597960	1.5156201	.6851416	1.4595522	.7111009	1.4062702	35
26	.6602136	1.5146614	.6855692	1.4586420	.7115390	1.4054044	34
27	.6606313	1.5137036	.6859969	1.4577326	.7119772	1.4045393	33
28	.6610492	1.5127466	.6864247	1.4568240	.7124157	1.4036749	32
29	.6614673	1.5117905	.6868528	1.4559161	.7128543	1.4028113	31
30	0.6618856	1.5108352	0.6872810	1.4550090	0.7132931	1.4019483	30
31	.6623040	1.5098807	.6877093	1.4541027	.7137320	1.4010860	29
32	.6627225	1.5089271	.6881379	1.4531971	.7141712	1.4002245	28
33	.6631413	1.5079743	.6885666	1.4522923	.7146106	1.3993636	27
34	.6635601	1.5070224	.6889955	1.4513883	.7150501	1.3985034	26
35	.6639792	1.5060713	.6894246	1.4504850	.7154898	1.3976440	25
36	.6643984	1.5051210	.6898538	1.4495825	.7159297	1.3967852	24
37	.6648178	1.5041716	.6902832	1.4486808	.7163698	1.3959272	23
38	.6652373	1.5032229	.6907128	1.4477798	.7168100	1.3950698	22
39	.6656570	1.5022751	.6911425	1.4468796	.7172505	1.3942131	21
40	0.6660769	1.5013282	0.6915725	1.4459801	0.7176911	1.3933571	20
41	.6664969	1.5003821	.6920026	1.4450814	.7181319	1.3925019	19
42	.6669171	1.4994367	.6024328	1.4441834	.7185729	1.3916473	18
43	.6673374	1.4984923	.6928633	1.4432862	.7190140	1.3907934	17
44	.6677580	1.4975486	.6932939	1.4423897	.7194554	1.3899401	16
45	.6681786	1.4966058	.6937247	1.4414940	.7198970	1.3890876	15
46	.6685995	1.4956637	.6941557	1.4405991	.7203387	1.3882358	14
47	.6690205	1.4947225	.6945868	1.4397049	.7207806	1.3873847	13
48	.6694417	1.4937822	.6950181	1.4388114	.7212227	1.3865342	12
49	.6698630	1.4928426	.6954496	1.4379187	.7216650	1.3856844	11
50	0.6702845	1.4919039	0.6958813	1.4370268	0.7221075	1.3848353	10
51	.6707061	1.4909659	.6963131	1.4361356	.7225502	1.3839869	9
52	.6711280	1.4900288	.6967451	1.4352451	.7229930	1.3831392	8
53	.6715500	1.4890925	.6971773	1.4343554	.7234361	1.3822922	7
54	.6719721	1.4881570	.6976097	1.4334664	.7238793	1.3814458	6
55	.6723944	1.4872223	.6980422	1.4325781	.7243227	1.3806001	5
56	.6728169	1.4862884	.6984749	1.4316906	.7247663	1.3797551	4
57	.6732396	1.4853554	.6989078	1.4308039	.7252101	1.3789108	3
58	.6736624	1.4844231	.6993409	1.4299178	.7256540	1.3780672	2
59	.6740854	1.4834916	.6997741	1.4290326	.7260982	1.3772242	1
60	0.6745085	1.4825610	0.7002075	1.4281480	0.7265425	1.3763819	0
′	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	′
	56°		55°		54°		

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	36°		37°		38°		′
	Tangent	Cotangent	Tangent	Cotangent	Tangent	Cotangent	
0	0.7265425	1.3763819	0.7535541	1.3270448	0.7812856	1.2799416	60
1	.7269871	1.3755403	.7540102	1.3262420	.7817542	1.2791745	59
2	.7274318	1.3746994	.7544666	1.3254397	.7822229	1.2784079	58
3	.7278767	1.3738591	.7549232	1.3246381	.7826919	1.2776419	57
4	.7283218	1.3730195	.7553799	1.3238371	.7831611	1.2768765	56
5	.7287671	1.3721806	.7558369	1.3230368	.7836305	1.2761116	55
6	.7292125	1.3713423	.7562941	1.3222370	.7841002	1.2753473	54
7	.7296582	1.3705047	.7567514	1.3214379	.7845700	1.2745835	53
8	.7301041	1.3696678	.7572090	1.3206393	.7850400	1.2738204	52
9	.7305501	1.3688315	.7576668	1.3198414	.7855103	1.2730578	51
10	0.7309963	1.3679959	0.7581248	1.3190441	0.7859808	1.2722957	50
11	.7314428	1.3671610	.7585829	1.3182474	.7864515	1.2715342	49
12	.7318894	1.3663267	.7590413	1.3174513	.7869224	1.2707733	48
13	.7323362	1.3654931	.7594999	1.3166559	.7873935	1.2700130	47
14	.7327832	1.3646602	.7599587	1.3158610	.7878649	1.2692532	46
15	.7332303	1.3638279	.7604177	1.3150668	.7883364	1.2684940	45
16	.7336777	1.3629963	.7608769	1.3142731	.7888082	1.2677353	44
17	.7341253	1.3621653	.7613363	1.3134801	.7892802	1.2669772	43
18	.7345730	1.3613350	.7617959	1.3126876	.7897524	1.2662196	42
19	.7350210	1.3605054	.7622557	1.3118958	.7902248	1.2654626	41
20	0.7354691	1.3596764	0.7627157	1.3111046	0.7906975	1.2647062	40
21	.7359174	1.3588481	.7631759	1.3103140	.7911703	1.2639503	39
22	.7363660	1.3580204	.7636363	1.3095239	.7916434	1.2631950	38
23	.7368147	1.3571934	.7640969	1.3087345	.7921167	1.2624402	37
24	.7372636	1.3563670	.7645577	1.3079457	.7925902	1.2616860	36
25	.7377127	1.3555413	.7650188	1.3071575	.7930640	1.2609323	35
26	.7381620	1.3547162	.7654800	1.3063699	.7935379	1.2601792	34
27	.7386115	1.3538918	.7659414	1.3055828	.7940121	1.2594267	33
28	.7390611	1.3530680	.7664031	1.3047964	.7944865	1.2586747	32
29	.7395110	1.3522449	.7668649	1.3040106	.7949611	1.2579232	31
30	0.7399611	1.3514224	0.7673270	1.3032254	0.7954359	1.2571723	30
31	.7404113	1.3506006	.7677893	1.3024408	.7959110	1.2564219	29
32	.7408618	1.3497794	.7682517	1.3016567	.7963862	1.2556721	28
33	.7413124	1.3489589	.7687144	1.3008733	.7968617	1.2549229	27
34	.7417633	1.3481390	.7691773	1.3000904	.7973374	1.2541742	26
35	.7422143	1.3473198	.7696404	1.2993081	.7978134	1.2534260	25
36	.7426655	1.3465011	.7701037	1.2985265	.7982895	1.2526784	24
37	.7431170	1.3456832	.7705672	1.2977454	.7987659	1.2519313	23
38	.7435686	1.3448658	.7710309	1.2969649	.7992425	1.2511848	22
39	.7440204	1.3440492	.7714948	1.2961850	.7997193	1.2504388	21
40	0.7444724	1.3432331	0.7719589	1.2954057	0.8001963	1.2496933	20
41	.7449246	1.3424177	.7724233	1.2946270	.8006736	1.2489484	19
42	.7453770	1.3416029	.7728878	1.2938488	.8011511	1.2482040	18
43	.7458296	1.3407888	.7733526	1.2930713	.8016288	1.2474602	17
44	.7462824	1.3399753	.7738176	1.2922943	.8021067	1.2467169	16
45	.7467354	1.3391624	.7742827	1.2915179	.8025849	1.2459742	15
46	.7471886	1.3383502	.7747481	1.2907421	.8030632	1.2452320	14
47	.7476420	1.3375386	.7752137	1.2899669	.8035418	1.2444903	13
48	.7480956	1.3367276	.7756795	1.2891922	.8040206	1.2437492	12
49	.7485494	1.3359172	.7761455	1.2884182	.8044997	1.2430086	11
50	0.7490033	1.3351075	0.7766118	1.2876447	0.8049790	1.2422685	10
51	.7494575	1.3342984	.7770782	1.2868718	.8054584	1.2415290	9
52	.7499119	1.3334900	.7775448	1.2860995	.8059382	1.2407900	8
53	.7503665	1.3326822	.7780117	1.2853277	.8064181	1.2400515	7
54	.7508212	1.3318750	.7784788	1.2845566	.8068983	1.2393136	6
55	.7512762	1.3310684	.7789460	1.2837860	.8073787	1.2385762	5
56	.7517314	1.3302624	.7794135	1.2830160	.8078593	1.2378393	4
57	.7521867	1.3294571	.7798812	1.2822466	.8083401	1.2371030	3
58	.7526423	1.3286524	.7803492	1.2814776	.8088212	1.2363672	2
59	.7530981	1.3278483	.7808173	1.2807094	.8093025	1.2356319	1
60	0.7535541	1.3270448	0.7812856	1.2799416	0.8097840	1.2348972	0
	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	
′	53°		52°		51°		′

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	39° Tangent	39° Cotangent	40° Tangent	40° Cotangent	41° Tangent	41° Cotangent	′
0	0.8097840	1.2348972	0.8390996	1.1917536	0.8692867	1.1503684	60
1	.8102658	1.2341629	.8395955	1.1910498	.8697976	1.1496928	59
2	.8107478	1.2334292	.8400915	1.1903465	.8703087	1.1490176	58
3	.8112300	1.2326961	.8405878	1.1896437	.8708200	1.1483429	57
4	.8117124	1.2319634	.8410844	1.1889414	.8713316	1.1476687	56
5	.8121951	1.2312313	.8415812	1.1882395	.8718435	1.1469949	55
6	.8126780	1.2304997	.8420782	1.1875382	.8723556	1.1463215	54
7	.8131611	1.2297687	.8425755	1.1868373	.8728680	1.1456486	53
8	.8136444	1.2290381	.8430730	1.1861369	.8733806	1.1449762	52
9	.8141280	1.2283081	.8435708	1.1854370	.8738935	1.1443041	51
10	0.8146118	1.2275786	0.8440688	1.1847376	0.8744067	1.1436326	50
11	.8150958	1.2268496	.8445670	1.1840387	.8749201	1.1429615	49
12	.8155801	1.2261211	.8450655	1.1833402	.8754338	1.1422908	48
13	.8160646	1.2253932	.8455643	1.1826422	.8759478	1.1416206	47
14	.8165493	1.2246658	.8460633	1.1819447	.8764620	1.1409508	46
15	.8170343	1.2239389	.8465625	1.1812477	.8769765	1.1402815	45
16	.8175195	1.2232125	.8470620	1.1805512	.8774912	1.1396126	44
17	.8180049	1.2224866	.8475617	1.1798551	.8780062	1.1389441	43
18	.8184905	1.2217613	.8480617	1.1791595	.8785215	1.1382761	42
19	.8189764	1.2210364	.8485619	1.1784644	.8790370	1.1376086	41
20	0.8194625	1.2203121	0.8490624	1.1777698	0.8795528	1.1369414	40
21	.8199488	1.2195883	.8495631	1.1770756	.8800688	1.1362747	39
22	.8204354	1.2188650	.8500640	1.1763820	.8805852	1.1356085	38
23	.8209222	1.2181422	.8505653	1.1756888	.8811017	1.1349427	37
24	.8214093	1.2174199	.8510667	1.1749960	.8816186	1.1342773	36
25	.8218965	1.2166982	.8515684	1.1743038	.8821357	1.1336124	35
26	.8223840	1.2159769	.8520704	1.1736120	.8826531	1.1329479	34
27	.8228718	1.2152562	.8525726	1.1729207	.8831707	1.1322839	33
28	.8233597	1.2145359	.8530750	1.1722298	.8836886	1.1316203	32
29	.8238479	1.2138162	.8535777	1.1715395	.8842068	1.1309571	31
30	0.8243364	1.2130970	0.8540807	1.1708496	0.8847253	1.1302944	30
31	.8248251	1.2123783	.8545839	1.1701601	.8852440	1.1296321	29
32	.8253140	1.2116601	.8550873	1.1694712	.8857630	1.1289702	28
33	.8258031	1.2109424	.8555910	1.1687827	.8862822	1.1283088	27
34	.8262925	1.2102252	.8560950	1.1680947	.8868017	1.1276478	26
35	.8267821	1.2095085	.8565992	1.1674071	.8873215	1.1269872	25
36	.8272719	1.2087924	.8571037	1.1667200	.8878415	1.1263271	24
37	.8277620	1.2080767	.8576084	1.1660334	.8883619	1.1256674	23
38	.8282523	1.2073615	.8581133	1.1653472	.8888825	1.1250081	22
39	.8287429	1.2066468	.8586185	1.1646615	.8894033	1.1243493	21
40	0.8292337	1.2059327	0.8591240	1.1639763	0.8899244	1.1236909	20
41	.8297247	1.2052190	.8596297	1.1632916	.8904458	1.1230329	19
42	.8302160	1.2045058	.8601357	1.1626073	.8909675	1.1223754	18
43	.8307075	1.2037932	.8606419	1.1619234	.8914894	1.1217183	17
44	.8311992	1.2030810	.8611484	1.1612400	.8920116	1.1210616	16
45	.8316912	1.2023693	.8616551	1.1605571	.8925341	1.1204053	15
46	.8321834	1.2016581	.8621621	1.1598747	.8930569	1.1197495	14
47	.8326759	1.2009475	.8626694	1.1591927	.8935799	1.1190941	13
48	.8331686	1.2002373	.8631768	1.1585112	.8941032	1.1184391	12
49	.8336615	1.1995276	.8636846	1.1578301	.8946268	1.1177846	11
50	0.8341547	1.1988184	0.8641926	1.1571495	0.8951506	1.1171305	10
51	.8346481	1.1981097	.8647009	1.1564693	.8956747	1.1164768	9
52	.8351418	1.1974015	.8652094	1.1557896	.8961991	1.1158235	8
53	.8356357	1.1966938	.8657181	1.1551104	.8967238	1.1151706	7
54	.8361298	1.1959866	.8662272	1.1544316	.8972487	1.1145182	6
55	.8366242	1.1952799	.8667365	1.1537532	.8977739	1.1138662	5
56	.8371188	1.1945736	.8672460	1.1530754	.8982994	1.1132146	4
57	.8376136	1.1938679	.8677558	1.1523979	.8988251	1.1125635	3
58	.8381087	1.1931626	.8682659	1.1517210	.8993512	1.1119127	2
59	.8386041	1.1924579	.8687762	1.1510445	.8998775	1.1112624	1
60	0.8390996	1.1917536	0.8692867	1.1503684	0.9004040	1.1106125	0
′	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	′
	50°		49°		48°		

TABLE XX (*Continued*)

Natural Tangents and Cotangents

′	42°		43°		44°		′
	Tangent	Cotangent	Tangent	Cotangent	Tangent	Cotangent	
0	0.9004040	1.1106125	0.9325151	1.0723687	0.9656888	1.0355303	60
1	.9009309	1.1099630	.9330591	1.0717435	.9662511	1.0349277	59
2	.9014580	1.1093140	.9336034	1.0711187	.9668137	1.0343254	58
3	.9019854	1.1086653	.9341479	1.0704943	.9673767	1.0337235	57
4	.9025131	1.1080171	.9346928	1.0698702	.9679399	1.0331220	56
5	.9030411	1.1073693	.9352380	1.0692466	.9685035	1.0325208	55
6	.9035693	1.1067219	.9357834	1.0686233	.9690674	1.0319199	54
7	.9040979	1.1060750	.9363292	1.0680004	.9696316	1.0313195	53
8	.9046267	1.1054284	.9368753	1.0673779	.9701962	1.0307194	52
9	.9051557	1.1047823	.9374216	1.0667558	.9707610	1.0301196	51
10	0.9056851	1.1041365	0.9379683	1.0661341	0.9713262	1.0295203	50
11	.9062147	1.1034913	.9385153	1.0655128	.9718917	1.0289212	49
12	.9067446	1.1028463	.9390625	1.0648918	.9724575	1.0283226	48
13	.9072748	1.1022019	.9396101	1.0642713	.9730236	1.0277243	47
14	.9078053	1.1015578	.9401579	1.0636511	.9735901	1.0271263	46
15	.9083360	1.1009141	.9407061	1.0630313	.9741569	1.0265287	45
16	.9088671	1.1002709	.9412545	1.0624119	.9747240	1.0259315	44
17	.9093984	1.0996281	.9418033	1.0617929	.9752914	1.0253346	43
18	.9099300	1.0989857	.9423523	1.0611742	.9758591	1.0247381	42
19	.9104619	1.0983436	.9429017	1.0605560	.9764272	1.0241419	41
20	0.9109940	1.0977020	0.9434513	1.0599381	0.9769956	1.0235461	40
21	.9115265	1.0970609	.9440013	1.0593206	.9775643	1.0229506	39
22	.9120592	1.0964201	.9445516	1.0587035	.9781333	1.0223555	38
23	.9125922	1.0957797	.9451021	1.0580867	.9787027	1.0217608	37
24	.9131255	1.0951397	.9456530	1.0574704	.9792724	1.0211664	36
25	.9136591	1.0945002	.9462042	1.0568544	.9798424	1.0205723	35
26	.9141929	1.0938610	.9467556	1.0562388	.9804127	1.0199786	34
27	.9147270	1.0932223	.9473074	1.0556235	.9809833	1.0193853	33
28	.9152615	1.0925840	.9478595	1.0550087	.9815543	1.0187923	32
29	.9157962	1.0919460	.9484119	1.0543942	.9821256	1.0181997	31
30	0.9163312	1.0913085	0.9489646	1.0537801	0.9826973	1.0176074	30
31	.9168665	1.0906714	.9495176	1.0531664	.9832692	1.0170155	29
32	.9174020	1.0900347	.9500709	1.0525531	.9838415	1.0164239	28
33	.9179379	1.0893984	.9506245	1.0519401	.9844141	1.0158326	27
34	.9184740	1.0887624	.9511784	1.0513275	.9849871	1.0152418	26
35	.9190104	1.0881269	.9517326	1.0507153	.9855603	1.0146512	25
36	.9195471	1.0874918	.9522871	1.0501034	.9861339	1.0140610	24
37	.9200841	1.0868571	.9528420	1.0494920	.9867079	1.0134712	23
38	.9206214	1.0862228	.9533971	1.0488809	.9872821	1.0128817	22
39	.9211590	1.0855889	.9539526	1.0482702	.9878567	1.0122925	21
40	0.9216969	1.0849554	0.9545083	1.0476598	0.9884316	1.0117038	20
41	.9222350	1.0843223	.9550644	1.0470498	.9890069	1.0111153	19
42	.9227734	1.0836896	.9556208	1.0464402	.9895825	1.0105272	18
43	.9233122	1.0830573	.9561774	1.0458310	.9901584	1.0099394	17
44	.9238512	1.0824254	.9567344	1.0452221	.9907346	1.0093520	16
45	.9243905	1.0817939	.9572917	1.0446136	.9913112	1.0087649	15
46	.9249301	1.0811628	.9578494	1.0440055	.9918881	1.0081782	14
47	.9254700	1.0805321	.9584073	1.0433977	.9924654	1.0075918	13
48	.9260102	1.0799018	.9589655	1.0427904	.9930429	1.0070058	12
49	.9265506	1.0792718	.9595241	1.0421833	.9936208	1.0064201	11
50	0.9270914	1.0786423	0.9600829	1.0415767	0.9941991	1.0058348	10
51	.9276324	1.0780132	.9606421	1.0409704	.9947777	1.0052497	9
52	.9281738	1.0773845	.9612016	1.0403645	.9953566	1.0046651	8
53	.9287154	1.0767561	.9617614	1.0397589	.9959358	1.0040808	7
54	.9292573	1.0761282	.9623215	1.0391538	.9965154	1.0034968	6
55	.9297996	1.0755006	.9628819	1.0385489	.9970953	1.0029131	5
56	.9303421	1.0748734	.9634427	1.0379445	.9976756	1.0023298	4
57	.9308849	1.0742467	.9640037	1.0373404	.9982562	1.0017469	3
58	.9314280	1.0736203	.9645651	1.0367367	.9988371	1.0011642	2
59	.9319714	1.0729943	.9651268	1.0361333	.9994184	1.0005819	1
60	0.9325151	1.0723687	0.9656888	1.0355303	1.0000000	1.0000000	0
′	Cotangent	Tangent	Cotangent	Tangent	Cotangent	Tangent	′
	47°		46°		45°		

TABLE XXI

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
1	1	1	1.0000000	1.0000000	1.000000000
2	4	8	1.4142136	1.2599210	.500000000
3	9	27	1.7320508	1.4422496	.333333333
4	16	64	2.0000000	1.5874011	.250000000
5	25	125	2.2360680	1.7099759	.200000000
6	36	216	2.4494897	1.8171206	.166666667
7	49	343	2.6457513	1.9129312	.142857143
8	64	512	2.8284271	2.0000000	.125000000
9	81	729	3.0000000	2.0800837	.111111111
10	100	1000	3.1622777	2.1544347	.100000000
11	121	1331	3.3166248	2.2239801	.090909091
12	144	1728	3.4641016	2.2894286	.083333333
13	169	2197	3.6055513	2.3513347	.076923077
14	196	2744	3.7416574	2.4101422	.071428571
15	225	3375	3.8729833	2.4662121	.066666667
16	256	4096	4.0000000	2.5198421	.062500000
17	289	4913	4.1231056	2.5712816	.058823529
18	324	5832	4.2426407	2.6207414	.055555556
19	361	6859	4.3588989	2.6684016	.052631579
20	400	8000	4.4721360	2.7144177	.050000000
21	441	9261	4.5825757	2.7589243	.047619048
22	484	10648	4.6904158	2.8020393	.045454545
23	529	12167	4.7958315	2.8438670	.043478261
24	576	13824	4.8989795	2.8844991	.041666667
25	625	15625	5.0000000	2.9240177	.040000000
26	676	17576	5.0990195	2.9624960	.038461538
27	729	19683	5.1961524	3.0000000	.037037037
28	784	21952	5.2915026	3.0365889	.035714286
29	841	24389	5.3851648	3.0723168	.034482759
30	900	27000	5.4772256	3.1072325	.033333333
31	961	29791	5.5677644	3.1413806	.032258065
32	1024	32768	5.6568542	3.1748021	.031250000
33	1089	35937	5.7445626	3.2075343	.030303030
34	1156	39304	5.8309519	3.2396118	.029411765
35	1225	42875	5.9160798	3.2710663	.028571429
36	1296	46656	6.0000000	3.3019272	.027777778
37	1369	50653	6.0827625	3.3322218	.027027027
38	1444	54872	6.1644140	3.3619754	.026315789
39	1521	59319	6.2449980	3.3912114	.025641026
40	1600	64000	6.3245553	3.4199519	.025000000
41	1681	68921	6.4031242	3.4482172	.024390244
42	1764	74088	6.4807407	3.4760266	.023809524
43	1849	79507	6.5574385	3.5033981	.023255814
44	1936	85184	6.6332496	3.5303483	.022727273
45	2025	91125	6.7082039	3.5568933	.022222222
46	2116	97336	6.7823300	3.5830479	.021739130
47	2209	103823	6.8556546	3.6088261	.021276600
48	2304	110592	6.9282032	3.6342411	.020833333
49	2401	117649	7.0000000	3.6593057	.020408163
50	2500	125000	7.0710678	3.6840314	.020000000
51	2601	132651	7.1414284	3.7084298	.019607843
52	2704	140608	7.2111026	3.7325111	.019230769
53	2809	148877	7.2801099	3.7562858	.018867925
54	2916	157464	7.3484692	3.7797631	.018518519
55	3025	166375	7.4161985	3.8029525	.018181818
56	3136	175616	7.4833148	3.8258624	.017857143
57	3249	185193	7.5498344	3.8485011	.017543860
58	3364	195112	7.6157731	3.8708766	.017241379
59	3481	205379	7.6811457	3.8929965	.016949153
60	3600	216000	7.7459667	3.9148676	.016666667
61	3721	226981	7.8102497	3.9364972	.016393443
62	3844	238328	7.8740079	3.9578915	.016129032

TABLE XXI (*Continued*)

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
63	3969	250047	7.9372539	3.9790571	.015873016
64	4096	262144	8.0000000	4.0000000	.015625000
65	4225	274625	8.0622577	4.0207256	.015384615
66	4356	287496	8.1240384	4.0412401	.015151515
67	4489	300763	8.1853528	4.0615480	.014925373
68	4624	314432	8.2462113	4.0816551	.014705882
69	4761	328509	8.3066239	4.1015661	.014492754
70	4900	343000	8.3666003	4.1212853	.014285714
71	5041	357911	8.4261498	4.1408178	.014084507
72	5184	373248	8.4852814	4.1601676	.013888889
73	5329	389017	8.5440037	4.1793390	.013698630
74	5476	405224	8.6023253	4.1983364	.013513514
75	5625	421875	8.6602540	4.2171633	.013333333
76	5776	438976	8.7177979	4.2358236	.013157895
77	5929	456533	8.7749644	4.2543210	.012987013
78	6084	474552	8.8317609	4.2726586	.012820513
79	6241	493039	8.8881944	4.2908404	.012658228
80	6400	512000	8.9442719	4.3088695	.012500000
81	6561	531441	9.0000000	4.3267487	.012345679
82	6724	551368	9.0553851	4.3444815	.012195122
83	6889	571787	9.1104336	4.3620707	.012048193
84	7056	592704	9.1651514	4.3795191	.011904762
85	7225	614125	9.2195445	4.3968296	.011764706
86	7396	636056	9.2736185	4.4140049	.011627907
87	7569	658503	9.3273791	4.4310476	.011494253
88	7744	681472	9.3808315	4.4479602	.011363636
89	7921	704969	9.4339811	4.4647451	.011235955
90	8100	729000	9.4868330	4.4814047	.011111111
91	8281	753571	9.5393920	4.4979414	.010989011
92	8464	778688	9.5916630	4.5143574	.010869565
93	8649	804357	9.6436508	4.5306549	.010752688
94	8836	830584	9.6953597	4.5468359	.010638298
95	9025	857375	9.7467943	4.5629026	.010526316
96	9216	884736	9.7979590	4.5788570	.010416667
97	9409	912673	9.8488578	4.5947009	.010309278
98	9604	941192	9.8994949	4.6104363	.010204082
99	9801	970299	9.9498744	4.6260650	.010101010
100	10000	1000000	10.0000000	4.6415888	.010000000
101	10201	1030301	10.0498756	4.6570095	.009900990
102	10404	1061208	10.0995049	4.6723287	.009803922
103	10609	1092727	10.1488916	4.6875482	.009708738
104	10816	1124864	10.1980390	4.7026694	.009615385
105	11025	1157625	10.2469508	4.7176940	.009523810
106	11236	1191016	10.2956301	4.7326235	.009433962
107	11449	1225043	10.3440804	4.7474594	.009345794
108	11664	1259712	10.3923048	4.7622032	.009259259
109	11881	1295029	10.4403065	4.7768562	.009174312
110	12100	1331000	10.4880885	4.7914199	.009090909
111	12321	1367631	10.5356538	4.8058955	.009009009
112	12544	1404928	10.5830052	4.8202845	.008928571
113	12769	1442897	10.6301458	4.8345881	.008849558
114	12996	1481544	10.6770783	4.8488076	.008771930
115	13225	1520875	10.7238053	4.8629442	.008695652
116	13456	1560896	10.7703296	4.8769990	.008620690
117	13689	1601613	10.8166538	4.8909732	.008547009
118	13924	1643032	10.8627805	4.9048681	.008474576
119	14161	1685159	10.9087121	4.9186847	.008403361
120	14400	1728000	10.9544512	4.9324242	.008333333
121	14641	1771561	11.0000000	4.9460874	.008264463
122	14884	1815848	11.0453610	4.9596757	.008196721
123	15129	1860867	11.0905365	4.9731898	.008130081
124	15376	1906624	11.1355287	4.9866310	.008064516

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
125	15625	1953125	11.1803399	5.0000000	.008000000
126	15876	2000376	11.2249722	5.0132979	.007936508
127	16129	2048383	11.2694277	5.0265257	.007874016
128	16384	2097152	11.3137085	5.0396842	.007812500
129	16641	2146689	11.3578167	5.0527743	.007751938
130	16900	2197000	11.4017543	5.0657970	.007692308
131	17161	2248091	11.4455231	5.0787531	.007633588
132	17424	2299968	11.4891253	5.0916434	.007575758
133	17689	2352637	11.5325626	5.1044687	.007518797
134	17956	2406104	11.5758369	5.1172299	.007462687
135	18225	2460375	11.6189500	5.1299278	.007407407
136	18496	2515456	11.6619038	5.1425632	.007352941
137	18769	2571353	11.7046999	5.1551367	.007299270
138	19044	2628072	11.7473401	5.1676493	.007246377
139	19321	2685619	11.7898261	5.1801015	.007194245
140	19600	2744000	11.8321596	5.1924941	.007142857
141	19881	2803221	11.8743421	5.2048279	.007092199
142	20164	2863288	11.9163753	5.2171034	.007042254
143	20449	2924207	11.9582607	5.2293215	.006993007
144	20736	2985984	12.0000000	5.2414828	.006944444
145	21025	3048625	12.0415946	5.2535879	.006896552
146	21316	3112136	12.0830460	5.2656374	.006849315
147	21609	3176523	12.1243557	5.2776321	.006802721
148	21904	3241792	12.1655251	5.2895725	.006756757
149	22201	3307949	12.2065556	5.3014592	.006711409
150	22500	3375000	12.2474487	5.3132928	.006666667
151	22801	3442951	12.2882057	5.3250740	.006622517
152	23104	3511808	12.3288280	5.3368033	.006578947
153	23409	3581577	12.3693169	5.3484812	.006535948
154	23716	3652264	12.4096736	5.3601084	.006493506
155	24025	3723875	12.4498996	5.3716854	.006451613
156	24336	3796416	12.4899960	5.3832126	.006410256
157	24649	3869893	12.5299641	5.3946907	.006369427
158	24964	3944312	12.5698051	5.4061202	.006329114
159	25281	4019679	12.6095202	5.4175015	.006289308
160	25600	4096000	12.6491106	5.4288352	.006250000
161	25921	4173281	12.6885775	5.4401218	.006211180
162	26244	4251528	12.7279221	5.4513618	.006172840
163	26569	4330747	12.7671453	5.4625556	.006134969
164	26896	4410944	12.8062485	5.4737037	.006097561
165	27225	4492125	12.8452326	5.4848066	.006060606
166	27556	4574296	12.8840987	5.4958647	.006024096
167	27889	4657463	12.9228480	5.5068784	.005988024
168	28224	4741632	12.9614814	5.5178484	.005952381
169	28561	4826809	13.0000000	5.5287748	.005917160
170	28900	4913000	13.0384048	5.5396583	.005882353
171	29241	5000211	13.0766968	5.5504991	.005847953
172	29584	5088448	13.1148770	5.5612978	.005813953
173	29929	5177717	13.1529464	5.5720546	.005780347
174	30276	5268024	13.1909060	5.5827702	.005747126
175	30625	5359375	13.2287566	5.5934447	.005714286
176	30976	5451776	13.2664992	5.6040787	.005681818
177	31329	5545233	13.3041347	5.6146724	.005649718
178	31684	5639752	13.3416641	5.6252263	.005617978
179	32041	5735339	13.3790882	5.6357408	.005586592
180	32400	5832000	13.4164079	5.6462162	.005555556
181	32761	5929741	13.4536240	5.6566528	.005524862
182	33124	6028568	13.4907376	5.6670511	.005494505
183	33489	6128487	13.5277493	5.6774114	.005464481
184	33856	6229504	13.5646600	5.6877340	.005434783
185	34225	6331625	13.6014705	5.6980192	.005405405
186	34596	6434856	13.6381817	5.7082675	.005376344

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
187	34969	6539203	13.6747943	5.7184791	.005347594
188	35344	6644672	13.7113092	5.7286543	.005319149
189	35721	6751269	13.7477271	5.7387936	.005291005
190	36100	6859000	13.7840488	5.7488971	.005263158
191	36481	6967871	13.8202750	5.7589652	.005235602
192	36864	7077888	13.8564065	5.7689982	.005208333
193	37249	7189057	13.8924440	5.7789966	.005181347
194	37636	7301384	13.9283883	5.7889604	.005154639
195	38025	7414875	13.9642400	5.7988900	.005128205
196	38416	7529536	14.0000000	5.8087857	.005102041
197	38809	7645373	14.0356688	5.8186479	.005076142
198	39204	7762392	14.0712473	5.8284767	.005050505
199	39601	7880599	14.1067360	5.8382725	.005025126
200	40000	8000000	14.1421356	5.8480355	.005000000
201	40401	8120601	14.1774469	5.8577660	.004975124
202	40804	8242408	14.2126704	5.8674643	.004950495
203	41209	8365427	14.2478068	5.8771307	.004926108
204	41616	8489664	14.2828569	5.8867653	.004901961
205	42025	8615125	14.3178211	5.8963685	.004878049
206	42436	8741816	14.3527001	5.9059406	.004854369
207	42849	8869743	14.3874946	5.9154817	.004830918
208	43264	8998912	14 4222051	5.9249921	.004807692
209	43681	9129329	14.4568323	5.9344721	.004784689
210	44100	9261000	14.4913767	5.9439220	.004761905
211	44521	9393931	14.5258390	5.9533418	.004739336
212	44944	9528128	14.5602198	5.9627320	.004716981
213	45369	9663597	14.5945195	5.9720926	.004694836
214	45796	9800344	14.6287388	5.9814240	.004672897
215	46225	9938375	14.6628783	5.9907264	.004651163
216	46656	10077696	14.6969385	6.0000000	.004629630
217	47089	10218313	14.7309199	6.0092450	.004608295
218	47524	10360232	14.7648231	6.0184617	.004587156
219	47961	10503459	14.7986486	6.0276502	.004566210
220	48400	10648000	14.8323970	6.0368107	.004545455
221	48841	10793861	14.8660687	6.0459435	.004524887
222	49284	10941048	14.8996644	6.0550489	.004504505
223	49729	11089567	14.9331845	6.0641270	.004484305
224	50176	11239424	14.9666295	6.0731779	.004464286
225	50625	11390625	15.0000000	6.0822020	.004444444
226	51076	11543176	15.0332964	6.0911994	.004424779
227	51529	11697083	15.0665192	6.1001702	.004405286
228	51984	11852352	15.0996689	6.1091147	.004385965
229	52441	12008989	15.1327460	6.1180332	.004366812
230	52900	12167000	15.1657509	6.1269257	.004347826
231	53361	12326391	15.1986842	6.1357924	.004329004
232	53824	12487168	15.2315462	6.1446337	.004310345
233	54289	12649337	15.2643375	6.1534495	.004291845
234	54756	12812904	15.2970585	6.1622401	.004273504
235	55225	12977875	15.3297097	6.1710058	.004255319
236	55696	13144256	15.3622915	6.1797466	.004237288
237	56169	13312053	15.3948043	6.1884628	.004219409
238	56644	13481272	15.4272486	6.1971544	.004201681
239	57121	13651919	15.4596248	6.2058218	.004184100
240	57600	13824000	15.4919334	6.2144650	.004166667
241	58081	13997521	15.5241747	6.2230843	.004149378
242	58564	14172488	15.5563492	6.2316797	.004132231
243	59049	14348907	15.5884573	6.2402515	.004115226
244	59536	14526784	15.6204994	6.2487998	.004098361
245	60025	14706125	15.6524758	6.2573248	.004081633
246	60516	14886936	15.6843871	6.2658266	.004065041
247	61009	15069223	15.7162336	6.2743054	.004048583
248	61504	15252992	15.7480157	6.2827613	.004032258

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
249	62001	15438249	15.7797338	6.2911946	.004016064
250	62500	15625000	15.8113883	6.2996053	.004000000
251	63001	15813251	15.8429795	6.3079935	.003984064
252	63504	16003008	15.8745079	6.3163596	.003968254
253	64009	16194277	15.9059737	6.3247035	.003952569
254	64516	16387064	15.9373775	6.3330256	.003937008
255	65025	16581375	15.9687194	6.3413257	.003921569
256	65536	16777216	16.0000000	6.3496042	.003906250
257	66049	16974593	16.0312195	6.3578611	.003891051
258	66564	17173512	16.0623784	6.3660968	.003875969
259	67081	17373979	16.0934769	6.3743111	.003861004
260	67600	17576000	16.1245155	6.3825043	.003846154
261	68121	17779581	16.1554944	6.3906765	.003831418
262	68644	17984728	16.1864141	6.3988279	.003816794
263	69169	18191447	16.2172747	6.4069585	.003802281
264	69696	18399744	16.2480768	6.4150687	.003787879
265	70225	18609625	16.2788206	6.4231583	.003773585
266	70756	18821096	16.3095064	6.4312276	.003759398
267	71289	19034163	16.3401346	6.4392767	.003745318
268	71824	19248832	16.3707055	6.4473057	.003731343
269	72361	19465109	16.4012195	6.4553148	.003717472
270	72900	19683000	16.4316767	6.4633041	.003703704
271	73441	19902511	16.4620776	6.4712736	.003690037
272	73984	20123648	16.4924225	6.4792236	.003676471
273	74529	20346417	16.5227116	6.4871541	.003663004
274	75076	20570824	16.5529454	6.4950653	.003649635
275	75625	20796875	16.5831240	6.5029572	.003636364
276	76176	21024576	16.6132477	6.5108300	.003623188
277	76729	21253933	16.6433170	6.5186839	.003610108
278	77284	21484952	16.6733320	6.5265189	.003597122
279	77841	21717639	16.7032931	6.5343351	.003584229
280	78400	21952000	16.7332005	6.5421326	.003571429
281	78961	22188041	16.7630546	6.5499116	.003558719
282	79524	22425768	16.7928556	6.5576722	.003546099
283	80089	22665187	16.8226038	6.5654144	.003533569
284	80656	22906304	16.8522995	6.5731385	.003521127
285	81225	23149125	16.8819430	6.5808443	.003508772
286	81796	23393656	16.9115345	6.5885323	.003496503
287	82369	23639903	16.9410743	6.5962023	.003484321
288	82944	23887872	16.9705627	6.6038545	.003472222
289	83521	24137569	17.0000000	6.6114890	.003460208
290	84100	24389000	17.0293864	6.6191060	.003448276
291	84681	24642171	17.0587221	6.6267054	.003436426
292	85264	24897088	17.0880075	6.6342874	.003424658
293	85849	25153757	17.1172428	6.6418522	.003412969
294	86436	25412184	17.1464282	6.6493998	.003401361
295	87025	25672375	17.1755640	6.6569302	.003389831
296	87616	25934336	17.2046505	6.6644437	.003378378
297	88209	26198073	17.2336879	6.6719403	.003367003
298	88804	26463592	17.2626765	6.6794200	.003355705
299	89401	26730899	17.2916165	6.6868831	.003344482
300	90000	27000000	17.3205081	6.6943295	.003333333
301	90601	27270901	17.3493516	6.7017593	.003322259
302	91204	27543608	17.3781472	6.7091729	.003311258
303	91809	27818127	17.4068952	6.7165700	.003300330
304	92416	28094464	17.4355958	6.7239508	.003289474
305	93025	28372625	17.4642492	6.7313155	.003278689
306	93636	28652616	17.4928557	6.7386641	.003267974
307	94249	28934443	17.5214155	6.7459967	.003257329
308	94864	29218112	17.5499288	6.7533134	.003246753
309	95481	29503629	17.5783958	6.7606143	.003236246
310	96100	29791000	17.6068169	6.7678995	.003225806

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
311	96721	30080231	17.6351921	6.7751690	.003215434
312	97344	30371328	17.6635217	6.7824229	.003205128
313	97969	30664297	17.6918060	6.7896613	.003194888
314	98596	30959144	17.7200451	6.7968844	.003184713
315	99225	31255875	17.7482393	6.8040921	.003174603
316	99856	31554496	17.7763888	6.8112847	.003164557
317	100489	31855013	17.8044938	6.8184620	.003154574
318	101124	32157432	17.8325545	6.8256242	.003144654
319	101761	32461759	17.8605711	6.8327714	.003134796
320	102400	32768000	17.8885438	6.8399037	.003125000
321	103041	33076161	17.9164729	6.8470213	.003115265
322	103684	33386248	17.9443584	6.8541240	.003105590
323	104329	33698267	17.9722008	6.8612120	.003095975
324	104976	34012224	18.0000000	6.8682855	.003086420
325	105625	34328125	18.0277564	6.8753443	.003076923
326	106276	34645976	18.0554701	6.8823888	.003067485
327	106929	34965783	18.0831413	6.8894188	.003058104
328	107584	35287552	18.1107703	6.8964345	.003048780
329	108241	35611289	18.1383571	6.9034359	.003039514
330	108900	35937000	18.1659021	6.9104232	.003030303
331	109561	36264691	18.1934054	6.9173964	.003021148
332	110224	36594368	18.2208672	6.9243556	.003012048
333	110889	36926037	18.2482876	6.9313008	.003003003
334	111556	37259704	18.2756669	6.9382321	.002994012
335	112225	37595375	18.3030052	6.9451496	.002985075
336	112896	37933056	18.3303028	6.9520533	.002976190
337	113569	38272753	18.3575598	6.9589434	.002967359
338	114244	38614472	18.3847763	6.9658198	.002958580
339	114921	38958219	18.4119526	6.9726826	.002949853
340	115600	39304000	18.4390889	6.9795321	.002941176
341	116281	39651821	18.4661853	6.9863681	.002932551
342	116964	40001688	18.4932420	6.9931906	.002923977
343	117649	40353607	18.5202592	7.0000000	.002915452
344	118336	40707584	18.5472370	7.0067962	.002906977
345	119025	41063625	18.5741756	7.0135791	.002898551
346	119716	41421736	18.6010752	7.0203490	.002890173
347	120409	41781923	18.6279360	7.0271058	.002881844
348	121104	42144192	18.6547581	7.0338497	.002873563
349	121801	42508549	18.6815417	7.0405806	.002865330
350	122500	42875000	18.7082869	7.0472987	.002857143
351	123201	43243551	18.7349940	7.0540041	.002849003
352	123904	43614208	18.7616630	7.0606967	.002840909
353	124609	43986977	18.7882942	7.0673767	.002832861
354	125316	44361864	18.8148877	7.0740440	.002824859
355	126025	44738875	18.8414437	7.0806988	.002816901
356	126736	45118016	18.8679623	7.0873411	.002808989
357	127449	45499293	18.8944436	7.0939709	.002801120
358	128164	45882712	18.9208879	7.1005885	.002793296
359	128881	46268279	18.9472953	7.1071937	.002785515
360	129600	46656000	18.9736660	7.1137866	.002777778
361	130321	47045881	19.0000000	7.1203674	.002770083
362	131044	47437928	19.0262976	7.1269360	.002762431
363	131769	47832147	19.0525589	7.1334925	.002754821
364	132496	48228544	19.0787840	7.1400370	.002747253
365	133225	48627125	19.1049732	7.1465695	.002739726
366	133956	49027896	19.1311265	7.1530901	.002732240
367	134689	49430863	19.1572441	7.1595988	.002724796
368	135424	49836032	19.1833261	7.1660957	.002717391
369	136161	50243409	19.2093727	7.1725809	.002710027
370	136900	50653000	19.2353841	7.1790544	.002702703
371	137641	51064811	19.2613603	7.1855162	.002695418
372	138384	51478848	19.2873015	7.1919663	.002688172

TABLE XXI (*Continued*)

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
373	139129	51895117	19.3132079	7.1984050	.002680965
374	139876	52313624	19.3390796	7.2048322	.002673797
375	140625	52734375	19.3649167	7.2112479	.002666667
376	141376	53157376	19.3907194	7.2176522	.002659574
377	142129	53582633	19.4164878	7.2240450	.002652520
378	142884	54010152	19.4422221	7.2304268	.002645503
379	143641	54439939	19.4679223	7.2367972	.002638522
380	144400	54872000	19.4935887	7.2431565	.002631579
381	145161	55306341	19.5192213	7.2495045	.002624672
382	145924	55742968	19.5448203	7.2558415	.002617801
383	146689	56181887	19.5703858	7.2621675	.002610966
384	147456	56623104	19.5959179	7.2684824	.002604167
385	148225	57066625	19.6214169	7.2747864	.002597403
386	148996	57512456	19.6468827	7.2810794	.002590674
387	149769	57960603	19.6723156	7.2873617	.002583979
388	150544	58411072	19.6977156	7.2936330	.002577320
389	151321	58863869	19.7230829	7.2998936	.002570694
390	152100	59319000	19.7484177	7.3061436	.002564103
391	152881	59776471	19.7737199	7.3123828	.002557545
392	153664	60236288	19.7989899	7.3186114	.002551020
393	154449	60698457	19.8242276	7.3248295	.002544529
394	155236	61162984	19.8494332	7.3310369	.002538071
395	156025	61629875	19.8746069	7.3372339	.002531646
396	156816	62099136	19.8997487	7.3434205	.002525253
397	157609	62570773	19.9248588	7.3495966	.002518892
398	158404	63044792	19.9499373	7.3557624	.002512563
399	159201	63521199	19.9749844	7.3619178	.002506266
400	160000	64000000	20.0000000	7.3680630	.002500000
401	160801	64481201	20.0249844	7.3741979	.002493766
402	161604	64964808	20.0499377	7.3803227	.002487562
403	162409	65450827	20.0748599	7.3864373	.002481390
404	163216	65939264	20.0997512	7.3925418	.002475248
405	164025	66430125	20.1246118	7.3986363	.002469136
406	164836	66923416	20.1494417	7.4047206	.002463054
407	165649	67419143	20.1742410	7.4107950	.002457002
408	166464	67917312	20.1990099	7.4168595	.002450980
409	167281	68417929	20.2237484	7.4229142	.002444988
410	168100	68921000	20.2484567	7.4289589	.002439024
411	168921	69426531	20.2731349	7.4349938	.002433090
412	169744	69934528	20.2977831	7.4410189	.002427184
413	170569	70444997	20.3224014	7.4470342	.002421308
414	171396	70957944	20.3469899	7.4530399	.002415459
415	172225	71473375	20.3715488	7.4590359	.002409639
416	173056	71991296	20.3960781	7.4650223	.002403846
417	173889	72511713	20.4205779	7.4709991	.002398082
418	174724	73034632	20.4450483	7.4769664	.002392344
419	175561	73560059	20.4694895	7.4829242	.002386635
420	176400	74088000	20.4939015	7.4888724	.002380952
421	177241	74618461	20.5182845	7.4948113	.002375297
422	178084	75151448	20.5426386	7.5007406	.002369668
423	178929	75686967	20.5669638	7.5066607	.002364066
424	179776	76225024	20 5912603	7.5125715	.002358491
425	180625	76765625	20.6155281	7.5184730	.002352941
426	181476	77308776	20.6397674	7.5243652	.002347418
427	182329	77854483	20.6639783	7.5302482	.002341920
428	183184	78402752	20.6881609	7.5361221	.002336449
429	184041	78953589	20.7123152	7.5419867	.002331002
430	184900	79507000	20.7364414	7.5478423	.002325581
431	185761	80062991	20.7605395	7.5536888	.002320186
432	186624	80621568	20.7846097	7.5595263	.002314815
423	187489	81182737	20.8086520	7.5653548	.002309469
434	188356	81746504	20.8326667	7.5711743	.002304147

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
435	189225	82312875	20.8566536	7.5769849	.002298851
436	190096	82881856	20.8806130	7.5827865	.002293578
437	190969	83453453	20.9045450	7.5885793	.002288330
438	191844	84027672	20.9284495	7.5943633	.002283105
439	192721	84604519	20.9523268	7.6001385	.002277904
440	193600	85184000	20.9761770	7.6059049	.002272727
441	194481	85766121	21.0000000	7.6116626	.002267574
442	195364	86350888	21.0237960	7.6174116	.002262443
443	196249	86938307	21.0475652	7.6231519	.002257336
444	197136	87528384	21.0713075	7.6288837	.002252252
445	198025	88121125	21.0950231	7.6346067	.002247191
446	198916	88716536	21.1187121	7.6403213	.002242152
447	199809	89314623	21.1423745	7.6460272	.002237136
448	200704	89915392	21.1660105	7.6517247	.002232143
449	201601	90518849	21.1896201	7.6574138	.002227171
450	202500	91125000	21.2132034	7.6630943	.002222222
451	203401	91733851	21.2367606	7.6687665	.002217295
452	204304	92345408	21.2602916	7.6744303	.002212389
453	205209	92959677	21.2837967	7.6800857	.002207506
454	206116	93576664	21.3072758	7.6857328	.002202643
455	207025	94196375	21.3307290	7.6913717	.002197802
456	207936	94818816	21.3541565	7.6970023	.002192982
457	208849	95443993	21.3775583	7.7026246	.002188184
458	209764	96071912	21.4009346	7.7082388	.002183406
459	210681	96702579	21.4242853	7.7138448	.002178649
460	211600	97336000	21.4476106	7.7194426	.002173913
461	212521	97972181	21.4709106	7.7250325	.002169197
462	213444	98611128	21.4941853	7.7306141	.002164502
463	214369	99252847	21.5174348	7.7361877	.002159827
464	215296	99897344	21.5406592	7.7417532	.002155172
465	216225	100544625	21.5638587	7.7473109	.002150538
466	217156	101194696	21.5870331	7.7528606	.002145923
467	218089	101847563	21.6101828	7.7584023	.002141328
468	219024	102503232	21.6333077	7.7639361	.002136752
469	219961	103161709	21.6564078	7.7694620	.002132196
470	220900	103823000	21.6794834	7.7749801	.002127660
471	221841	104487111	21.7025344	7.7804904	.002123142
472	222784	105154048	21.7255610	7.7859928	.002118644
473	223729	105823817	21.7485632	7.7914875	.002114165
474	224676	106496424	21.7715411	7.7969745	.002109705
475	225625	107171875	21.7944947	7.8024538	.002105263
476	226576	107850176	21.8174242	7.8079254	.002100840
477	227529	108531333	21.8403297	7.8133892	.002096436
478	228484	109215352	21 8632111	7.8188456	.002092050
479	229441	109902239	21.8860686	7.8242942	.002087683
480	230400	110592000	21.9089023	7.8297353	.002083333
481	231361	111284641	21.9317122	7.8351688	.002079002
482	232324	111980168	21.9544984	7.8405949	.002074689
483	233289	112678587	21.9772610	7.8460134	.002070393
484	234256	113379904	22.0000000	7.8514244	.002066116
485	235225	114084125	22.0227155	7.8568281	.002061856
486	236196	114791256	22.0454077	7.8622242	.002057613
487	237169	115501303	22.0680765	7.8676130	.002053388
488	238144	116214272	22.0907220	7.8729944	.002049180
489	239121	116930169	22.1133444	7.8783684	.002044990
490	240100	117649000	22.1359436	7.8837352	.002040816
491	241081	118370771	22.1585198	7.8890946	.002036660
492	242064	119095488	22.1810730	7.8944468	.002032520
493	243049	119823157	22.2036033	7.8997917	.002028398
494	244036	120553784	22.2261108	7.9051294	.002024291
495	245025	121287375	22.2485955	7.9104599	.002020202
496	246016	122023936	22.2710575	7.9157832	.002016129

TABLE XXI (*Continued*)

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
497	247009	122763473	22.2934968	7.9210994	.002012072
498	248004	123505992	22.3159136	7.9264085	.002008032
499	249001	124251499	22.3383079	7.9317104	.002004008
500	250000	125000000	22.3606798	7.9370053	.002000000
501	251001	125751501	22.3830293	7.9422931	.001996008
502	252004	126506008	22.4053565	7.9475739	.001992032
503	253009	127263527	22.4276615	7.9528477	.001988072
504	254016	128024064	22.4499443	7.9581144	.001984127
505	255025	128787625	22.4722051	7.9633743	.001980198
506	256036	129554216	22.4944438	7.9686271	.001976285
507	257049	130323843	22.5166605	7.9738731	.001972387
508	258064	131096512	22.5388553	7.9791122	.001968504
509	259081	131872229	22.5610283	7.9843444	.001964637
510	260100	132651000	22.5831796	7.9895697	.001960784
511	261121	133432831	22.6053091	7.9947883	.001956947
512	262144	134217728	22.6274170	8.0000000	.001953125
513	263169	135005697	22.6495033	8.0052049	.001949318
514	264196	135796744	22.6715681	8.0104032	.001945525
515	265225	136590875	22.6936114	8.0155946	.001941748
516	266256	137388096	22.7156334	8.0207794	.001937984
517	267289	138188413	22.7376340	8.0259574	.001934236
518	268324	138991832	22.7596134	8.0311287	.001930502
519	269361	139798359	22.7815715	8.0362935	.001926782
520	270400	140608000	22.8035085	8.0414515	.001923077
521	271441	141420761	22.8254244	8.0466030	.001919386
522	272484	142236648	22.8473193	8.0517479	.001915709
523	273529	143055667	22.8691933	8.0568862	.001912046
524	274576	143877824	22.8910463	8.0620180	.001908397
525	275625	144703125	22.9128785	8.0671432	.001904762
526	276676	145531576	22.9346899	8.0722620	.001901141
527	277729	146363183	22.9564806	8.0773743	.001897533
528	278784	147197952	22.9782506	8.0824800	.001893939
529	279841	148035889	23.0000000	8.0875794	.001890359
530	280900	148877000	23.0217289	8.0926723	.001886792
531	281961	149721291	23.0434372	8.0977589	.001883239
532	283024	150568768	23.0651252	8.1028390	.001879699
533	284089	151419437	23.0867928	8.1079128	.001876173
534	285156	152273304	23.1084400	8.1129803	.001872659
535	286225	153130375	23.1300670	8.1180414	.001869159
536	287296	153990656	23.1516738	8.1230962	.001865672
537	288369	154854153	23.1732605	8.1281447	.001862197
538	289444	155720872	23.1948270	8.1331870	.001858736
539	290521	156590819	23.2163735	8.1382230	.001855288
540	291600	157464000	23.2379001	8.1432529	.001851852
541	292681	158340421	23.2594067	8.1482765	.001848429
542	293764	159220088	23.2808935	8.1532939	.001845018
543	294849	160103007	23.3023604	8.1583051	.001841621
544	295936	160989184	23.3238076	8.1633102	.001838235
545	297025	161878625	23.3452351	8.1683092	.001834862
546	298116	162771336	23.3666429	8.1733020	.001831502
547	299209	163667323	23.3880311	8.1782888	.001828154
548	300304	164566592	23.4093998	8.1832695	.001824818
549	301401	165469149	23.4307490	8.1882441	.001821494
550	302500	166375000	23.4520788	8.1932127	.001818182
551	303601	167284151	23.4733892	8.1981752	.001814882
552	304704	168196608	23.4946802	8.2031319	.001811594
553	305809	169112377	23.5159520	8.2080825	.001808318
554	306916	170031464	23.5372046	8.2130271	.001805054
555	308025	170953875	23.5584380	8.2179657	.001801802
556	309136	171879616	23.5796522	8.2228985	.001798561
557	310249	172808693	23.6008474	8.2278254	.001795332
558	311364	173741112	23.6220236	8.2327463	.001792115

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
559	312481	174676879	23.6431808	8.2376614	.001788909
560	313600	175616000	23.6643191	8.2425706	.001785714
561	314721	176558481	23.6854386	8.2474740	.001782531
562	315844	177504328	23.7065392	8.2523715	.001779359
563	316969	178453547	23.7276210	8.2572633	.001776199
564	318096	179406144	23.7486842	8.2621492	.001773050
565	319225	180362125	23.7697286	8.2670294	.001769912
566	320356	181321496	23.7907545	8.2719039	.001766784
567	321489	182284263	23.8117618	8.2767726	.001763668
568	322624	183250432	23.8327506	8.2816355	.001760563
569	323761	184220000	23.8537209	8.2864928	.001757469
570	324900	185193000	23.8746728	8.2913444	.001754386
571	326041	186169411	23.8956063	8.2961903	.001751313
572	327184	187149248	23 9165215	8.3010304	.001748252
573	328329	188132517	23.9374184	8.3058651	.001745201
574	329476	189119224	23.9582971	8.3106941	.001742160
575	330625	190109375	23.9791576	8.3155175	.001739130
576	331776	191102976	24.0000000	8.3203353	.001736111
577	332929	192100033	24.0208243	8.3251475	.001733102
578	334084	193100552	24.0416306	8.3299542	.001730104
579	335241	194104539	24.0624188	8.3347553	.001727116
580	336400	195112000	24.0831891	8.3395509	.001724138
581	337561	196122941	24.1039416	8.3443410	.001721170
582	338724	197137368	24.1246762	8.3491256	.001718213
583	339889	198155287	24.1453929	8.3539047	.001715266
584	341056	199176704	24.1660919	8.3586784	.001712329
585	342225	200201625	24.1867732	8.3634466	.001709402
586	343396	201230056	24.2074369	8.3682095	.001706485
587	344569	202262003	24.2280829	8.3729668	.001703578
588	345744	203297472	24.2487113	8.3777188	.001700680
589	346921	204336469	24.2693222	8.3824653	.001697793
590	348100	205379000	24.2899156	8.3872065	.001694915
591	349281	206425071	24.3104916	8.3919423	.001692047
592	350464	207474688	24.3310501	8.3966729	.001689189
593	351649	208527857	24.3515913	8.4013981	.001686341
594	352836	209584584	24.3721152	8.4061180	.001683502
595	354025	210644875	24.3926218	8.4108326	.001680672
596	355216	211708736	24.4131112	8.4155419	.001677852
597	356409	212776173	24.4335834	8.4202460	.001675042
598	357604	213847192	24.4540385	8.4249448	.001672241
599	358801	214921799	24.4744765	8.4296383	.001669449
600	360000	216000000	24.4948974	8.4343267	001666667
601	361201	217081801	24.5153013	8.4390098	.001663894
602	362404	218167208	24.5356883	8.4436877	.001661130
603	363609	219256227	24.5560583	8.4483605	.001658375
604	364816	220348864	24.5764115	8.4530281	.001655629
605	366025	221445125	24.5967478	8.4576906	.001652893
606	367236	222545016	24.6170673	8.4623479	.001650165
607	368449	223648543	24.6373700	8.4670001	.001647446
608	369664	224755712	24.6576560	8.4716471	.001644737
609	370881	225866529	24.6779254	8.4762892	.001642036
610	372100	226981000	24.6981781	8.4809261	.001639344
611	373221	228099131	24.7184142	8.4855579	.001636661
612	374544	229220928	24.7386338	8.4901848	001633987
613	375769	230346397	24.7588368	8.4948065	001631321
614	376996	231475544	24.7790234	8.4994233	.001628664
615	378225	232608375	24.7991935	8.5040350	.001626016
616	379456	233744896	24.8193473	8.5086417	.001623377
617	380689	234885113	24.8394847	8.5132435	.001620746
618	381924	236029032	24.8596058	8.5178403	.001618123
619	383161	237176659	24.8797106	8.5224321	.001615509
620	384400	238328000	24.8997992	8.5270189	.001612903

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
621	385641	239483061	24.9198716	8.5316009	.001610306
622	386884	240641848	24.9399278	8.5361780	.001607717
623	388129	241804367	24.9599679	8.5407501	.001605136
624	389376	242970624	24.9799920	8.5453173	.001602564
625	390625	244140625	25.0000000	8.5498797	.001600000
626	391876	245314376	25.0199920	8.5544372	.001597444
627	393129	246491883	25.0399681	8.5589899	.001594896
628	394384	247673152	25.0599282	8.5635377	.001592357
629	395641	248858189	25.0798724	8.5680807	.001589825
630	396900	250047000	25.0998008	8.5726189	.001587302
631	398161	251239591	25.1197134	8.5771523	.001584786
632	399424	252435968	25.1396102	8.5816809	.001582278
633	400689	253636137	25.1594913	8.5862047	.001579779
634	401956	254840104	25.1793566	8.5907238	.001577287
635	403225	256047875	25.1992063	8.5952380	.001574803
636	404496	257259456	25.2190404	8.5997476	.001572327
637	405769	258474853	25.2388589	8.6042525	.001569859
638	407044	259694072	25.2586619	8.6087526	.001567398
639	408321	260917119	25.2784493	8.6132480	.001564945
640	409600	262144000	25.2982213	8.6177388	.001562500
641	410881	263374721	25.3179778	8.6222248	.001560062
642	412164	264609288	25.3377189	8.6267063	.001557632
643	413449	265847707	25.3574447	8.6311830	.001555210
644	414736	267089984	25.3771551	8.6356551	.001552795
645	416025	268336125	25.3968502	8.6401226	.001550388
646	417316	269586136	25.4165301	8.6445855	.001547988
647	418609	270840023	25.4361947	8.6490437	.001545595
648	419904	272097792	25.4558441	8.6534974	.001543210
649	421201	273359449	25.4754784	8.6579465	.001540832
650	422500	274625000	25.4950976	8.6623911	.001538462
651	423801	275894451	25.5147016	8.6668310	.001536098
652	425104	277167808	25.5342907	8.6712665	.001533742
653	426409	278445077	25.5538647	8.6756974	.001531394
654	427716	279726264	25.5734237	8.6801237	.001529052
655	429025	281011375	25.5929678	8.6845456	.001526718
656	430336	282300416	25.6124969	8.6889630	.001524390
657	431649	283593393	25.6320112	8.6933759	.001522070
658	432964	284890312	25.6515107	8.6977843	.001519757
659	434281	286191179	25.6709953	8.7021882	.001517451
660	435600	287496000	25.6904652	8.7065877	.001515152
661	436921	288804781	25.7099203	8.7109827	.001512859
662	438244	290117528	25.7293607	8.7153734	.001510574
663	439569	291434247	25.7487864	8.7197596	.001508296
664	440896	292754944	25.7681975	8.7241414	.001506024
665	442225	294079625	25.7875939	8.7285187	.001503759
666	443556	295408296	25.8069758	8.7328918	.001501502
667	444889	296740963	25.8263431	8.7372604	.001499250
668	446224	298077632	25.8456960	8.7416246	.001497006
669	447561	299418309	25.8650343	8.7459846	.001494768
670	448900	300763000	25.8843582	8.7503401	.001492537
671	450241	302111711	25.9036677	8.7546913	.001490313
672	451584	303464448	25.9229628	8.7590383	.001488095
673	452929	304821217	25.9422435	8.7633809	.001485884
674	454276	306182024	25.9615100	8.7677192	.001483680
675	455625	307546875	25.9807621	8.7720532	.001481481
676	456976	308915776	26.0000000	8.7763830	.001479290
677	458329	310288733	26.0192237	8.7807084	.001477105
678	459684	311665752	26.0384331	8.7850296	.001474926
679	461041	313046839	26.0576284	8.7893466	.001472754
680	462400	314432000	26.0768096	8.7936593	.001470588
681	463761	315821241	26.0959767	8.7979679	.001468429
682	465124	317214568	26.1151297	8.8022721	.001466276

TABLE XXI (*Continued*)

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
683	466489	318611987	26.1342687	8.8065722	.001464129
684	467856	320013504	26.1533937	8.8108681	.001461988
685	469225	321419125	26.1725047	8.8151598	.001459854
686	470596	322828856	26.1916017	8.8194474	.001457726
687	471969	324242703	26.2106848	8.8237307	.001455604
688	473344	325660672	26.2297541	8.8280099	.001453488
689	474721	327082769	26.2488095	8.8322850	.001451379
690	476100	328509000	26.2678511	8.8365559	.001449275
691	477481	329939371	26.2868789	8.8408227	.001447178
692	478864	331373888	26.3058929	8.8450854	.001445087
693	480249	332812557	26.3248932	8.8493440	.001443001
694	481636	334255384	26.3438797	8.8535985	.001440922
695	483025	335702375	26.3628527	8.8578489	.001438849
696	484416	337153536	26.3818119	8.8620952	.001436782
697	485809	338608873	26.4007576	8.8663375	.001434720
698	487204	340068392	26.4196896	8.8705757	.001432665
699	488601	341532099	26.4386081	8.8748099	.001430615
700	490000	343000000	26.4575131	8.8790400	.001428571
701	491401	344472101	26.4764046	8.8832661	.001426534
702	492804	345948408	26.4952826	8.8874882	.001424501
703	494209	347428927	26.5141472	8.8917063	.001422475
704	495616	348913664	26.5329983	8.8959204	.001420455
705	497025	350402625	26.5518361	8.9001304	.001418440
706	498436	351895816	26.5706605	8.9043366	.001416431
707	499849	353393243	26.5894716	8.9085387	.001414427
708	501264	354894912	26.6082694	8.9127369	.001412429
709	502681	356400829	26.6270539	8.9169311	.001410437
710	504100	357911000	26.6458252	8.9211214	.001408451
711	505521	359425431	26.6645833	8.9253078	.001406470
712	506944	360944128	26.6833281	8.9294902	.001404494
713	508369	362467097	26.7020598	8.9336687	.001402525
714	509796	363994344	26.7207784	8.9378433	.001400560
715	511225	365525875	26.7394839	8.9420140	.001398601
716	512656	367061696	26.7581763	8.9461809	.001396648
717	514089	368601813	26.7768557	8.9503438	.001394700
718	515524	370146232	26.7955220	8.9545029	.001392758
719	516961	371694959	26.8141754	8.9586581	.001390821
720	518400	373248000	26.8328157	8.9628095	.001388889
721	519841	374805361	26.8514432	8.9669570	.001386963
722	521284	376367048	26.8700577	8.9711007	.001385042
723	522729	377933067	26.8886593	8.9752406	.001383126
724	524176	379503424	26.9072481	8.9793766	.001381215
725	525625	381078125	26.9258240	8.9835089	.001379310
726	527076	382657176	26.9443872	8.9876373	.001377410
727	528529	384240583	26.9629375	8.9917620	.001375516
728	529984	385828352	26.9814751	8.9958829	.001373626
729	531441	387420489	27.0000000	9.0000000	.001371742
730	532900	389017000	27.0185122	9.0041134	.001369863
731	534361	390617891	27.0370117	9.0082229	.001367989
732	535824	392223168	27.0554985	9.0123288	.001366120
733	537289	393832837	27.0739727	9.0164309	.001364256
734	538756	395446904	27.0924344	9.0205293	.001362398
735	540225	397065375	27.1108834	9.0246239	.001360544
736	541696	398688256	27.1293199	9.0287149	.001358696
737	543169	400315553	27.1477439	9.0328021	.001356852
738	544644	401947272	27.1661554	9.0368857	.001355014
739	546121	403583419	27.1845544	9.0409655	.001353180
740	547600	405224000	27.2029410	9.0450419	.001351351
741	549081	406869021	27.2213152	9.0491142	.001349528
742	550564	408518488	27.2396769	9.0531831	.001347709
743	552049	410172407	27.2580263	9.0572482	.001345895
744	553536	411830784	27.2763634	9.0613098	.001344086

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
745	555025	413493625	27.2946881	9.0653677	.001342282
746	556516	415160936	27.3130006	9.0694220	.001340483
747	558009	416832723	27.3313007	9.0734726	.001338688
748	559504	418508992	27.3495887	9.0775197	.001336898
749	561001	420189749	27.3678644	9.0815631	.001335113
750	562500	421875000	27.3861279	9.0856030	.001333333
751	564001	423564751	27.4043792	9.0896392	.001331558
752	565504	425259008	27.4226184	9.0936719	.001329787
753	567009	426957777	27.4408455	9.0977010	.001328021
754	568516	428661064	27.4590604	9.1017265	.001326260
755	570025	430368875	27.4772633	9.1057485	.001324503
756	571536	432081216	27.4954542	9.1097669	.001322751
757	573049	433798093	27.5136330	9.1137818	.001321004
758	574564	435519512	27.5317998	9.1177931	.001319261
759	576081	437245479	27.5499546	9.1218010	.001317523
760	577600	438976000	27.5680975	9.1258053	.001315789
761	579121	440711081	27.5862284	9.1298061	.001314060
762	580644	442450728	27.6043475	9.1338034	.001312336
763	582169	444194947	27.6224546	9.1377971	.001310616
764	583696	445943744	27.6405499	9.1417874	.001308901
765	585225	447697125	27.6586334	9.1457742	.001307190
766	586756	449455096	27.6767050	9.1497576	.001305483
767	588289	451217663	27.6947648	9.1537375	.001303781
768	589824	452984832	27.7128129	9.1577139	.001302083
769	591361	454756609	27.7308492	9.1616869	.001300390
770	592900	456533000	27.7488739	9.1656565	.001298701
771	594441	458314011	27.7668868	9.1696225	.001297017
772	595984	460099648	27.7848880	9.1735852	.001295337
773	597529	461889917	27.8028775	9.1775445	.001293661
774	599076	463684824	27.8208555	9.1815003	.001291990
775	600625	465484375	27.8388218	9.1854527	.001290323
776	602176	467288576	27.8567766	9.1894018	.001288660
777	603729	469097433	27.8747197	9.1933474	.001287001
778	605284	470910952	27.8926514	9.1972897	.001285347
779	606841	472729139	27.9105715	9.2012286	.001283697
780	608400	474552000	27.9284801	9.2051641	.001282051
781	609961	476379541	27.9463772	9.2090962	.001280410
782	611524	478211768	27.9642629	9.2130250	.001278772
783	613089	480048687	27.9821372	9.2169505	.001277139
784	614656	481890304	28.0000000	9.2208726	.001275510
785	616225	483736625	28.0178515	9.2247914	.001273885
786	617796	485587656	28.0356915	9.2287068	.001272265
787	619369	487443403	28.0535203	9.2326189	.001270648
788	620944	489303872	28.0713377	9.2365277	.001269036
789	622521	491169069	28.0891438	9.2404333	.001267427
790	624100	493039000	28.1069386	9.2443355	.001265823
791	625681	494913671	28.1247222	9.2482344	.001264223
792	627264	496793088	28.1424946	9.2521300	.001262626
793	628849	498677257	28.1602557	9.2560224	.001261034
794	630436	500566184	28.1780056	9.2599114	.001259446
795	632025	502459875	28.1957444	9.2637973	.001257862
796	633616	504358336	28.2134720	9.2676798	.001256281
797	635209	506261573	28.2311884	9.2715592	.001254705
798	636804	508169592	28.2488938	9.2754352	.001253133
799	638401	510082399	28.2665881	9.2793081	.001251564
800	640000	512000000	28.2842712	9.2831777	.001250000
801	641601	513922401	28.3019434	9.2870440	.001248439
802	643204	515849608	28.3196045	9.2909072	.001246883
803	644809	517781627	28.3372546	9.2947671	.001245330
804	646416	519718464	28.3548938	9.2986239	.001243781
805	648025	521660125	28.3725219	9.3024775	.001242236
806	649636	523606616	28.3901391	9.3063278	.001240695

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
807	651249	525557943	28.4077454	9.3101750	.001239157
808	652864	527514112	28.4253408	9.3140190	.001237624
809	654481	529475129	28.4429253	9.3178599	.001236094
810	656100	531441000	28.4604989	9.3216975	.001234568
811	657721	533411731	28.4780617	9.3255320	.001233046
812	659344	535387328	28.4956137	9.3293634	.001231527
813	660969	537367797	28.5131549	9.3331916	.001230012
814	662596	539353144	28.5306852	9.3370167	.001228501
815	664225	541343375	28.5482048	9.3408386	.001226994
816	665856	543338496	28.5657137	9.3446575	.001225490
817	667489	545338513	28.5832119	9.3484731	.001223990
818	669124	547343432	28.6006993	9.3522857	.001222494
819	670761	549353259	28.6181760	9.3560952	.001221001
820	672400	551368000	28.6356421	9.3599016	.001219512
821	674041	553387661	28.6530976	9.3637049	.001218027
822	675684	555412248	28.6705424	9.3675051	.001216545
823	677329	557441767	28.6879766	9.3713022	.001215067
824	678976	559476224	28.7054002	9.3750963	.001213592
825	680625	561515625	28.7228132	9.3788873	.001212121
826	682276	563559976	28.7402157	9.3826752	.001210654
827	683929	565609283	28.7576077	9.3864600	.001209190
828	685584	567663552	28.7749891	9.3902419	.001207729
829	687241	569722789	28.7923601	9.3940206	.001206273
830	688900	571787000	28.8097206	9.3977964	.001204819
831	690561	573856191	28.8270706	9.4015691	.001203369
832	692224	575930368	28.8444102	9.4053387	.001201923
833	693889	578009537	28.8617394	9.4091054	.001200480
834	695556	580093704	28.8790582	9.4128690	.001199041
835	697225	582182875	28.8963666	9.4166297	.001197605
836	698896	584277056	28.9136646	9.4203873	.001196172
837	700569	586376253	28.9309523	9.4241420	.001194743
838	702244	588480472	28.9482297	9.4278936	.001193317
839	703921	590589719	28.9654967	9.4316423	.001191895
840	705600	592704000	28.9827535	9.4353880	.001190476
841	707281	594823321	29.0000000	9.4391307	.001189061
842	708964	596947688	29.0172363	9.4428704	.001187648
843	710649	599077107	29.0344623	9.4466072	.001186240
844	712336	601211584	29.0516781	9.4503410	.001184834
845	714025	603351125	29.0688837	9.4540719	.001183432
846	715716	605495736	29.0860791	9.4577999	.001182033
847	717409	607645423	29.1032644	9.4615249	.001180638
848	719104	609800192	29.1204396	9.4652470	.001179245
849	720801	611960049	29.1376046	9.4689661	.001177856
850	722500	614125000	29.1547595	9.4726824	.001176471
851	724201	616295051	29.1719043	9.4763957	.001175088
852	725904	618470208	29.1890390	9.4801061	.001173709
853	727609	620650477	29.2061637	9.4838136	.001172333
854	729316	622835864	29.2232784	9.4875182	.001170960
855	731025	625026375	29.2403830	9.4912200	.001169591
856	732736	627222016	29.2574777	9.4949188	.001168224
857	734449	629422793	29.2745623	9.4986147	.001166861
858	736164	631628712	29.2916370	9.5023078	.001165501
859	737881	633839779	29.3087018	9.5059980	.001164144
860	739600	636056000	29.3257566	9.5096854	.001162791
861	741321	638277381	29.3428015	9.5133699	.001161440
862	743044	640503928	29.3598365	9.5170515	.001160093
863	744769	642735647	29.3768616	9.5207303	.001158749
864	746496	644972544	29.3938769	9.5244063	.001157407
865	748225	647214625	29.4108823	9.5280794	.001156069
866	749956	649461896	29.4278779	9.5317497	.001154734
867	751689	651714363	29.4448637	9.5354172	.001153403
868	753424	653972032	29.4618397	9.5390818	.001152074

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
869	755161	656234909	29.4788059	9.5427437	.001150748
870	756900	658503000	29.4957624	9.5464027	.001149425
871	758641	660776311	29.5127091	9.5500589	.001148106
872	760384	663054848	29.5296461	9.5537123	.001146789
873	762129	665338617	29.5465734	9.5573630	.001145475
874	763876	667627624	29.5634910	9.5610108	.001144165
875	765625	669921875	29.5803989	9.5646559	.001142857
876	767376	672221376	29.5972972	9.5682982	.001141553
877	769129	674526133	29.6141858	9.5719377	.001140251
878	770884	676836152	29.6310648	9.5755745	.001138952
879	772641	679151439	29.6479342	9.5792085	.001137656
880	774400	681472000	29.6647939	9.5828397	.001136364
881	776161	683797841	29.6816442	9.5864682	.001135074
882	777924	686128968	29.6984848	9.5900939	.001133787
883	779689	688465387	29.7153159	9.5937169	.001132503
884	781456	690807104	29.7321375	9.5973373	.001131222
885	783225	693154125	29.7489496	9.6009548	.001129944
886	784996	695506456	29.7657521	9.6045696	.001128668
887	786769	697864103	29.7825452	9.6081817	.001127396
888	788544	700227072	29.7993289	9.6117911	.001126126
889	790321	702595369	29.8161030	9.6153977	.001124859
890	792100	704969000	29.8328678	9.6190017	.001123596
891	793881	707347971	29.8496231	9.6226030	.001122334
892	795664	709732288	29.8663690	9.6262016	.001121076
893	797449	712121957	29.8831056	9.6297975	.001119821
894	799236	714516984	29.8998328	9.6333907	.001118568
895	801025	716917375	29.9165506	9.6369812	.001117318
896	802816	719323136	29.9332591	9.6405690	.001116071
897	804609	721734273	29.9499583	9.6441542	.001114827
898	806404	724150792	29.9666481	9.6477367	.001113586
899	808201	726572699	29.9833287	9.6513166	.001112347
900	810000	729000000	30.0000000	9.6548938	.001111111
901	811801	731432701	30.0166620	9.6584684	.001109878
902	813604	733870808	30.0333148	9.6620403	.001108647
903	815409	736314327	30.0499584	9.6656096	.001107420
904	817216	738763264	30.0665928	9.6691762	.001106195
905	819025	741217625	30.0832179	9.6727403	.001104972
906	820836	743677416	30.0998339	9.6763017	.001103753
907	822649	746142643	30.1164407	9.6798604	.001102536
908	824464	748613312	30.1330383	9.6834166	.001101322
909	826281	751089429	30.1496269	9.6869701	.001100110
910	828100	753571000	30.1662063	9.6905211	.001098901
911	829921	756058031	30.1827765	9.6940694	.001097695
912	831744	758550528	30.1993377	9.6976151	.001096491
913	833569	761048497	30.2158899	9.7011583	.001095290
914	835396	763551944	30.2324329	9.7046989	.001094092
915	837225	766060875	30.2489669	9.7082369	.001092896
916	839056	768575296	30.2654919	9.7117723	.001091703
917	840889	771095213	30.2820079	9.7153051	.001090513
918	842724	773620632	30.2985148	9.7188354	.001089325
919	844561	776151559	30.3150128	9.7223631	.001088139
920	846400	778688000	30.3315018	9.7258883	.001086957
921	848241	781229961	30.3479818	9.7294109	.001085776
922	850084	783777448	30.3644529	9.7329309	.001084599
923	851929	786330467	30.3809151	9.7364484	.001083423
924	853776	788889024	30.3973683	9.7399634	.001082251
925	855625	791453125	30.4138127	9.7434758	.001081081
926	857476	794022776	30.4302481	9.7469857	.001079914
927	859329	796597983	30.4466747	9.7504930	.001078749
928	861184	799178752	30.4630924	9.7539979	.001077586
929	863041	801765089	30.4795013	9.7575002	.001076426
930	864900	804357000	30.4959014	9.7610001	.001075269

TABLE XXI (*Continued*)

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, AND RECIPROCALS

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
931	866761	806954491	30.5122926	9.7644974	.001074114
932	868624	809557568	30.5286750	9.7679922	.001072961
933	870489	812166237	30.5450487	9.7714845	.001071811
934	872356	814780504	30.5614136	9.7749743	.001070664
935	874225	817400375	30.5777697	9.7784616	.001069519
936	876096	820025856	30.5941171	9.7819466	.001068376
937	877969	822656953	30.6104557	9.7854288	.001067236
938	879844	825293672	30.6267857	9.7889087	.001066098
939	881721	827936019	30.6431069	9.7923861	.001064963
940	883600	830584000	30.6594194	9.7958611	.001063830
941	885481	833237621	30.6757233	9.7993336	.001062699
942	887364	835896888	30.6920185	9.8028036	.001061571
943	889249	838561807	30.7083051	9.8062711	.001060445
944	891136	841232384	30.7245830	9.8097362	.001059322
945	893025	843908625	30.7408523	9.8131989	.001058201
946	894916	846590536	30.7571130	9.8166591	.001057082
947	896809	849278123	30.7733651	9.8201169	.001055966
948	898704	851971392	30.7896086	9.8235723	.001054852
949	900601	854670349	30.8058436	9.8270252	.001053741
950	902500	857375000	30.8220700	9.8304757	.001052632
951	904401	860085351	30.8382879	9.8339238	.001051525
952	906304	862801408	30.8544972	9.8373695	.001050420
953	908209	865523177	30.8706981	9.8408127	.001049318
954	910116	868250664	30.8868904	9.8442536	.001048218
955	912025	870983875	30.9030743	9.8476920	.001047120
956	913936	873722816	30.9192497	9.8511280	.001046025
957	915849	876467493	30.9354166	9.8545617	.001044932
958	917764	879217912	30.9515751	9.8579929	.001043841
959	919681	881974079	30.9677251	9.8614218	.001042753
960	921600	884736000	30.9838668	9.8648483	.001041667
961	923521	887503681	31.0000000	9.8682724	.001040583
962	925444	890277128	31.0161248	9.8716941	.001039501
963	927369	893056347	31.0322413	9.8751135	.001038422
964	929296	895841344	31.0483494	9.8785305	.001037344
965	931225	898632125	31.0644491	9.8819451	.001036269
966	933156	901428696	31.0805405	9.8853574	.001035197
967	935089	904231063	31.0966236	9.8887673	.001034126
968	937024	907039232	31.1126984	9.8921749	.001033058
969	938961	909853209	31.1287648	9.8955801	.001031992
970	940900	912673000	31.1448230	9.8989830	.001030928
971	942841	915498611	31.1608729	9.9023835	.001029866
972	944784	918330048	31.1769145	9.9057817	.001028807
973	946729	921167317	31.1929479	9.9091776	.001027749
974	948676	924010424	31.2089731	9.9125712	.001026694
975	950625	926859375	31.2249900	9.9159624	.001025641
976	952576	929714176	31.2409987	9.9193513	.001024590
977	954529	932574833	31.2569992	9.9227379	.001023541
978	956484	935441352	31.2729915	9.9261222	.001022495
979	958441	938313739	31.2889757	9.9295042	.001021450
980	960400	941192000	31.3049517	9.9328839	.001020408
981	962361	944076141	31.3209195	9.9362613	.001019368
982	964324	946966168	31.3368792	9.9396363	.001018330
983	966289	949862087	31.3528308	9.9430092	.001017294
984	968256	952763904	31.3687743	9.9463797	.001016260
985	970225	955671625	31.3847097	9.9497479	.001015228
986	972196	958585256	31.4006369	9.9531138	.001014199
987	974169	961504803	31.4165561	9.9564775	.001013171
988	976144	964430272	31.4324673	9.9598389	.001012146
989	978121	967361669	31.4483704	9.9631981	.001011122
990	980100	970299000	31.4642654	9.9665549	.001010101
991	982081	973242271	31.4801525	9.9699095	.001009082
992	984064	976191488	31.4960315	9.9732619	.001008065

TABLE XXI (*Continued*)

Squares, Cubes, Square Roots, Cube Roots, and Reciprocals

No.	Squares.	Cubes.	Square Roots.	Cube Roots.	Reciprocals.
993	986049	979146657	31.5119025	9.9766120	.001007049
994	988036	982107784	31.5277655	9.9799599	.001006036
995	990025	985074875	31.5436206	9.9833055	.001005025
996	992016	988047936	31.5594677	9.9866488	.001004016
997	994009	991026973	31.5753068	9.9899900	.001003009
998	996004	994011992	31.5911380	9.9933289	.001002004
999	998001	997002999	31.6069613	9.9966656	.001001001
1000	1000000	1000000000	31.6227766	10.0000000	.001000000
1001	1002001	1003003001	31.6385840	10.0033322	.0009990010
1002	1004004	1006012008	31.6543836	10.0066622	.0009980040
1003	1006009	1009027027	31.6701752	10.0099899	.0009970090
1004	1008016	1012048064	31.6859590	10.0133155	.0009960159
1005	1010025	1015075125	31.7017349	10.0166389	.0009950249
1006	1012036	1018108216	31.7175030	10.0199601	.0009940358
1007	1014049	1021147343	31.7332633	10.0232791	.0009930487
1008	1016064	1024192512	31.7490157	10.0265958	.0009920635
1009	1018081	1027243729	31.7647603	10.0299104	.0009910803
1010	1020100	1030301000	31.7804972	10.0332228	.0009900990
1011	1022121	1033364331	31.7962262	10.0365330	.0009891197
1012	1024144	1036433728	31.8119474	10.0398410	.0009881423
1013	1026169	1039509197	31.8276609	10.0431469	.0009871668
1014	1028196	1042590744	31.8433666	10.0464506	.0009861933
1015	1030225	1045678375	31.8590646	10.0497521	.0009852217
1016	1032256	1048772096	31.8747549	10.0530514	.0009842520
1017	1034289	1051871913	31.8904374	10.0563485	.0009832842
1018	1036324	1054977832	31.9061123	10.0596435	.0009823183
1019	1038361	1058089859	31.9217794	10.0629364	.0009813543
1020	1040400	1061208000	31.9374388	10.0662271	.0009803922
1021	1042441	1064332261	31.9530906	10.0695156	.0009794319
1022	1044484	1067462648	31.9687347	10.0728020	.0009784736
1023	1046529	1070599167	31.9843712	10.0760863	.0009775171
1024	1048576	1073741824	32.0000000	10.0793684	.0009765625
1025	1050625	1076890625	32.0156212	10.0826484	.0009756098
1026	1052676	1080045576	32.0312348	10.0859262	.0009746589
1027	1054729	1083206683	32.0468407	10.0892019	.0009737098
1028	1056784	1086373952	32.0624391	10.0924755	.0009727626
1029	1058841	1089547389	32.0780298	10.0957469	.0009718173
1030	1060900	1092727000	32.0936131	10.0990163	.0009708738
1031	1062961	1095912791	32.1091887	10.1022835	.0009699321
1032	1065024	1099104768	32.1247568	10.1055487	.0009689922
1033	1067089	1102302937	32.1403173	10.1088117	.0009680542
1034	1069156	1105507304	32.1558704	10.1120726	.0009671180
1035	1071225	1108717875	32.1714159	10.1153314	.0009661836
1036	1073296	1111934656	32.1869539	10.1185882	.0009652510
1037	1075369	1115157653	32.2024844	10.1218428	.0009643202
1038	1077444	1118386872	32.2180074	10.1250953	.0009633911
1039	1079521	1121622319	32.2335229	10.1283457	.0009624639
1040	1081600	1124864000	32.2490310	10.1315941	.0009615385
1041	1083681	1128111921	32.2645316	10.1348403	.0009606148
1042	1085764	1131366088	32.2800248	10.1380845	.0009596929
1043	1087849	1134626507	32.2955105	10.1413266	.0009587738
1044	1089936	1137893184	32.3109888	10.1445667	.0009578544
1045	1092025	1141166125	32.3264598	10.1478047	.0009569378
1046	1094116	1144445336	32.3419233	10.1510406	.0009560229
1047	1096209	1147730823	32.3573794	10.1542744	.0009551098
1048	1098304	1151022592	32.3728281	10.1575062	.0009541985
1049	1100401	1154320649	32.3882695	10.1607359	.0009532888
1050	1102500	1157625000	32.4037035	10.1639636	.0009523810
1051	1104601	1160935651	32.4191301	10.1671893	.0009514748
1052	1106704	1164252608	32.4345495	10.1704129	.0009505703
1053	1108809	1167575877	32.4499615	10.1736344	.0009496676
1054	1110916	1170905464	32.4653662	10.1768539	.0009487666

TABLE XXII

Logarithms of Numbers

No. 100 L. 000.] [No. 109 L. 040.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
1	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
2	8600	9026	9451	9876	0300	0724	1147	1570	1993	2415	424
3	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	420
4	7033	7451	7868	8284	8700	9116	9532	9947	0361	0775	416
5	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
6	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
7	9384	9789	0195	0600	1004	1408	1812	2216	2619	3021	404
8	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
9	7426 04	7825	8223	8620	9017	9414	9811	0207	0602	0998	397

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
434	43.4	86.8	130.2	173.6	217.0	260.4	303.8	347.2	390.6
433	43.3	86.6	129.9	173.2	216.5	259.8	303.1	346.4	389.7
432	43 2	86.4	129.6	172.8	216.0	259.2	302.4	345.6	388.8
431	43.1	86.2	129.3	172.4	215.5	258.6	301.7	344.8	387.9
430	43.0	86.0	129.0	172 0	215.0	258.0	301.0	344.0	387.0
429	42.9	85.8	128.7	171.6	214.5	257.4	300.3	343.2	386.1
428	42.8	85.6	128.4	171.2	214.0	256.8	299.6	342.4	385.2
427	42 7	85.4	128.1	170.8	213.5	256.2	298.9	341.6	384.3
426	42.6	85.2	127.8	170.4	213.0	255.6	298.2	340.8	383.4
425	42.5	85.0	127.5	170.0	212.5	255.0	297.5	340.0	382.5
424	42.4	84.8	127.2	169.6	212.0	254.4	296.8	339.2	381.6
423	42.3	84.6	126.9	169.2	211.5	253.8	296.1	338.4	380.7
422	42.2	84.4	126.6	168.8	211.0	253.2	295.4	337.6	379.8
421	42.1	84.2	126.3	168.4	210.5	252.6	294.7	336.8	378.9
420	42.0	84.0	126.0	168.0	210.0	252.0	294.0	336.0	378.0
419	41.9	83.8	125.7	167.6	209.5	251.4	293.3	335.2	377.1
418	41.8	83.6	125.4	167.2	209.0	250.8	292.6	334.4	376.2
417	41.7	83.4	125.1	166.8	208.5	250.2	291.9	333.6	375.3
416	41.6	83.2	124.8	166.4	208.0	249.6	291.2	332.8	374.4
415	41.5	83.0	124.5	166.0	207.5	249.0	290.5	332.0	373.5
414	41.4	82.8	124.2	165.6	207.0	248.4	289.8	331.2	372.6
413	41.3	82.6	123.9	165.2	206.5	247.8	289.1	330.4	371.7
412	41.2	82.4	123.6	164.8	206.0	247.2	288.4	329.6	370.8
411	41.1	82.2	123.3	164.4	205.5	246.6	287.7	328.8	369.9
410	41.0	82.0	123.0	164.0	205.0	246.0	287.0	328.0	369.0
409	40.9	81.8	122.7	163.6	204.5	245.4	286.3	327.2	368.1
408	40.8	81.6	122.4	163.2	204.0	244.8	285.6	326.4	367.2
407	40.7	81.4	122.1	162.8	203.5	244.2	284.9	325.6	366.3
406	40.6	81.2	121.8	162.4	203.0	243 6	284.2	324.8	365.4
405	40.5	81.0	121.5	162.0	202.5	243.0	283.5	324.0	364.5
404	40.4	80.8	121.2	161.6	202.0	242.4	282.8	323.2	363.6
403	40.3	80.6	120.9	161.2	201.5	241.8	282.1	322.4	362.7
402	40.2	80.4	120 6	160.8	201.0	241 2	281.4	321.6	361.8
401	40.1	80.2	120.3	160.4	200 5	240 6	280.7	320.8	360.9
400	40.0	80 0	120.0	160.0	200.0	240.0	280.0	320.0	360.0
399	39.9	79 8	119.7	159.6	199.5	239.4	279 3	319.2	359.1
398	39.8	79.6	119.4	159.2	199.0	238.8	278.6	318.4	358.2
397	39.7	79.4	119.1	158.8	198.5	238.2	277.9	317.6	357.3
396	39.6	79.2	118.8	158.4	198.0	237.6	277.2	316.8	356.4
395	39.5	79.0	118.5	158.0	197.5	237.0	276.5	316.0	355.5

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 110 L. 041.] [No. 119 L. 078.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
1	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	390
2	9218	9606	9993	0380	0766	1153	1538	1924	2309	2694	386
3	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	383
4	6905	7286	7666	8046	8426	8805	9185	9563	9942	0320	379
5	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
6	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	373
7	8186	8557	8928	9298	9668	0038	0407	0776	1145	1514	370
8	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
9	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
395	39.5	79.0	118.5	158.0	197.5	237.0	276.5	316.0	355.5
394	39.4	78.8	118.2	157.6	197.0	236.4	275.8	315.2	354.6
393	39.3	78.6	117.9	157.2	196.5	235.8	275.1	314.4	353.7
392	39.2	78.4	117.6	156.8	196.0	235.2	274.4	313.6	352.8
391	39.1	78.2	117.3	156.4	195.5	234.6	273.7	312.8	351.9
390	39.0	78.0	117.0	156.0	195.0	234.0	273.0	312.0	351.0
389	38.9	77.8	116.7	155.6	194.5	233.4	272.3	311.2	350.1
388	38.8	77.6	116.4	155.2	194.0	232.8	271.6	310.4	349.2
387	38.7	77.4	116.1	154.8	193.5	232.2	270.9	309.6	348.3
386	38.6	77.2	115.8	154.4	193.0	231.6	270.2	308.8	347.4
385	38.5	77.0	115.5	154.0	192.5	231.0	269.5	308.0	346.5
384	38.4	76.8	115.2	153.6	192.0	230.4	268.8	307.2	345.6
383	38.3	76.6	114.9	153.2	191.5	229.8	268.1	306.4	344.7
382	38.2	76.4	114.6	152.8	191.0	229.2	267.4	305.6	343.8
381	38.1	76.2	114.3	152.4	190.5	228.6	266.7	304.8	342.9
380	38.0	76.0	114.0	152.0	190.0	228.0	266.0	304.0	342.0
379	37.9	75.8	113.7	151.6	189.5	227.4	265.3	303.2	341.1
378	37.8	75.6	113.4	151.2	189.0	226.8	264.6	302.4	340.2
377	37.7	75.4	113.1	150.8	188.5	226.2	263.9	301.6	339.3
376	37.6	75.2	112.8	150.4	188.0	225.6	263.2	300.8	338.4
375	37.5	75.0	112.5	150.0	187.5	225.0	262.5	300.0	337.5
374	37.4	74.8	112.2	149.6	187.0	224.4	261.8	299.2	336.6
373	37.3	74.6	111.9	149.2	186.5	223.8	261.1	298.4	335.7
372	37.2	74.4	111.6	148.8	186.0	223.2	260.4	297.6	334.8
371	37.1	74.2	111.3	148.4	185.5	222.6	259.7	296.8	333.9
370	37.0	74.0	111.0	148.0	185.0	222.0	259.0	296.0	333.0
369	36.9	73.8	110.7	147.6	184.5	221.4	258.3	295.2	332.1
368	36.8	73.6	110.4	147.2	184.0	220.8	257.6	294.4	331.2
367	36.7	73.4	110.1	146.8	183.5	220.2	256.9	293.6	330.3
366	36.6	73.2	109.8	146.4	183.0	219.6	256.2	292.8	329.4
365	36.5	73.0	109.5	146.0	182.5	219.0	255.7	292.0	328.5
364	36.4	72.8	109.2	145.6	182.0	218.4	254.8	291.2	327.6
363	36.3	72.6	108.9	145.2	181.5	217.8	254.1	290.4	326.7
362	36.2	72.4	108.6	144.8	181.0	217.2	253.4	289.6	325.8
361	36.1	72.2	108.3	144.4	180.5	216.6	252.7	288.8	324.9
360	36.0	72.0	108.0	144.0	180.0	216.0	252.0	288.0	324.0
359	35.9	71.8	107.7	143.6	179.5	215.4	251.3	287.2	323.1
358	35.8	71.6	107.4	143.2	179.0	214.8	250.6	286.4	322.2
357	35.7	71.4	107.1	142.8	178.5	214.2	249.9	285.6	321.3
356	35.6	71.2	106.8	142.4	178.0	213.6	249.2	284.8	320.4

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 120 L. 079.] [No. 134 L. 130.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
120	079181	9543	9904	0266	0626	0987	1347	1707	2067	2426	360
1	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
2	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
3	9905	0258	0611	0963	1315	1667	2018	2370	2721	3071	352
4	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
5	6910	7257	7604	7951	8298	8644	8990	9335	9681	0026	346
6	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
7	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	341
8	7210	7549	7888	8227	8565	8903	9241	9579	9916	0253	338
9	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
1	7271	7603	7934	8265	8595	8926	9256	9586	9915	0245	330
2	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
3	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
4	7105	7429	7753	8076	8399	8722	9045	9368	9690	0012	323
	13										

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
355	35.5	71.0	106.5	142.0	177.5	213.0	248.5	284.0	319.5
354	35.4	70.8	106.2	141.6	177.0	212.4	247.8	283.2	318.6
353	35.3	70.6	105.9	141.2	176.5	211.8	247.1	282.4	317.7
352	35.2	70.4	105.6	140.8	176.0	211.2	246.4	281.6	316.8
351	35.1	70.2	105.3	140.4	175.5	210.6	245.7	280.8	315.9
350	35.0	70.0	105.0	140.0	175.0	210.0	245.0	280.0	315.0
349	34.9	69.8	104.7	139.6	174.5	209.4	244.3	279.2	314.1
348	34.8	69.6	104.4	139.2	174.0	208.8	243.6	278.4	313.2
347	34.7	69.4	104.1	138.8	173.5	208.2	242.9	277.6	312.3
346	34.6	69.2	103.8	138.4	173.0	207.6	242.2	276.8	311.4
345	34.5	69.0	103.5	138.0	172.5	207.0	241.5	276.0	310.5
344	34.4	68.8	103.2	137.6	172.0	206.4	240.8	275.2	309.6
343	34.3	68.6	102.9	137.2	171.5	205.8	240.1	274.4	308.7
342	34.2	68.4	102.6	136.8	171.0	205.2	239.4	273.6	307.8
341	34.1	68.2	102.3	136.4	170.5	204.6	238.7	272.8	306.9
340	34.0	68.0	102.0	136.0	170.0	204.0	238.0	272.0	306.0
339	33.9	67.8	101.7	135.6	169.5	203.4	237.3	271.2	305.1
338	33.8	67.6	101.4	135.2	169.0	202.8	236.6	270.4	304.2
337	33.7	67.4	101.1	134.8	168.5	202.2	235.9	269.6	303.3
336	33.6	67.2	100.8	134.4	168.0	201.6	235.2	268.8	302.4
335	33.5	67.0	100.5	134.0	167.5	201.0	234.5	268.0	301.5
334	33.4	66.8	100.2	133.6	167.0	200.4	233.8	267.2	300.6
333	33.3	66.6	99.9	133.2	166.5	199.8	233.1	266.4	299.7
332	33.2	66.4	99.6	132.8	166.0	199.2	232.4	265.6	298.8
331	33.1	66.2	99.3	132.4	165.5	198.6	231.7	264.8	297.9
330	33.0	66.0	99.0	132.0	165.0	198.0	231.0	264.0	297.0
329	32.9	65.8	98.7	131.6	164.5	197.4	230.3	263.2	296.1
328	32.8	65.6	98.4	131.2	164.0	196.8	229.6	262.4	295.2
327	32.7	65.4	98.1	130.8	163.5	196.2	228.9	261.6	294.3
326	32.6	65.2	97.8	130.4	163.0	195.6	228.2	260.8	293.4
325	32.5	65.0	97.5	130.0	162.5	195.0	227.5	260.0	292.5
324	32.4	64.8	97.2	129.6	162.0	194.4	226.8	259.2	291.6
323	32.3	64.6	96.9	129.2	161.5	193.8	226.1	258.4	290.7
322	32.2	64.4	96.6	128.8	161.0	193.2	225.4	257.6	289.8

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 135 L. 130.] [No. 149 L. 175.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
6	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
7	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	316
8	9879	0194	0508	0822	1136	1450	1763	2076	2389	2702	314
9	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	6128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
1	9219	9527	9835	0142	0449	0756	1063	1370	1676	1982	307
2	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
3	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
4	8362	8664	8965	9266	9567	9868	0168	0469	0769	1068	301
5	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
6	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
7	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
8	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
9	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
321	32.1	64.2	96.3	128.4	160.5	192.6	224.7	256.8	288.9
320	32.0	64.0	96.0	128.0	160.0	192.0	224.0	256.0	288.0
319	31.9	63.8	95.7	127.6	159.5	191.4	223.3	255.2	287.1
318	31.8	63.6	95.4	127.2	159.0	190.8	222.6	254.4	286.2
317	31.7	63.4	95.1	126.8	158.5	190.2	221.9	253.6	285.3
316	31.6	63.2	94.8	126.4	158.0	189.6	221.2	252.8	284.4
315	31.5	63.0	94.5	126.0	157.5	189.0	220.5	252.0	283.5
314	31.4	62.8	94.2	125.6	157.0	188.4	219.8	251.2	282.6
313	31.3	62.6	93.9	125.2	156.5	187.8	219.1	250.4	281.7
312	31.2	62.4	93.6	124.8	156.0	187.2	218.4	249.6	280.8
311	31.1	62.2	93.3	124.4	155.5	186.6	217.7	248.8	279.9
310	31.0	62.0	93.0	124.0	155.0	186.0	217.0	248.0	279.0
309	30.9	61.8	92.7	123.6	154.5	185.4	216.3	247.2	278.1
308	30.8	61.6	92.4	123.2	154.0	184.8	215.6	246.4	277.2
307	30.7	61.4	92.1	122.8	153.5	184.2	214.9	245.6	276.3
306	30.6	61.2	91.8	122.4	153.0	183.6	214.2	244.8	275.4
305	30.5	61.0	91.5	122.0	152.5	183.0	213.5	244.0	274.5
304	30.4	60.8	91.2	121.6	152.0	182.4	212.8	243.2	273.6
303	30.3	60.6	90.9	121.2	151.5	181.8	212.1	242.4	272.7
302	30.2	60.4	90.6	120.8	151.0	181.2	211.4	241.6	271.8
301	30.1	60.2	90.3	120.4	150.5	180.6	210.7	240.8	270.9
300	30.0	60.0	90.0	120.0	150.0	180.0	210.0	240.0	270.0
299	29.9	59.8	89.7	119.6	149.5	179.4	209.3	239.2	269.1
298	29.8	59.6	89.4	119.2	149.0	178.8	208.6	238.4	268.2
297	29.7	59.4	89.1	118.8	148.5	178.2	207.9	237.6	267.3
296	29.6	59.2	88.8	118.4	148.0	177.6	207.2	236.8	266.4
295	29.5	59.0	88.5	118.0	147.5	177.0	206.5	236.0	265.5
294	29.4	58.8	88.2	117.6	147.0	176.4	205.8	235.2	264.6
293	29.3	58.6	87.9	117.2	146.5	175.8	205.1	234.4	263.7
292	29.2	58.4	87.6	116.8	146.0	175.2	204.4	233.6	262.8
291	29.1	58.2	87.3	116.4	145.5	174.6	203.7	232.8	261.9
290	29.0	58.0	87.0	116.0	145.0	174.0	203.0	232.0	261.0
289	28.9	57.8	86.7	115.6	144.5	173.4	202.3	231.2	260.1
288	28.8	57.6	86.4	115.2	144.0	172.8	201.6	230.4	259.2
287	28.7	57.4	86.1	114.8	143.5	172.2	200.9	229.6	258.3
286	28.6	57.2	85.8	114.4	143.0	171.6	200.2	228.8	257.4

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 150 L. 176.] [No. 169 L. 230.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
1	8977	9264	9552	9839							
					0126	0413	0699	0986	1272	1558	287
2	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
3	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
4	7521	7803	8084	8366	8647	8928	9209	9490	9771		
										0051	281
5	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
6	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
7	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
8	8657	8932	9206	9481	9755						
						0029	0303	0577	0850	1124	274
9	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
1	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
2	9515	9783									
			0051	0319	0586	0853	1121	1388	1654	1921	267
3	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
4	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
5	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
6	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
7	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
8	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
9	7887	8144	8400	8657	8913	9170	9426	9682	9938		
	23									0193	256

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
285	28.5	57.0	85.5	114.0	142.5	171.0	199.5	228.0	256.5
284	28.4	56.8	85.2	113.6	142.0	170.4	198.8	227.2	255.6
283	28.3	56.6	84.9	113.2	141.5	169.8	198.1	226.4	254.7
282	28.2	56.4	84.6	112.8	141.0	169.2	197.4	225.6	253.8
281	28.1	56.2	84.3	112 4	140.5	168.6	196.7	224.8	252.9
280	28.0	56.0	84.0	112.0	140.0	168.0	196.0	224.0	252.0
279	27.9	55.8	83.7	111.6	139.5	167.4	195.3	223.2	251.1
278	27.8	55.6	83.4	111.2	139.0	166.8	194.6	222.4	250.2
277	27.7	55 4	83.1	110 8	138.5	166.2	193.9	221.6	249.3
276	27.6	55.2	82.8	110.4	138.0	165.6	193.2	220.8	248.4
275	27.5	55.0	82.5	110.0	137.5	165.0	192.5	220.0	247.5
274	27.4	54.8	82.2	109.6	137.0	164.4	191.8	219.2	246.6
273	27.3	54.6	81.9	109.2	136.5	163.8	191.1	218.4	245.7
272	27.2	54.4	81.6	108.8	136.0	163.2	190.4	217.6	244.8
271	27.1	54.2	81.3	108.4	135.5	162.6	189.7	216.8	243.9
270	27.0	54.0	81.0	108.0	135.0	162.0	189.0	216.0	243.0
269	26 9	53.8	80.7	107.6	134.5	161.4	188.3	215.2	242.1
268	26.8	53.6	80.4	107.2	134.0	160.8	187.6	214.4	241.2
267	26.7	53.4	80.1	106.8	133.5	160.2	186.9	213.6	240.3
266	26.6	53.2	79.8	106.4	133.0	159.6	186.2	212.8	239.4
265	26.5	53.0	79.5	106.0	132.5	159.0	185.5	212.0	238.5
264	26.4	52.8	79.2	105.6	132.0	158.4	184.8	211.2	237.6
263	26.3	52.6	78.9	105.2	131.5	157.8	184.1	210.4	236.7
262	26.2	52.4	78.6	104.8	131.0	157.2	183.4	209.6	235.8
261	26.1	52.2	78.3	104.4	130.5	156.6	182.7	208.8	234.9
260	26.0	52.0	78.0	104.0	130.0	156.0	182.0	208.0	234.0
259	25.9	51.8	77.7	103.6	129.5	155.4	181.3	207.2	233.1
258	25.8	51.6	77.4	103.2	129.0	154.8	180 6	206.4	232.2
257	25.7	51.4	77.1	102.8	128.5	154.2	179.9	205.6	231.3
256	25.6	51.2	76.8	102.4	128.0	153.6	179.2	204.8	230.4
255	25.5	51.0	76.5	102.0	127.5	153.0	178.5	204.0	229.5

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 170 L. 230.] [No. 189 L. 278.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	255
1	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
2	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
3	8046	8297	8548	8799	9049	9299	9550	9800	0050	0300	250
4	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
5	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
6	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
7	7973	8219	8464	8709	8954	9198	9443	9687	9932	0176	245
8	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
9	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	5273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
1	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
2	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
3	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
4	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
5	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
6	9513	9746	9980	0213	0446	0679	0912	1144	1377	1609	233
7	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
8	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
9	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
255	25.5	51.0	76.5	102.0	127.5	153.0	178.5	204.0	229.5
254	25.4	50.8	76.2	101.6	127.0	152.4	177.8	203.2	228.6
253	25.3	50.6	75.9	101.2	126.5	151.8	177.1	202.4	227.7
252	25.2	50.4	75.6	100.8	126.0	151.2	176.4	201.6	226.8
251	25.1	50.2	75.3	100.4	125.5	150.6	175.7	200.8	225.9
250	25.0	50.0	75.0	100.0	125.0	150.0	175.0	200.0	225.0
249	24.9	49.8	74.7	99.6	124.5	149.4	174.3	199.2	224.1
248	24.8	49.6	74.4	99.2	124.0	148.8	173.6	198.4	223.2
247	24.7	49.4	74.1	98.8	123.5	148.2	172.9	197.6	222.3
246	24.6	49.2	73.8	98.4	123.0	147.6	172.2	196.8	221.4
245	24.5	49.0	73.5	98.0	122.5	147.0	171.5	196.0	220.5
244	24.4	48.8	73.2	97.6	122.0	146.4	170.8	195.2	219.6
243	24.3	48.6	72.9	97.2	121.5	145.8	170.1	194.4	218.7
242	24.2	48.4	72.6	96.8	121.0	145.2	169.4	193.6	217.8
241	24.1	48.2	72.3	96.4	120.5	144.6	168.7	192.8	216.9
240	24.0	48.0	72.0	96.0	120.0	144.0	168.0	192.0	216.0
239	23.9	47.8	71.7	95.6	119.5	143.4	167.3	191.2	215.1
238	23.8	47.6	71.4	95.2	119.0	142.8	166.6	190.4	214.2
237	23.7	47.4	71.1	94.8	118.5	142.2	165.9	189.6	213.3
236	23.6	47.2	70.8	94.4	118.0	141.6	165.2	188.8	212.4
235	23.5	47.0	70.5	94.0	117.5	141.0	164.5	188.0	211.5
234	23.4	46.8	70.2	93.6	117.0	140.4	163.8	187.2	210.6
233	23.3	46.6	69.9	93.2	116.5	139.8	163.1	186.4	209.7
232	23.2	46.4	69.6	92.8	116.0	139.2	162.4	185.6	208.8
231	23.1	46.2	69.3	92.4	115.5	138.6	161.7	184.8	207.9
230	23.0	46.0	69.0	92.0	115.0	138.0	161.0	184.0	207.0
229	22.9	45.8	68.7	91.6	114.5	137.4	160.3	183.2	206.1
228	22.8	45.6	68.4	91.2	114.0	136.8	159.6	182.4	205.2
227	22.7	45.4	68.1	90.8	113.5	136.2	158.9	181.6	204.3
226	22.6	45.2	67.8	90.4	113.0	135.6	158.2	180.8	203.4

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 190 L. 278.] [No. 214 L. 332.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
190	278754	8982	9211	9439	9667	9895	0123	0351	0578	0806	228
1	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
2	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
3	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
4	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
5	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
6	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
7	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
8	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
9	8853	9071	9289	9507	9725	9943	0161	0378	0595	0813	218
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
1	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
2	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
3	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
4	9630	9843	0056	0268	0481	0693	0906	1118	1330	1542	212
5	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
6	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
7	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
8	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
9	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
1	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
2	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
3	8380	8583	8787	8991	9194	9398	9601	9805	0008	0211	203
4	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
225	22.5	45.0	67.5	90.0	112.5	135.0	157.5	180.0	202.5
224	22.4	44.8	67.2	89.6	112.0	134.4	156.8	179.2	201.6
223	22.3	44.6	66.9	89.2	111.5	133.8	156.1	178.4	200.7
222	22.2	44.4	66.6	88.8	111.0	133.2	155.4	177.6	199.8
221	22.1	44.2	66.3	88.4	110.5	132.6	154.7	176.8	198.9
220	22.0	44.0	66.0	88.0	110.0	132.0	154.0	176.0	198.0
219	21.9	43.8	65.7	87.6	109.5	131.4	153.3	175.2	197.1
218	21.8	43.6	65.4	87.2	109.0	130.8	152.6	174.4	196.2
217	21.7	43.4	65.1	86.8	108.5	130.2	151.9	173.6	195.3
216	21.6	43.2	64.8	86.4	108.0	129.6	151.2	172.8	194.4
215	21.5	43.0	64.5	86.0	107.5	129.0	150.5	172.0	193.5
214	21.4	42.8	64.2	85.6	107.0	128.4	149.8	171.2	192.6
213	21.3	42.6	63.9	85.2	106.5	127.8	149.1	170.4	191.7
212	21.2	42.4	63.6	84.8	106.0	127.2	148.4	169.6	190.8
211	21.1	42.2	63.3	84.4	105.5	126.6	147.7	168.8	189.9
210	21.0	42.0	63.0	84.0	105.0	126.0	147.0	168.0	189.0
209	20.9	41.8	62.7	83.6	104.5	125.4	146.3	167.2	188.1
208	20.8	41.6	62.4	83.2	104.0	124.8	145.6	166.4	187.2
207	20.7	41.4	62.1	82.8	103.5	124.2	144.9	165.6	186.3
206	20.6	41.2	61.8	82.4	103.0	123.6	144.2	164.8	185.4
205	20.5	41.0	61.5	82.0	102.5	123.0	143.5	164.0	184.5
204	20.4	40.8	61.2	81.6	102.0	122.4	142.8	163.2	183.6
203	20.3	40.6	60.9	81.2	101.5	121.8	142.1	162.4	182.7
202	20.2	40.4	60.6	80.8	101.0	121.2	141.4	161.6	181.8

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 215 L. 332.] [No. 239 L. 380.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
215	332438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
6	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
7	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
8	8456	8656	8855	9054	9253	9451	9650	9849			
									0047	0246	199
9	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
220	2423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
1	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
2	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
3	8305	8500	8694	8889	9083	9278	9472	9666	9860		
										0054	194
4	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
5	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
6	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
7	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
8	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
9	9835										
		0025	0215	0404	0593	0783	0972	1161	1350	1539	189
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
1	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
2	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
3	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
4	9216	9401	9587	9772	9958						
						0143	0328	0513	0698	0883	185
5	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
6	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
7	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
8	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
9	8398	8580	8761	8943	9124	9306	9487	9668	9849		
	38									0030	181

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
202	20.2	40.4	60.6	80.8	101.0	121.2	141.4	161.6	181.8
201	20.1	40.2	60.3	80.4	100.5	120.6	140.7	160.8	180.9
200	20.0	40.0	60.0	80.0	100.0	120.0	140.0	160.0	180.0
199	19.9	39.8	59.7	79.6	99.5	119.4	139.3	159.2	179.1
198	19.8	39.6	59.4	79.2	99.0	118.8	138.6	158.4	178.2
197	19.7	39.4	59.1	78.8	98.5	118.2	137.9	157.6	177.3
196	19.6	39.2	58.8	78.4	98.0	117.6	137.2	156.8	176.4
195	19.5	39.0	58.5	78.0	97.5	117.0	136.5	156.0	175.5
194	19.4	38.8	58.2	77.6	97.0	116.4	135.8	155.2	174.6
193	19.3	38.6	57.9	77.2	96.5	115.8	135.1	154.4	173.7
192	19.2	38.4	57.6	76.8	96.0	115.2	134.4	153.6	172.8
191	19.1	38.2	57.3	76.4	95.5	114.6	133.7	152.8	171.9
190	19.0	38.0	57.0	76.0	95.0	114.0	133.0	152.0	171.0
189	18.9	37.8	56.7	75.6	94.5	113.4	132.3	151.2	170.1
188	18.8	37.6	56.4	75.2	94.0	112.8	131.6	150.4	169.2
187	18.7	37 4	56.1	74.8	93.5	112.2	130.9	149.6	168.3
186	18.6	37.2	55.8	74.4	93.0	111.6	130.2	148.8	167.4
185	18.5	37.0	55.5	74.0	92.5	111.0	129.5	148.0	166.5
184	18.4	36.8	55.2	73.6	92.0	110.4	128.8	147.2	165.6
183	18.3	36.6	54.9	73.2	91.5	109.8	128.1	146.4	164.7
182	18.2	36.4	54.6	72.8	91.0	109.2	127.4	145.6	163.8
181	18.1	36.2	54.3	72.4	90.5	108.6	126.7	144.8	162 9
180	18.0	36.0	54.0	72.0	90.0	108.0	126.0	144.0	162.0
179	17.9	35.8	53.7	71.6	89.5	107.4	125.3	143.2	161.1

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 240 L. 380.] [No. 269 L. 431.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
1	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
2	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
3	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
4	7390	7568	7746	7924	8101	8279	8456	8634	8811	8989	178
5	9166	9343	9520	9698	9875	0051	0228	0405	0582	0759	177
6	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
7	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
8	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
9	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
1	9674	9847	0020	0192	0365	0538	0711	0883	1056	1228	173
2	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
3	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
4	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
5	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
6	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
7	9933	0102	0271	0440	0609	0777	0946	1114	1283	1451	169
8	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
9	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	4973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
1	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
2	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
3	9956	0121	0286	0451	0616	0781	0945	1110	1275	1439	165
4	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
5	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
6	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
7	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
8	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
9	9752 43	9914	0075	0236	0398	0559	0720	0881	1042	1203	161

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
178	17.8	35.6	53.4	71.2	89.0	106.8	124.6	142.4	160.2
177	17.7	35.4	53.1	70.8	88.5	106.2	123.9	141.6	159.3
176	17.6	35.2	52.8	70.4	88.0	105.6	123.2	140.8	158.4
175	17.5	35.0	52.5	70.0	87.5	105.0	122.5	140.0	157.5
174	17.4	34.8	52.2	69.6	87.0	104.4	121.8	139.2	156.6
173	17.3	34.6	51.9	69.2	86.5	103.8	121.1	138.4	155.7
172	17.2	34.4	51.6	68.8	86.0	103.2	120.4	137.6	154.8
171	17.1	34.2	51.3	68.4	85.5	102.6	119.7	136.8	153.9
170	17.0	34.0	51.0	68.0	85.0	102.0	119.0	136.0	153.0
169	16.9	33.8	50.7	67.6	84.5	101.4	118.3	135.2	152.1
168	16.8	33.6	50.4	67.2	84.0	100.8	117.6	134.4	151.2
167	16.7	33.4	50.1	66.8	83.5	100.2	116.9	133.6	150.3
166	16.6	33.2	49.8	66.4	83.0	99.6	116.2	132.8	149.4
165	16.5	33.0	49.5	66.0	82.5	99.0	115.5	132.0	148.5
164	16.4	32.8	49.2	65.6	82.0	98.4	114.8	131.2	147.6
163	16.3	32.6	48.9	65.2	81.5	97.8	114.1	130.4	146.7
162	16.2	32.4	48.5	64.8	81.0	97.2	113.4	129.6	145.8
161	16.1	32.2	48.3	64.4	80.5	96.6	112.7	128.8	144.9

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 270 L. 431.] [No. 299 L. 476.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
1	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
2	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
3	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
4	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
5	9333	9491	9648	9806	9964	0122	0279	0437	0594	0752	158
6	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
7	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
8	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
9	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
1	8706	8861	9015	9170	9324	9478	9633	9787	9941	0095	154
2	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
3	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
4	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
5	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
6	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
7	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
8	9392	9543	9694	9845	9995	0146	0296	0447	0597	0748	151
9	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
1	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
2	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
3	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
4	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
5	9822	9969	0116	0263	0410	05[illegible]	0704	0851	0998	1145	147
6	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
7	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
8	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
9	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
161	16.1	32.2	48.3	64.4	80.5	96.6	112.7	128.8	144.9
160	16.0	32.0	48.0	64.0	80.0	96.0	112.0	128.0	144.0
159	15.9	31.8	47.7	63.6	79.5	95.4	111.3	127.2	143.1
158	15.8	31.6	47.4	63.2	79.0	94.8	110.6	126.4	142.2
157	15.7	31.4	47.1	62.8	78.5	94.2	109.9	125.6	141.3
156	15.6	31.2	46.8	62.4	78.0	93.6	109.2	124.8	140.4
155	15.5	31.0	46.5	62.0	77.5	93.0	108.5	124.0	139.5
154	15.4	30.8	46.2	61.6	77.0	92.4	107.8	123.2	138.6
153	15.3	30.6	45.9	61.2	76.5	91.8	107.1	122.4	137.7
152	15.2	30.4	45.6	60.8	76.0	91.2	106.4	121.6	136.8
151	15.1	30.2	45.3	60.4	75.5	90.6	105.7	120.8	135.9
150	15.0	30.0	45.0	60.0	75.0	90.0	105.0	120.0	135.0
149	14.9	29.8	44.7	59.6	74.5	89.4	104.3	119.2	134.1
148	14.8	29.6	44.4	59.2	74.0	88.8	103.6	118.4	133.2
147	14.7	29.4	44.1	58.8	73.5	88.2	102.9	117.6	132.3
146	14.6	29.2	43.8	58.4	73.0	87.6	102.2	116.8	131.4
145	14.5	29.0	43.5	58.0	72.5	87.0	101.5	116.0	130.5
144	14.4	28.8	43.2	57.6	72.0	86.4	100.8	115.2	129.6
143	14.3	28.6	42.9	57.2	71.5	85.8	100.1	114.4	128.7
142	14.2	28.4	42.6	56.8	71.0	85.2	99.4	113.6	127.8
141	14.1	28.2	42.3	56.4	70.5	84.6	98.7	112.8	126.9
140	14.0	28.0	42.0	56.0	70.0	84.0	98.0	112.0	126.0

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 300 L. 477.] [No. 339 L. 531.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
300	477121	7266	7411	7555	7700	7844	7989	8133	8278	8422	145
1	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
2	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
3	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
4	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
5	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
6	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
7	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
8	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
9	9958	0099	0239	0380	0520	0661	0801	0941	1081	1222	140
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
1	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
2	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
3	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
4	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
5	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
6	9687	9824	9962	0099	0236	0374	0511	0648	0785	0922	137
7	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
8	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
9	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
1	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
2	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
3	9203	9337	9471	9606	9740	9874	0009	0143	0277	0411	134
4	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
5	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	134
6	3218	3351	3484	3617	3750	3883	4016	4149	4282	4415	133
7	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
8	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
9	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
1	9828	9959	0090	0221	0353	0484	0615	0745	0876	1007	131
2	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
3	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
4	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
5	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
6	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
7	7630	7759	7888	8016	8145	8274	8402	8531	8660	8788	129
8	8917	9045	9174	9302	9430	9559	9687	9815	9943	0072	128
9	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351	128

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
139	13.9	27.8	41.7	55.6	69.5	83.4	97.3	111.2	125.1
138	13.8	27.6	41.4	55.2	69.0	82.8	96.6	110.4	124.2
137	13.7	27.4	41.1	54.8	68.5	82.2	95.9	109.6	123.3
136	13.6	27.2	40.8	54.4	68.0	81.6	95.2	108.8	122.4
135	13.5	27.0	40.5	54.0	67.5	81.0	94.5	108.0	121.5
134	13.4	26.8	40.2	53.6	67.0	80.4	93.8	107.2	120.6
133	13.3	26.6	39.9	53.2	66.5	79.8	93.1	106.4	119.7
132	13.2	26.4	39.6	52.8	66.0	79.2	92.4	105.6	118.8
131	13.1	26.2	39.3	52.4	65.5	78.6	91.7	104.8	117.9
130	13.0	26.0	39.0	52.0	65.0	78.0	91.0	104.0	117.0
129	12.9	25.8	38.7	51.6	64.5	77.4	90.3	103.2	116.1
128	12.8	25.6	38.4	51.2	64.0	76.8	89.6	102.4	115.2
127	12.7	25.4	38.1	50.8	63.5	76.2	88.9	101.6	114.3

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 340 L. 531.] [No. 379 L. 579.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
340	531479	1607	1734	1862	1990	2117	2245	2372	2500	2627	128
1	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
2	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
3	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
4	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
5	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
6	9076	9202	9327	9452	9578	9703	9829	9954	0079	0204	125
7	540329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
8	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
9	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	4068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
1	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
2	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
3	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
4	9003	9126	9249	9371	9494	9616	9739	9861	9984	0106	123
5	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
6	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
7	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
8	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
9	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
1	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
2	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
3	9907	0026	0146	0265	0385	0504	0624	0743	0863	0982	119
4	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
5	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
6	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
7	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
8	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
9	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
1	9374	9491	9608	9725	9842	9959	0076	0193	0309	0426	117
2	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
3	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
4	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
5	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
6	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
7	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
8	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
9	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
128	12.8	25.6	38.4	51.2	64.0	76.8	89.6	102.4	115.2
127	12.7	25.4	38.1	50.8	63.5	76.2	88.9	101.6	114.3
126	12.6	25.2	37.8	50.4	63.0	75.6	88.2	100.8	113.4
125	12.5	25.0	37.5	50.0	62.5	75.0	87.5	100.0	112.5
124	12.4	24.8	37.2	49.6	62.0	74.4	86.8	99.2	111.6
123	12.3	24.6	36.9	49.2	61.5	73.8	86.1	98.4	110.7
122	12.2	24.4	36.6	48.8	61.0	73.2	85.4	97.6	109.8
121	12.1	24.2	36.3	48.4	60.5	72.6	84.7	96.8	108.9
120	12.0	24.0	36.0	48.0	60.0	72.0	84.0	96.0	108.0
119	11.9	23.8	35.7	47.6	59.5	71.4	83.3	95.2	107.1

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 380. L. 579.] [No. 414 L. 617.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
380	579784	9898	0012	0126	0241	0355	0469	0583	0697	0811	114
1	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950	
2	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	
3	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	
4	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
5	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	
6	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	
7	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
8	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	
9	9950	0061	0173	0284	0396	0507	0619	0730	0842	0953	
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066	
1	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
2	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	
3	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	
4	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	
5	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
6	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	
7	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	
8	9883	9992	0101	0210	0319	0428	0537	0646	0755	0864	109
9	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951	
400	2060	2169	2277	2386	2494	2603	2711	2819	2928	3036	
1	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
2	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	
3	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	
4	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	
5	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
6	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	
7	9594	9701	9808	9914	0021	0128	0234	0341	0447	0554	
8	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617	
9	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	
1	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	
2	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	
3	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
4	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
118	11.8	23.6	35.4	47.2	59.0	70.8	82.6	94.4	106.2
117	11.7	23.4	35.1	46.8	58.5	70.2	81.9	93.6	105.3
116	11.6	23.2	34.8	46.4	58.0	69.6	81.2	92.8	104.4
115	11.5	23.0	34.5	46.0	57.5	69.0	80.5	92.0	103.5
114	11.4	22.8	34.2	45.6	57.0	68.4	79.8	91.2	102.6
113	11.3	22.6	33.9	45.2	56.5	67.8	79.1	90.4	101.7
112	11.2	22.4	33.6	44.8	56.0	67.2	78.4	89.6	100.8
111	11.1	22.2	33.3	44.4	55.5	66.6	77.7	88.8	99.9
110	11.0	22.0	33.0	44.0	55.0	66.0	77.0	88.0	99.0
109	10.9	21.8	32.7	43.6	54.5	65.4	76.3	87.2	98.1
108	10.8	21.6	32.4	43.2	54.0	64.8	75.6	86.4	97.2
107	10.7	21.4	32.1	42.8	53.5	64.2	74.9	85.6	96.3
106	10.6	21.2	31.8	42.4	53.0	63.6	74.2	84.8	95.4
105	10.5	21.0	31.5	42.0	52.5	63.0	73.5	84.0	94.5
104	10.4	20.8	31.2	41.6	52.0	62.4	72.8	83.2	93.6

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 415 L. 618.] [No. 459 L. 662

N.	0	1	2	3	4	5	6	7	8	9	Diff.
415	618048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
6	9093	9198	9302	9406	9511	9615	9719	9824	9928	0032	
7	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072	104
8	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	
9	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	
420	3249	3353	3456	3559	3663	3766	3869	3973	4076	4179	
1	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
2	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	
3	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	
4	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	
5	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
6	9410	9512	9613	9715	9817	9919	0021	0123	0224	0326	
7	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342	
8	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	
9	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	101
1	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	
2	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	
3	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	
4	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	100
5	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	
6	9486	9586	9686	9785	9885	9984	0084	0183	0283	0382	
7	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375	
8	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	
9	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340	
1	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	
2	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	
3	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
4	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	
5	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	
6	9335	9432	9530	9627	9724	9821	9919	0016	0113	0210	
7	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181	
8	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
9	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	
450	3213	3309	3405	3502	3598	3695	3791	3888	3984	4080	
1	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	
2	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
3	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	
4	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	
5	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	
6	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	
7	9916	0011	0106	0201	0296	0391	0486	0581	0676	0771	95
8	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	
9	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
105	10.5	21.0	31.5	42.0	52.5	63.0	73.5	84.0	94.5
104	10.4	20.8	31.2	41.6	52.0	62.4	72.8	83.2	93.6
103	10.3	20.6	30.9	41.2	51.5	61.8	72.1	82.4	92.7
102	10.2	20.4	30.6	40.8	51.0	61.2	71.4	81.6	91.8
101	10.1	20.2	30.3	40.4	50.5	60.6	70.7	80.8	90.9
100	10.0	20.0	30.0	40.0	50.0	60.0	70.0	80.0	90.0
99	9.9	19.8	29.7	39.6	49.5	59.4	69.3	79.2	89.1

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 460 L. 662.] [No. 499 L. 698.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
460	662758	2852	2947	3041	3135	3230	3324	3418	3512	3607	
1	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	
2	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
3	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	
4	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	
5	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	
6	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	
7	9317	9410	9503	9596	9689	9782	9875	9967	0060	0153	93
8	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080	
9	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	
1	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	
2	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
3	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	
4	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	
5	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	
6	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	
7	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
8	9428	9519	9610	9700	9791	9882	9973	0063	0154	0245	
9	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151	
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	
1	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	
2	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
3	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	
4	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	
5	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	
6	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	
7	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
8	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	
9	9309	9398	9486	9575	9664	9753	9841	9930	0019	0107	
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993	
1	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	
2	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	
3	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
4	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	
5	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	
6	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	
7	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	
8	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	
9	8100	8188	8275	8362	8449	8535	8622	8709	8796	8883	87

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
98	9.8	19.6	29.4	39.2	49.0	58.8	68.6	78.4	88.2
97	9.7	19.4	29.1	38.8	48.5	58.2	67.9	77.6	87.3
96	9.6	19.2	28.8	38.4	48.0	57.6	67.2	76.8	86.4
95	9.5	19.0	28.5	38.0	47.5	57.0	66.5	76.0	85.5
94	9.4	18.8	28.2	37.6	47.0	56.4	65.8	75.2	84.6
93	9.3	18.6	27.9	37.2	46.5	55.8	65.1	74.4	83.7
92	9.2	18.4	27.6	36.8	46.0	55.2	64.4	73.6	82.8
91	9.1	18.2	27.3	36.4	45.5	54.6	63.7	72.8	81.9
90	9.0	18.0	27.0	36.0	45.0	54.0	63.0	72.0	81.0
89	8.9	17.8	26.7	35.6	44.5	53.4	62.3	71.2	80.1
88	8.8	17.6	26.4	35.2	44.0	52.8	61.6	70.4	79.2
87	8.7	17.4	26.1	34.8	43.5	52.2	60.9	69.6	78.3
86	8.6	17.2	25.8	34.4	43.0	51.6	60.2	68.8	77.4

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 500 L. 698.] [No. 544 L. 736.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
500	698970	9057	9144	9231	9317	9404	9491	9578	9664	9751	
1	9838	9924	0011	0098	0184	0271	0358	0444	0531	0617	
2	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482	
3	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	
4	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	
5	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
6	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	
7	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	
8	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	
9	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	
1	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
2	9270	9355	9440	9524	9609	9694	9779	9863	9948	0033	
3	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879	
4	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	
5	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	
6	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	
7	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
8	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	
9	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	
1	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	
2	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	
3	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
4	9331	9414	9497	9580	9663	9745	9828	9911	9994	0077	
5	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903	
6	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	
7	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	
8	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	
9	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	
1	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	
2	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	
3	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	
4	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	
5	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	
6	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
7	9974	0055	0136	0217	0298	0378	0459	0540	0621	0702	
8	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	
9	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	
540	2394	2474	2555	2635	2715	2796	2876	2956	3037	3117	
1	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	
2	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	
3	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
4	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
87	8.7	17.4	26.1	34.8	43.5	52.2	60.9	69.6	78.3
86	8.6	17.2	25.8	34.4	43.0	51.6	60.2	68.8	77.4
85	8.5	17.0	25.5	34.0	42.5	51.0	59.5	68.0	76.5
84	8.4	16.8	25.2	33.6	42.0	50.4	58.8	67.2	75.6

TABLE XXII (*Continued*)

LOGARITHMS OF NUMBERS

No. 545 L. 736.] [No. 584 L. 767.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
545	736397	6476	6556	6635	6715	6795	6874	6954	7034	7113	
6	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	
7	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	
8	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	
9	9572	9651	9731	9810	9889	9968					
							0047	0126	0205	0284	79
550	740363	0442	0521	0600	0678	0757	0836	0915	0994	1073	
1	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	
2	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	
3	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	
4	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	
5	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	
6	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
7	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	
8	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	
9	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885	
1	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	
2	9736	9814	9891	9968							
					0045	0123	0200	0277	0354	0431	
3	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202	
4	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
5	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	
6	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	
7	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	
8	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	
9	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560	
1	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
2	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	
3	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	
4	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	
5	9668	9743	9819	9894	9970						
						0045	0121	0196	0272	0347	
6	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101	
7	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	
8	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
9	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	
580	3428	3503	3578	3653	3727	3802	3877	3952	4027	4101	
1	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	
2	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	
3	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	
4	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	

PROPORTIONAL PARTS

Diff.	1	2	3	4	5	6	7	8	9
83	8.3	16.6	24.9	33.2	41.5	49.8	58.1	66.4	74.7
82	8.2	16.4	24.6	32.8	41.0	49.2	57.4	65.6	73.8
81	8.1	16.2	24.3	32.4	40.5	48.6	56.7	64.8	72.9
80	8.0	16.0	24.0	32.0	40.0	48.0	56.0	64.0	72.0
79	7.9	15.8	23.7	31.6	39.5	47.4	55.3	63.2	71.1
78	7.8	15.6	23.4	31.2	39.0	46.8	54.6	62.4	70.2
77	7.7	15.4	23.1	30.8	38.5	46.2	53.9	61.6	69.3
76	7.6	15.2	22.8	30.4	38.0	45.6	53.2	60.8	68.4
75	7.5	15.0	22.5	30.0	37.5	45.0	52.5	60.0	67.5
74	7.4	14.8	22.2	29.6	37.0	44.4	51.8	59.2	66.6

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 585 L. 767.] [No. 629 L. 799.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
585	767156	7230	7304	7379	7453	7527	7601	7675	7749	7823	
6	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
7	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	
8	9377	9451	9525	9599	9673	9746	9820	9894	9968	0042	
9	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778	
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	
1	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	
2	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	
3	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	
4	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
5	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	
6	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	
7	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	
8	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	
9	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	
600	8151	8224	8296	8368	8441	8513	8585	8658	8730	8802	
1	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	
2	9596	9669	9741	9813	9885	9957	0029	0101	0173	0245	
3	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965	72
4	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	
5	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	
6	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	
7	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	
8	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	
9	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970	
1	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
2	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	
3	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	
4	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	
5	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	
6	9581	9651	9722	9792	9863	9933	0004	0074	0144	0215	
7	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918	
8	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	
9	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	
620	2392	2462	2532	2602	2672	2742	2812	2882	2952	3022	70
1	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	
2	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	
3	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	
4	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	
5	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	
6	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	
7	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	
8	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	
9	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
75	7.5	15.0	22.5	30.0	37.5	45.0	52.5	60.0	67.5
74	7.4	14.8	22.2	29.6	37.0	44.4	51.8	59.2	66.6
73	7.3	14.6	21.9	29.2	36.5	43.8	51.1	58.4	65.7
72	7.2	14.4	21.6	28.8	36.0	43.2	50.4	57.6	64.8
71	7.1	14.2	21.3	28.4	35.5	42.6	49.7	56.8	63.9
70	7.0	14.0	21.0	28.0	35.0	42.0	49.0	56.0	63.0
69	6.9	13.8	20.7	27.6	34.5	41.4	48.3	55.2	62.1

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 630 L. 799.] [No. 674 L. 829.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
630	799341	9409	9478	9547	9616	9685	9754	9823	9892	9961	
1	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648	
2	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	
3	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	
4	2089	2158	2226	2205	2363	2432	2500	2568	2637	2705	
5	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	
6	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	
7	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	
8	4821	4889	4957	5025	5093	5161	5229	5297	5365	5453	68
9	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	
640	806180	6248	6316	6384	6451	6519	6587	6655	6723	6790	
1	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	
2	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	
3	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	
4	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	
5	9560	9627	9694	9762	9829	9896	9964				
								0031	0098	0165	
6	810233	0300	0367	0434	0501	0569	0636	0703	0770	0837	
7	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
8	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	
9	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	
650	2913	2980	3047	3114	3181	3247	3314	3381	3448	3514	
1	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	
2	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	
3	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	
4	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	
5	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	
6	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	
7	7565	7631	7698	7764	7830	7896	7962	8028	8004	8160	
8	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	
9	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	9544	9610	9676	9741	9807	9873	9939				
								0004	0070	0136	
1	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792	
2	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	
3	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	
4	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	
5	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	
6	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	
7	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	
8	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
9	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658	
1	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	
2	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	
3	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	
4	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
68	6.8	13.6	20.4	27.2	34.0	40.8	47.6	54.4	61.2
67	6.7	13.4	20.1	26.8	33.5	40.2	46.9	53.6	60.3
66	6.6	13.2	19.8	26.4	33.0	39.6	46.2	52.8	59.4
65	6.5	13.0	19.5	26.0	32.5	39.0	45.5	52.0	58.5
64	6.4	12.8	19.2	25.6	32.0	38.4	44.8	51.2	57.6

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 675 L. 829.] No. 719 L. 857.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
675	829304	9368	9432	9497	9561	9625	9690	9754	9818	9882	
6	9947	0011	0075	0139	0204	0268	0332	0396	0460	0525	
7	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	
8	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
9	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	
1	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	
2	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	
3	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	
4	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	
5	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	
6	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	
7	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	
8	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	
9	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	8849	8912	8975	9038	9101	9164	9227	9289	9352	9415	
1	9478	9541	9604	9667	9729	9792	9855	9918	9981	0043	
2	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671	
3	0733	0796	0859	0021	0984	1046	1109	1172	1234	1297	
4	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	
5	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	
6	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	
7	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	
8	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	
9	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	
700	5098	5160	5222	5284	5346	5408	5470	5532	5594	5656	
1	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
2	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	
3	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	
4	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	
5	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	
6	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	
7	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	
8	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585	
9	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	
1	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	
2	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
3	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	
4	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	
5	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	
6	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	
7	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	
8	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	
9	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
65	6.5	13.0	19.5	26.0	32.5	39.0	45.5	52.0	58.5
64	6.4	12.8	19.2	25.6	32.0	38.4	44.8	51.2	57.6
63	6.3	12.6	18.9	25.2	31.5	37.8	44.1	50.4	56.7
62	6.2	12.4	18.6	24.8	31.0	37.2	43.4	49.6	55.8
61	6.1	12.2	18.3	24.4	30.5	36.6	42.7	48.8	54.9
60	6.0	12.0	18.0	24.0	30.0	36.0	42.0	48.0	54.0

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 720 L. 857.] [No. 764 L. 883.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
720	857332	7393	7453	7513	7574	7634	7694	7755	7815	7875	
1	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	
2	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	
3	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
4	9739	9799	9859	9918	9978	0038	0098	0158	0218	0278	
5	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877	
6	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	
7	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	
8	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	
9	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	
1	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	
2	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	
3	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	
4	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	
5	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
6	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	
7	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	
8	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	
9	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	
740	9232	9290	9349	9408	9466	9525	9584	9642	9701	9760	
1	9818	9877	9935	9994	0053	0111	0170	0228	0287	0345	
2	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930	
3	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	
4	1573	1631	1690	1748	1806	1865	1923	1981	2040	2008	
5	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	
6	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	
7	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	
8	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
9	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	
750	5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	
1	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	
2	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	
3	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	
4	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	
5	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	
6	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	
7	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	
8	9669	9726	9784	9841	9898	9956	0013	0070	0127	0185	
9	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756	
760	0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	
1	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
2	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	
3	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	
4	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
59	5.9	11.8	17.7	23.6	29.5	35.4	41.3	47.2	53.1
58	5.8	11.6	17.4	23.2	29.0	34.8	40.6	46.4	52.2
57	5.7	11.4	17.1	22.8	28.5	34.2	39.9	45.6	51.3
56	5.6	11.2	16.8	22.4	28.0	33.6	39.2	44.8	50.4

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 765 L. 883.] [No. 809 L. 908.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
765	883661	3718	3775	3832	3888	3945	4002	4059	4115	4172	
6	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	
7	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	
8	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	
9	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998	
1	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	
2	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	
3	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	
4	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	
5	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
6	9862	9918	9974	0030	0086	0141	0197	0253	0309	0365	
7	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924	
8	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	
9	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	
780	2095	2150	2206	2262	2317	2373	2429	2484	2540	2595	
1	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	
2	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	
3	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	
4	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	
5	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	
6	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	
7	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	
8	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	
9	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	
1	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	
2	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	
3	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	
4	9821	9875	9930	9985	0039	0094	0149	0203	0258	0312	
5	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859	
6	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	
7	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	
8	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	
9	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	
800	3090	3144	3199	3253	3307	3361	3416	3470	3524	3578	
1	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	
2	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	
3	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	
4	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
5	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	
6	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	
7	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	
8	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	
9	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
57	5.7	11.4	17.1	22.8	28.5	34.2	39.9	45.6	51.3
56	5.6	11.2	16.8	22.4	28.0	33.6	39.2	44.8	50.4
55	5.5	11.0	16.5	22.0	27.5	33.0	38.5	44.0	49.5
54	5.4	10.8	16.2	21.6	27.0	32.4	37.8	43.2	48.6

TABLE XXII (*Continued*)

LOGARITHMS OF NUMBERS

No. 810 L. 908.] **[No. 854 L. 931.**

N.	0	1	2	3	4	5	6	7	8	9	Diff.
810	908485	8539	8592	8646	8699	8753	8807	8860	8914	8967	
1	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	
2	9556	9610	9663	9716	9770	9823	9877	9930	9984	0037	
3	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571	
4	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	
5	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	
6	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	
7	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	
8	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	
9	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
820	3814	3867	3920	3973	4026	4079	4132	4184	4237	4290	
1	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	
2	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	
3	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	
4	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	
5	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	
6	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	
7	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	
8	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	
9	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	
1	9601	9653	9706	9758	9810	9862	9914	9967	0019	0071	
2	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593	
3	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	
4	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
5	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	
6	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	
7	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	
8	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	
9	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	
840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744	
1	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	
2	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	
3	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	
4	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	
5	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	
6	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	
7	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	
8	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	
9	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	
850	9419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
1	9930	9981	0032	0083	0134	0185	0236	0287	0338	0389	
2	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	
3	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	
4	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	

PROPORTIONAL PARTS.

Diff.	1	2	3	4	5	6	7	8	9
53	5.3	10.6	15.9	21.2	26.5	31.8	37.1	42.4	47.7
52	5.2	10.4	15.6	20.8	26.0	31.2	36.4	41.6	46.8
51	5.1	10.2	15.3	20.4	25.5	30.6	35.7	40.8	45.9
50	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 855 L. 931.] **[No. 899 L. 954.**

N.	0	1	2	3	4	5	6	7	8	9	Diff.
855	931966	2017	2068	2118	2169	2220	2271	2322	2372	2423	
6	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	
7	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	
8	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	
9	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	
860	4498	4549	4599	4650	4700	4751	4801	4852	4902	4953	
1	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	
2	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	
3	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	
4	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	
5	7016	7066	7116	7167	7217	7267	7317	7367	7418	7468	
6	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	
7	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
8	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	
9	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968	
1	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	
2	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	
3	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	
4	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	
5	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	
6	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	
7	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	
8	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	
9	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	
1	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	
2	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	
3	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	
4	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	
5	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	
6	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
7	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	
8	8413	8462	8511	8560	8608	8657	8706	8755	8804	8853	
9	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829	
1	9878	9926	9975	0024	0073	0121	0170	0219	0267	0316	
2	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	
3	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	
4	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	
5	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	
6	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	
7	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	
8	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	
9	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
51	5.1	10.2	15.3	20.4	25.5	30.6	35.7	40.8	45.9
50	5.0	10.0	15.0	20.0	25.0	30.0	35.0	40.0	45.0
49	4.9	9.8	14.7	19.6	24.5	29.4	34.3	39.2	44.1
48	4.8	9.6	14.4	19.2	24.0	28.8	33.6	38.4	43.2

TABLE XXII (*Continued*)

Logarithms of Numbers

No 900 L. 954.] [No. 944 L. 975.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677	
1	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	
2	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	
3	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	
4	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	
5	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	**48**
6	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	
7	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	
8	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	
9	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	
1	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	
2	9995	0042	0090	0138	0185	0233	0280	0328	0376	0423	
3	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	
4	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	
5	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	
6	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	
7	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	
8	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	
9	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	
1	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	
2	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	
3	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	
4	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	**47**
5	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	
6	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	
7	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	
8	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	
9	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	
1	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	
2	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	
3	9882	9928	9975	0021	0068	0114	0161	0207	0254	0300	
4	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765	
5	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	
6	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	
7	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	
8	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	
9	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	
940	3128	3174	3220	3266	3313	3359	3405	3451	3497	3543	
1	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	
2	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	
3	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	
4	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	**46**

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
47	4.7	9.4	14.1	18.8	23.5	28.2	32.9	37.6	42.3
46	4.6	9.2	13.8	18.4	23.0	27.6	32.2	36.8	41.4

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 945 L. 975.] [No. 989 L. 995.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
945	975432	5478	5524	5570	5616	5662	5707	5753	5799	5845	
6	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	
7	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	
8	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	
9	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	
950	7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	
1	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	
2	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	
3	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	
4	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	
5	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	
6	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	
7	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	
8	1366	1411	1456	1501	1517	1592	1637	1683	1728	1773	
9	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	
1	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	
2	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	
3	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	
4	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	
5	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
6	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	
7	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	
8	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	
9	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	
1	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	
2	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	
3	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	
4	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	
5	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	
6	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	
7	9895	9939	9983	0028	0072	0117	0161	0206	0250	0294	
8	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	
9	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	
980	1226	1270	1315	1359	1403	1448	1492	1536	1580	1625	
1	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	
2	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	
3	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	
4	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	
5	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	
6	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	
7	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
8	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	
9	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	

Proportional Parts.

Diff.	1	2	3	4	5	6	7	8	9
46	4.6	9.2	13.8	18.4	23.0	27.6	32.2	36.8	41.4
45	4.5	9.0	13.5	18.0	22.5	27.0	31.5	36.0	40.5
44	4.4	8.8	13.2	17.6	22.0	26.4	30.8	35.2	39.6
43	4.3	8.6	12.9	17.2	21.5	25.8	30.1	34.4	38.7

TABLE XXII (*Continued*)

Logarithms of Numbers

No. 990 L. 995.] [No. 999 L. 999.

N.	0	1	2	3	4	5	6	7	8	9	Diff.
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030	
1	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
2	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	
3	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	
4	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	
5	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	
6	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	
7	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	
8	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	
9	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43

Logarithms of Numbers from 1 to 100.

N.	Log.	N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

TABLE XXIII

Logarithmic Sines, Cosines, Tangents, and Cotangents

0°		Sine.	q−l		Tang.	Cotang.	q+l	D1″	Cosine.	179°
″	′		4.685				15.314			′
0	0	Inf. neg.	575	575	Inf. neg.	Inf. pos.	425		ten	60
60	1	6.463726	575	575	6.463726	13.536274	425		ten	59
120	2	.764756	575	575	.764756	.235244	425		ten	58
180	3	6.940847	575	575	6.940847	13.059153	425		ten	57
240	4	7.065786	575	575	7.065786	12.934214	425		ten	56
300	5	.162696	575	575	.162696	.837304	425		ten	55
360	6	.241877	575	575	.241878	.758122	425	.02	9.999999	54
420	7	.308824	575	575	.308825	.691175	425	.00	.999999	53
480	8	.366816	574	576	.366817	.633183	424	.00	.999999	52
540	9	.417968	574	576	.417970	.582030	424	.00	.999999	51
600	10	.463726	574	576	.463727	.536273	424	.02	.999998	50
660	11	7.505118	574	576	7.505120	12.494880	424	.00	9.999998	49
720	12	.542906	574	577	.542909	.457091	423	.02	.999997	48
780	13	.577668	574	577	.577672	.422328	423	.00	.999997	47
840	14	.609853	574	577	.609857	.390143	423	.02	.999996	46
900	15	.639816	573	578	.639820	.360180	422	.00	.999996	45
960	16	.667845	573	578	.667849	.332151	422	.02	.999995	44
1020	17	.694173	573	578	.694179	.305821	422	.00	.999995	43
1080	18	.718997	573	579	.719003	.280997	421	.02	.999994	42
1140	19	.742478	573	579	.742484	.257516	421	.02	.999993	41
1200	20	.764754	572	580	.764761	.235239	420	.00	.999993	40
1260	21	7.785943	572	580	7.785951	12.214049	420	.02	9.999992	39
1320	22	.806146	572	581	.806155	.193845	419	.02	.999991	38
1380	23	.825451	572	581	.825460	.174540	419	.02	.999990	37
1440	24	.843934	571	582	.843944	.156056	418	.02	.999989	36
1500	25	.861662	571	583	.861674	.138326	417	.00	.999989	35
1560	26	.878695	571	583	.878708	.121292	417	.02	.999988	34
1620	27	.895085	570	584	.895099	.104901	416	.02	.999987	33
1680	28	.910879	570	584	.910894	.089106	416	.02	.999986	32
1740	29	.926119	570	585	.926134	.073866	415	.02	.999985	31
1800	30	.940842	569	586	.940858	.059142	414	.03	.999983	30
1860	31	7.955082	569	587	7.955100	12.044900	413	.02	9.999982	29
1920	32	.968870	569	587	.968889	.031111	413	.02	.999981	28
1980	33	.982238	568	588	.982253	.017747	412	.02	.999980	27
2040	34	7.995198	568	589	7.995219	12.004781	411	.02	.999979	26
2100	35	8.007787	567	590	8.007809	11.992191	410	.03	.999977	25
2160	36	.020021	567	591	.020044	.979956	409	.02	.999976	24
2220	37	.031919	566	592	.031945	.968055	408	.02	.999975	23
2280	38	.043501	566	593	.043527	.956473	407	.03	.999973	22
2340	39	.054781	566	593	.054809	.945191	407	.02	.999972	21
2400	40	.065776	565	594	.065806	.934194	406	.02	.999971	20
2460	41	8.076500	565	595	8.076531	11.923469	405	.03	9.999969	19
2520	42	.086965	564	596	.086997	.913003	404	.02	.999968	18
2580	43	.097183	564	598	.097217	.902783	402	.03	.999966	17
2640	44	.107167	563	599	.107203	.892797	401	.03	.999964	16
2700	45	.116926	562	600	.116963	.883037	400	.02	.999963	15
2760	46	.126471	562	601	.126510	.873490	399	.03	.999961	14
2820	47	.135810	561	602	.135851	.864149	398	.03	.999959	13
2880	48	.144953	561	603	.144996	.855004	397	.02	.999958	12
2940	49	.153907	560	604	.153952	.846048	396	.03	.999956	11
3000	50	.162681	560	605	.162727	.837273	395	.03	.999954	10
3060	51	8.171280	559	607	8.171328	11.828672	393	.03	9.999952	9
3120	52	.179713	558	608	.179763	.820237	392	.03	.999950	8
3180	53	.187985	558	609	.188036	.811964	391	.03	.999948	7
3240	54	.196102	557	611	.196156	.803844	389	.03	.999946	6
3300	55	.204070	556	612	.204126	.795874	388	.03	.999944	5
3360	56	.211895	556	613	.211953	.788047	387	.03	.999942	4
3420	57	.219581	555	615	.219641	.780359	385	.03	.999940	3
3480	58	.227134	554	616	.227195	.772805	384	.03	.999938	2
3540	59	.234557	554	618	.234621	.765379	382	.03	.999936	1
3600	60	8.241855	553	619	8.241921	11.758079	381	.03	9.999934	0
″	′		4.685				15.314			′
90°		Cosine.	q−l		Cotang.	Tang.	q+l	D1″	Sine.	89°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

1°		Sine.	$q-l$		Tang.	Cotang.	$q+l$	D1''	Cosine.	178°
''	'		4.685				15.314			'
3600	0	8.241855	553	619	8.241921	11.758079	381		9.999934	60
3660	1	.249033	552	620	.249102	.750898	380	.03	.999932	59
3720	2	.256094	551	622	.256165	.743835	378	.05	.999929	58
3780	3	.263042	551	623	.263115	.736885	377	.03	.999927	57
3840	4	.269881	550	625	.269956	.730044	375	.03	.999925	56
3900	5	.276614	549	627	.276691	.723309	373	.05	.999922	55
3960	6	.283243	548	628	.283323	.716677	372	.03	.999920	54
4020	7	.289773	547	630	.289856	.710144	370	.03	.999918	53
4080	8	.296207	546	632	.296292	.703708	368	.05	.999915	52
4140	9	.302546	546	633	.302634	.697366	367	.03	.999913	51
4200	10	.308794	545	635	.308884	.691116	365	.05	.999910	50
4260	11	8.314954	544	637	8.315046	11.684954	363	.05	9.999907	49
4320	12	.321027	543	638	.321122	.678878	362	.03	.999905	48
4380	13	.327016	542	640	.327114	.672886	360	.05	.999902	47
4440	14	.332924	541	642	.333025	.666975	358	.05	.999899	46
4500	15	.338753	540	644	.338856	.661144	356	.03	.999897	45
4560	16	.344504	539	646	.344610	.655390	354	.05	.999894	44
4620	17	.350181	539	648	.350289	.649711	352	.05	.999891	43
4680	18	.355783	538	649	.355895	.644105	351	.05	.999888	42
4740	19	.361315	537	651	.361430	.638570	349	.05	.999885	41
4800	20	.366777	536	653	.366895	.633105	347	.05	.999882	40
4860	21	8.372171	535	655	8.372292	11.627708	345	.05	9.999879	39
4920	22	.377499	534	657	.377622	.622378	343	.05	.999876	38
4980	23	.382762	533	659	.382889	.617111	341	.05	.999873	37
5040	24	.387962	532	661	.388092	.611908	339	.05	.999870	36
5100	25	.393101	531	663	.393234	.606766	337	.05	.999867	35
5160	26	.398179	530	666	.398315	.601685	334	.05	.999864	34
5220	27	.403199	529	668	.403338	.596662	332	.05	.999861	33
5280	28	.408161	527	670	.408304	.591696	330	.05	.999858	32
5340	29	.413068	526	672	.413213	.586787	328	.07	.999854	31
5400	30	.417919	525	674	.418068	.581932	326	.05	.999851	30
5460	31	8.422717	524	676	8.422869	11.577131	324	.05	9.999848	29
5520	32	.427462	523	679	.427618	.572382	321	.07	.999844	28
5580	33	.432156	522	681	.432315	.567685	319	.05	.999841	27
5640	34	.436800	521	683	.436962	.563038	317	.05	.999838	26
5700	35	.441394	520	685	.441560	.558440	315	.07	.999834	25
5760	36	.445941	518	688	.446110	.553890	312	.05	.999831	24
5820	37	.450440	517	690	.450613	.549387	310	.07	.999827	23
5880	38	.454893	516	693	.455070	.544930	307	.05	.999824	22
5940	39	.459301	515	695	.459481	.540519	305	.07	.999820	21
6000	40	.463665	514	697	.463849	.536151	303	.07	.999816	20
6060	41	8.467985	512	700	8.468172	11.531828	300	.05	9.999813	19
6120	42	.472263	511	702	.472454	.527546	298	.07	.999809	18
6180	43	.476498	510	705	.476693	.523307	295	.07	.999805	17
6240	44	.480693	509	707	.480892	.519108	293	.07	.999801	16
6300	45	.484848	507	710	.485050	.514950	290	.07	.999797	15
6360	46	.488963	506	713	.489170	.510830	287	.05	.999794	14
6420	47	.493040	505	715	.493250	.506750	285	.07	.999790	13
6480	48	.497078	503	718	.497293	.502707	282	.07	.999786	12
6540	49	.501080	502	720	.501298	.498702	280	.07	.999782	11
6600	50	.505045	501	723	.505267	.494733	277	.07	.999778	10
6660	51	8.508974	499	726	8.509200	11.490800	274	.07	9.999774	9
6720	52	.512867	498	729	.513098	.486902	271	.08	.999769	8
6780	53	.516726	497	731	.516961	.483039	269	.07	.999765	7
6840	54	.520551	495	734	.520790	.479210	266	.07	.999761	6
6900	55	.524343	494	737	.524586	.475414	263	.07	.999757	5
6960	56	.528102	492	740	.528349	.471651	260	.07	.999753	4
7020	57	.531828	491	743	.532080	.467920	257	.08	.999748	3
7080	58	.535523	490	745	.535779	.464221	255	.07	.999744	2
7140	59	.539186	488	748	.539447	.460553	252	.07	.999740	1
7200	60	8.542819	487	751	8.543084	11.456916	249	.08	9.999735	0
''	'		4.685				15.314			'
91°		Cosine.	$q-l$		Cotang.	Tang.	$q+l$	D1''	Sine.	88°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

2°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	177°
0′	8.542819	60.05	9.999735	.07	8.543084	60.12	11.456916	60′
1	.546422	59.55	.999731	.08	.546091	59.62	.453309	59
2	.549995	59.07	.999726	.07	.550268	59.15	.449732	58
3	.553539	58.58	.999722	.08	.553817	58.65	.446183	57
4	.557054	58.10	.999717	.07	.557336	58.20	.442664	56
5	.560540	57.65	.999713	.08	.560828	57.72	.439172	55
6	.563999	57.20	.999708	.07	.564291	57.27	.435709	54
7	.567431	56.75	.999704	.08	.567727	56.83	.432273	53
8	.570836	56.30	.999699	.08	.571137	56.38	.428863	52
9	.574214	55.87	.999694	.08	.574520	55.95	.425480	51
10	.577566	55.43	.999689	.07	.577877	55.52	.422123	50
11	8.580892	55.02	9.999685	.08	8.581208	55.10	11.418792	49
12	.584193	54.60	.999680	.08	.584514	54.68	.415486	48
13	.587469	54.20	.999675	.08	.587795	54.27	.412205	47
14	.590721	53.78	.999670	.08	.591051	53.87	.408949	46
15	.593948	53.40	.999665	.08	.594283	53.48	.405717	45
16	.597152	53.00	.999660	.08	.597492	53.08	.402508	44
17	.600332	52.62	.999655	.08	.600677	52.70	.399323	43
18	.603489	52.23	.999650	.08	.603839	52.32	.396161	42
19	606623	51.85	.999645	.08	.606978	51.93	.393022	41
20	.609734	51.48	.999640	.08	.610094	51.58	.389906	40
21	8.612823	51.13	9.999635	.10	8.613189	51.22	11.386811	39
22	.615891	50.77	.999629	.08	.616262	50.85	.383738	38
23	.618937	50.42	.999624	.08	.619313	50.50	.380687	37
24	.621962	50.05	.999619	.08	.622343	50.15	.377657	36
25	.624965	49.72	.999614	.10	.625352	49.80	.374648	35
26	.627948	49.38	.999608	.08	.628340	49.47	.371660	34
27	.630911	49.05	.999603	.10	.631308	49.13	.368692	33
28	.633854	48.70	.999597	.08	.634256	48.80	.365744	32
29	.636776	48.40	.999592	.10	.637184	48.48	.362816	31
30	.639680	48.05	.999586	.08	.640093	48.15	.359907	30
31	8.642563	47.75	9.999581	.10	8.642982	47.85	11.357018	29
32	.645428	47.43	.999575	.08	.645853	47.52	.354147	28
33	.648274	47.13	.999570	.10	.648704	47.22	.351296	27
34	.651102	46.82	.999564	.10	.651537	46.92	.348463	26
35	.653911	46.52	.999558	.08	.654352	46.62	.345648	25
36	.656702	46.22	.999553	.10	.657149	46.32	.342851	24
37	.659475	45.92	.999547	.10	.659928	46.02	.340072	23
38	.662230	45.63	.999541	.10	.662689	45.73	.337311	22
39	.664968	45.35	.999535	.10	.665433	45.45	.334567	21
40	.667689	45.07	.999529	.08	.668160	45.17	.331840	20
41	8.670393	44.78	9.999524	.10	8.670870	44.88	11.329130	19
42	.673080	44.52	.999518	.10	.673563	44.60	.326437	18
43	.675751	44.23	.999512	.10	.676239	44.35	.323761	17
44	.678405	43.97	.999506	.10	.678900	44.07	.321100	16
45	.681043	43.70	.999500	.12	.681544	43.80	.318456	15
46	.683665	43.45	.999493	.10	.684172	43.53	.315828	14
47	.686272	43.18	.999487	.10	.686784	43.28	.313216	13
48	.688863	42.92	.999481	.10	.689381	43.03	.310619	12
49	.691438	42.67	.999475	.10	.691963	42.77	.308037	11
50	.693998	42.42	.999469	.10	.694529	42.53	.305471	10
51	8.696543	42.17	9.999463	.12	8.697081	42.27	11.302919	9
52	.699073	41.93	.999456	.10	.699617	42.03	.300383	8
53	.701589	41.68	.999450	.12	.702139	41.78	.297861	7
54	.704090	41.45	.999443	.10	.704646	41.57	.295354	6
55	.706577	41.20	.999437	.10	.707140	41.30	.292860	5
56	.709049	40.97	.999431	.12	.709618	41.08	.290382	4
57	.711507	40.75	.999424	.10	.712083	40.85	.287917	3
58	.713952	40.52	.999418	.12	.714534	40.63	.285466	2
59	.716383	40.28	.999411	.12	.716972	40.40	.283028	1
60′	8.718800		9.999404		8.719396		11.280604	0′
92°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	87°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

3°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	176°
0′	8.718800	40.07	9.999404	.10	8.719396	40.17	11.280604	60′
1	.721204	39.85	.999398	.12	.721806	39.97	.278194	59
2	.723595	39.62	.999391	.12	.724204	39.73	.275796	58
3	.725972	39.42	.999384	.10	.726588	39.52	.273412	57
4	.728337	39.18	.999378	.12	.728959	39.30	.271041	56
5	.730688	38.98	.999371	.12	.731317	39.10	.268683	55
6	.733027	38.78	.999364	.12	.733663	38.88	.266337	54
7	.735354	38.55	.999357	.12	.735996	38.68	.264004	53
8	.737667	38.37	.999350	.12	.738317	38.48	.261683	52
9	.739969	38.17	.999343	.12	.740626	38.27	.259374	51
10	.742259	37.95	.999336	.12	.742922	38.08	.257078	50
11	8.744536	37.77	9.999329	.12	8.745207	37.87	11.254793	49
12	.746802	37.55	.999322	.12	.747479	37.68	.252521	48
13	.749055	37.37	.999315	.12	.749740	37.48	.250260	47
14	.751297	37.18	.999308	.12	.751989	37.30	.248011	46
15	.753528	36.98	.999301	.12	.754227	37.10	.245773	45
16	.755747	36.80	.999294	.12	.756453	36.92	.243547	44
17	.757955	36.60	.999287	.13	.758668	36.73	.241332	43
18	.760151	36.43	.999279	.12	.760872	36.55	.239128	42
19	.762337	36.23	.999272	.12	.763065	36.35	.236935	41
20	.764511	36.07	.999265	.13	.765246	36.18	.234754	40
21	8.766675	35.88	9.999257	.12	8.767417	36.02	11.232583	39
22	.768828	35.70	.999250	.13	.769578	35.82	.230422	38
23	.770970	35.52	.999242	.12	.771727	35.65	.228273	37
24	.773101	35.37	.999235	.13	.773866	35.48	.226134	36
25	.775223	35.17	.999227	.12	.775995	35.32	.224005	35
26	.777333	35.02	.999220	.13	.778114	35.13	.221886	34
27	.779434	34.83	.999212	.12	.780222	34.97	.219778	33
28	.781524	34.68	.999205	.13	.782320	34.80	.217680	32
29	.783605	34.50	.999197	.13	.784408	34.63	.215592	31
30	.785675	34.35	.999189	.13	.786486	34.47	.213514	30
31	8.787736	34.18	9.999181	.12	8.788554	34.32	11.211446	29
32	.789787	34.02	.999174	.13	.790613	34.15	.209387	28
33	.791828	33.85	.999166	.13	.792662	33.98	.207338	27
34	.793859	33.70	.999158	.13	.794701	33.83	.205299	26
35	.795881	33.55	.999150	.13	.796731	33.68	.203269	25
36	.797894	33.38	.999142	.13	.798752	33.52	.201248	24
37	.799897	33.25	.999134	.13	.800763	33.37	.199237	23
38	.801892	33.07	.999126	.13	.802765	33.22	.197235	22
39	.803876	32.93	.999118	.13	.804758	33.07	.195242	21
40	.805852	32.78	.999110	.13	.806742	32.92	.193258	20
41	8.807819	32.63	9.999102	.13	8.808717	32.77	11.191283	19
42	.809777	32.48	.999094	.13	.810683	32.63	.189317	18
43	.811726	32.35	.999086	.15	.812641	32.47	.187359	17
44	.813667	32.20	.999077	.13	.814589	32.33	.185411	16
45	.815599	32.05	.999069	.13	.816529	32.20	.183471	15
46	.817522	31.90	.999061	.13	.818461	32.05	.181539	14
47	.819436	31.78	.999053	.15	.820384	31.90	.179616	13
48	.821343	31.62	.999044	.13	.822298	31.78	.177702	12
49	.823240	31.50	.999036	.15	.824205	31.63	.175795	11
50	.825130	31.35	.999027	.13	.826103	31.48	.173897	10
51	8.827011	31.22	9.999019	.15	8.827992	31.37	11.172008	9
52	.828884	31.08	.999010	.13	.829874	31.23	.170126	8
53	.830749	30.97	.999002	.15	.831748	31.08	.168252	7
54	.832607	30.82	.998993	.15	.833613	30.97	.166387	6
55	.834456	30.68	.998984	.13	.835471	30.83	.164529	5
56	.836297	30.55	.998976	.15	.837321	30.70	.162679	4
57	.838130	30.43	.998967	.15	.839163	30.58	.160837	3
58	.839956	30.30	.998958	.13	.840998	30.45	.159002	2
59	.841774	30.18	.998950	.15	.842825	30.32	.157175	1
60′	8.843585		9.998941		8.844644		11.155356	0′
93°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	86°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

4°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	175°
0′	8.843585	30.03	9.998941	.15	8.844644	30.18	11.155356	60′
1	.845387	29.93	.998932	.15	.846455	30.08	.153545	59
2	.847183	29.80	.998923	.15	.848260	29.95	.151740	58
3	.848971	29.67	.998914	.15	.850057	29.82	.149943	57
4	.850751	29.57	.998905	.15	.851846	29.70	.148154	56
5	.852525	29.43	.998896	.15	.853628	29.58	.146372	55
6	.854291	29.30	.998887	.15	.855403	29.47	.144597	54
7	.856049	29.20	.998878	.15	.857171	29.35	.142829	53
8	.857801	29.08	.998869	.15	.858932	29.23	.141068	52
9	.859546	28.95	.998860	.15	.860686	29.12	.139314	51
10	.861283	28.85	.998851	.17	.862433	29.00	.137567	50
11	8.863014	28.73	9.998841	.15	8.864173	28.88	11.135827	49
12	.864738	28.62	.998832	.15	.865906	28.77	.134094	48
13	.866455	28.50	.998823	.17	.867632	28.65	.132368	47
14	.868165	28.38	.998813	.15	.869351	28.55	.130649	46
15	.869868	28.28	.998804	.15	.871064	28.43	.128936	45
16	.871565	28.17	.998795	.17	.872770	28.32	.127230	44
17	.873255	28.05	.998785	.15	.874469	28.22	.125531	43
18	.874938	27.95	.998776	.17	.876162	28.12	.123838	42
19	.876615	27.83	.998766	.15	.877849	28.00	.122151	41
20	.878285	27.73	.998757	.17	.879529	27.88	.120471	40
21	8.879949	27.63	9.998747	.15	8.881202	27.78	11.118798	39
22	.881607	27.52	.998738	.17	.882869	27.68	.117131	38
23	.883258	27.42	.998728	.17	.884530	27.58	.115470	37
24	.884903	27.32	.998718	.17	.886185	27.47	.113815	36
25	.886542	27.20	.998708	.15	.887833	27.38	.112167	35
26	.888174	27.12	.998699	.17	.889476	27.27	.110524	34
27	.889801	27.00	.998689	.17	.891112	27.17	.108888	33
28	.891421	26.90	.998679	.17	.892742	27.07	.107258	32
29	.893035	26.80	.998669	.17	.894366	26.97	.105634	31
30	.894643	26.72	.998659	.17	.895984	26.87	.104016	30
31	8.896246	26.60	9.998649	.17	8.897596	26.78	11.102404	29
32	.897842	26.50	.998639	.17	.899203	26.67	.100797	28
33	.899432	26.42	.998629	.17	.900803	26.58	.099197	27
34	.901017	26.32	.998619	.17	.902398	26.48	.097602	26
35	.902596	26.22	.998609	.17	.903987	26.38	.096013	25
36	.904169	26.12	.998599	.17	.905570	26.28	.094430	24
37	.905736	26.02	.998589	.18	.907147	26.20	.092853	23
38	.907297	25.93	.998578	.17	.908719	26.10	.091281	22
39	.908853	25.85	.998568	.17	.910285	26.02	.089715	21
40	.910404	25.75	.998558	.17	.911846	25.92	.088154	20
41	8.911949	25.65	9.998548	.18	8.913401	25.83	11.086599	19
42	.913488	25.57	.998537	.17	.914951	25.73	.085049	18
43	.915022	25.47	.998527	.18	.916495	25.65	.083505	17
44	.916550	25.38	.998516	.17	.918034	25.57	.081966	16
45	.918073	25.30	.998506	.18	.919568	25.47	.080432	15
46	.919591	25.20	.998495	.17	.921096	25.38	.078904	14
47	.921103	25.12	.998485	.18	.922619	25.28	.077381	13
48	.922610	25.03	.998474	.17	.924136	25.22	.075864	12
49	.924112	24.95	.998464	.18	.925649	25.12	.074351	11
50	.925609	24.85	.998453	.18	.927156	25.03	.072844	10
51	8.927100	24.78	9.998442	.18	8.928658	24.95	11.071342	9
52	.928587	24.68	.998431	.17	.930155	24.87	.069845	8
53	.930068	24.60	.998421	.18	.931647	24.78	.068353	7
54	.931544	24.52	.998410	.18	.933134	24.70	.066866	6
55	.933015	24.43	.998399	.18	.934616	24.62	.065384	5
56	.934481	24.35	.998388	.18	.936093	24.53	.063907	4
57	.935942	24.27	.998377	.18	.937565	24.45	.062435	3
58	.937398	24.20	.998366	.18	.939032	24.37	.060968	2
59	.938850	24.10	.998355	.18	.940494	24.30	.059506	1
60′	8.940296		9.998344		8.941952		11.058048	0′
94°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	85°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

5°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	174°
0′	8.940296	24.03	9.998344	.18	8.941952	24.20	11.058048	60′
1	.941738	23.93	.998333	.18	.943404	24.13	.056596	59
2	.943174	23.87	.998322	.18	.944852	24.05	.055148	58
3	.944606	23.80	.998311	.18	.946295	23.98	.053705	57
4	.946034	23.70	.998300	.18	.947734	23.90	.052266	56
5	.947456	23.63	.998289	.20	.949168	23.82	.050832	55
6	.948874	23.55	.998277	.18	.950597	23 73	.049403	54
7	.950287	23.48	.998266	.18	.952021	23.67	.047979	53
8	.951696	23.40	.998255	.20	.953441	23.58	.046559	52
9	.953100	23.32	.998243	.18	.954856	23.52	.045144	51
10	.954499	23.25	.998232	.20	.956267	23.45	.043733	50
11	8.955894	23.17	9.998220	.18	8.957674	23.35	11.042326	49
12	.957284	23.10	.998209	.20	.959075	23.30	.040925	48
13	.958670	23 03	.998197	.18	.960473	23.22	.039527	47
14	.960052	22.95	.998186	.20	.961866	23.15	.038134	46
15	.961429	22.87	.998174	.18	.963255	23.07	.036745	45
16	.962801	22.82	.998163	.20	.964639	23.00	.035361	44
17	.964170	22.73	.998151	.20	.966019	22.92	.033981	43
18	.965534	22.65	.998139	.18	.967394	22.87	.032606	42
19	.966893	22.60	.998128	.20	.968766	22.78	.031234	41
20	.968249	22.52	.998116	.20	.970133	22.72	.029867	40
21	8.969600	22.45	9.998104	.20	8.971496	22.65	11.028504	39
22	.970947	22.37	.998092	.20	.972855	22.57	.027145	38
23	.972289	22.32	.998080	.20	.974209	22.52	.025791	37
24	.973628	22.23	.998068	.20	.975560	22.43	.024410	36
25	.974962	22.18	.998056	.20	.976906	22.37	.023094	35
26	.976293	22.10	.998044	.20	.978248	22.30	.021752	34
27	.977619	22.03	.998032	.20	.979586	22.25	.020414	33
28	.978941	21.97	.998020	.20	.980921	22.17	.019079	32
29	.980259	21.90	.998008	.20	.982251	22.10	.017749	31
30	.981573	21.83	.997996	.20	.983577	22.03	.016423	30
31	8.982883	21.77	9.997984	.20	8.984899	21.97	11.015101	29
32	.984189	21.70	.997972	.22	.986217	21.92	.013783	28
33	.985491	21.63	.997959	.20	.987532	21.83	.012468	27
34	.986789	21.57	.997947	.20	.988842	21.78	.011158	26
35	.988083	21.52	.997935	.22	.990149	21.70	.009851	25
36	.989374	21.43	.997922	.20	.991451	21.65	.008549	24
37	.990660	21.38	.997910	.22	.992750	21.58	.007250	23
38	.991943	21.32	.997897	.20	.994045	21.53	.005955	22
39	.993222	21.25	.997885	.22	.995337	21.45	.004663	21
40	.994497	21.18	.997872	.20	.996624	21.40	.003376	20
41	8.995768	21.13	9.997860	.22	8.997908	21.33	11.002092	19
42	.997036	21.05	.997847	.20	8.999188	21.28	11.000812	18
43	.998299	21.02	.997835	.22	9.000465	21.22	10.999535	17
44	8.999560	20.93	.997822	.22	.001738	21.15	.998262	16
45	9.000816	20.88	.997809	.20	.003007	21.08	.996993	15
46	.002069	20.82	.997797	.22	.004272	21.03	.995728	14
47	.003318	20.75	.997784	.22	.005534	20.97	.994466	13
48	.004563	20.70	.997771	.22	.006792	20.92	.993208	12
49	.005805	20.65	.997758	.22	.008047	20.85	.991953	11
50	.007044	20.57	.997745	.22	.009298	20.80	.990702	10
51	9.008278	20.53	9.997732	.22	9.010546	20.73	10.989454	9
52	.009510	20.45	.997719	.22	.011790	20.68	.988210	8
53	.010737	20.42	.997706	.22	.013031	20.62	.986969	7
54	.011962	20.33	.997693	.22	.014268	20.57	.985732	6
55	.013182	20.30	.997680	.22	.015502	20.50	.984498	5
56	.014400	20.22	.997667	.22	.016732	20.45	.983268	4
57	.015618	20.18	.997654	.22	.017959	20.40	.982041	3
58	.016824	20.12	.997641	.22	.019183	20.33	.980817	2
59	.018031	20.07	.997628	.23	.020403	20.28	.979597	1
60′	9.019235		9.997614		9.021620		10.978380	0′
95°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	84°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

6°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	173°
0'	9.019235	20.00	9.997614	.22	9.021620	20.23	10.978380	60'
1	.020435	19.95	.997601	.22	.022834	20.17	.977166	59
2	.021632	19.88	.997588	.23	.024044	20.12	.975956	58
3	.022825	19.85	.997574	.22	.025251	20.07	.974749	57
4	.024016	19.78	.997561	.23	.026455	20.00	.973545	56
5	.025203	19.72	.997547	.22	.027655	19.95	.972345	55
6	.026386	19.68	.997534	.23	.028852	19.90	.971148	54
7	.027567	19.62	.997520	.22	.030046	19.85	.969954	53
8	.028744	19.57	.997507	.23	.031237	19.80	.968763	52
9	.029918	19.52	.997493	.22	.032425	19.73	.967575	51
10	.031089	19.47	.997480	.23	.033609	19.70	.966391	50
11	9.032257	19.40	9.997466	.23	9.034791	19.63	10.965209	49
12	.033421	19.35	.997452	.22	.035969	19.58	.964031	48
13	.034582	19.32	.997439	.23	.037144	19.53	.962856	47
14	.035741	19.25	.997425	.23	.038316	19.48	.961684	46
15	.036896	19.20	.997411	.23	.039485	19.43	.960515	45
16	.038048	19.15	.997397	.23	.040651	19.37	.959349	44
17	.039197	19.08	.997383	.23	.041813	19.33	.958187	43
18	.040342	19.05	.997369	.23	.042973	19.28	.957027	42
19	.041485	19.00	.997355	.23	.044130	19.23	.955870	41
20	.042625	18.95	.997341	.23	.045284	19.17	.954716	40
21	9.043762	18.88	9.997327	.23	9.046434	19.13	10.953566	39
22	.044895	18.85	.997313	.23	.047582	19.08	.952418	38
23	.046026	18.80	.997299	.23	.048727	19.03	.951273	37
24	.047154	18.75	.997285	.23	.049869	18.98	.950131	36
25	.048279	18.68	.997271	.23	.051008	18.93	.948992	35
26	.049400	18.65	.997257	.25	.052144	18.88	.947856	34
27	.050519	18.60	.997242	.23	.053277	18.83	.946723	33
28	.051635	18.57	.997228	.23	.054407	18.80	.945593	32
29	.052749	18.50	.997214	.25	.055535	18.73	.944465	31
30	.053859	18.45	.997199	.23	.056659	18.70	.943341	30
31	9.054966	18.42	9.997185	.25	9.057781	18.65	10.942219	29
32	.056071	18.35	.997170	.23	.058900	18.60	.941100	28
33	.057172	18.32	.997156	.25	.060016	18.57	.939984	27
34	.058271	18.27	.997141	.23	.061130	18.50	.938870	26
35	.059367	18.22	.997127	.25	.062240	18.47	.937760	25
36	.060460	18.18	.997112	.23	.063348	18.42	.936652	24
37	.061551	18.13	.997098	.25	.064453	18.38	.935547	23
38	.062639	18.08	.997083	.25	.065556	18.32	.934444	22
39	.063724	18.03	.997068	.25	.066655	18.28	.933345	21
40	.064806	17.98	.997053	.23	.067752	18.23	.932248	20
41	9.065885	17.95	9.997039	.25	9.068846	18.20	10.931154	19
42	.066962	17.90	.997024	.25	.069938	18.15	.930062	18
43	.068036	17.85	.997009	.25	.071027	18.10	.928973	17
44	.069107	17.82	.996994	.25	.072113	18.07	.927887	16
45	.070176	17.77	.996979	.25	.073197	18.02	.926803	15
46	.071242	17.73	.996964	.25	.074278	17.97	.925722	14
47	.072306	17.67	.996949	.25	.075356	17.93	.924644	13
48	.073366	17.63	.996934	.25	.076432	17.88	.923568	12
49	.074424	17.60	.996919	.25	.077505	17.85	.922495	11
50	.075480	17.55	.996904	.25	.078576	17.80	.921424	10
51	9.076533	17.50	9.996889	.25	9.079644	17.77	10.920356	9
52	.077583	17.47	.996874	.27	.080710	17.72	.919290	8
53	.078631	17.42	.996858	.25	.081773	17.67	.918227	7
54	.079676	17.38	.996843	.27	.082833	17.63	.917167	6
55	.080719	17.33	.996828	.27	.083891	17.60	.916109	5
56	.081759	17.30	.996812	.25	.084947	17.55	.915053	4
57	.082797	17.25	.996797	.25	.086000	17.50	.914000	3
58	.083832	17.20	.996782	.27	.087050	17.47	.912950	2
59	.084864	17.17	.996766	.25	.088098	17.43	.911902	1
60'	9.085894		9.996751		9.089144		10.910856	0'
96°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	83°

TABLE XXIII (*Continued*)

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS

7°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	172°
0′	9.085894	17.13	9.996751	.27	9.089144	17.38	10.910856	60′
1	.086922	17.08	.996735	.25	.090167	17.35	.909813	59
2	.087947	17.05	.996720	.27	.091228	17.30	.908772	58
3	.088970	17.00	.996704	.27	.092266	17.27	.907734	57
4	.089990	16.97	.996688	.25	.093302	17.23	.906698	56
5	.091008	16.93	.996673	.27	.094336	17.18	.905664	55
6	.092024	16.88	.996657	.27	.095367	17.13	.904633	54
7	.093037	16.83	.996641	.27	.096395	17.12	.903605	53
8	.094047	16.82	.996625	.25	.097422	17.07	.902578	52
9	.095056	16.77	.996610	.27	.098446	17.03	.901554	51
10	.096062	16.72	.996594	.27	.099468	16.98	.900532	50
11	9.097065	16.68	9.996578	.27	9.100487	16.95	10.899513	49
12	.098066	16.65	.996562	.27	.101504	16.92	.898496	48
13	.099065	16.62	.996546	.27	.102519	16.88	.897481	47
14	.100062	16.57	.996530	.27	.103532	16.83	.896468	46
15	.101056	16.53	.996514	.27	.104542	16.80	.895458	45
16	.102048	16.48	.996498	.27	.105550	16.77	.894450	44
17	.103037	16.47	.996482	.28	.106556	16.72	.893444	43
18	.104025	16.42	.996465	.27	.107559	16.68	.892441	42
19	.105010	16.37	.996449	.27	.108560	16.65	.891440	41
20	.105992	16.35	.996433	.27	.109559	16.62	.890441	40
21	9.106973	16.30	9.996417	.28	9.110556	16.58	10.889444	39
22	.107951	16.27	.996400	.27	.111551	16.53	.888449	38
23	.108927	16.23	.996384	.27	.112543	16.50	.887457	37
24	.109901	16.20	.996368	.28	.113533	16.47	.886467	36
25	.110873	16.15	.996351	.27	.114521	16.43	.885479	35
26	.111842	16.12	.996335	.28	.115507	16.40	.884493	34
27	.112809	16.08	.996318	.27	.116491	16.35	.883509	33
28	.113774	16.05	.996302	.28	.117472	16.33	.882528	32
29	.114737	16.02	.996285	.27	.118452	16.28	.881548	31
30	.115698	15.97	.996269	.28	.119429	16.25	.880571	30
31	9.116656	15.95	9.996252	.28	9.120404	16.22	10.879596	29
32	.117613	15.90	.996235	.27	.121377	16.18	.878623	28
33	.118567	15.87	.996219	.28	.122348	16.15	.877652	27
34	.119519	15.83	.996202	.28	.123317	16.12	.876683	26
35	.120469	15.80	.996185	.28	.124284	16.08	.875716	25
36	.121417	15.75	.996168	.28	.125249	16.03	.874751	24
37	.122362	15.73	.996151	.28	.126211	16.02	.873789	23
38	.123306	15.70	.996134	.28	.127172	15.97	.872828	22
39	.124248	15.65	.996117	.28	.128130	15.95	.871870	21
40	.125187	15.63	.996100	.28	.129087	15.90	.870913	20
41	9.126125	15.58	9.996083	.28	9.130041	15.88	10.869959	19
42	.127060	15.55	.996066	.28	.130994	15.83	.869006	18
43	.127993	15.53	.996049	.28	.131944	15.82	.868056	17
44	.128925	15.48	.996032	.28	.132893	15.77	.867107	16
45	.129854	15.45	.996015	.28	.133839	15.75	.866161	15
46	.130781	15.42	.995998	.30	.134784	15.70	.865216	14
47	.131706	15.40	.995980	.28	.135726	15.68	.864274	13
48	.132630	15.35	.995963	.28	.136667	15.63	.863333	12
49	.133551	15.32	.995946	.30	.137605	15.62	.862395	11
50	.134470	15.28	.995928	.28	.138542	15.57	.861458	10
51	9.135387	15.27	9.995911	.28	9.139476	15.55	10.860524	9
52	.136303	15.22	.995894	.30	.140409	15.52	.859591	8
53	.137216	15.20	.995876	.28	.141340	15.48	.858660	7
54	.138128	15.15	.995859	.30	.142269	15.45	.857731	6
55	.139037	15.12	.995841	.30	.143196	15.42	.856804	5
56	.139944	15.10	.995823	.28	.144121	15.38	.855879	4
57	.140850	15.07	.995806	.30	.145044	15.37	.854956	3
58	.141754	15.02	.995788	.28	.145966	15.32	.854034	2
59	.142655	15.00	.995771	.30	.146885	15.30	.853115	1
60′	9.143555		9.995753		9.147803		10.852197	0′
97°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	82°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

8°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	171°
0′	9.143555	14.97	9.995753	.30	9.147803	15.25	10.852197	60′
1	.144453	14.93	.995735	.30	.148718	15.23	.851282	59
2	.145349	14.90	.995717	.30	.149632	15.20	.850368	58
3	.146243	14.88	.995699	.30	.150544	15.17	.849456	57
4	.147136	14.83	.995681	.28	.151454	15.15	.848546	56
5	.148026	14.82	.995664	.30	.152363	15.10	.847637	55
6	.148915	14.78	.995646	.30	.153269	15.08	.846731	54
7	.149802	14.73	.995628	.30	.154174	15.05	.845826	53
8	.150686	14.72	.995610	.32	.155077	15.02	.844923	52
9	.151569	14.70	.995591	.30	.155978	14.98	.844022	51
10	.152451	14.65	.995573	.30	.156877	14.97	.843123	50
11	9.153330	14.63	9.995555	.30	9.157775	14.93	10.842225	49
12	.154208	14.58	.995537	.30	.158671	14.90	.841329	48
13	.155083	14.57	.995519	.30	.159565	14.87	.840435	47
14	.155957	14.55	.995501	.32	.160457	14.83	.839543	46
15	.156830	14.50	.995482	.30	.161347	14.82	.838653	45
16	.157700	14.48	.995464	.30	.162236	14.78	.837764	44
17	.158569	14.43	.995446	.32	.163123	14.75	.836877	43
18	.159435	14.43	.995427	.30	.164008	14.73	.835992	42
19	.160301	14.38	.995409	.32	.164892	14.70	.835108	41
20	.161164	14.35	.995390	.30	.165774	14.67	.834226	40
21	9.162025	14.33	9.995372	.32	9.166654	14.63	10.833346	39
22	.162885	14.30	.995353	.32	.167532	14.62	.832468	38
23	.163743	14.28	.995334	.30	.168409	14.58	.831591	37
24	.164600	14.23	.995316	.32	.169284	14.55	.830716	36
25	.165454	14.22	.995297	.32	.170157	14.53	.829843	35
26	.166307	14.20	.995278	.30	.171029	14.50	.828971	34
27	.167159	14.15	.995260	.32	.171899	14.47	.828101	33
28	.168008	14.13	.995241	.32	.172767	14.45	.827233	32
29	.168856	14.10	.995222	.32	.173634	14.42	.826366	31
30	.169702	14.08	.995203	.32	.174499	14.38	.825501	30
31	9.170547	14.03	9.995184	.32	9.175362	14.37	10.824638	29
32	.171389	14.02	.995165	.32	.176224	14.33	.823776	28
33	.172230	14.00	.995146	.32	.177084	14.30	.822916	27
34	.173070	13.97	.995127	.32	.177942	14.28	.822058	26
35	.173908	13.93	.995108	.32	.178799	14 27	.821201	25
36	.174744	13.90	.995089	.32	.179655	14.22	.820345	24
37	.175578	13.88	.995070	.32	.180508	14.20	.819492	23
38	.176411	13.85	.995051	.32	.181360	14.18	.818640	22
39	.177242	13.83	.995032	.32	.182211	14.13	.817789	21
40	.178072	13.80	.995013	.33	.183059	14.13	.816941	20
41	9.178900	13.77	9.994993	.32	9.183907	14.08	10.816093	19
42	.179726	13.75	.994974	.32	.184752	14.08	.815248	18
43	.180551	13.72	.994955	.33	.185597	14.03	.814403	17
44	.181374	13.70	.994935	.32	.186439	14.02	.813561	16
45	.182196	13.67	.994916	.33	.187280	14.00	.812720	15
46	.183016	13.63	.994896	.32	.188120	13.97	.811880	14
47	.183834	13.62	.994877	.33	.188958	13.93	.811042	13
48	.184651	13.58	.994857	.32	.189794	13.92	.810206	12
49	.185466	13.57	.994838	.33	.190629	13.88	.809371	11
50	.186280	13.53	.994818	.33	.191462	13.87	.808538	10
51	9.187092	13.52	9.994798	.32	9.192294	13.83	10.807706	9
52	.187903	13.48	.994779	.33	.193124	13.82	.806876	8
53	.188712	13.45	.994759	.33	.193953	13.78	.806047	7
54	.189519	13.43	.994739	.32	.194780	13.77	.805220	6
55	.190325	13.42	.994720	.33	.195606	13.73	.804394	5
56	.191130	13.38	.994700	.33	.196430	13.72	.803570	4
57	.191933	13.35	.994680	.33	.197253	13.68	.802747	3
58	.192734	13.33	.994660	.33	.198074	13.67	.801926	2
59	.193534	13.30	.994640	.33	.198894	13.65	.801106	1
60′	9.194332		9.994620		9.199713		10.800287	0′
98°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	81°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

9°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	170°
0′	9.194332	13.28	9.994620	.33	9.199713	13.60	10.800287	60′
1	.195129	13.27	.994600	.33	.200529	13.60	.799471	59
2	.195925	13.23	.994580	.33	.201345	13.57	.798655	58
3	.196719	13.20	.994560	.33	.202159	13.53	.797841	57
4	.197511	13.18	.994540	.35	.202971	13.52	.797029	56
5	.198302	13.15	.994519	.33	.203782	13.50	.796218	55
6	.199091	13.13	.994499	.33	.204592	13.47	.795408	54
7	.199879	13.12	.994479	.33	.205400	13.45	.794600	53
8	.200666	13.08	.994459	.35	.206207	13.43	.793793	52
9	.201451	13.05	.994438	.33	.207013	13.40	.792987	51
10	.202234	13.05	.994418	.33	.207817	13.37	.792183	50
11	9.203017	13.00	9.994398	.35	9.208619	13.35	10.791381	49
12	.203797	13.00	.994377	.33	.209420	13.33	.790580	48
13	.204577	12.95	.994357	.35	.210220	13.30	.789780	47
14	.205354	12.95	.994336	.33	.211018	13.28	.788982	46
15	.206131	12.92	.994316	.35	.211815	13.27	.788185	45
16	.206906	12.88	.994295	.35	.212611	13.23	.787389	44
17	.207679	12.88	.994274	.33	.213405	13.22	.786595	43
18	.208452	12.83	.994254	.35	.214198	13.18	.785802	42
19	.209222	12.83	.994233	.35	.214989	13.18	.785011	41
20	.209992	12.80	.994212	.35	.215780	13.13	.784220	40
21	9.210760	12.77	9.994191	.33	9.216568	13.13	10.783432	39
22	.211526	12.75	.994171	.35	.217356	13.10	.782644	38
23	.212291	12.73	.994150	.35	.218142	13.07	.781858	37
24	.213055	12.72	.994129	.35	.218926	13.07	.781074	36
25	.213818	12.68	.994108	.35	.219710	13.03	.780290	35
26	.214579	12.65	.994087	.35	.220492	13.00	.779508	34
27	.215338	12.65	.994066	.35	.221272	13.00	.778728	33
28	.216097	12.62	.994045	.35	.222052	12.97	.777948	32
29	.216854	12.58	.994024	.35	.222830	12.95	.777170	31
30	.217609	12.57	.994003	.35	.223607	12.92	.776393	30
31	9.218363	12.55	9.993982	.37	9.224382	12.90	10.775618	29
32	.219116	12.53	.993960	.35	.225156	12.88	.774844	28
33	.219868	12.50	.993939	.35	.225929	12.85	.774071	27
34	.220618	12.48	.993918	.35	.226700	12.85	.773300	26
35	.221367	12.47	.993897	.37	.227471	12.80	.772529	25
36	.222115	12.43	.993875	.35	.228239	12.80	.771761	24
37	.222861	12.42	.993854	.37	.229007	12.77	.770993	23
38	.223606	12.38	.993832	.35	.229773	12.77	.770227	22
39	.224349	12.38	.993811	.37	.230539	12.72	.769461	21
40	.225092	12.35	.993789	.35	.231302	12.72	.768698	20
41	9.225833	12.33	9.993768	.37	9.232065	12.68	10.767935	19
42	.226573	12.30	.993746	.35	.232826	12.67	.767174	18
43	.227311	12.28	.993725	.37	.233586	12.65	.766414	17
44	.228048	12.27	.993703	.37	.234345	12.63	.765655	16
45	.228784	12.23	.993681	.35	.235103	12.60	.764897	15
46	.229518	12.23	.993660	.37	.235859	12.58	.764141	14
47	.230252	12.20	.993638	.37	.236614	12.57	.763386	13
48	.230984	12.18	.993616	.37	.237368	12.53	.762632	12
49	.231715	12.15	.993594	.37	.238120	12.53	.761880	11
50	.232444	12.13	.993572	.37	.238872	12.50	.761128	10
51	9.233172	12.12	9.993550	.37	9.239622	12.48	10.760378	9
52	.233899	12.10	.993528	.37	.240371	12.45	.759629	8
53	.234625	12.07	.993506	.37	.241118	12.45	.758882	7
54	.235349	12.07	.993484	.37	.241865	12.42	.758135	6
55	.236073	12.03	.993462	.37	.242610	12.40	.757390	5
56	.236795	12.00	.993440	.37	.243354	12.38	.756646	4
57	.237515	12.00	.993418	.37	.244097	12.37	.755903	[illegible]
58	.238235	11.97	.993396	.37	.244839	12.33	.755161	[illegible]
59	.238958	11.95	.993374	.38	.245579	12.33	.754421	1
60′	9.239670		9.993351		9.246319		10.753681	0′
99°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	80°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

10°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	169°
0′	9.239670	11.93	9.993351	.37	9.246319	12.30	10.753681	60′
1	.240386	11.92	.993329	.37	.247057	12.28	.752943	59
2	.241101	11.88	.993307	.38	.247794	12.27	.752206	58
3	.241814	11.87	.993284	.37	.248530	12.23	.751470	57
4	.242526	11.85	.993262	.37	.249264	12.23	.750736	56
5	.243237	11.83	.993240	.38	.249998	12.20	.750002	55
6	.243947	11.82	.993217	.37	.250730	12.18	.749270	54
7	.244656	11.78	.993195	.38	.251461	12.17	.748539	53
8	.245363	11.77	.993172	.38	.252191	12.15	.747809	52
9	.246069	11.77	.993149	.37	.252920	12.13	.747080	51
10	.246775	11.72	.993127	.38	.253648	12.10	.746352	50
11	9.247478	11.72	9.993104	.38	9.254374	12.10	10.745626	49
12	.248181	11.70	.993081	.37	.255100	12.07	.744900	48
13	.248883	11.67	.993059	.38	.255824	12.05	.744176	47
14	.249583	11.65	.993036	.38	.256547	12.03	.743453	46
15	.250282	11.63	.993013	.38	.257269	12.02	.742731	45
16	.250980	11.62	.992990	.38	.257990	12.00	.742010	44
17	.251677	11.60	.992967	.38	.258710	11.98	.741290	43
18	.252373	11.57	.992944	.38	.259429	11.95	.740571	42
19	.253067	11.57	.992921	.38	.260146	11.95	.739854	41
20	.253761	11.53	.992898	.38	.260863	11.92	.739137	40
21	9.254453	11.52	9.992875	.38	9.261578	11.90	10.738422	39
22	.255144	11.50	.992852	.38	.262292	11.88	.737708	38
23	.255834	11.48	.992829	.38	.263005	11.87	.736995	37
24	.256523	11.47	.992806	.38	.263717	11.85	.736283	36
25	.257211	11.45	.992783	.40	.264428	11.83	.735572	35
26	.257898	11.42	.992759	.38	.265138	11.82	.734862	34
27	.258583	11.42	.992736	.38	.265847	11.80	.734153	33
28	.259268	11.38	.992713	.38	.266555	11.77	.733445	32
29	.259951	11.37	.992690	.40	.267261	11.77	.732739	31
30	.260633	11.35	.992666	.38	.267967	11.73	.732033	30
31	9.261314	11.33	9.992643	.40	9.268671	11.73	10.731329	29
32	.261994	11.32	.992619	.38	.269375	11.70	.730625	28
33	.262673	11.30	.992596	.40	.270077	11.70	.729923	27
34	.263351	11.27	.992572	.38	.270779	11.67	.729221	26
35	.264027	11.27	.992549	.40	.271479	11.65	.728521	25
36	.264703	11.23	.992525	.40	.272178	11.63	.727822	24
37	.265377	11.23	.992501	.38	.272876	11.62	.727124	23
38	.266051	11.20	.992478	.40	.273573	11.60	.726427	22
39	.266723	11.20	.992454	.40	.274269	11.58	.725731	21
40	.267395	11.17	.992430	.40	.274964	11.57	.725036	20
41	9.268065	11.15	9.992406	.40	9.275658	11.55	10.724342	19
42	.268734	11.13	.992382	.38	.276351	11.53	.723649	18
43	.269402	11.12	.992359	.40	.277043	11.52	.722957	17
44	.270069	11.10	.992335	.40	.277734	11.50	.722266	16
45	.270735	11.08	.992311	.40	.278424	11.48	.721576	15
46	.271400	11.07	.992287	.40	.279113	11.47	.720887	14
47	.272064	11.03	.992263	.40	.279801	11.45	.720199	13
48	.272726	11.03	.992239	.42	.280488	11.43	.719512	12
49	.273388	11.02	.992214	.40	.281174	11.40	.718826	11
50	.274049	10.98	.992190	.40	.281858	11.40	.718142	10
51	9.274708	10.98	9.992166	.40	9.282542	11.38	10.717458	9
52	.275367	10.97	.992142	.40	.283225	11.37	.716775	8
53	.276025	10.93	.992118	.42	.283907	11.35	.716093	7
54	.276681	10.93	.992093	.40	.284588	11.33	.715412	6
55	.277337	10.90	.992069	.42	.285268	11.32	.714732	5
56	.277991	10.90	.992044	.40	.285947	11.28	.714053	4
57	.278645	10.87	.992020	.40	.286624	11.28	.713376	3
58	.279297	10.85	.991996	.42	.287301	11.27	.712699	2
59	.279948	10.85	.991971	.40	.287977	11.25	.712023	1
60′	9.280599		9.991947		9.288652		10.711348	0′
100°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	79°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

11°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	168°
0′	9.280599	10.82	9.991947	.42	9.288652	11.23	10.711348	60′
1	.281248	10.82	.991922	.42	.289326	11.22	.710674	59
2	.281897	10.78	.991897	.40	.289999	11.20	.710001	58
3	.282544	10.77	.991873	.42	.290671	11.18	.709329	57
4	.283190	10.77	.991848	.42	.291342	11.18	.708658	56
5	.283836	10.73	.991823	.40	.292013	11.15	.707987	55
6	.284480	10.73	.991799	.42	.292682	11.13	.707318	54
7	.285124	10.70	.991774	.42	.293350	11.12	.706650	53
8	.285766	10.70	.991749	.42	.294017	11.12	.705983	52
9	.286408	10.67	.991724	.42	.294684	11.08	.705316	51
10	.287048	10.67	.991699	.42	.295349	11.07	.704651	50
11	9.287688	10.63	9.991674	.42	9.296013	11.07	10.703987	49
12	.288326	10.63	.991649	.42	.296677	11.03	.703323	48
13	.288964	10.60	.991624	.42	.297339	11.03	.702661	47
14	.289600	10.60	.991599	.42	.298001	11.02	.701999	46
15	.290236	10.57	.991574	.42	.298662	11.00	.701338	45
16	.290870	10.57	.991549	.42	.299322	10.97	.700678	44
17	.291504	10.55	.991524	.43	.299980	10.97	.700020	43
18	.292137	10.52	.991498	.42	.300638	10.95	.699362	42
19	.292768	10.52	.991473	.42	.301295	10.93	.698705	41
20	.293399	10.50	.991448	.43	.301951	10.93	.698049	40
21	9.294029	10.48	9.991422	.42	9.302607	10.90	10.697393	39
22	.294658	10.47	.991397	.42	.303261	10.88	.696739	38
23	.295286	10.45	.991372	.43	.303914	10.88	.696086	37
24	.295913	10.43	.991346	.42	.304567	10.85	.695433	36
25	.296539	10.42	.991321	.43	.305218	10.85	.694782	35
26	.297164	10.40	.991295	.42	.305869	10.83	.694131	34
27	.297788	10.40	.991270	.43	.306519	10.82	.693481	33
28	.298412	10.37	.991244	.43	.307168	10.80	.692832	32
29	.299034	10.35	.991218	.42	.307816	10.78	.692184	31
30	.299655	10.35	.991193	.43	.308463	10.77	.691537	30
31	9.300276	10.32	9.991167	.43	9.309109	10.75	10.690891	29
32	.300895	10.32	.991141	.43	.309754	10.75	.690246	28
33	.301514	10.30	.991115	.42	.310399	10.72	.689601	27
34	.302132	10.27	.991090	.43	.311042	10.72	.688958	26
35	.302748	10.27	.991064	.43	.311685	10.70	.688315	25
36	.303364	10.25	.991038	.43	.312327	10.68	.687673	24
37	.303979	10.23	.991012	.43	.312968	10.67	.687032	23
38	.304593	10.23	.990986	.43	.313608	10.65	.686392	22
39	.305207	10.20	.990960	.43	.314247	10.63	.685753	21
40	.305819	10.18	.990934	.43	.314885	10.63	.685115	20
41	9.306430	10.18	9.990908	.43	9.315523	10.60	10.684477	19
42	.307041	10.15	.990882	.45	.316159	10.60	.683841	18
43	.307650	10.15	.990855	.43	.316795	10.58	.683205	17
44	.308259	10.13	.990829	.43	.317430	10.57	.682570	16
45	.308867	10.12	.990803	.43	.318064	10.55	.681936	15
46	.309474	10.10	.990777	.45	.318697	10.55	.681303	14
47	.310080	10.08	.990750	.43	319330	10.52	.680670	13
48	.310685	10.07	.990724	.45	.319961	10.52	.680039	12
49	.311289	10.07	.990697	.43	.320592	10.50	.679408	11
50	.311893	10.03	.990671	.43	.321222	10.48	.678778	10
51	9.312495	10.03	9.990645	.45	9.321851	10.47	10.678149	9
52	.313097	10.02	.990618	.45	.322479	10.45	.677521	8
53	.313698	9.98	.990591	.43	.323106	10.45	.676894	7
54	.314297	10.00	.990565	.45	.323733	10.42	.676267	6
55	.314897	9.97	.990538	.45	.324358	10.42	.675642	5
56	.315495	9.95	.990511	.43	.324983	10.40	.675017	4
57	.316092	9.95	.990485	.45	.325607	10.40	.674393	3
58	.316689	9.92	.990458	.45	.326231	10.37	.673769	2
59	.317284	9.92	.990431	.45	.326853	10.37	.673147	1
60′	9.317879		9.990404		9.327475		10.672525	0′
101°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	78°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

12°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	167°
0′	9.317879	9.90	9.990404	.43	9.327475	10.33	10.672525	60′
1	.318473	9.88	.990378	.45	.328095	10.33	.671905	59
2	.319066	9.87	.990351	.45	.328715	10.32	.671285	58
3	.319658	9.85	.990324	.45	.329334	10.32	.670666	57
4	.320249	9.85	.990297	.45	.329953	10.28	.670047	56
5	.320840	9.83	.990270	.45	.330570	10.28	.669430	55
6	.321430	9.82	.990243	.47	.331187	10.27	.668813	54
7	.322019	9.80	.990215	.45	.331803	10.25	.668197	53
8	.322607	9.78	.990188	.45	.332418	10.25	.667582	52
9	.323194	9.77	.990161	.45	.333033	10.22	.666967	51
10	.323780	9.77	.990134	.45	.333646	10.22	.666354	50
11	9.324366	9.73	9.990107	.47	9.334259	10.20	10.665741	49
12	.324950	9.73	.990079	.45	.334871	10.18	.665129	48
13	.325534	9.72	.990052	.45	.335482	10.18	.664518	47
14	.326117	9.72	.990025	.47	.336093	10.15	.663907	46
15	.326700	9.68	.989997	.45	.336702	10.15	.663298	45
16	.327281	9.68	.989970	.47	.337311	10.13	.662689	44
17	.327862	9.67	.989942	.45	.337919	10.13	.662081	43
18	.328442	9.65	.989915	.47	.338527	10.10	.661473	42
19	.329021	9.63	.989887	.45	.339133	10.10	.660867	41
20	.329599	9.62	.989860	.47	.339739	10.08	.660261	40
21	9.330176	9.62	9.989832	.47	9.340344	10.07	10.659656	39
22	.330753	9.60	.989804	.45	.340948	10.07	.659052	38
23	.331329	9.57	.989777	.47	.341552	10.05	.658448	37
24	.331903	9.58	.989749	.47	.342155	10.03	.657845	36
25	.332478	9.55	.989721	.47	.342757	10.02	.657243	35
26	.333051	9.55	.989693	.47	.343358	10.00	.656642	34
27	.333624	9.52	.989665	.47	.343958	10.00	.656042	33
28	.334195	9.53	.989637	.45	.344558	9.98	.655442	32
29	.334767	9.50	.989610	.47	.345157	9.97	.654843	31
30	.335337	9.48	.989582	.48	.345755	9.97	.654245	30
31	9.335906	9.48	9.989553	.47	9.346353	9.93	10.653647	29
32	.336475	9.47	.989525	.47	.346949	9.93	.653051	28
33	.337043	9.45	.989497	.47	.347545	9.93	.652455	27
34	.337610	9.43	.989469	.47	.348141	9.90	.651859	26
35	.338176	9.43	.989441	.47	.348735	9.90	.651265	25
36	.338742	9.42	.989413	.47	.349329	9.88	.650671	24
37	.339307	9.40	.989385	.48	.349922	9.87	.650078	23
38	.339871	9.38	.989356	.47	.350514	9.87	.649486	22
39	.340434	9.37	.989328	.47	.351106	9.85	.648894	21
40	.340996	9.37	.989300	.48	.351697	9.83	.648303	20
41	9.341558	9.35	9.989271	.47	9.352287	9.82	10.647713	19
42	.342119	9.33	.989243	.48	.352876	9.82	.647124	18
43	.342679	9.33	.989214	.47	.353465	9.80	.646535	17
44	.343239	9.30	.989186	.48	.354053	9.78	.645947	16
45	.343797	9 30	.989157	.48	.354640	9.78	.645360	15
46	.344355	9.28	.989128	.47	.355227	9.77	.644773	14
47	.344912	9.28	.989100	.48	.355813	9.75	.644187	13
48	.345469	9.25	.989071	.48	.356398	9.73	.643602	12
49	.346024	9.25	.989042	.47	.356982	9.73	.643018	11
50	.346579	9.25	.989014	.48	.357566	9.72	.642434	10
51	9.347134	9.22	9.988985	.48	9.358149	9.70	10.641851	9
52	.347687	9.22	.988956	.48	.358731	9.70	.641269	8
53	.348240	9.20	.988927	.48	.359313	9.67	.640687	7
54	.348792	9.18	.988898	.48	.359893	9.68	.640107	6
55	.349343	9.17	.988869	.48	.360474	9.65	.639526	5
56	.349893	9.17	.988840	.48	.361053	9.65	.638947	4
57	.350443	9.15	.988811	.48	.361632	9.63	.638368	3
58	.350992	9.13	.988782	.48	.362210	9.62	.637790	2
59	.351540	9.13	.988753	.48	.362787	9.62	.637213	1
60′	9.352088		9.988724		9.363364		10.636636	0′
102°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	77°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

13°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	166°
0′	9.352088	9.12	9.988724	.48	9.363364	9.60	10.636636	60′
1	.352635	9.10	.988695	.48	.363940	9.58	.636060	59
2	.353181	9.08	.988666	.50	.364515	9.58	.635485	58
3	.353726	9.08	.988636	.48	.365090	9.57	.634910	57
4	.354271	9.07	.988607	.48	.365664	9.55	.634336	56
5	.354815	9.05	.988578	.50	.366237	9 55	.633763	55
6	.355358	9.05	.988548	.48	.366810	9.53	.633190	54
7	.355901	9.03	.988519	.50	.367382	9.52	.632618	53
8	.356443	9.02	.988489	.48	.367953	9.52	.632047	52
9	.356984	9.00	.988460	.50	.368524	9.50	.631476	51
10	.357524	9.00	.988430	.48	.369094	9.48	.630906	50
11	9.358064	8.98	9.988401	.50	9.369663	9.48	10.630337	49
12	.358603	8.97	.988371	.48	.370232	9.45	.629768	48
13	.359141	8.95	.988342	.50	.370799	9.47	.629201	47
14	.359678	8.95	.988312	.50	.371367	9.43	.628633	46
15	.360215	8.95	.988282	.50	.371933	9.43	.628067	45
16	.360752	8.92	.988252	.48	.372499	9.42	.627501	44
17	.361287	8.92	.988223	.50	.373064	9.42	.626936	43
18	.361822	8.90	.988193	.50	.373629	9.40	.626371	42
19	.362356	8.88	.988163	.50	.374193	9.38	.625807	41
20	.362889	8.88	.988133	.50	.374756	9.38	.625244	40
21	9.363422	8.87	9.988103	.50	9.375319	9.37	10.624681	39
22	.363954	8.85	.988073	.50	.375881	9.35	.624119	38
23	.364485	8.85	.988043	.50	.376142	9.35	.623558	37
24	.365016	8.83	.988013	.50	.377003	9.33	.622997	36
25	.365546	8.82	.987983	.50	.377563	9.32	.622437	35
26	.366075	8.82	.987953	.52	.378122	9.32	.621878	34
27	.366604	8.78	.987922	.50	.378681	9.30	.621319	33
28	.367131	8.80	.987892	.50	.379239	9.30	.620761	32
29	.367659	8.77	.987862	.50	.319797	9.28	.620203	31
30	.368185	8.77	.987832	.52	.380354	9.27	.619646	30
31	9.368711	8.75	9.987801	.50	9.380910	9.27	10.619090	29
32	.369236	8.75	.987771	.52	.381466	9.23	.618534	28
33	.369761	8.72	.987740	.50	.382020	9.25	.617980	27
34	.370285	8.72	.987710	.52	.382575	9.23	.617425	26
35	.370808	8 70	.987679	.50	.383129	9.22	.616871	25
36	.371330	8.70	.987649	.52	.383682	9.20	.616318	24
37	.371852	8 68	.987618	.50	.384234	9.20	.615766	23
38	.372373	8.68	.987588	.52	.384786	9.18	.615214	22
39	.372894	8.67	.987557	.52	.385337	9.18	.614663	21
40	.373414	8.65	.987526	.50	.385888	9.17	.614112	20
41	9.373933	8.65	9.987496	.52	9.386438	9.15	10.613562	19
42	.374452	8.63	.987465	.52	.386987	9.15	.613013	18
43	.374970	8.62	.987434	.52	.387536	9.13	.612464	17
44	.375487	8.60	.987403	.52	.388084	9.12	.611916	16
45	.376003	8.60	.987372	.52	.388631	9.12	.611369	15
46	.376519	8.60	.987341	.52	.389178	9.10	.610822	14
47	.377035	8.57	.987310	.52	.389724	9.10	.610276	13
48	.377549	8.57	.987279	.52	.390270	9.08	.609730	12
49	.378063	8.57	.987248	.52	.390815	9.08	.609185	11
50	.378577	8.53	.987217	.52	.391360	9.05	.608640	10
51	9·379089	8.53	9.987186	.52	9.391903	9.07	10.608097	9
52	.379601	8.53	.987155	.52	.392447	9.03	.607553	8
53	.380113	8.52	.987124	.53	.392989	9.03	.607011	7
54	.380624	8.50	.987092	.52	.393531	9.03	.606469	6
55	.381134	8.48	.987061	.52	.394073	9.02	.605927	5
56	.381643	8.48	.987030	.53	.394614	9.00	.605386	4
57	.382152	8.48	.986998	.52	.395154	9.00	.604846	3
58	.382661	8.45	.986967	.52	.395694	8.98	.604306	2
59	.383168	8.45	.986936	.53	.396233	8.97	.603767	1
60′	9.383675		9.986904		9.396771		10.603229	0′
103°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	76°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

14°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	165°
0′	9.383675	8.45	9.986904	.52	9.396771	8.97	10.603229	60′
1	.384182	8.42	.986873	.53	.397309	8.95	.602691	59
2	.384687	8.42	.986841	.53	.397846	8.95	.602154	58
3	.385192	8.42	.986809	.52	.398383	8.93	.601617	57
4	.385697	8.40	.986778	.53	.398919	8.93	.601081	56
5	.386201	8.38	.986746	.53	.399455	8.92	.600545	55
6	.386704	8.38	.986714	.52	.399990	8.90	.600010	54
7	.387207	8.37	.986683	.53	.400524	8.90	.599476	53
8	.387709	8.35	.986651	.53	.401058	8.88	.598942	52
9	.388210	8.35	.986619	.53	.401591	8.88	.598409	51
10	.388711	8.33	.986587	.53	.402124	8.87	.597876	50
11	9.389211	8.33	9.986555	.53	9.402656	8.85	10.597344	49
12	.389711	8.32	.986523	.53	.403187	8.85	.596813	48
13	.390210	8.30	.986491	.53	.403718	8.85	.596282	47
14	.390708	8.30	.986459	.53	.404249	8.82	.595751	46
15	.391206	8.28	.986427	.53	.404778	8.83	.595222	45
16	.391703	8.27	.986395	.53	.405308	8.80	.594692	44
17	.392199	8.27	.986363	.53	.405836	8.80	.594164	43
18	.392695	8.27	.986331	.53	.406364	8.80	.593636	42
19	.393191	8.23	.986299	.55	.406892	8.78	.593108	41
20	.393685	8.23	.986266	.53	.407419	8.77	.592581	40
21	9.394179	8.23	9.986234	.53	9.407945	8.77	10.592055	39
22	.394673	8.22	.986202	.55	.408471	8.75	.591529	38
23	.395166	8.20	.986169	.53	.408996	8.75	.591004	37
24	.395658	8.20	.986137	.55	.409521	8.73	.590479	36
25	.396150	8.18	.986104	.53	.410045	8.73	.589955	35
26	.396641	8.18	.986072	.55	.410569	8.72	.589431	34
27	.397132	8.15	.986039	.53	.411092	8.72	.588908	33
28	.397621	8.17	.986007	.55	.411615	8.70	.588385	32
29	.398111	8.15	.985974	.53	.412137	8.68	.587863	31
30	.398600	8.13	.985942	.55	.412658	8.68	.587342	30
31	9.399088	8.12	9.985909	.55	9.413179	8.67	10.586821	29
32	.399575	8.12	.985876	.55	.413699	8.67	.586301	28
33	.400062	8.12	.985843	.53	.414219	8.65	.585781	27
34	.400549	8.10	.985811	.55	.414738	8.65	.585262	26
35	.401035	8.08	.985778	.55	.415257	8.63	.584743	25
36	.401520	8.08	.985745	.55	.415775	8.63	.584225	24
37	.402005	8.07	.985712	.55	.416293	8.62	.583707	23
38	.402489	8.05	.985679	.55	.416810	8.60	.583190	22
39	.402972	8.05	.985646	.55	.417326	8.60	.582674	21
40	.403455	8.05	.985613	.55	.417842	8.60	.582158	20
41	9.403938	8.03	9.985580	.55	9.418358	8.58	10.581642	19
42	.404420	8.02	.985547	.55	.418873	8.57	.581127	18
43	.404901	8.02	.985514	.57	.419387	8.57	.580613	17
44	.405382	8.00	.985480	.55	.419901	8.57	.580099	16
45	.405862	7.98	.985447	.55	.420415	8.53	.579585	15
46	.406341	7.98	.985414	.55	.420927	8.55	.579073	14
47	.406820	7.98	.985381	.57	.421440	8.53	.578560	13
48	.407299	7.97	.985347	.55	.421952	8.52	.578048	12
49	.407777	7.95	.985314	.57	.422463	8.52	.577537	11
50	.408254	7.95	.985280	.55	.422974	8.50	.577026	10
51	9.408731	7.93	9.985247	.57	9.423484	8.48	10.576516	9
52	.409207	7.93	.985213	.55	.423993	8.50	.576007	8
53	.409682	7.92	.985180	.57	.424503	8.47	.575497	7
54	.410157	7.92	.985146	.55	.425011	8.47	.574989	6
55	.410632	7.90	.985113	.57	.425519	8.47	.574481	5
56	.411106	7.88	.985079	.57	.426027	8.45	.573973	4
57	.411579	7.88	.985045	.57	.426534	8.45	.573466	3
58	.412052	7.87	.985011	.55	.427041	8.43	.572959	2
59	.412524	7.87	.984978	.57	.427547	8.42	.572453	1
60′	9.412996		9.984944		9.428052		10.571948	0′
104°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	75°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

15°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	164°
0'	9.412996	7.85	9.984944	.57	9.428052	8.43	10.571948	60'
1	.413467	7.85	.984910	.57	.428558	8.40	.571442	59
2	.413938	7.83	.984876	.57	.429062	8.40	.570938	58
3	.414408	7.83	.984842	.57	.429566	8.40	.570434	57
4	.414878	7.82	.984808	.57	.430070	8.38	.569930	56
5	.415347	7.80	.984774	.57	.430573	8.37	.569427	55
6	.415815	7.80	.984740	.57	.431075	8.37	.568925	54
7	.416283	7.80	.984706	.57	.431577	8.37	.568423	53
8	.416751	7.77	.984672	.57	.432079	8.35	.567921	52
9	.417217	7.78	.984638	.58	.432580	8.33	.567420	51
10	.417684	7.77	.984603	.57	.433080	8.33	.566920	50
11	9.418150	7.75	9.984569	.57	9.433580	8.33	10.566420	49
12	.418615	7.73	.984535	.58	.434080	8.32	.565920	48
13	.419079	7.75	.984500	.57	.434579	8.32	.565421	47
14	.419544	7.72	.984466	.57	.435078	8.30	.564922	46
15	.420007	7.72	.984432	.58	.435576	8.28	.564424	45
16	.420470	7.72	.984397	.57	.436073	8.28	.563927	44
17	.420933	7.70	.984363	.58	.436570	8.28	.563430	43
18	.421395	7.70	.984328	.57	.437067	8.27	.562933	42
19	.421857	7.68	.984294	.58	.437563	8.27	.562437	41
20	.422318	7.67	.984259	.58	.438059	8.25	.561941	40
21	9.422778	7.67	9.984224	.57	9.438554	8.23	10.561446	39
22	.423238	7.65	.984190	.58	.439048	8.25	.560952	38
23	.423697	7.65	.984155	.58	.439543	8.22	.560457	37
24	.424156	7.65	.984120	.58	.440036	8.22	.559964	36
25	.424615	7.63	.984085	.58	.440529	8.22	.559471	35
26	.425073	7.62	.984050	.58	.441022	8.20	.558978	34
27	.425530	7.62	.984015	.57	.441514	8.20	.558486	33
28	425987	7.60	.983981	.58	.442006	8.18	.557994	32
29	.426443	7 60	.983946	.58	.442497	8.18	.557503	31
30	.426899	7.58	.983911	.60	.442988	8.18	557012	30
31	9.427354	7.58	9.983875	.58	9.443479	8.15	10.556521	29
32	.427809	7.57	.983840	.58	.443968	8.17	.556032	28
33	.428263	7.57	.983805	.58	.444458	8.15	.555542	27
34	.428717	7.55	.983770	.58	.444947	8.13	.555053	26
35	.429170	7.55	.983735	.58	.445435	8.13	.554565	25
36	.429623	7.53	.983700	.60	.445923	8.13	.554077	24
37	.430075	7.53	.983664	.58	.446411	8.12	.553589	23
38	.430527	7.52	.983629	.58	.446898	8.10	.553102	22
39	.430978	7.52	.983594	.60	.447384	8.10	.552616	21
40	.431429	7.50	.983558	.58	.447870	8.10	.552130	20
41	9.431879	7.50	9.983523	.60	9.448356	8.08	10.551644	19
42	.432329	7.48	.983487	.58	.448841	8.08	.551159	18
43	.432778	7.47	.983452	.60	.449326	8.07	.550674	17
44	.433226	7.48	.983416	.58	.449810	8.07	.550190	16
45	.433675	7.45	.983381	.60	.450294	8.05	.549706	15
46	.434122	7.45	.983345	.60	.450777	8.05	.549223	14
47	.434569	7.45	.983309	.60	.451260	8.05	.548740	13
48	.435016	7.43	.983273	.58	.451743	8.03	.548257	12
49	.435462	7.43	.983238	.60	.452225	8.02	.547775	11
50	.435908	7.42	.983202	.60	.452706	8.02	.547294	10
51	9.436353	7.42	9.983166	.60	9.453187	8.02	10.546813	9
52	.436798	7.40	.983130	.60	.453668	8.00	.546332	8
53	.437242	7.40	.983094	.60	.454148	8.00	.545852	7
54	.437686	7.38	.983058	.60	.454628	7.98	.545372	6
55	.438129	7.38	.983022	.60	.455107	7.98	.544893	5
56	.438572	7.37	.982986	.60	.455586	7.97	.544414	4
57	.439014	7.37	.982950	.60	.456064	7.97	.543936	3
58	.439456	7.35	.982914	.60	.456542	7.95	.543458	2
59	.439897	7.35	.982878	.60	.457019	7.95	.542981	1
60'	9.440338		9.982842		9.457496		10.542504	0'
105°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	74°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

16°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	163°
0′	9.440338	7.33	9.982842	.62	9.457496	7.95	10.542504	60′
1	.440778	7.33	.982805	.60	.457973	7.93	.542027	59
2	.441218	7.33	.982769	.60	.458449	7.93	.541551	58
3	.441658	7.30	.982733	.62	.458925	7.92	.541075	57
4	.442096	7.32	.982696	.60	.459400	7.92	.540600	56
5	.442535	7.30	.982660	.60	.459875	7.90	.540125	55
6	.442973	7.28	.982624	.62	.460349	7.90	.539651	54
7	.443410	7.28	.982587	.60	.460823	7.90	.539177	53
8	.443847	7.28	.982551	.62	.461297	7.88	.538703	52
9	.444284	7.27	.982514	.62	.461770	7.87	.538230	51
10	.444720	7.25	.982477	.60	.462242	7.88	.537758	50
11	9.445155	7.25	9.982441	.62	9.462715	7.85	10.537285	49
12	.445590	7.25	.982404	.62	.463186	7.87	.536814	48
13	.446025	7.23	.982367	.60	.463658	7.83	.536342	47
14	.446459	7.23	.982331	.62	.464128	7.85	.535872	46
15	.446893	7.22	.982294	.62	.464599	7.83	.535401	45
16	.447326	7.22	.982257	.62	.465069	7.83	.534931	44
17	.447759	7.20	.982220	.62	.465539	7.82	.534461	43
18	.448191	7.20	.982183	.62	.466008	7.82	.533992	42
19	.448623	7.18	.982146	.62	.466477	7.80	.533523	41
20	.449054	7.18	.982109	.62	.466945	7.80	.533055	40
21	9.449485	7.17	9.982072	.62	9.467413	7.78	10.532587	39
22	.449915	7.17	.982035	.62	.467880	7.78	.532120	38
23	.450345	7.17	.981998	.62	.468347	7.78	.531653	37
24	.450775	7.15	.981961	.62	.468814	7.77	.531186	36
25	.451204	7.13	.981924	.63	.469280	7.77	.530720	35
26	.451632	7.13	.981886	.62	.469746	7.75	.530254	34
27	.452060	7.13	.981849	.62	.470211	7.75	.529789	33
28	.452488	7.12	.981812	.63	.470676	7.75	.529324	32
29	.452915	7.12	.981774	.62	.471141	7.73	.528859	31
30	.453342	7.10	.981737	.62	.471605	7.73	.528395	30
31	9.453768	7.10	9.981700	.63	9.472069	7.72	10.527931	29
32	.454194	7.08	.981662	.62	.472532	7.72	.527468	28
33	.454619	7.08	.981625	.63	.472995	7.70	.527005	27
34	.455044	7.08	.981587	.63	.473457	7.70	.526543	26
35	.455469	7.07	.981549	.62	.473919	7.70	.526081	25
36	.455893	7.05	.981512	.63	.474381	7.68	.525619	24
37	.456316	7.05	.981474	.63	.474842	7.68	.525158	23
38	.456739	7.05	.981436	.62	.475303	7.67	.524697	22
39	.457162	7.03	.981399	.63	.475763	7.67	.524237	21
40	.457584	7.03	.981361	.63	.476223	7.67	.523777	20
41	9.458006	7.02	9.981323	63	9.476683	7.65	10.523317	19
42	.458427	7.02	.981285	.63	.477142	7.65	.522858	18
43	.458848	7.00	.981247	.63	.477601	7.63	.522399	17
44	.459268	7.00	.981209	.63	.478059	7.63	.521941	16
45	.459688	7.00	.981171	.63	.478517	7.63	.521483	15
46	.460108	6.98	.981133	.63	.478975	7.62	.521025	14
47	.460527	6.98	.981095	.63	.479432	7.62	.520568	13
48	.460946	6.97	.981057	.63	.479889	7.60	.520111	12
49	.461364	6.97	.981019	.63	.480345	7.60	.519655	11
50	.461782	6.95	.980981	.65	.480801	7.60	.519199	10
51	9.462199	6.95	9.980942	.63	9.481257	7.58	10.518743	9
52	.462616	6.93	.980904	.63	.481712	7.58	.518288	8
53	.463032	6.93	.980866	.65	.482167	7.57	.517833	7
54	.463448	6.93	.980827	.63	.482621	7.57	.517379	6
55	.463864	6.92	.980789	.65	.483075	7.57	.516925	5
56	.464279	6.92	.980750	.63	.483529	7.55	.516471	4
57	.464694	6.90	.980712	.65	.483982	7.55	.516018	3
58	.465108	6.90	.980673	.63	.484435	7.53	.515565	2
59	.465522	6.88	.980635	.65	.484887	7.53	.515118	1
60′	9.465935		9.980596		9.485339		10.514661	0′
106°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	73°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

17°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	162°
0'	9.465935	6.88	9.980596	.63	9.485339	7.53	10.514661	60'
1	.466348	6.88	.980558	.65	.485791	7.52	.514209	59
2	.466761	6.87	.980519	.65	.486242	7.52	.513758	58
3	.467173	6.87	.980480	.63	.486693	7.50	.513307	57
4	.467585	6.85	.980442	.65	.487143	7.50	.512857	56
5	.467996	6.85	.980403	.65	.487593	7.50	.512407	55
6	.468407	6.83	.980364	.65	.488043	7.48	.511957	54
7	.468817	6.83	.980325	.65	.488492	7.48	.511508	53
8	.469227	6.83	.980286	.65	.488941	7.48	.511059	52
9	.469637	6.82	.980247	.65	.489390	7.47	.510610	51
10	.470046	6.82	.980208	.65	.489838	7.47	.510162	50
11	9.470455	6.80	9.980169	.65	9.490286	7.45	10.509714	49
12	.470863	6.80	.980130	.65	.490733	7.45	.509267	48
13	.471271	6.80	.980091	.65	.491180	7.45	.508820	47
14	.471679	6.78	.980052	.67	.491627	7.43	.508373	46
15	.472086	6.77	.980012	.65	.492073	7.43	.507927	45
16	.472492	6.77	.979973	.65	.492519	7.43	.507481	44
17	.472898	6.77	.979934	.65	.492965	7.42	.507035	43
18	.473304	6.77	.979895	.67	.493410	7.40	.506590	42
19	.473710	6.75	.979855	.65	.493854	7.42	.506146	41
20	.474115	6.73	.979816	.67	.494299	7.40	.505701	40
21	9.474519	6.73	9.979776	.65	9.494743	7.38	10.505257	39
22	.474923	6.73	.979737	.67	.495186	7.40	.504814	38
23	.475327	6.72	.979697	.65	.495630	7.38	.504370	37
24	.475730	6.72	.979658	.67	.496073	7.37	.503927	36
25	.476133	6.72	.979618	.65	.496515	7.37	.503485	35
26	.476536	6.70	.979579	.67	.496957	7.37	.503043	34
27	.476938	6.70	.979539	.67	.497399	7.37	.502601	33
28	.477340	6.68	.979499	.67	.497841	7.35	.502159	32
29	.477741	6.68	.979459	.65	.498282	7.33	.501718	31
30	.478142	6.67	.979420	.67	.498722	7.35	.501278	30
31	9.478542	6.67	9.979380	.67	9.499163	7.33	10.500837	29
32	.478942	6.67	.979340	.67	.499603	7.32	.500397	28
33	.479342	6.65	.979300	.67	.500042	7.32	.499958	27
34	.479741	6.65	.979260	.67	.500481	7.32	.499519	26
35	.480140	6.65	.979220	.67	.500920	7.32	.499080	25
36	.480539	6.63	.979180	.67	.501359	7.30	.498641	24
37	.480937	6.62	.979140	.67	.501797	7.30	.498203	23
38	.481334	6.62	.979100	.68	.502235	7.28	.497765	22
39	.481731	6.62	.979059	.67	.502672	7.28	.497328	21
40	.482128	6.62	.979019	.67	.503109	7.28	.496891	20
41	9.482525	6.60	9.978979	.67	9.503546	7.27	10.496454	19
42	.482921	6.58	.978939	.68	.503982	7.27	.496018	18
43	.483316	6.60	.978898	.67	.504418	7.27	.495582	17
44	.483712	6.58	.978858	.68	.504854	7.25	.495146	16
45	.484107	6.57	.978817	.67	.505289	7.25	.494711	15
46	.484501	6.57	.978777	.67	.505724	7.25	.494276	14
47	.484895	6.57	.978737	.68	.506159	7.23	.493841	13
48	.485289	6.55	.978696	.68	.506593	7.23	.493407	12
49	.485682	6.55	.978655	.67	.507027	7.22	.492973	11
50	.486075	6.53	.978615	.68	.507460	7.22	.492540	10
51	9.486467	6.55	9.978574	.68	9.507893	7.22	10.492107	9
52	.486860	6.52	.978533	.67	.508326	7.22	.491674	8
53	.487251	6.53	.978493	.68	.508759	7.20	.491241	7
54	.487643	6.52	.978452	.68	.509191	7.18	.490809	6
55	.488034	6.50	.978411	.68	.509622	7.20	.490378	5
56	.488424	6.50	.978370	.68	.510054	7.18	.489946	4
57	.488814	6.50	.978329	.68	.510485	7.18	.489515	3
58	.489204	6.48	.978288	.68	.510916	7.17	.489084	2
59	.489593	6.48	.978247	.68	.511346	7.17	.488654	1
60'	9.489982		9.978206		9.511776		10.488224	0'
107°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	72°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

18°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	161°
0'	9.489982	6.48	9.978206	.68	9.511776	7.17	10.488224	60
1	.490371	6.47	.978165	.68	.512206	7.15	.487794	59
2	.490759	6.47	.978124	.68	.512635	7.15	.487365	58
3	.491147	6.47	.978083	.68	.513064	7.15	.486936	57
4	.491535	6.45	.978042	.68	.513493	7.13	.486507	56
5	.491922	6.43	.978001	.70	.513921	7.13	.486079	55
6	.492308	6.45	.977959	.68	.514349	7.13	.485651	54
7	.492695	6.43	.977918	.68	.514777	7.12	.485223	53
8	.493081	6.42	.977877	.70	.515204	7.12	.484796	52
9	.493466	6.42	.977835	.68	.515631	7.10	.484369	51
10	.493851	6.42	.977794	.70	.516057	7.12	.483943	50
11	9.494236	6.42	9.977752	.68	9.516484	7.10	10.483516	49
12	.494621	6.40	.977711	.70	.516910	7.08	.483090	48
13	.495005	6.38	.977669	.68	.517335	7.10	.482665	47
14	.495388	6.40	.977628	.70	.517761	7.08	.482239	46
15	.495772	6.37	.977586	.70	.518186	7.07	.481814	45
16	.496154	6.38	.977544	.68	.518610	7.07	.481390	44
17	.496537	6.37	.977503	.70	.519034	7.07	.480966	43
18	.496919	6.37	.977461	.70	.519458	7.07	.480542	42
19	.497301	6.35	.977419	.70	.519882	7.05	.480118	41
20	.497682	6.37	.977377	.70	.520305	7.05	.479695	40
21	9.498064	6.33	9.977335	.70	9.520728	7.05	10.479272	39
22	.498444	6.35	.977293	.70	.521151	7.03	.478849	38
23	.498825	6.32	.977251	.70	.521573	7.03	.478427	37
24	.499204	6.33	.977209	.70	.521995	7.03	.478005	36
25	.499584	6.32	.977167	.70	.522417	7.02	.477583	35
26	.499963	6.32	.977125	.70	.522838	7.02	.477162	34
27	.500342	6.32	.977083	.70	.523259	7.02	.476741	33
28	.500721	6.30	.977041	.70	.523680	7.00	.476320	32
29	.501099	6.28	.976999	.70	.524100	7.00	.475900	31
30	.501476	6.30	.976957	.72	.524520	7.00	.475480	30
31	9.501854	6.28	9.976914	.70	9.524940	6.98	10.475060	29
32	.502231	6.27	.976872	.70	.525359	6.98	.474641	28
33	.502607	6.28	.976830	.72	.525778	6.98	.474222	27
34	.502984	6.27	.976787	.70	.526197	6.97	.473803	26
35	.503360	6.25	.976745	.72	.526615	6.97	.473385	25
36	.503735	6.25	.976702	.70	.527033	6.97	.472967	24
37	.504110	6.25	.976660	.72	.527451	6.95	.472549	23
38	.504485	6.25	.976617	.72	.527868	6.95	.472132	22
39	.504860	6.23	.976574	.70	.528285	6.95	.471715	21
40	.505234	6.23	.976532	.72	.528702	6.95	.471298	20
41	9.505608	6.22	9.976489	.72	9.529119	6.93	10.470881	19
42	.505981	6.22	.976446	.70	.529535	6.93	.470465	18
43	.506354	6.22	.976404	.72	.529951	6.92	.470049	17
44	.506727	6.20	.976361	.72	.530366	6.92	.469634	16
45	.507099	6.20	.976318	.72	.530781	6.92	.469219	15
46	.507471	6.20	.976275	.72	.531196	6.92	.468804	14
47	.507843	6.18	.976232	.72	.531611	6.90	.468389	13
48	.508214	6.18	.976189	.72	.532025	6.90	.467975	12
49	.508585	6.18	.976146	.72	.532439	6.90	.467561	11
50	.508956	6.17	.976103	.72	.532853	6.88	.467147	10
51	9.509326	6.17	9.976060	.72	9.533266	6.88	10.466734	9
52	.509696	6.15	.976017	.72	.533679	6.88	.466321	8
53	.510065	6.15	.975974	.73	.534092	6.87	.465908	7
54	.510434	6.15	.975930	.72	.534504	6.87	.465496	6
55	.510803	6.15	.975887	.72	.534916	6.87	.465084	5
56	.511172	6.13	.975844	.73	.535328	6.85	.464672	4
57	.511540	6.12	.975800	.72	.535739	6.85	.464261	3
58	.511907	6.13	.975757	.72	.536150	6.85	.463850	2
59	.512275	6.12	.975714	.73	.536561	6.85	.463439	1
60'	9.512642		9.975670		9.536972		10.463028	0'
108°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	71°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

19°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	160°
0'	9.512642	6.12	9.975670	.72	9.536972	6.83	10.463028	60'
1	.513009	6.10	.975627	.73	.537382	6.83	.462618	59
2	.513375	6.10	.975583	.73	.537792	6.83	.462208	58
3	.513741	6.10	.975539	.72	.538202	6.82	.461798	57
4	.514107	6.08	.975496	.73	.538611	6.82	.461389	56
5	.514472	6.08	.975452	.73	.539020	6.82	.460980	55
6	.514837	6.08	.975408	.72	.539429	6.80	.460571	54
7	.515202	6.07	.975365	.73	.539837	6.80	.460163	53
8	.515566	6.07	.975321	.73	.540245	6.80	.459755	52
9	.515930	6.07	.975277	.73	.540653	6.80	.459347	51
10	.516294	6.05	.975233	.73	.541061	6.78	.458939	50
11	9.516657	6.05	9.975189	.73	9.541468	6.78	10.458532	49
12	.517020	6.03	.975145	.73	.541875	6.77	.458125	48
13	.517382	6.05	.975101	.73	.542281	6.78	.457719	47
14	.517745	6.03	.975057	.73	.542688	6.77	.457312	46
15	.518107	6.02	.975013	.73	.543094	6.75	.456906	45
16	.518468	6.02	.974969	.73	.543499	6.77	.456501	44
17	.518829	6.02	.974925	.75	.543905	6.75	.456095	43
18	.519190	6.02	.974880	.73	.544310	6.75	.455690	42
19	.519551	6.00	.974836	.73	.544715	6.73	.455285	41
20	.519911	6.00	.974792	.73	.545119	6.75	.454881	40
21	9.520271	6.00	9.974748	.75	9.545524	6.73	10.454476	39
22	.520631	5.98	.974703	.73	.545928	6.72	.454072	38
23	.520990	5.98	.974659	.75	.546331	6.73	.453669	37
24	.521349	5.97	.974614	.73	.546735	6.72	.453265	36
25	.521707	5.98	.974570	.75	.547138	6.70	.452862	35
26	.522066	5.97	.974525	.73	.547540	6.72	.452460	34
27	.522424	5.95	.974481	.75	.547943	6.70	.452057	33
28	.522781	5.95	.974436	.75	.548345	6.70	.451655	32
29	.523138	5.95	.974391	.73	.548747	6.70	.451253	31
30	.523495	5.95	.974347	.75	.549149	6.68	.450851	30
31	9.523852	5.93	9.974302	.75	9.549550	6.68	10.450450	29
32	.524208	5.93	.974257	.75	.549951	6.68	.450049	28
33	.524564	5.93	.974212	.75	.550352	6.67	.449648	27
34	.524920	5.92	.974167	.75	.550752	6.68	.449248	26
35	.525275	5.92	.974122	.75	.551153	6.65	.448847	25
36	.525630	5.90	.974077	.75	.551552	6.67	.448448	24
37	.525984	5.92	.974032	.75	.551952	6.65	.448048	23
38	.526339	5.90	.973987	.75	.552351	6.65	.447649	22
39	.526693	5.88	.973942	.75	.552750	6.65	.447250	21
40	.527046	5.90	.973897	.75	.553149	6.65	.446851	20
41	9.527400	5.88	9.973852	.75	9.553548	6.63	10.446452	19
42	.527753	5.87	.973807	.77	.553946	6.63	.446054	18
43	.528105	5.88	.973761	.75	.554344	6.62	.445656	17
44	.528458	5.87	.973716	.75	.554741	6.63	.445259	16
45	.528810	5.85	.973671	.77	.555139	6.62	.444861	15
46	.529161	5.87	.973625	.75	.555536	6.62	.444464	14
47	.529513	5.85	.973580	.75	.555933	6.60	.444067	13
48	.529864	5.85	.973535	.77	.556329	6.60	.443671	12
49	.530215	5.83	.973489	.75	.556725	6.60	.443275	11
50	.530565	5.83	.973444	.77	.557121	6.60	.442879	10
51	9.530915	5.83	9.973398	.77	9.557517	6.60	10.442483	9
52	.531265	5.82	.973352	.75	.557913	6.58	.442087	8
53	.531614	5.82	.973307	.77	.558308	6.58	.441692	7
54	.531963	5.82	.973261	.77	.558703	6.57	.441297	6
55	.532312	5.82	.973215	.77	.559097	6.57	.440903	5
56	.532661	5.80	.973169	.75	.559491	6.57	.440509	4
57	.533009	5.80	.973124	.77	.559885	6.57	.440115	3
58	.533357	5.78	.973078	.77	.560279	6.57	.439721	2
59	.533704	5.80	.973032	.77	.560673	6.55	.439327	1
60'	9.534052		9.972986		9.561066		10.438934	0'
109°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	70°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

20°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	159°
0′	9.534052	5.78	9.972986	.77	9.561066	6.55	10.438934	60′
1	.534399	5.77	.972940	.77	.561459	6.53	.438541	59
2	.534745	5.78	.972894	.77	.561851	6.55	.438149	58
3	.535092	5.77	.972848	.77	.562244	6.53	.437756	57
4	.535438	5.75	.972802	.78	.562636	6.53	.437364	56
5	.535783	5.77	.972755	.77	.563028	6.52	.436972	55
6	.536129	5.75	.972709	.77	.563419	6.53	.436581	54
7	.536474	5.73	.972663	.77	.563811	6.52	.436189	53
8	.536818	5.75	.972617	.78	.564202	6.52	.435798	52
9	.537163	5.73	.972570	.77	.564593	6.50	.435407	51
10	.537507	5.73	.972524	.77	.564983	6.50	.435017	50
11	9.537851	5.72	9.972478	.78	9.565373	6.50	10.434627	49
12	.538194	5.73	.972431	.77	.565763	6.50	.434237	48
13	.538538	5.70	.972385	.78	.566153	6.48	.433847	47
14	.538880	5.72	.972338	.78	.566542	6.50	.433458	46
15	.539223	5.70	.972291	.77	.566932	6.47	.433068	45
16	.539565	5.70	.972245	.78	.567320	6.48	.432680	44
17	.539907	5.68	.972198	.78	.567709	6.48	.432291	43
18	.540249	5.68	.972151	.77	.568098	6.47	.431902	42
19	.540590	5.68	.972105	.78	.568486	6.45	.431514	41
20	.540931	5.68	.972058	.78	.568873	6.47	.431127	40
21	9.541272	5.68	9.972011	.78	9.569261	6.45	10.430739	39
22	.541613	5.67	.971964	.78	.569648	6.45	.430352	38
23	.541953	5.67	.971917	.78	.570035	6.45	.429965	37
24	.542293	5.65	.971870	.78	.570422	6.45	.429578	36
25	.542632	5.65	.971823	.78	.570809	6.43	.429191	35
26	.542971	5.65	.971776	.78	.571195	6.43	.428805	34
27	.543310	5.65	.971729	.78	.571581	6.43	.428419	33
28	.543649	5.63	.971682	.78	.571967	6.42	.428033	32
29	.543987	5.63	.971635	.78	.572352	6.43	.427648	31
30	.544325	5.63	.971588	.80	.572738	6.42	.427262	30
31	9.544663	5.62	9.971540	.78	9.573123	6.40	10.426877	29
32	.545000	5.63	.971493	.78	.573507	6.42	.426493	28
33	.545338	5.60	.971446	.80	.573892	6.40	.426108	27
34	.545674	5.62	.971398	.78	.574276	6.40	.425724	26
35	.546011	5.60	.971351	.80	.574660	6.40	.425340	25
36	.546347	5.60	.971303	.78	.575044	6.38	.424956	24
37	.546683	5.60	.971256	.80	.575427	6.38	.424573	23
38	.547019	5.58	.971208	.78	.575810	6.38	.424190	22
39	.547354	5.58	.971161	.80	.576193	6.38	.423807	21
40	.547689	5.58	.971113	.78	.576576	6.38	.423424	20
41	9.548024	5.58	9.971066	.80	9.576959	6.37	10.423041	19
42	.548359	5.57	.971018	.80	.577341	6.37	.422659	18
43	.548693	5.57	.970970	.80	.577723	6.35	.422277	17
44	.549027	5.55	.970922	.80	.578104	6.37	.421896	16
45	.549360	5.55	.970874	.78	.578486	6.35	.421514	15
46	.549693	5.55	.970827	.80	.578867	6.35	.421133	14
47	.550026	5.55	.970779	.80	.579248	6.35	.420752	13
48	.550359	5.55	.970731	.80	.579629	6.33	.420371	12
49	.550692	5.53	.970683	.80	.580009	6.33	.419991	11
50	.551024	5.53	.970635	.82	.580389	6.33	.419611	10
51	9.551356	5.52	9.970586	.80	9.580769	6.33	10.419231	9
52	.551687	5.52	.970538	.80	.581149	6.32	.418851	8
53	.552018	5.52	.970490	.80	.581528	6.32	.418472	7
54	.552349	5.52	.970442	.80	.581907	6.32	.418093	6
55	.552680	5.50	.970394	.82	.582286	6.32	.417714	5
56	.553010	5.52	.970345	.80	.582665	6.32	.417335	4
57	.553341	5.48	.970297	.80	.583044	6.30	.416956	3
58	.553670	5.50	.970249	.82	.583422	6.30	.416578	2
59	.554000	5.48	.970200	.80	.583800	6.28	.416200	1
60′	9.554329		9.970152		9.584177		10.415823	0′
110°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	69°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

21°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	158°
0′	9.554329	5.48	9.970152	.82	9.584177	6.30	10.415823	60′
1	.554658	5.48	.970103	.80	.584555	6.28	.415445	59
2	.554987	5.47	.970055	.82	.584932	6.28	.415068	58
3	.555315	5.47	.970006	.82	.585309	6.28	.414691	57
4	.555643	5.47	.969957	.80	.585686	6.27	.414314	56
5	.555971	5.47	.969909	.82	.586062	6.28	.413938	55
6	.556299	5.45	.969860	.82	.586439	6.27	.413561	54
7	.556626	5.45	.969811	.82	.586815	6.25	.413185	53
8	.556953	5.45	.969762	.80	.587190	6.27	.412810	52
9	.557280	5.43	.969714	.82	.587566	6.25	.412434	51
10	.557606	5.43	.969665	.82	.587941	6.25	.412059	50
11	9.557932	5.43	9.969616	.82	9.588316	6.25	10.411684	49
12	.558258	5.42	.969567	.82	.588691	6.25	.411309	48
13	.558583	5.43	.969518	.82	.589066	6.23	.410934	47
14	.558909	5.42	.969469	.82	.589440	6.23	.410560	46
15	.559234	5.40	.969420	.83	.589814	6.23	.410186	45
16	.559558	5.42	.969370	.82	.590188	6.23	.409812	44
17	.559883	5.40	.969321	.82	.590562	6.22	.409438	43
18	.560207	5.40	.969272	.82	.590935	6.22	.409065	42
19	.560531	5.40	.969223	.83	.591308	6.22	.408692	41
20	.560855	5.38	.969173	.82	.591681	6.22	.408319	40
21	9.561178	5.38	9.969124	.82	9.592054	6.20	10.407946	39
22	.561501	5.38	.969075	.83	.592426	6.22	.407574	38
23	.561824	5.37	.969025	.82	.592799	6.20	.407201	37
24	.562146	5.37	.968976	.83	.593171	6.18	.406829	36
25	.562468	5.37	.968926	.82	.593542	6.20	.406458	35
26	.562790	5.37	.968877	.83	.593914	6.18	.406086	34
27	.563112	5.35	.968827	.83	.594285	6.18	.405715	33
28	.563433	5.37	.968777	.82	.594656	6.18	.405344	32
29	.563755	5.33	.968728	.83	.595027	6.18	.404973	31
30	.564075	5.35	.968678	.83	.595398	6.17	.404602	30
31	9.564396	5.33	9.968628	.83	9.595768	6.17	10.404232	29
32	.564716	5.33	.968578	.83	.596138	6.17	.403862	28
33	.565036	5.33	.968528	.82	.596508	6.17	.403492	27
34	.565356	5.33	.968479	.83	.596878	6.15	.403122	26
35	.565676	5.32	.968429	.83	.597247	6.15	.402753	25
36	.565995	5.32	.968379	.83	.597616	6.15	.402384	24
37	.566314	5.30	.968329	.85	.597985	6.15	.402015	23
38	.566632	5.32	.968278	.83	.598354	6.13	.401646	22
39	.566951	5.30	.968228	.83	.598722	6.15	.401278	21
40	.567269	5.30	.968178	.83	.599091	6.13	.400909	20
41	9.567587	5.28	9.968128	.83	9.599459	6.13	10.400541	19
42	.567904	5.30	.968078	.85	.599827	6.12	.400173	18
43	.568222	5.28	.968027	.83	.600194	6.13	.399806	17
44	.568539	5.28	.967977	.83	.600562	6.12	.399438	16
45	.568856	5.27	.967927	.85	.600929	6.12	.399071	15
46	.569172	5.27	.967876	.83	.601296	6.12	.398704	14
47	.569488	5.27	.967826	.85	.601663	6.10	.398337	13
48	.569804	5.27	.967775	.83	.602029	6.10	.397971	12
49	.570120	5.25	.967725	.85	.602395	6.10	.397605	11
50	.570435	5.27	.967674	.83	.602761	6.10	.397239	10
51	9.570751	5.25	9.967624	.85	9.603127	6.10	10.396873	9
52	.571066	5.23	.967573	.85	.603493	6.08	.396507	8
53	.571380	5.25	.967522	.85	.603858	6.08	.396142	7
54	.571695	5.23	.967471	.83	.604223	6.08	.395777	6
55	.572009	5.23	.967421	.85	.604588	6.08	.395412	5
56	.572323	5.22	.967370	.85	.604953	6.07	.395047	4
57	.572636	5.23	.967319	.85	.605317	6.08	.394683	3
58	.572950	5.22	.967268	.85	.605682	6.07	.394318	2
59	.573263	5.20	.967217	.85	.606046	6.07	.393954	1
60′	9.573575		9.967166		9.606410		10.393590	0′
111°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	68°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

22°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	157°
0′	9.573575	5.22	9.967166	.85	9.606410	6.05	10.393590	60′
1	.573888	5.20	.967115	.85	.606773	6.07	.393227	59
2	.574200	5.20	.967064	.85	.607137	6.05	.392863	58
3	.574512	5.20	.967013	.87	.607500	6.05	.392500	57
4	.574824	5.20	.966961	.85	.607863	6.03	.392137	56
5	.575136	5.18	.966910	.85	.608225	6.05	.391775	55
6	.575447	5.18	.966859	.85	.608588	6.03	.391412	54
7	.575758	5.18	.966808	.87	.608950	6.03	.391050	53
8	.576069	5.17	.966756	.85	.609312	6.03	.390688	52
9	.576379	5.17	.966705	.87	.609674	6.03	.390326	51
10	.576689	5.17	.966653	.85	.610036	6.02	.389964	50
11	9.576999	5.17	9.966602	.87	9.610397	6.03	10.389603	49
12	.577309	5.15	.966550	.85	.610759	6.02	.389241	48
13	.577618	5.15	.966499	.87	.611120	6.00	.388880	47
14	.577927	5.15	.966447	.87	.611480	6.02	.388520	46
15	.578236	5.15	.966395	.85	.611841	6.00	.388159	45
16	.578545	5.13	.966344	.87	.612201	6.00	.387799	44
17	.578853	5.15	.966292	.87	.612561	6.00	.387439	43
18	.579162	5.13	.966240	.87	.612921	6.00	.387079	42
19	.579470	5.12	.966188	.87	.613281	6.00	.386719	41
20	.579777	5.13	.966136	.85	.613641	5.98	.386359	40
21	9.580085	5.12	9.966085	.87	9.614000	5.98	10.386000	39
22	.580392	5.12	.966033	.87	.614359	5.98	.385641	38
23	.580699	5.10	.965981	.87	.614718	5.98	.385282	37
24	.581005	5.12	.965929	.88	.615077	5.97	.384923	36
25	.581312	5.10	.965876	.87	.615435	5.97	.384565	35
26	.581618	5.10	.965824	.87	.615793	5.97	.384207	34
27	.581924	5.08	.965772	.87	.616151	5.97	.383849	33
28	.582229	5.10	.965720	.87	.616509	5.97	.383491	32
29	.582535	5.08	.965668	.88	.616867	5.95	.383133	31
30	.582840	5.08	.965615	.87	.617224	5.97	.382776	30
31	9.583145	5.07	9.965563	.87	9.617582	5.95	10.382418	29
32	.583449	5.08	.965511	.88	.617939	5.93	.382061	28
33	.583754	5.07	.965458	.87	.618295	5.95	.381705	27
34	.584058	5.05	.965406	.88	.618652	5.93	.381348	26
35	.584361	5.07	.965353	.87	.619008	5.93	.380992	25
36	.584665	5.05	.965301	.88	.619364	5.93	.380636	24
37	.584968	5.07	.965248	.88	.619720	5.93	.380280	23
38	.585272	5.03	.965195	.87	.620076	5.93	.379924	22
39	.585574	5.05	.965143	.88	.620432	5.92	.379568	21
40	.585877	5.03	.965090	.88	.620787	5.92	.379213	20
41	9.586179	5.05	9.965037	.88	9.621142	5.92	10.378858	19
42	.586482	5.02	.964984	.88	.621497	5.92	.378503	18
43	.586783	5.03	.964931	.87	.621852	5.92	.378148	17
44	.587085	5.02	.964879	.88	.622207	5.90	.377793	16
45	.587386	5.03	.964826	.88	.622561	5.90	.377439	15
46	.587688	5.02	.964773	.88	.622915	5.90	.377085	14
47	.587989	5.00	.964720	.90	.623269	5.90	.376731	13
48	.588289	5.02	.964666	.88	.623623	5.88	.376377	12
49	.588590	5.00	.964613	.88	.623976	5.90	.376024	11
50	.588890	5.00	.964560	.88	.624330	5.88	.375670	10
51	9.589190	4.98	9.964507	.88	9.624683	5.88	10.375317	9
52	.589489	5.00	.964454	.90	.625036	5.87	.374964	8
53	.589789	4.98	.964400	.88	.625388	5.88	.374612	7
54	.590088	4.98	.964347	.88	.625741	5.87	.374259	6
55	.590387	4.98	.964294	.90	.626093	5.87	.373907	5
56	.590686	4.97	.964240	.88	.626445	5.87	.373555	4
57	.590984	4.97	.964187	.90	.626797	5.87	.373203	3
58	.591282	4.97	.964133	.88	.627149	5.87	.372851	2
59	.591580	4.97	.964080	.90	.627501	5.85	.372499	1
60′	9.591878		9.964026		9.627852		10.372148	0′
112°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	67°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

23°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	156°
0'	9.591878	4.97	9.964026	.90	9.627852	5.85	10.372148	60'
1	.592176	4.95	.963972	.88	.628203	5.85	.371797	59
2	.592473	4.95	.963919	.90	.628554	5.85	.371446	58
3	.592770	4.95	.963865	.90	.628905	5.83	.371095	57
4	.593067	4.93	.963811	.90	.629255	5.85	.370745	56
5	.593362	4.93	.963757	.88	.629606	5.83	.370394	55
6	.593659	4.93	.963704	.90	.629956	5.83	.370044	54
7	.593955	4.93	.963650	.90	.630306	5.83	.369694	53
8	.594251	4.93	.963596	.90	.630656	5.82	.369344	52
9	.594547	4.92	.963542	.90	.631005	5.83	.368995	51
10	.594842	4.92	.963488	.90	.631355	5.82	.368645	50
11	9.595137	4.92	9.963434	.92	9.631704	5.82	10.368296	49
12	.595432	4.92	.963379	.90	.632053	5.82	.367947	48
13	.595727	4.90	.963325	.90	.632402	5.80	.367598	47
14	.596021	4.90	.963271	.90	.632750	5.82	.367250	46
15	.596315	4.90	.963217	.90	.633099	5.80	.366901	45
16	.596609	4.90	.963163	.92	.633447	5.80	.366553	44
17	.596903	4.88	.963108	.90	.633795	5 80	.366205	43
18	.597196	4.90	.963054	.92	.634143	5.78	.365857	42
19	.597490	4.88	.962999	.90	.634490	5.80	.365510	41
20	.597783	4.87	.962945	.92	.634838	5.78	.365162	40
21	9.598075	4.88	9.962890	.90	9.635185	5.78	10.364815	39
22	.598368	4.87	.962836	.92	.635532	5.78	.364468	38
23	.598660	4.87	.962781	.90	.635879	5.78	.364121	37
24	.598952	4.87	.962727	.92	.636226	5.77	.363774	36
25	.599244	4.87	.962672	.92	.636572	5.78	.363428	35
26	.599536	4.85	.962617	.92	.636919	5.77	.363081	34
27	.599827	4.85	.962562	.90	.637265	5.77	.362735	33
28	.600118	4.85	.962508	.92	.637611	5.75	.362389	32
29	.600409	4.85	.962453	.92	.637956	5.77	.362044	31
30	.600700	4.83	.962398	.92	.638302	5.75	.361698	30
31	9.600990	4.83	9.962343	.92	9.638647	5.75	10.361353	29
32	.601280	4.83	.962288	.92	.638992	5.75	.361008	28
33	.601570	4.83	.962233	.92	.639337	5.75	.360663	27
34	.601860	4.83	.962178	.92	.639682	5.75	.360318	26
35	.602150	4.82	.962123	.93	.640027	5.73	.359973	25
36	.602439	4.82	.962067	.92	.640371	5.75	.359629	24
37	.602728	4.82	.962012	.92	.640716	5.73	.359284	23
38	.603017	4.80	.961957	.92	.641060	5.73	.358940	22
39	.603305	4.82	.961902	.93	.641404	5.72	.358596	21
40	.603594	4.80	.961846	.92	.641747	5 73	.358253	20
41	9.603882	4 80	9.961791	.93	9.642091	5.72	10.357909	19
42	.604170	4.78	.961735	.92	.642434	5.72	.357566	18
43	.604457	4.80	.961680	.93	.642777	5.72	.357223	17
44	.604745	4.78	.961624	.92	.643120	5.72	.356880	16
45	.605032	4.78	.961569	.93	.643463	5.72	.356537	15
46	.605319	4.78	.961513	.92	.643806	5.70	.356194	14
47	.605606	4.77	.961458	.93	.644148	5.70	.355852	13
48	.605892	4.78	.961402	.93	.644490	5.70	.355510	12
49	.606179	4.77	.961346	.93	.644832	5.70	.355168	11
50	.606465	4.77	.961290	.92	.645174	5.70	.354826	10
51	9.606751	4.75	9.961235	.93	9.645516	5.68	10.354484	9
52	.607036	4.77	.961179	.93	.645857	5.70	.354143	8
53	.607322	4.75	.961123	.93	.646199	5.68	.353801	7
54	.607607	4.75	.961067	.93	.646540	5.68	.353460	6
55	.607892	4.75	.961011	.93	.646881	5.68	.353119	5
56	.608177	4.73	.960955	.93	.647222	5.67	.352778	4
57	.608461	4.73	.960899	.93	.647562	5.68	.352438	3
58	.608745	4.73	.960843	.95	.647903	5.67	.352097	2
59	.609029	4.73	.960786	.93	.648243	5.67	.351757	1
60'	9.609313		9.960730		9.648583		10.351417	0'
113°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	66°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

24°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	155°
0′	9.609313	4.73	9.960730	.93	9.648583	5.67	10.351417	60′
1	.609597	4.72	.960674	.93	.648923	5.67	.351077	59
2	.609880	4.73	.960618	.95	.649263	5.65	.350737	58
3	.610164	4.72	.960561	.93	.649602	5.67	.350398	57
4	.610447	4.70	.960505	.95	.649942	5.65	.350058	56
5	.610729	4.72	.960448	.93	.650281	5.65	.349719	55
6	.611012	4.70	.960392	.95	.650620	5.65	.349380	54
7	.611294	4.70	.960335	.93	.650959	5.63	.349041	53
8	.611576	4.70	.960279	.95	.651297	5.65	.348703	52
9	.611858	4.70	.960222	.95	.651636	5.63	.348364	51
10	.612140	4.68	.960165	.93	.651974	5.63	.348026	50
11	9.612421	4.68	9.960109	.95	9.652312	5.63	10.347688	49
12	.612702	4 68	.960052	.95	.652650	5.63	.347350	48
13	.612983	4.68	.959995	.95	.652988	5.63	.347012	47
14	.613264	4 68	.959938	.93	.653326	5.62	.346674	46
15	.613545	4 67	.959882	.95	.653663	5.62	.346337	45
16	.613825	4.67	.959825	.95	.654000	5.62	.346000	44
17	.614105	4.67	.959768	.95	.654337	5.62	.345663	43
18	.614385	4.67	.959711	.95	.654674	5.62	.345326	42
19	.614665	4.65	.959654	.97	.655011	5.62	.344989	41
20	.614944	4.65	.959596	.95	.655348	5.60	.344652	40
21	9.615223	4.65	9.959539	.95	9.655684	5.60	10.344316	39
22	.615502	4.65	.959482	.95	.656020	5.60	.343980	38
23	.615781	4.65	.959425	.95	.656356	5 60	.343644	37
24	.616060	4.63	.959368	.97	.656692	5.60	.343308	36
25	.616338	4.63	.959310	.95	.657028	5.60	.342972	35
26	.616616	4.63	.959253	.97	.657364	5.58	.342636	34
27	.616894	4.63	.959195	.95	.657699	5.58	.342301	33
28	.617172	4.63	.959138	.97	.658034	5.58	.341966	32
29	.617450	4.62	.959080	.95	.658369	5.58	.341631	31
30	.617727	4.62	.959023	.97	.658704	5.58	.341296	30
31	9 618004	4.62	9.958965	.95	9.659039	5.57	10.340961	29
32	.618281	4.62	.958908	.97	.659373	5.58	.340627	28
33	.618558	4.60	.958850	.97	.659708	5.57	.340292	27
34	.618834	4.60	.958792	.97	.660042	5.57	.339958	26
35	.619110	4.60	.958734	.95	.660376	5.57	.339624	25
36	.619386	4.60	.958677	.97	.660710	5.55	.339290	24
37	.619662	4.60	.958619	.97	.661043	5.57	.338957	23
38	.619933	4.58	.958561	.97	.661377	5.55	.338623	22
39	.620213	4.58	.958503	.97	.661710	5.55	.338290	21
40	.620488	4.58	.958445	.97	.662043	5.55	.337957	20
41	9.620763	4.58	9.958387	.97	9.662376	5.55	10.337624	19
42	.621038	4.58	.958329	.97	.662709	5.55	.337291	18
43	.621313	4.57	.958271	.97	.663042	5.55	.336958	17
44	.621587	4.57	.958213	.98	.663375	5.53	.336625	16
45	.621861	4.57	.958154	.97	.663707	5.53	.336293	15
46	.622135	4.57	.958096	.97	.664039	5.53	.335961	14
47	.622409	4.55	.958038	.98	.664371	5.53	.335629	13
48	.622682	4.57	.957979	.97	.664703	5.53	.335297	12
49	.622956	4.55	.957921	.97	.665035	5.52	.334965	11
50	.623229	4.55	.957863	.98	.665366	5.53	.334634	10
51	9.623502	4.53	9.957804	.97	9.665698	5.52	10.334302	9
52	.623774	4.55	.957746	.98	.666029	5.52	.333971	8
53	.624047	4.53	.957687	.98	.666360	5.52	.333640	7
54	.624319	4.53	.957628	.97	.666691	5.50	.333309	6
55	.624591	4.53	.957570	.98	.667021	5.52	.332979	5
56	.624863	4.53	.957511	.98	.667352	5.50	.332648	4
57	.625135	4.52	.957452	.98	.667682	5.52	.332318	3
58	.625406	4.52	.957393	.97	.668013	5.50	.331987	2
59	.625677	4.52	.957335	.98	.668343	5.50	.331657	1
60′	9.625948		9.957276		9.668673		10.331327	0′
114°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	65°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

25°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	154°
0′	9.625948	4.52	9.957276	.98	9.668673	5.48	10.331327	60′
1	.626219	4.52	.957217	.98	.669002	5.50	.330998	59
2	.626490	4.50	.957158	.98	.669332	5.48	.330668	58
3	.626760	4.50	.957099	.98	.669661	5.50	.330339	57
4	.627030	4.50	.957040	.98	.669991	5.48	.330009	56
5	.627300	4.50	.956981	1.00	.670320	5.48	.329680	55
6	.627570	4.50	.956921	.98	.670649	5.47	.329351	54
7	.627840	4.48	.956862	.98	.670977	5.48	.329023	53
8	.628109	4.48	.956803	.98	.671306	5.48	.328694	52
9	.628378	4.48	.956744	1.00	.671635	5.47	.328365	51
10	.628647	4.48	.956684	.98	.671963	5.47	.328037	50
11	9.628916	4.48	9.956625	.98	9.672291	5.47	10.327709	49
12	.629185	4.47	.956566	1.00	.672619	5.47	.327381	48
13	.629453	4.47	.956506	.98	.672947	5.45	.327053	47
14	.629721	4.47	.956447	1.00	.673274	5.47	.326726	46
15	.629989	4.47	.956387	1.00	.673602	5.45	.326398	45
16	.630257	4.45	.956327	.98	.673929	5.47	.326071	44
17	.630524	4.47	.956268	1.00	.674257	5.45	.325743	43
18	.630792	4.45	.956208	1.00	.674584	5.45	.325416	42
19	.631059	4.45	.956148	.98	.674911	5.43	.325089	41
20	.631326	4.45	.956089	1.00	.675237	5.45	.324763	40
21	9.631593	4.43	9.956029	1.00	9.675564	5.43	10.324436	39
22	.631859	4.43	.955969	1.00	.675890	5.45	.324110	38
23	.632125	4.45	.955909	1.00	.676217	5.43	.323783	37
24	.632392	4.43	.955849	1.00	.676543	5.43	.323457	36
25	.632658	4.42	.955789	1.00	.676869	5.42	.323131	35
26	.632923	4.43	.955729	1.00	.677194	5.43	.322806	34
27	.633189	4.42	.955669	1.00	.677520	5.43	.322480	33
28	.633454	4.42	.955609	.98	.677846	5.42	.322154	32
29	.633719	4.42	.955548	1.00	.678171	5.42	.321829	31
30	.633984	4.42	.955488	1.00	.678496	5.42	.321504	30
31	9.634249	4.42	9.955428	1.00	9.678821	5.42	10.321179	29
32	.634514	4.40	.955368	1.02	.679146	5.42	.320854	28
33	.634778	4.40	.955307	1.00	.679471	5.40	.320529	27
34	.635042	4.40	.955247	1.02	.679795	5.42	.320205	26
35	.635306	4.40	.955186	1.00	.680120	5.40	.319880	25
36	.635570	4.40	.955126	1.02	.680444	5.40	.319556	24
37	.635834	4.38	.955065	1.00	.680768	5.40	.319232	23
38	.636097	4.38	.955005	1.02	.681092	5.40	.318908	22
39	.636360	4.38	.954944	1.02	.681416	5.40	.318584	21
40	.636623	4.38	.954883	1.00	.681740	5.38	.318260	20
41	9.636886	4.37	9.954823	1.02	9.682063	5.40	10.317937	19
42	.637148	4.38	.954762	1.02	.682387	5.38	.317613	18
43	.637411	4.37	.954701	1.02	.682710	5.38	.317290	17
44	.637673	4.37	.954640	1.02	.683033	5.38	.316967	16
45	.637935	4.37	.954579	1.02	.683356	5.38	.316644	15
46	.638197	4.35	.954518	1.02	.683679	5.37	.316321	14
47	.638458	4.37	.954457	1.02	.684001	5.38	.315999	13
48	.638720	4.35	.954396	1.02	.684324	5.37	.315676	12
49	.638981	4.35	.954335	1.02	.684646	5.37	.315354	11
50	.639242	4.35	.954274	1.02	.684968	5.37	.315032	10
51	9.639503	4.35	9.954213	1.02	9.685290	5.37	10.314710	9
52	.639764	4.33	.954152	1.03	.685612	5.37	.314388	8
53	.640024	4.33	.954090	1.02	.685934	5.35	.314066	7
54	.640284	4.33	.954029	1.02	.686255	5.37	.313745	6
55	.640544	4.33	.953968	1.03	.686577	5.35	.313423	5
56	.640804	4.33	.953906	1.02	.686898	5.35	.313102	4
57	.641064	4.33	.953845	1.03	.687219	5.35	.312781	3
58	.641324	4.32	.953783	1.02	.687540	5.35	.312460	2
59	.641583	4.32	.953722	1.03	.687861	5.35	.312139	1
60′	9.641842		9.953660		9.688182		10.311818	0′
115° Cosine.		D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	64°

TABLE XXIII (*Continued*)

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS

26°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	153°
0′	9.641842	4.32	9.953660	1.02	9.688182	5.33	10.311818	60′
1	.642101	4.32	.953599	1.03	.688502	5.35	.311498	59
2	.642360	4.30	.953537	1.03	.688823	5.33	.311177	58
3	.642618	4.32	.953475	1.03	.689143	5.33	.310857	57
4	.642877	4.30	.953413	1.02	.689463	5.33	.310537	56
5	.643135	4.30	.953352	1.03	.689783	5.33	.310217	55
6	.643393	4.28	.953290	1.03	.690103	5.33	.309897	54
7	.643650	4.30	.953228	1.03	.690423	5.32	.309577	53
8	.643908	4.28	.953166	1.03	.690742	5.33	.309258	52
9	.644165	4.30	.953104	1.03	.691062	5.32	.308938	51
10	.644423	4.28	.953042	1.03	.691881	5.32	.308619	50
11	9.644680	4.27	9.952980	1.03	9.691700	5.32	10.308300	49
12	.644936	4.28	.952918	1.05	.692019	5.32	.307981	48
13	.645193	4.28	.952855	1.03	.692338	5.30	.307662	47
14	.645450	4.27	.952793	1.03	.692656	5.32	.307344	46
15	.645706	4.27	.952731	1.03	.692975	5.30	.307025	45
16	.645962	4.27	.952669	1.05	.693293	5.32	.306707	44
17	.646218	4.27	.952606	1.03	.693612	5.30	.306388	43
18	.646474	4.25	.952544	1.05	.693930	5.30	.306070	42
19	.646729	4.25	.952481	1.03	.694248	5.30	.305752	41
20	.646984	4.27	.952419	1.05	.694566	5.28	.305434	40
21	9.647240	4.23	9.952356	1.03	9.694883	5.30	10.305117	39
22	.647494	4.25	.952294	1.05	.695201	5.28	.304799	38
23	.647749	4.25	.952231	1.05	.695518	5.30	.304482	37
24	.648004	4.23	.952168	1.03	.695836	5.28	.304164	36
25	.648258	4.23	.952106	1.05	.696153	5.28	.303847	35
26	.648512	4.23	.952043	1.05	.696470	5.28	.303530	34
27	.648766	4.23	.951980	1.05	.696787	5.27	.303213	33
28	.649020	4.23	.951917	1.05	.697103	5.28	.302897	32
29	.649274	4.22	.951854	1.05	.697420	5.27	.302580	31
30	.649527	4.23	.951791	1.05	.697736	5.28	.302264	30
31	9.649781	4.22	9.951728	1.05	9.698053	5.27	10.301947	29
32	.650034	4.22	.951665	1.05	.698369	5.27	.301631	28
33	.650287	4.20	.951602	1.05	.698685	5.27	.301315	27
34	.650539	4.22	.951539	1.05	.699001	5.25	.300999	26
35	.650792	4.20	.951476	1.07	.699316	5.27	.300684	25
36	.651044	4.22	.951412	1.05	.699632	5.25	.300368	24
37	.651297	4.20	.951349	1.05	.699947	5.27	.300053	23
38	.651549	4.18	.951286	1.07	.700263	5.25	.299737	22
39	.651800	4.20	.951222	1.05	.700578	5.25	.299422	21
40	.652052	4.20	.951159	1.05	.700893	5.25	.299107	20
41	9.652304	4.18	9.951096	1.07	9.701208	5.25	10.298792	19
42	.652555	4.18	.951032	1.07	.701523	5.23	.298477	18
43	.652806	4.18	.950968	1.05	.701837	5.25	.298163	17
44	.653057	4.18	.950905	1.07	.702152	5.23	.297848	16
45	.653308	4.17	.950841	1.05	.702466	5.25	.297534	15
46	.653558	4.17	.950778	1.07	.702781	5.23	.297219	14
47	.653808	4.18	.950714	1.07	.703095	5.23	.296905	13
48	.654059	4.17	.950650	1.07	.703409	5.22	.296591	12
49	.654309	4.15	.950586	1.07	.703722	5.23	.296278	11
50	.654558	4.17	.950522	1.07	.704036	5.23	.295964	10
51	9.654808	4.17	9.950458	1.07	9.704350	5.22	10.295650	9
52	.655058	4.15	.950394	1.07	.704663	5.22	.295337	8
53	.655307	4.15	.950330	1.07	.704976	5.23	.295024	7
54	.655556	4.15	.950266	1.07	.705290	5.22	.294710	6
55	.655805	4.15	.950202	1.07	.705603	5.22	.294397	5
56	.656054	4.13	.950138	1.07	.705916	5.20	.294084	4
57	656302	4.15	.950074	1.07	.706228	5.22	.293772	3
58	.656551	4.13	.950010	1.08	.706541	5.22	.293459	2
59	.656799	4.13	.949945	1.07	.706854	5.20	.293146	1
60′	9.657047		9.949881		9.707166		10.292834	0′
116°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	63°

TABLE XXIII (*Continued*)

LOGARITHMIC SINES, COSINES, TANGENTS, AND COTANGENTS

27°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	152°
0'	9.657047	4.13	9.949881	1.08	9.707166	5.20	10.292834	60'
1	.657295	4.12	.949816	1.07	.707478	5.20	.292522	59
2	.657542	4.13	.949752	1.07	.707790	5.20	.292210	58
3	.657790	4.12	.949688	1.08	.708102	5.20	.291898	57
4	.658087	4.12	.949623	1.08	.708414	5.20	.291586	56
5	.658284	4.12	.949558	1.07	.708726	5.18	.291274	55
6	.658531	4.12	.949494	1.08	.709037	5.20	.290963	54
7	.658778	4.12	.949429	1.08	.709349	5.18	.290651	53
8	.659025	4.10	.949364	1.07	.709660	5.18	.290340	52
9	.659271	4.10	.949300	1.08	.709971	5.18	.290029	51
10	.650517	4.10	.949235	1.08	.710282	5.18	.289718	50
11	9.659763	4.10	9.949170	1.08	9.710593	5.18	10.289407	49
12	.660009	4.10	.949105	1.08	.710904	5.18	.289096	48
13	.660255	4.10	.949040	1.08	.711215	5.17	.288785	47
14	.660501	4.08	.948975	1.08	.711525	5.18	.288475	46
15	.660746	4.08	.948910	1.08	.711836	5.17	.288164	45
16	.660991	4.08	.948845	1.08	.712146	5.17	.287854	44
17	.661236	4.08	.948780	1.08	.712456	5.17	.287544	43
18	.661481	4.08	.948715	1.08	.712766	5.17	.287234	42
19	.661726	4.07	.948650	1.10	.713076	5.17	.286924	41
20	.661970	4.07	.948584	1.08	.713386	5.17	.286614	40
21	9.662214	4.08	9.948519	1.08	9.713696	5.15	10.286304	39
22	.662459	4.07	.948454	1.10	.714005	5.15	.285995	38
23	.662703	4.05	.948388	1.08	.714314	5.17	.285686	37
24	.662946	4.07	.948323	1.10	.714624	5.15	.285376	36
25	.663190	4.05	.948257	1.08	.714933	5.15	.285067	35
26	.663433	4.07	.948192	1.10	.715242	5.15	.284758	34
27	.663677	4.05	.948126	1.10	.715551	5.15	.284449	33
28	.663920	4.05	.948060	1.08	.715860	5.13	.284140	32
29	.664163	4.05	.947995	1.10	.716168	5.15	.283832	31
30	.664406	4.03	.947929	1.10	.716477	5.13	.283523	30
31	9.664648	4.05	9.947863	1.10	9.716785	5.13	10.283215	29
32	.664891	4.03	.947797	1.10	.717093	5.13	.282907	28
33	.665133	4.03	.947731	1.10	.717401	5.13	.282599	27
34	.665375	4.03	.947665	1.08	.717709	5.13	.282291	26
35	.665617	4.03	.947600	1.12	.718017	5.13	.281983	25
36	.665859	4.02	.947533	1.10	.718325	5.13	.281675	24
37	.666100	4.03	.947467	1.10	.718633	5.12	.281367	23
38	.666342	4.02	.947401	1.10	.718940	5.13	.281060	22
39	.666583	4.02	.947335	1.10	.719248	5.12	.280752	21
40	.666824	4.02	.947269	1.10	.719555	5.12	.280445	20
41	9.667065	4.00	9.947203	1.12	9.719862	5.12	10.280138	19
42	.667305	4.02	.947136	1.10	.720169	5.12	.279831	18
43	.667546	4.00	.947070	1.10	.720476	5.12	.279524	17
44	667786	4.02	.947004	1.12	.720783	5.10	.279217	16
45	.668027	4.00	.946937	1.10	.721089	5.12	.278911	15
46	.668267	3.98	.946871	1.12	.721396	5.10	.278604	14
47	.668506	4.00	.946804	1.10	.721702	5.12	.278298	13
48	.668746	4.00	.946738	1.12	.722009	5.10	.277991	12
49	.668986	3.98	.946671	1.12	.722315	5.10	.277685	11
50	.669225	3.98	.946604	1.10	.722621	5.10	.277379	10
51	9.669464	3.98	9.946538	1.12	9.722927	5.08	10.277073	9
52	.669703	3.98	.946471	1.12	.723232	5.10	.276768	8
53	.669942	3.98	.946404	1.12	.723538	5.10	.276462	7
54	.670181	3.97	.946337	1.12	.723844	5.08	.276156	6
55	.670419	3.98	.946270	1.12	.724149	5.08	.275851	5
56	.670658	3.97	.946203	1.12	.724454	5.10	.275546	4
57	.670896	3.97	.946136	1.12	.724760	5.08	.275240	3
58	.671134	3.97	.946069	1.12	.725065	5.08	.274935	2
59	.671372	3.95	.946002	1.12	.725370	5.07	.274630	1
60'	9.671609		9.945935		9.725674		10.274326	0'
117°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	62°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

28°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	151°
0′	9.671609	3.97	9.945935	1.12	9.725674	5.08	10.274326	60′
1	.671847	3.95	.945868	1.13	.725979	5.08	.274021	59
2	.672084	3.95	.945800	1.12	.726284	5.07	.273716	58
3	.672321	3.95	.945733	1.12	.726588	5.07	.273412	57
4	.672558	3.95	.945666	1.13	.726892	5.08	.273108	56
5	.672795	3.95	.945598	1.12	.727197	5.07	.272803	55
6	.673032	3.93	.945531	1.12	.727501	5.07	.272499	54
7	.673268	3.95	.945464	1.13	.727805	5.07	.272195	53
8	.673505	3.93	.945396	1.13	.728109	5.05	.271891	52
9	.673741	3.93	.945328	1.12	.728412	5.07	.271588	51
10	.673977	3.93	.945261	1.13	.728716	5.07	.271284	50
11	9.674213	3.92	9.945193	1.13	9.729020	5.05	10.270980	49
12	.674448	3.93	.945125	1.12	.729323	5.05	.270677	48
13	.674684	3.92	.945058	1.13	.729626	5.05	.270374	47
14	.674919	3.93	.944990	1.13	.729929	5.07	.270071	46
15	.675155	3.92	.944922	1.13	.730233	5.03	.269767	45
16	.675390	3.90	.944854	1.13	.730535	5.05	.269465	44
17	.675624	3.92	.944786	1.13	.730838	5.05	.269162	43
18	.675859	3.92	.944718	1.13	.731141	5.05	.268859	42
19	.676094	3.90	.944650	1.13	.731444	5.03	.268556	41
20	.676328	3.90	.944582	1.13	.731746	5.03	.268254	40
21	9.676562	3.90	9.944514	1.13	9.732048	5.05	10.267952	39
22	.676796	3.90	.944446	1.15	.732351	5.03	.267649	38
23	.677030	3.90	.944377	1.13	.732653	5.03	.267347	37
24	.677264	3.90	.944309	1.13	.732955	5.03	.267045	36
25	.677498	3.88	.944241	1.15	.733257	5.02	.266743	35
26	.677731	3.88	.944172	1.13	.733558	5.03	.266442	34
27	.677964	3.88	.944104	1.13	.733860	5.03	.266140	33
28	.678197	3.88	.944036	1.15	.734162	5.02	.265838	32
29	.678430	3.88	.943967	1.13	.734463	5.02	.265537	31
30	.678663	3.87	.943899	1.15	.734764	5.03	.265236	30
31	9.678895	3.88	9.943830	1.15	9.735066	5.02	10.264934	29
32	.679128	3.87	.943761	1.13	.735367	5.02	.264633	28
33	.679360	3.87	.943693	1.15	.735668	5.02	.264332	27
34	.679592	3.87	.943624	1.15	.735969	5.00	.264031	26
35	.679824	3.87	.943555	1.15	.736269	5.02	.263731	25
36	.680056	3.87	.943486	1.15	.736570	5.00	.263430	24
37	.680288	3.85	.943417	1.15	.736870	5.02	.263130	23
38	.680519	3.85	.943348	1.15	.737171	5.00	.262829	22
39	.680750	3.87	.943279	1.15	.737471	5.00	.262529	21
40	.680982	3.85	.943210	1.15	.737771	5.00	.262229	20
41	9.681213	3.83	9.943141	1.15	9.738071	5.00	10.261929	19
42	.681443	3.85	.943072	1.15	.738371	5.00	.261629	18
43	.681674	3.85	.943003	1.15	.738671	5.00	.261329	17
44	.681905	3.83	.942934	1.17	.738971	5.00	.261029	16
45	.682135	3.83	.942864	1.15	.739271	4.98	.260729	15
46	.682365	3.83	.942795	1.15	.739570	5.00	.260430	14
47	.682595	3.83	.942726	1.17	.739870	4.98	.260130	13
48	.682825	3.83	.942656	1.15	.740169	4.98	.259831	12
49	.683055	3.82	.942587	1.17	.740468	4.98	.259532	11
50	.683284	3.83	.942517	1.15	.740767	4.98	.259233	10
51	9.683514	3.82	9.942448	1.17	9.741066	4.98	10.258934	9
52	.683743	3.82	.942378	1.17	.741365	4.98	.258635	8
53	.683972	3.82	.942308	1.15	.741664	4.97	.258336	7
54	.684201	3.82	.942239	1.17	.741962	4.98	.258038	6
55	.684430	3.80	.942169	1.17	.742261	4.97	.257739	5
56	.684658	3.82	.942099	1.17	.742559	4.98	.257441	4
57	.684887	3.80	.942029	1.17	.742858	4.97	.257142	3
58	.685115	3.80	.941959	1.17	.743156	4.97	.256844	2
59	.685343	3.80	.941889	1.17	.743454	4.97	.256546	1
60′	9.685571		9.941819		9.743752		10.256248	0′
118°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	61°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

29°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	150°
0′	9.685571	3.80	9.941819	1.17	9.743752	4.97	10.256248	60′
1	.685799	3.80	.941749	1.17	.744050	4.97	.255950	59
2	.686027	3.78	.941679	1.17	.744348	4.95	.255652	58
3	.686254	3.80	.941609	1.17	.744645	4.97	.255355	57
4	.686482	3.78	.941539	1.17	.744943	4.95	.255057	56
5	.686709	3.78	.941469	1.18	.745240	4.97	.254760	55
6	.686936	3.78	.941398	1.17	.745538	4.95	.254462	54
7	.687163	3.77	.941328	1.17	.745835	4.95	.254165	53
8	.687389	3.78	.941258	1.18	.746132	4.95	.253868	52
9	.687616	3.78	.941187	1.17	.746429	4.95	.253571	51
10	.687843	3.77	.941117	1.18	.746726	4.95	.253274	50
11	9.688069	3.77	9.941046	1.18	9.747023	4.93	10.252977	49
12	.688295	3.77	.940975	1.17	.747319	4.95	.252681	48
13	.688521	3.77	.940905	1.18	.747616	4.95	.252384	47
14	.688747	3.75	.940834	1.18	.747913	4.93	.252087	46
15	.688972	3.77	.940763	1.17	.748209	4.93	.251791	45
16	.689198	3.75	.940693	1.18	.748505	4.93	.251495	44
17	.689423	3.75	.940622	1.18	.748801	4.93	.251199	43
18	.689648	3.75	.940551	1.18	.749097	4.93	.250903	42
19	.689873	3.75	.940480	1.18	.749393	4.93	.250607	41
20	.690098	3.75	.940409	1.18	.749689	4.93	.250311	40
21	9.690323	3.75	9.940338	1.18	9.749985	4.93	10.250015	39
22	.690548	3.73	.940267	1.18	.750281	4.92	.249719	38
23	.690772	3.73	.940196	1.18	.750576	4.93	.249424	37
24	.690996	3.73	.940125	1.18	.750872	4.92	.249128	36
25	.691220	3.73	.940054	1.20	.751167	4.92	.248833	35
26	.691444	3.73	.939982	1.18	.751462	4.92	.248538	34
27	.691668	3.73	.939911	1.18	.751757	4.92	.248243	33
28	.691892	3.72	.939840	1.20	.752052	4.92	.247948	32
29	.692115	3.73	.939768	1.18	.752347	4.92	.247653	31
30	.692339	3.72	.939697	1.20	.752642	4.92	.247358	30
31	9.692562	3.72	9.939625	1.18	9.752937	4.90	10.247063	29
32	.692785	3.72	.939554	1.20	.753231	4.92	.246769	28
33	.693008	3.72	.939482	1.20	.753526	4.90	.246474	27
34	.693231	3.70	.939410	1.18	.753820	4.92	.246180	26
35	.693453	3.72	.939339	1.20	.754115	4.90	.245885	25
36	.693676	3.70	.939267	1.20	.754409	4.90	.245591	24
37	.693898	3.70	.939195	1.20	.754703	4.90	.245297	23
38	.694120	3.70	.939123	1.18	.754997	4.90	.245003	22
39	.694342	3.70	.939052	1.20	.755291	4.90	.244709	21
40	.694564	3.70	.938980	1.20	.755585	4.88	.244415	20
41	9.694786	3.68	9.938908	1.20	9.755878	4.90	10.244122	19
42	.695007	3.70	.938836	1.22	.756172	4.88	.243828	18
43	.695229	3.68	.938763	1.20	.756465	4.90	.243535	17
44	.695450	3.68	.938691	1.20	.756759	4.88	.243241	16
45	.695671	3.68	.938619	1.20	.757052	4.88	.242948	15
46	.695892	3.68	.938547	1.20	.757345	4.88	.242655	14
47	.696113	3.68	.938475	1.22	.757638	4.88	.242362	13
48	.696334	3.67	.938402	1.20	.757931	4.88	.242069	12
49	.696554	3.68	.938330	1.20	.758224	4.88	.241776	11
50	.696775	3.67	.938258	1.22	.758517	4.88	.241483	10
51	9.696995	3.67	9.938185	1.20	9.758810	4.87	10.241190	9
52	.697215	3.67	.938113	1.22	.759102	4.88	.240898	8
53	.697435	3.65	.938040	1.22	.759395	4.87	.240605	7
54	.697654	3.67	.937967	1.20	.759687	4.87	.240313	6
55	.697874	3.67	.937895	1.22	.759979	4.88	.240021	5
56	.698094	3.65	.937822	1.22	.760272	4.87	.239728	4
57	.698313	3.65	.937749	1.22	.760564	4.87	.239436	3
58	.698532	3.65	.937676	1.20	.760856	4.87	.239144	2
59	.698751	3.65	.937604	1.22	.761148	4.85	.238852	1
60′	9.698970		9.937531		9.761439		10.238561	0′
119°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	60°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

30°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	149°
0′	9.698970	3.65	9.937531	1.22	9.761439	4.87	10.238561	60
1	.699189	3.63	.937458	1.22	.761731	4.87	.238269	59
2	.699407	3.65	.937385	1.22	.762023	4.85	.237977	58
3	.699626	3.63	.937312	1.23	.762314	4.87	.237686	57
4	.699844	3.63	.937238	1.22	.762606	4.85	.237394	56
5	.700062	3.63	.937165	1.22	.762897	4.85	.237103	55
6	.700280	3.63	.937092	1.22	.763188	4.85	.236812	54
7	.700498	3.63	.937019	1.22	.763479	4.85	.236521	53
8	.700716	3.62	.936946	1.23	.763770	4.85	.236230	52
9	.700933	3.63	.936872	1.22	.764061	4.85	.235939	51
10	.701151	3.62	.936799	1.23	.764352	4.85	.235648	50
11	9.701368	3.62	9.936725	1.22	9.764643	4.83	10.235357	49
12	.701585	3.62	.936652	1.23	.764933	4.85	.235067	48
13	.701802	3.62	.936578	1.22	.765224	4.83	.234776	47
14	.702019	3.62	.936505	1.23	.765514	4.85	.234486	46
15	.702236	3.60	.936431	1.23	.765805	4.83	.234195	45
16	.702452	3.62	.936357	1.22	.766095	4.83	.233905	44
17	.702669	3.60	.936284	1.23	.766385	4.83	.233615	43
18	.702885	3.60	.936210	1.23	.766675	4.83	.233325	42
19	.703101	3.60	.936136	1.23	.766965	4.83	.233035	41
20	.703317	3.60	.936062	1.23	.767255	4.83	.232745	40
21	9.703533	3.60	9.935988	1.23	9.767545	4.82	10.232455	39
22	.703749	3.58	.935914	1.23	.767834	4.83	.232166	38
23	.703964	3.58	.935840	1.23	.768124	4.83	.231876	37
24	.704179	3.60	.935766	1.23	.768414	4.82	.231586	36
25	.704395	3.58	.935692	1.23	.768703	4.82	.231297	35
26	.704610	3.58	.935618	1.25	.768992	4.82	.231008	34
27	.704825	3.58	.935543	1.23	.769281	4.83	.230719	33
28	.705040	3.57	.935469	1.23	.769571	4.82	.230429	32
29	.705254	3.58	.935395	1.25	.769860	4.80	.230140	31
30	.705469	3.57	.935320	1.23	.770148	4.82	.229852	30
31	9.705683	3.58	9.935246	1.25	9.770437	4.82	10.229563	29
32	.705898	3.57	.935171	1.23	.770726	4.82	.229274	28
33	.706112	3.57	.935097	1.25	.771015	4.80	.228985	27
34	.706326	3.55	.935022	1.23	.771303	4.82	.228697	26
35	.706539	3.57	.934948	1.25	.771592	4.80	.228408	25
36	.706753	3.57	.934873	1.25	.771880	4.80	.228120	24
37	.706967	3.55	.934798	1.25	.772168	4.82	.227832	23
38	.707180	3.55	.934723	1.23	.772457	4.80	.227543	22
39	.707393	3.55	.934649	1.25	.772745	4.80	.227255	21
40	.707606	3.55	.934574	1.25	.773033	4.80	.226967	20
41	9.707819	3.55	9.934499	1.25	9.773321	4.78	10.226679	19
42	.708032	3.55	.934424	1.25	.773608	4.80	.226392	18
43	.708245	3.55	.934349	1.25	.773896	4.80	.226104	17
44	.708458	3.53	.934274	1.25	.774184	4.78	.225816	16
45	.708670	3.53	.934199	1.27	.774471	4.80	.225529	15
46	.708882	3.53	.934123	1.25	.774759	4.78	.225241	14
47	.709094	3.53	.934048	1.25	.775046	4.78	.224954	13
48	.709306	3.53	.933973	1.25	.775333	4.80	.224667	12
49	.709518	3.53	.933898	1.27	.775621	4.78	.224379	11
50	.709730	3.52	.933822	1.25	.775908	4.78	.224092	10
51	9.709941	3.53	9.933747	1.27	9.776195	4.78	10.223805	9
52	.710153	3.52	.933671	1.25	.776482	4.77	.223518	8
53	.710364	3.52	.933596	1.27	.776768	4.78	.223232	7
54	.710575	3.52	.933520	1.25	.777055	4.78	.222945	6
55	.710786	3.52	.933445	1.27	.777342	4.77	.222658	5
56	.710997	3.52	.933369	1.27	.777628	4.78	.222372	4
57	.711208	3.52	.933293	1.27	.777915	4.77	.222085	3
58	.711419	3.50	.933217	1.27	.778201	4.78	.221799	2
59	.711629	3.50	.933141	1.25	.778488	4.77	.221512	1
60′	9.711839		9.933066		9.778774		10.221226	0′
120°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	59°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

31°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	148°
0′	9.711839	3.52	9.933066	1.27	9.778774	4.77	10.221226	60′
1	.712050	3.50	.932990	1.27	.779060	4.77	.220940	59
2	.712260	3.48	.932914	1.27	.779346	4.77	.220654	58
3	.712469	3.50	.932838	1.27	.779632	4.77	.220368	57
4	.712679	3.50	.932762	1.28	.779918	4.75	.220082	56
5	.712889	3.48	.932685	1.27	.780203	4.77	.219797	55
6	.713098	3.50	.932609	1.27	.780489	4.77	.219511	54
7	.713308	3.48	.932533	1.27	.780775	4.75	.219225	53
8	.713517	3.48	.932457	1.28	.781060	4.77	.218940	52
9	.713726	3.48	.932380	1.27	.781346	4.75	.218654	51
10	.713935	3.48	.932304	1.27	.781631	4.75	.218369	50
11	9.714144	3.47	9.932228	1.28	9.781916	4.75	10.218084	49
12	.714352	3.48	.932151	1.27	.782201	4.75	.217799	48
13	.714561	3.47	.932075	1.28	.782486	4.75	.217514	47
14	.714769	3.48	.931998	1.28	.782771	4.75	.217229	46
15	.714978	3.47	.931921	1.27	.783056	4.75	.216944	45
16	.715186	3.47	.931845	1.28	.783341	4.75	.216659	44
17	.715394	3.47	.931768	1.28	.783626	4.73	.216374	43
18	.715602	3.45	.931691	1.28	.783910	4.75	.216090	42
19	.715809	3.47	.931614	1.28	.784195	4.73	.215805	41
20	.716017	3.45	.931537	1.28	.784479	4.75	.215521	40
21	9.716224	3.47	9.931460	1.28	9.784764	4.73	10.215236	39
22	.716432	3.45	.931383	1.28	.785048	4.73	.214952	38
23	.716639	3.45	.931306	1.28	.785332	4.73	.214668	37
24	.716846	3.45	.931229	1.28	.785616	4.73	.214384	36
25	.717053	3.43	.931152	1.28	.785900	4.73	.214100	35
26	.717259	3.45	.931075	1.28	.786184	4.73	.213816	34
27	717466	3.45	.930998	1.28	.786468	4.73	.213532	33
28	.717673	3.43	.930921	1.30	.786752	4.73	.213248	32
29	.717879	3.43	.930843	1.28	.787036	4.72	.212964	31
30	.718085	3.43	.930766	1.30	.787319	4.73	.212681	30
31	9.718291	3.43	9.930688	1.28	9.787603	4.72	10.212397	29
32	.718497	3.43	.930611	1.30	.787886	4.73	.212114	28
33	.718703	3.43	.930533	1.28	.788170	4.72	.211830	27
34	.718909	3.42	.930456	1.30	.788453	4.72	.211547	26
35	.719114	3.43	.930378	1.30	.788736	4.72	.211264	25
36	.719320	3.42	.930300	1.28	.789019	4.72	.210981	24
37	.719525	3.42	.930223	1.30	.789302	4.72	.210698	23
38	.719730	3.42	.930145	1.30	.789585	4.72	.210415	22
39	.719935	3.42	.930067	1.30	.789868	4.72	.210132	21
40	.720140	3.42	.929989	1.30	.790151	4.72	.209849	20
41	9.720345	3.40	9.929911	1.30	9.790434	4.70	10.209566	19
42	.720549	3.42	.929833	1.30	.790716	4.72	.209284	18
43	.720754	3.40	.929755	1.30	.790999	4.70	.209001	17
44	.720958	3.40	.929677	1.30	.791281	4.70	.208719	16
45	.721162	3.40	.929599	1.30	.791563	4.72	.208437	15
46	.721366	3.40	.929521	1.32	.791846	4.70	.208154	14
47	.721570	3.40	.929442	1.30	.792128	4.70	.207872	13
48	.721774	3.40	.929364	1.30	.792410	4.70	.207590	12
49	.721978	3.38	.929286	1.32	.792692	4.70	.207308	11
50	.722181	3.40	.929207	1.30	.792974	4.70	.207026	10
51	9.722385	3.38	9.929129	1.32	9.793256	4.70	10.206744	9
52	.722588	3.38	.929050	1.30	.793538	4.68	.206462	8
53	.722791	3.38	.928972	1.32	.793819	4.70	.206181	7
54	.722994	3.38	.928893	1.30	.794101	4.70	.205899	6
55	.723197	3.38	.928815	1.32	.794383	4.68	..205617	5
56	.723400	3.38	.928736	1.32	.794664	4.70	.205336	4
57	.723603	3.37	.928657	1.32	.794946	4.68	.205054	3
58	.723805	3.37	.928578	1.32	.795227	4.68	.204773	2
59	.724007	3.38	.928499	1.32	.795508	4.68	.204492	1
60′	9.724210		9.928420		9.795789		10.204211	0′
121°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	58°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

32°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	147°
0'	9.724210	3.37	9.928420	1.30	9.795789	4.68	10.204211	60
1	.724412	3.37	.928342	1.32	.796070	4.68	.203930	59
2	.724614	3.37	.928263	1.33	.796351	4.68	.203649	58
3	.724816	3.35	.928183	1.32	.796632	4.68	.203368	57
4	.725017	3.37	.928104	1.32	.796913	4.68	.203087	56
5	.725219	3.35	.928025	1.32	.797194	4.67	.202806	55
6	.725420	3.37	.927946	1.32	.797474	4.68	.202526	54
7	.725622	3.35	.927867	1.33	.797755	4.68	.202245	53
8	.725823	3.35	.927787	1.32	.798036	4.67	.201964	52
9	.726024	3.35	.927708	1.32	.798316	4.67	.201684	51
10	.726225	3.35	.927629	1.33	.798596	4.68	.201404	50
11	9.726426	3.33	9.927549	1.32	9.798877	4.67	10.201123	49
12	.726626	3.35	.927470	1.33	.799157	4.67	.200843	48
13	.726827	3.33	.927390	1.33	.799437	4.67	.200563	47
14	.727027	3.35	.927310	1.32	.799717	4.67	.200283	46
15	.727228	3.33	.927231	1.33	.799997	4.67	.200003	45
16	.727428	3 33	.927151	1.33	.800277	4.67	.199723	44
17	.727628	3.33	.927071	1.33	.800557	4.65	.199443	43
18	.727828	3.32	.926991	1.33	.800836	4.67	.199164	42
19	.728027	3.33	.926911	1.33	.801116	4.67	.198884	41
20	.728227	3.33	.926831	1.33	.801396	4.65	.198604	40
21	9.728427	3.32	9.926751	1.33	9.801675	4.67	10.198325	39
22	.728626	3.32	.926671	1.33	.801955	4 65	.198045	38
23	.728825	3.32	.926591	1.33	.802234	4.65	.197766	37
24	.729024	3.32	.926511	1.33	.802513	4.65	.197487	36
25	.729223	3.32	.926431	1.33	.802792	4.67	.197208	35
26	.729422	3.32	.926351	1.35	.803072	4.65	.196928	34
27	.729621	3.32	.926270	1.33	.803351	4.65	.196649	33
28	.729820	3.30	.926190	1.33	.803630	4.65	.196370	32
29	.730018	3.32	.926110	1.35	.803909	4.63	.196091	31
30	.730217	3.30	.926029	1.33	.804187	4.65	.195813	30
31	9.730415	3.30	9.925949	1.35	9.804466	4.65	10.195534	29
32	.730613	3.30	.925868	1.33	.804745	4.63	.195255	28
33	.730811	3.30	.925788	1.35	.805023	4 65	.194977	27
34	.731009	3.28	.925707	1.35	.805302	4.63	.194698	26
35	.731206	3.30	.925626	1.35	.805580	4.65	.194420	25
36	.731404	3.30	.925545	1.33	.805859	4 63	.194141	24
37	.731602	3.28	.925465	1.35	.806137	4.63	.193863	23
38	.731799	3.28	.925384	1.35	.806415	4.63	.193585	22
39	.731996	3.28	.925303	1.35	.806693	4.63	.193307	21
40	.732193	3.28	.925222	1.35	.806971	4.63	.193029	20
41	9.732390	3.28	9.925141	1.35	9.807249	4.63	10.192751	19
42	.732587	3.28	.925060	1.35	.807527	4.63	.192473	18
43	.732784	3.27	.924979	1·37	.807805	4.63	.192195	17
44	.732980	3.28	.924897	1.35	.808083	4.63	.191917	16
45	.733177	3.27	.924816	1.35	.808361	4.62	.191639	15
46	.733373	3.27	.924735	1.35	.808638	4.63	.191362	14
47	.733569	3.27	.924654	1.37	.808916	4.62	.191084	13
48	.733765	3.27	.924572	1.35	.809193	4.63	.190807	12
49	.733961	3.27	.924491	1.37	.809471	4.62	.190529	11
50	.734157	3.27	.924409	1.35	.809748	4.62	.190252	10
51	9.734353	3.27	9.924328	1.37	9.810025	4.62	10.189975	9
52	.734549	3.25	.924246	1.37	.810302	4.63	.189698	8
53	.734744	3.25	.924164	1.35	.810580	4.62	.189420	7
54	.734939	3.27	.924083	1.37	.810857	4.62	.189143	6
55	.735135	3.25	.924001	1.37	.811134	4.60	.188866	5
56	.735330	3.25	.923919	1.37	.811410	4.62	.188590	4
57	.735525	3.23	.923837	1 37	.811687	4.62	.188313	3
58	.735719	3.25	.923755	1.37	.811964	4.62	.188036	2
59	.735914	3.25	.923673	1.37	.812241	4.60	.187759	1
60'	9.736109		9.923591		9.812517		10.187483	0
122°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	57°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

33°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	146°
0′	9.736109	3.23	9.923591	1.37	9.812517	4.62	10.187483	60′
1	.736303	3.25	.923509	1.37	.812794	4.60	.187206	59
2	.736498	3.23	.923427	1.37	.813070	4.62	.186930	58
3	.736692	3.23	.923345	1.37	.813347	4.60	.186653	57
4	.736886	3.23	.923263	1.37	.813623	4.60	.186377	56
5	.737080	3.23	.923181	1.38	.813899	4.62	.186101	55
6	.737274	3.22	.923098	1.37	.814176	4.60	.185824	54
7	.737467	3.23	.923016	1.38	.814452	4.60	.185548	53
8	.737661	3.23	.922933	1.37	.814728	4.60	.185272	52
9	.737855	3.22	.922851	1.38	.815004	4.60	.184996	51
10	.738048	3.22	.922768	1.37	.815280	4.58	.184720	50
11	9.738241	3.22	9.922686	1.38	9.815555	4.60	10.184445	49
12	.738434	3.22	.922603	1.38	.815831	4.60	.184169	48
13	.738627	3.22	.922520	1.37	.816107	4.58	.183893	47
14	.738820	3.22	.922438	1.38	.816382	4.60	.183618	46
15	.739013	3.22	.922355	1.38	.816658	4.58	.183342	45
16	.739206	3.20	.922272	1.38	.816933	4.60	.183067	44
17	.739398	3.20	.922189	1.38	.817209	4.58	.182791	43
18	.739590	3.22	.922106	1.38	.817484	4.58	.182516	42
19	.739783	3.20	.922023	1.38	.817759	4.60	.182241	41
20	.739975	3.20	.921940	1.38	.818035	4.58	.181965	40
21	9.740167	3.20	9.921857	1.38	9.818310	4.58	10.181690	39
22	.740359	3.18	.921774	1.38	.818585	4.58	.181415	38
23	.740550	3.20	.921691	1.40	.818860	4.58	.181140	37
24	.740742	3.20	.921607	1.38	.819135	4.58	.180865	36
25	.740934	3.18	.921524	1.38	.819410	4.57	.180590	35
26	.741125	3.18	.921441	1.40	.819684	4.58	.180316	34
27	.741316	3.20	.921357	1.38	.819959	4.58	.180041	33
28	.741508	3.18	.921274	1.40	.820234	4.57	.179766	32
29	.741699	3.17	.921190	1.38	.820508	4.58	.179492	31
30	.741889	3.18	.921107	1.40	.820783	4.57	.179217	30
31	9.742080	3.18	9.921023	1.40	9.821057	4.58	10.178943	29
32	.742271	3.18	.920939	1.38	.821332	4.57	.178668	28
33	.742462	3.17	.920856	1.40	.821606	4.57	.178394	27
34	.742652	3.17	.920772	1.40	.821880	4.57	.178120	26
35	.742842	3.18	.920688	1.40	.822154	4.58	.177846	25
36	.743033	3.17	.920604	1.40	.822429	4.57	.177571	24
37	.743223	3.17	.920520	1.40	.822703	4.57	.177297	23
38	.743413	3.15	.920436	1.40	.822977	4.57	.177023	22
39	.743602	3.17	.920352	1.40	.823251	4.55	.176749	21
40	.743792	3.17	.920268	1.40	.823524	4.57	.176476	20
41	9.743982	3.15	9.920184	1.42	9.823798	4.57	10.176202	19
42	.744171	3.17	.920099	1.40	.824072	4.55	.175928	18
43	.744361	3.15	.920015	1.40	.824345	4.57	.175655	17
44	.744550	3.15	.919931	1.42	.824619	4.57	.175381	16
45	.744739	3.15	.919846	1.40	.824893	4.55	.175107	15
46	.744928	3.15	.919762	1.42	.825166	4.55	.174834	14
47	.745117	3.15	.919677	1.40	.825439	4.57	.174561	13
48	.745306	3.13	.919593	1.42	.825713	4.55	.174287	12
49	.745494	3.15	.919508	1.40	.825986	4.55	.174014	11
50	.745683	3.13	.919424	1.42	.826259	4.55	.173741	10
51	9.745871	3.15	9.919339	1.42	9.826532	4.55	10.173468	9
52	.746060	3.13	.919254	1.42	.826805	4.55	.173195	8
53	.746248	3.13	.919169	1.40	.827078	4.55	.172922	7
54	.746436	3.13	.919085	1.42	.827351	4.55	.172649	6
55	.746624	3.13	.919000	1.42	.827624	4.55	.172376	5
56	.746812	3.12	.918915	1.42	.827897	4.55	.172103	4
57	.746999	3.13	.918830	1.42	.828170	4.53	.171830	3
58	.747187	3.12	.918745	1.43	.828442	4.55	.171558	2
59	.747374	3.13	.918659	1.42	.828715	4.53	.171285	1
60′	9.747562		9.918574		9.828987		10.171013	0′
123°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	56°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

34°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	145°
0′	9.747562	3.12	9.918574	1.42	9.828987	4.55	10.171013	60
1	.747749	3.12	.918489	1.42	.829260	4.53	.170740	59
2	.747936	3.12	.918404	1.43	.829532	4.55	.170468	58
3	.748123	3.12	.918318	1.42	.829805	4.53	.170195	57
4	.748310	3.12	.918233	1.43	.830077	4.53	.169923	56
5	.748497	3.10	.918147	1.42	.830349	4.53	.169651	55
6	.748683	3.12	.918062	1.43	.830621	4.53	.169379	54
7	.748870	3.10	.917976	1.42	.830893	4.53	.169107	53
8	.749056	3.12	.917891	1.43	.831165	4.53	.168835	52
9	.749243	3.10	.917805	1.43	.831437	4.53	.168563	51
10	.749429	3.10	.917719	1.42	.831709	4.53	.168291	50
11	9.749615	3.10	9.917634	1.43	9.831981	4.53	10.168019	49
12	.749801	3.10	.917548	1.43	.832253	4.53	.167747	48
13	.749987	3.08	.917462	1.43	.832525	4.52	.167475	47
14	.750172	3.10	.917376	1.43	.832796	4.53	.167204	46
15	.750358	3.08	.917290	1.43	.833068	4.52	.166932	45
16	.750543	3.10	.917204	1.43	.833339	4.53	.166661	44
17	.750729	3.08	.917118	1.43	.833611	4.52	.166389	43
18	.750914	3.08	.917032	1.43	.833882	4.53	.166118	42
19	.751099	3.08	.916946	1.45	.834154	4.52	.165846	41
20	.751284	3.08	.916859	1.43	.834425	4.52	.165575	40
21	9.751469	3.08	9.916773	1.43	9.834696	4.52	10.165304	39
22	.751654	3.08	.916687	1.45	.834967	4.52	.165033	38
23	.751839	3.07	.916600	1.43	.835238	4.52	.164762	37
24	.752023	3.08	.916514	1.45	.835509	4.52	.164491	36
25	.752208	3.07	.916427	1.43	.835780	4.52	.164220	35
26	.752392	3.07	.916341	1.45	.836051	4.52	.163949	34
27	.752576	3.07	.916254	1.45	.836322	4.52	.163678	33
28	.752760	3.07	.916167	1.43	.836593	4.52	.163407	32
29	.752944	3.07	.916081	1.45	.836864	4.50	.163136	31
30	.753128	3.07	.915994	1.45	.837134	4.52	.162866	30
31	9.753312	3.05	9.915907	1.45	9.837405	4.50	10.162595	29
32	.753495	3.07	.915820	1.45	.837675	4.52	.162325	28
33	.753679	3.05	.915733	1.45	.837946	4.50	.162054	27
34	.753862	3.07	.915646	1.45	.838216	4.52	.161784	26
35	.754046	3.05	.915559	1.45	.838487	4.50	.161513	25
36	.754229	3.05	.915472	1.45	.838757	4.50	.161243	24
37	.754412	3.05	.915385	1.47	.839027	4.50	.160973	23
38	.754595	3.05	.915297	1.45	.839297	4.52	.160703	22
39	.754778	3.03	.915210	1.45	.839568	4.50	.160432	21
40	.754960	3.05	.915123	1.47	.839838	4.50	.160162	20
41	9.755143	3.05	9.915035	1.45	9.840108	4.50	10.159892	19
42	.755326	3.03	.914948	1.47	.840378	4.50	.159622	18
43	.755508	3.03	.914860	1.45	.840648	4.48	.159352	17
44	.755690	3.03	.914773	1.47	.840917	4.50	.159083	16
45	.755872	3.03	.914685	1.45	.841187	4.50	.158813	15
46	.756054	3.03	.914598	1.47	.841457	4.50	.158543	14
47	.756236	3.03	.914510	1.47	.841727	4.48	.158273	13
48	.756418	3.03	.914422	1.47	.841996	4.50	.158004	12
49	.756600	3.03	.914334	1.47	.842266	4.48	.157734	11
50	.756782	3.02	.914246	1.47	.842535	4.50	.157465	10
51	9.756963	3.02	9.914158	1.47	9.842805	4.48	10.157195	9
52	.757144	3.03	.914070	1.47	.843074	4.48	.156926	8
53	.757326	3.02	.913982	1.47	.843343	4.48	.156657	7
54	.757507	3.02	.913894	1.47	.843612	4.50	.156388	6
55	.757688	3.02	.913806	1.47	.843882	4.48	.156118	5
56	.757869	3.02	.913718	1.47	.844151	4.48	.155849	4
57	.758050	3.00	.913630	1.48	.844420	4.48	.155580	3
58	.758230	3.02	.913541	1.47	.844689	4.48	.155311	2
59	.758411	3.00	.913453	1.47	.844958	4.48	.155042	1
60′	9.758591		9.913365		9.845227		10.154773	0′
124°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	55°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

35°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	144°
0′	9.758591	3.02	9.913365	1.48	9.845227	4.48	10.154773	60′
1	.758772	3.00	.913276	1.48	.845496	4.47	.154504	59
2	.758952	3.00	.913187	1.47	.845764	4.48	.154236	58
3	.759132	3.00	.913099	1.48	.846033	4.48	.153967	57
4	.759312	3.00	.913010	1.47	.846302	4.47	.153698	56
5	.759492	3.00	.912922	1.48	.846570	4.48	.153430	55
6	.759672	3.00	.912833	1.48	.846839	4.48	.153161	54
7	.759852	2.98	.912744	1.48	.847108	4.47	.152892	53
8	.760031	3.00	.912655	1.48	.847376	4.47	.152624	52
9	.760211	2.98	.912566	1.48	.847644	4.48	.152356	51
10	.760390	2.98	.912477	1.48	.847913	4.47	.152087	50
11	9.760569	2.98	9.912388	1.48	9.848181	4.47	10.151819	49
12	.760748	2.98	.912299	1.48	.848449	4.47	.151551	48
13	.760927	2.98	.912210	1.48	.848717	4.48	.151283	47
14	.761106	2.98	.912121	1.50	.848986	4.47	.151014	46
15	.761285	2.98	.912031	1.48	.849254	4.47	.150746	45
16	.761464	2.97	.911942	1.48	.849522	4.47	.150478	44
17	.761642	2.98	.911853	1.50	.849790	4.45	.150210	43
18	.761821	2.97	.911763	1.48	.850057	4.47	.149943	42
19	.761999	2.97	.911674	1.50	.850325	4.47	.149675	41
20	.762177	2.98	.911584	1.48	.850593	4.47	.149407	40
21	9.762356	2.97	9.911495	1.50	9.850861	4.47	10.149139	39
22	.762534	2.97	.911405	1.50	.851129	4.45	.148871	38
23	.762712	2.95	.911315	1.48	.851396	4.47	.148604	37
24	.762889	2.97	.911226	1.50	.851664	4.45	.148336	36
25	.763067	2.97	.911136	1.50	.851931	4.47	.148069	35
26	.763245	2.95	.911046	1.50	.852199	4.45	.147801	34
27	.763422	2.97	.910956	1.50	.852466	4.45	.147534	33
28	.763600	2.95	.910866	1.50	.852733	4.47	.147267	32
29	.763777	2.95	.910776	1.50	.853001	4.45	.146999	31
30	.763954	2.95	.910686	1.50	.853268	4.45	.146732	30
31	9.764131	2.95	9.910596	1.50	9.853535	4.45	10.146465	29
32	.764308	2.95	.910506	1.52	.853802	4.45	.146198	28
33	.764485	2.95	.910415	1.50	.854069	4.45	.145931	27
34	.764662	2.93	.910325	1.50	.854336	4.45	.145664	26
35	.764838	2.95	.910235	1.52	.854603	4.45	.145397	25
36	.765015	2.93	.910144	1.50	.854870	4.45	.145130	24
37	.765191	2.93	.910054	1.52	.855137	4.45	.144863	23
38	.765367	2.95	.909963	1.50	.855404	4.45	.144596	22
39	.765544	2.93	.909873	1.52	.855671	4.45	.144329	21
40	.765720	2.93	.909782	1.52	.855938	4.43	.144062	20
41	9.765896	2.93	9.909691	1.50	9.856204	4.45	10.143796	19
42	.766072	2.92	.909601	1.52	.856471	4.43	.143529	18
43	.766247	2.93	.909510	1.52	.856737	4.45	.143263	17
44	.766423	2.92	.909419	1.52	.857004	4.43	.142996	16
45	.766598	2.93	.909328	1.52	.857270	4.45	.142730	15
46	.766774	2.92	.909237	1.52	.857537	4.43	.142463	14
47	.766949	2.92	.909146	1.52	.857803	4.43	.142197	13
48	.767124	2.93	.909055	1.52	.858069	4.45	.141931	12
49	.767300	2.92	.908964	1.52	.858336	4.43	.141664	11
50	.767475	2.90	.908873	1.53	.858602	4.43	.141398	10
51	9.767649	2.92	9.908781	1.52	9.858868	4.43	10.141132	9
52	.767824	2.92	.908690	1.52	.859134	4.43	.140866	8
53	.767999	2.90	.908599	1.53	.859400	4.43	.140600	7
54	.768173	2.92	.908507	1.52	.859666	4.43	.140334	6
55	.768348	2.90	.908416	1.53	.859932	4.43	.140068	5
56	.768522	2.92	.908324	1.52	.860198	4.43	.139802	4
57	.768697	2.90	.908233	1.53	.860464	4.43	.139536	3
58	.768871	2.90	.908141	1.53	.860730	4.42	.139270	2
59	.769045	2.90	.908049	1.52	.860995	4.43	.139005	1
60′	9.769219		9.907958		9.861261		10.138739	0′
125°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	54°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

36°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	143°
0′	9.769219	2.90	9.907958	1.53	9.861261	4.43	10.138739	60′
1	.769393	2.88	.907866	1.53	.861527	4.42	.138473	59
2	.769566	2.90	.907774	1.53	.861792	4.43	.138208	58
3	.769740	2.88	.907682	1.53	.862058	4.42	.137942	57
4	.769913	2.90	.907590	1.53	.862323	4.43	.137677	56
5	.770087	2.88	.907498	1.53	.862589	4.42	.137411	55
6	.770260	2.88	.907406	1.53	.862854	4.42	.137146	54
7	.770433	2.88	.907314	1.53	.863119	4.43	.136881	53
8	.770606	2.88	.907222	1.55	.863385	4.42	.136615	52
9	.770779	2.88	.907129	1.58	.863650	4.42	.136350	51
10	.770952	2.88	.907037	1.53	.863915	4.42	.136085	50
11	9.771125	2.88	9.906945	1.55	9.864180	4.42	10.135820	49
12	.771298	2.87	.906852	1.58	.864445	4.42	.135555	48
13	.771470	2.88	.906760	1.55	.864710	4.42	.135290	47
14	.771643	2.87	.906667	1.58	.864975	4.42	.135025	46
15	.771815	2.87	.906575	1.55	.865240	4.42	.134760	45
16	.771987	2.87	.906482	1.55	.865505	4.42	.134495	44
17	.772159	2.87	.906389	1.55	.865770	4.42	.134230	43
18	.772331	2.87	.906296	1.53	.866035	4.42	.133965	42
19	.772503	2 87	.906204	1.55	.866300	4.40	.133700	41
20	.772675	2.87	.906111	1.55	.866564	4.42	.133436	40
21	9.772847	2.85	9.906018	1.55	9.866829	4.42	10.133171	39
22	.773018	2.87	.905925	1.55	.867094	4.40	.132906	38
23	.773190	2.85	.905832	1.55	.867358	4.42	.132642	37
24	.773361	2.87	.905739	1.57	.867623	4.40	.132377	36
25	.773533	2.85	.905645	1.55	.867887	4.42	.132113	35
26	.773704	2.85	.905552	1.55	.868152	4.40	.131848	34
27	.773875	2.85	.905459	1.55	.868416	4.40	.131584	33
28	.774046	2.85	.905366	1.57	.868680	4 42	.131320	32
29	.774217	2.85	.905272	1.55	.868945	4.40	.131055	31
30	.774388	2.83	.905179	1.57	.869209	4.40	.130791	30
31	9.774558	2.85	9.905085	1.55	9.869473	4.40	10.130527	29
32	.774729	2.83	.904992	1.57	.869737	4.40	.130263	28
33	.774899	2.85	.904898	1.57	.870001	4.40	.129999	27
34	.775070	2.83	.904804	1.55	.870265	4.40	.129735	26
35	.775240	2.83	.904711	1.57	.870529	4.40	.129471	25
36	.775410	2.83	.904617	1.57	.870793	4.40	.129207	24
37	.775580	2.83	.904523	1.57	.871057	4.40	.128943	23
38	.775750	2.83	.904429	1.57	.871321	4.40	.128679	22
39	.775920	2.83	.904335	1.57	.871585	4.40	.128415	21
40	.776090	2.82	.904241	1.57	.871849	4.38	.128151	20
41	9.776259	2.83	9.904147	1.57	9.872112	4.40	10.127888	19
42	.776429	2.82	.904053	1.57	.872376	4.40	.127624	18
43	.776598	2.83	.903959	1.58	.872640	4.38	.127360	17
44	.776768	2.82	.903864	1.57	.872903	4.40	.127097	16
45	.776937	2.82	.903770	1.57	.873167	4.38	.126833	15
46	.777106	2.82	.903676	1.58	.873430	4.40	.126570	14
47	.777275	2.82	.903581	1.57	.873694	4.38	.126306	13
48	.777444	2.82	.903487	1.58	.873957	4.38	.126043	12
49	.777613	2.80	.903392	1.57	.874220	4.40	.125780	11
50	.777781	2.82	.903298	1.58	.874484	4.38	.125516	10
51	9.777950	2.82	9.903203	1.58	9.874747	4.38	10.125253	9
52	.778119	2.80	.903108	1.57	.875010	4.38	.124990	8
53	.778287	2.80	.903014	1.58	.875273	4.40	.124727	7
54	.778455	2.82	.902919	1.58	.875537	4.38	.124463	6
55	.778624	2.80	.902824	1.58	.875800	4.38	.124200	5
56	.778792	2.80	.902729	1.58	.876063	4.38	.123937	4
57	.778960	2.80	.902634	1.58	.876326	4.38	.123674	3
58	.779128	2.78	.902539	1.58	.876589	4.38	.123411	2
59	.779295	2.80	.902444	1.58	.876852	4.37	.123148	1
60′	9.779463		9.902349		9.877114		10.122886	0′
126°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	53°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

37°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	142°
0′	9.779463	2.80	9.902349	1.60	9.877114	4.38	10.122886	60
1	.779631	2.78	.902253	1.58	.877377	4.38	.122623	59
2	.779798	2.80	.902158	1.58	.877640	4.38	.122360	58
3	.779966	2.78	.902063	1.60	.877903	4.37	.122097	57
4	.780133	2.78	.901967	1.58	.878165	4.38	.121835	56
5	.780300	2.78	.901872	1.60	.878428	4.38	.121572	55
6	.780467	2.78	.901776	1.58	.878691	4.37	.121309	54
7	.780634	2.78	.901681	1.60	.878953	4.38	.121047	53
8	.780801	2.78	.901585	1.58	.879216	4.37	.120784	52
9	.780968	2.77	.901490	1.60	.879478	4.38	.120522	51
10	.781134	2.78	.901394	1.60	.879741	4.37	.120259	50
11	9.781301	2.78	9.901298	1.60	9.880003	4.37	10.119997	49
12	.781468	2.77	.901202	1.60	.880265	4.38	.119735	48
13	.781634	2.77	.901106	1.60	.880528	4.37	.119472	47
14	.781800	2.77	.901010	1.60	.880790	4.37	.119210	46
15	.781966	2.77	.900914	1.60	.881052	4.37	.118948	45
16	.782132	2.77	.900818	1.60	.881314	4.38	.118686	44
17	.782298	2.77	.900722	1.60	.881577	4.37	.118423	43
18	.782464	2.77	.900626	1.62	.881839	4.37	.118161	42
19	.782630	2.77	.900529	1.60	.882101	4.37	.117899	41
20	.782796	2.75	.900433	1.60	.882363	4.37	.117637	40
21	9.782961	2.77	9.900337	1.62	9.882625	4.37	10.117375	39
22	.783127	2.75	.900240	1.60	.882887	4.35	.117113	38
23	.783292	2.77	.900144	1.62	.883148	4.37	.116852	37
24	.783458	2.75	.900047	1.60	.883410	4.37	.116590	36
25	.783623	2.75	.899951	1.62	.883672	4.37	.116328	35
26	.783788	2.75	.899854	1.62	.883934	4.37	.116066	34
27	.783953	2.75	.899757	1.62	.884196	4.35	.115804	33
28	.784118	2.73	.899660	1.60	.884457	4.37	.115543	32
29	.784282	2.75	.899564	1.62	.884719	4.35	.115281	31
30	.784447	2.75	.899467	1.62	.884980	4.37	.115020	30
31	9.784612	2.73	9.899370	1.62	9.885242	4.37	10.114758	29
32	.784776	2.75	.899273	1.62	.885504	4.35	.114496	28
33	.784941	2.73	.899176	1.63	.885765	4.35	.114235	27
34	.785105	2.73	.899078	1.62	.886026	4.37	.113974	26
35	.785269	2.73	.898981	1.62	.886288	4.35	.113712	25
36	.785433	2.73	.898884	1.62	.886549	4.37	.113451	24
37	.785597	2.73	.898787	1.63	.886811	4.35	.113189	23
38	.785761	2.73	.898689	1.62	.887072	4.35	.112928	22
39	.785925	2.73	.898592	1.63	.887333	4.35	.112667	21
40	.786089	2.72	.898494	1.62	.887594	4.35	.112406	20
41	9.786252	2.73	9.898397	1.63	9.887855	4.35	10.112145	19
42	.786416	2.72	.898209	1.62	.888116	4.37	.111884	18
43	.786579	2.72	.898202	1.63	.888378	4.35	.111622	17
44	.786742	2.73	.898104	1.63	.888630	4.35	.111361	16
45	.786906	2.72	.898006	1.63	.888900	4.35	.111100	15
46	.787069	2.72	.897908	1.63	.889161	4.33	.110839	14
47	.787232	2.72	.897810	1.63	.889421	4.35	.110579	13
48	.787395	2.70	.897712	1.63	.889682	4.35	.110318	12
49	.787557	2.72	.897614	1.63	.889943	4.35	.110057	11
50	.787720	2.72	.897516	1.63	.890204	4.35	.109796	10
51	9.787883	2.70	9.897418	1.63	9.890465	4.33	10.109535	9
52	.788045	2.72	.897320	1.63	.890725	4.35	.109275	8
53	.788208	2.70	.897222	1.65	.890986	4.35	.109014	7
54	.788370	2.70	.897123	1.63	.891247	4.33	.108753	6
55	.788532	2.70	.897025	1.65	.891507	4.35	.108493	5
56	.788694	2.70	.896926	1.63	.891768	4.33	.108232	4
57	.788856	2.70	.896828	1.65	.892028	4.35	.107972	3
58	.789018	2.70	.896729	1.63	.892289	4.33	.107711	2
59	.789180	2.70	.896631	1.65	.892549	4.35	.107451	1
60′	9.789342		9.896532		9.892810		10.107190	0′
127°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	52°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

38°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	141°
0′	9.789342	2.70	9.896532	1.65	9.892810	4.33	10.107190	60′
1	.789504	2.68	.896433	1.63	.893070	4.35	.106930	59
2	.789665	2.70	.896335	1.65	.893331	4.33	.106669	58
3	.789827	2.68	.896236	1.65	.893591	4.33	.106409	57
4	.789988	2.68	.896137	1.65	.893851	4.33	.106149	56
5	.790149	2.68	.896038	1.65	.894111	4.35	.105889	55
6	.790310	2.68	.895939	1.65	.894372	4.33	.105628	54
7	.790471	2.68	.895840	1.65	.894632	4.33	.105368	53
8	.790632	2.68	.895741	1.67	.894892	4.33	.105108	52
9	.790793	2.68	.895641	1.65	.895152	4.33	.104848	51
10	.790954	2.68	.895542	1.65	.895412	4.33	.104588	50
11	9.791115	2.67	9.895443	1.67	9.895672	4.33	10.104328	49
12	.791275	2.63	.895343	1.65	.895932	4.33	.104068	48
13	.791436	2.67	.895244	1.65	.896192	4.33	.103808	47
14	.791596	2.68	.895145	1.67	.896452	4.33	.103548	46
15	.791757	2.67	.895045	1.67	.896712	4.32	.103288	45
16	.791917	2.67	.894945	1.65	.896971	4.33	.103029	44
17	.792077	2.67	.894846	1.67	.897231	4.33	.102769	43
18	.792237	2.67	.894746	1.67	.897491	4.33	.102509	42
19	.792397	2.67	.894646	1.67	.897751	4.32	.102249	41
20	.792557	2.65	.894546	1.67	.898010	4.33	.101990	40
21	9.792716	2.67	9.894446	1.67	9.898270	4.33	10.101730	39
22	.792876	2.65	.894346	1.67	.898530	4.32	.101470	38
23	.793035	2.67	.894246	1.67	.898789	4.33	.101211	37
24	.793195	2.65	.894146	1.67	.899049	4.32	.100951	36
25	.793354	2.67	.894046	1.67	.899308	4.83	.100692	35
26	.793514	2.65	.893946	1.67	.899568	4.32	.100432	34
27	.793673	2.65	.893846	1.68	.899827	4.33	.100173	33
28	.793832	2.65	.893745	1.67	.900087	4.32	.099913	32
29	.793991	2.65	.893645	1.68	.900346	4.32	.099654	31
30	.794150	2.63	.893544	1.67	.900605	4.32	.099395	30
31	9.794308	2.65	9.893444	1.68	9.900864	4.33	10.099136	29
32	.794467	2.65	.893343	1.67	.901124	4.32	.098876	28
33	.794626	2.63	.893243	1.68	.901383	4.32	.098617	27
34	.794784	2.63	.893142	1.68	.901642	4.32	.098358	26
35	.794942	2.65	.893041	1.68	.901901	4.32	.098099	25
36	.795101	2.63	.892940	1.68	.902160	4.33	.097840	24
37	.795259	2.63	.892839	1.67	.902420	4.32	.097580	23
38	.795417	2.63	.892739	1.68	.902679	4.32	.097321	22
39	.795575	2.63	.892638	1.70	.902938	4.32	.097062	21
40	.795733	2.63	.892536	1.68	.903197	4.32	.096803	20
41	9.795891	2.63	9.892435	1.68	9.903456	4.30	10.096544	19
42	.796049	2.62	.892334	1.68	.903714	4.32	.096286	18
43	.796206	2.63	.892233	1.68	.903973	4.32	.096027	17
44	.796364	2.62	.892132	1.70	.904232	4.32	.095768	16
45	.796521	2.63	.892030	1.68	.904491	4.32	.095509	15
46	.796679	2.62	.891929	1.70	.904750	4.30	.095250	14
47	.796836	2.62	.891827	1.68	.905008	4.32	.094992	13
48	.796993	2.62	.891726	1.70	.905267	4.32	.094733	12
49	.797150	2.62	.891624	1.68	.905526	4.32	.094474	11
50	.797307	2.62	.891523	1.70	.905785	4.30	.094215	10
51	9.797464	2.62	9.891421	1.70	9.906043	4.32	10.093957	9
52	.797621	2.60	.891319	1.70	.906302	4.30	.093698	8
53	.797777	2.62	.891217	1.70	.906560	4.32	.093440	7
54	.797934	2.62	.891115	1.70	.906819	4.30	.093181	6
55	.798091	2.60	.891013	1.70	.907077	4.32	.092923	5
56	.798247	2.60	.890911	1.70	.907336	4.30	.092664	4
57	.798403	2.62	.890809	1.70	.907594	4.22	.092406	3
58	.798560	2.60	.890707	1.70	.907853	4.30	.092147	2
59	.798716	2.60	.890605	1.70	.908111	4.30	.091889	1
60′	9.798872		9.890503		9.908369		10.091631	0′
128°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	51°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

39°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	140°
0′	9.798872	2.60	9.890508	1.72	9.908369	4.32	10.091631	60′
1	.799028	2.60	.890400	1.70	.908628	4.30	.091372	59
2	.799184	2.58	.890298	1.72	.908886	4.30	.091114	58
3	.799339	2.60	.890195	1.70	.909144	4.30	.090856	57
4	.799495	2.60	.890093	1.72	.909402	4.30	.090598	56
5	.799651	2.58	.889990	1.70	.909660	4.30	.090340	55
6	.799806	2.60	.889888	1.72	.909918	4.32	.090082	54
7	.799962	2.58	.889785	1.72	.910177	4.30	.089823	53
8	.800117	2.58	.889682	1.72	.910435	4.30	.089565	52
9	.800272	2.58	.889579	1.70	.910693	4.30	.089307	51
10	.800427	2.58	.889477	1.72	.910951	4.30	.089049	50
11	9.800582	2.58	9.889374	1.72	9.911209	4.30	10.088791	49
12	.800737	2.58	.889271	1.72	.911467	4.30	.088533	48
13	.800892	2.58	.889168	1.73	.911725	4.28	.088275	47
14	.801047	2.57	.889064	1.72	.911982	4.30	.088018	46
15	.801201	2.58	.888961	1.72	.912240	4.30	.087760	45
16	.801356	2.58	.888858	1.72	.912498	4.30	.087502	44
17	.801511	2.57	.888755	1.73	.912756	4.30	.087244	43
18	.801665	2.57	.888651	1.72	.913014	4.28	.086986	42
19	.801819	2.57	.888548	1.73	.913271	4.30	.086729	41
20	.801973	2.58	.888444	1.72	.913529	4.30	.086471	40
21	9.802128	2.57	9.888341	1.73	9.913787	4.28	10.086213	39
22	.802282	2.57	.888237	1.72	.914044	4.30	.085956	38
23	.802436	2.55	.888134	1.73	.914302	4.30	.085698	37
24	.802589	2.57	.888030	1.73	.914560	4.28	.085440	36
25	.802743	2.57	.887926	1.73	.914817	4.30	.085183	35
26	.802897	2.55	.887822	1.73	.915075	4.28	.084925	34
27	.803050	2.57	.887718	1.73	.915332	4.30	.084668	33
28	.803204	2.55	.887614	1.73	.915590	4.28	.084410	32
29	.803357	2.57	.887510	1.73	.915847	4.28	.084153	31
30	.803511	2.55	.887406	1.73	.916104	4.30	.083896	30
31	9.803664	2.55	9.887302	1.73	9.916362	4.28	10.083638	29
32	.803817	2.55	.887198	1.75	.916619	4.30	.083381	28
33	.803970	2.55	.887093	1.73	.916877	4.28	.083123	27
34	.804123	2.55	.886989	1.73	.917134	4.28	.082866	26
35	.804276	2.55	.886885	1.75	.917391	4.28	.082609	25
36	.804428	2.55	.886780	1.73	.917648	4.30	.082352	24
37	.804581	2.55	.886676	1.75	.917906	4.28	.082094	23
38	.804734	2.53	.886571	1.75	.918163	4.28	.081837	22
39	.804886	2.55	.886466	1.73	.918420	4.28	.081580	21
40	.805039	2.53	.886362	1.75	.918677	4.28	.081323	20
41	9.805191	2.53	9.886257	1.75	9.918934	4.28	10.081066	19
42	.805343	2.53	.886152	1.75	.919191	4.28	.080809	18
43	.805495	2.53	.886047	1.75	.919448	4.28	.080552	17
44	.805647	2.53	.885942	1.75	.919705	4.28	.080295	16
45	.805799	2.53	.885837	1.75	.919962	4.28	.080038	15
46	.805951	2.53	.885732	1.75	.920219	4.28	.079781	14
47	.806103	2.52	.885627	1.75	.920476	4.28	.079524	13
48	.806254	2.53	.885522	1.77	.920733	4.28	.079267	12
49	.806406	2.52	.885416	1.75	.920990	4.28	.079010	11
50	.806557	2.53	.885311	1.77	.921247	4.27	.078753	10
51	9.806709	2.52	9.885205	1.75	9.921503	4.28	10.078497	9
52	.806860	2.52	.885100	1.77	.921760	4.28	.078240	8
53	.807011	2.53	.884994	1.75	.922017	4.28	.077983	7
54	.807163	2.52	.884889	1.77	.922274	4.27	.077726	6
55	.807314	2.52	.884783	1.77	.922530	4.28	.077470	5
56	.807465	2.50	.884677	1.75	.922787	4.28	.077213	4
57	.807615	2.52	.884572	1.77	.923044	4.27	.076956	3
58	.807766	2.52	.884466	1.77	.923300	4.28	.076700	2
59	.807917	2.50	.884360	1.77	.923557	4.28	.076443	1
60′	9.808067		9.884254		9.923814		10.076186	0′
129°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	50°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

40°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	139°
0'	9.808067	2.52	9.884254	1.77	9.923814	4.27	10.076186	60'
1	.808218	2.50	.884148	1.77	.924070	4.28	.075930	59
2	.808368	2.52	.884042	1.77	.924327	4.27	.075673	58
3	.808519	2.50	.883936	1.78	.924583	4.28	.075417	57
4	.808669	2.50	.883829	1.77	.924840	4.27	.075160	56
5	.808819	2.50	.883723	1.77	.925096	4.27	.074904	55
6	.808969	2.50	.883617	1.78	.925352	4.28	.074648	54
7	.809119	2.50	.883510	1.77	.925609	4.27	.074391	53
8	.809269	2.50	.883404	1.78	.925865	4.28	.074135	52
9	.809419	2.50	.883297	1.77	.926122	4.27	.073878	51
10	.809569	2.48	.883191	1.78	.926378	4.27	.073622	50
11	9.809718	2.50	9.883084	1.78	9.926634	4.27	10.073366	49
12	.809868	2.48	.882977	1.77	.926890	4.28	.073110	48
13	.810017	2.50	.882871	1.78	.927147	4.27	.072853	47
14	.810167	2.48	.882764	1.78	.927403	4.27	.072597	46
15	.810316	2.48	.882657	1.78	.927659	4.27	.072341	45
16	.810465	2.48	.882550	1.78	.927915	4.27	.072085	44
17	.810614	2.48	.882443	1.78	.928171	4.27	.071829	43
18	.810763	2.48	.882336	1.78	.928427	4.28	.071573	42
19	.810912	2.48	.882229	1.80	.928684	4.27	.071316	41
20	.811061	2.48	.882121	1.78	.928940	4.27	.071060	40
21	9.811210	2.47	9.882014	1.78	9.929196	4.27	10.070804	39
22	.811358	2.48	.881907	1.80	.929452	4.27	.070548	38
23	.811507	2.47	.881799	1.78	.929708	4.27	.070292	37
24	.811655	2.48	.881692	1.80	.929964	4.27	.070036	36
25	.811804	2.47	.881584	1.78	.930220	4.25	.069780	35
26	.811952	2.47	.881477	1.80	.930475	4.27	.069525	34
27	.812100	2.47	.881369	1.80	.930731	4.27	.069269	33
28	.812248	2.47	.881261	1.80	.930987	4.27	.069013	32
29	.812396	2.47	.881153	1.78	.931243	4.27	.068757	31
30	.812544	2.47	.881046	1.80	.931499	4.27	.068501	30
31	9.812692	2.47	9.880938	1.80	9.931755	4.25	10.068245	29
32	.812840	2.47	.880830	1.80	.932010	4.27	.067990	28
33	.812988	2.45	.880722	1.82	.932266	4.27	.067734	27
34	.813135	2.47	.880613	1.80	.932522	4.27	.067478	26
35	.813283	2.45	.880505	1.80	.932778	4.25	.067222	25
36	.813430	2.47	.880397	1.80	.933033	4.27	.066967	24
37	.813578	2.45	.880289	1.82	.933289	4.27	.066711	23
38	.813725	2.45	.880180	1.80	.933545	4.25	.066455	22
39	.813872	2.45	.880072	1.82	.933800	4.27	.066200	21
40	.814019	2.45	.879963	1.80	.934056	4.25	.065944	20
41	9.814166	2.45	9.879855	1.82	9.934311	4.27	10.065689	19
42	.814313	2.45	.879746	1.82	.934567	4.25	.065433	18
43	.814460	2.45	.879637	1.80	.934822	4.27	.065178	17
44	.814607	2.43	.879529	1.82	.935078	4.25	.064922	16
45	.814753	2.45	.879420	1.82	.935333	4.27	.064667	15
46	.814900	2.43	.879311	1.82	.935589	4.25	.064411	14
47	.815046	2.45	.879202	1.82	.935844	4.27	.064156	13
48	.815193	2.43	.879093	1.82	.936100	4.25	.063900	12
49	.815339	2.43	.878984	1.82	.936355	4.27	.063645	11
50	.815485	2.45	.878875	1.82	.936611	4.25	.063389	10
51	9.815632	2.43	9.878766	1.83	9.936866	4.25	10.063134	9
52	.815778	2.43	.878656	1.82	.937121	4.27	.062879	8
53	.815924	2.42	.878547	1.82	.937377	4.25	.062623	7
54	.816069	2.43	.878438	1.83	.937632	4.25	.062368	6
55	.816215	2.43	.878328	1.82	.937887	4.25	.062113	5
56	.816361	2.43	.878219	1.83	.938142	4.27	.061858	4
57	.816507	2.42	.878109	1.83	.938398	4.25	.061602	3
58	.816652	2.43	.877999	1.82	.938653	4.25	.061347	2
59	.816798	2.42	.877890	1.83	.938908	4.25	.061092	1
60	9.816943		9.877780		9.939163		10.060837	0'
130°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	49°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

41°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	138°
0	9.816943	2.42	9.877780	1.83	9.939163	4.25	10.060837	60
1	.817088	2.42	.877670	1.83	.939418	4.25	.060582	59
2	.817233	2.43	.877560	1.83	.939673	4.25	.060327	58
3	.817379	2.42	.877450	1.83	.939928	4.25	.060072	57
4	.817524	2.40	.877340	1.83	.940183	4.27	.059817	56
5	.817668	2.42	.877230	1.83	.940439	4.25	.059561	55
6	.817813	2.42	.877120	1.83	.940694	4.25	.059306	54
7	.817958	2.42	.877010	1.85	.940949	4.25	.059051	53
8	.818103	2.40	.876899	1.83	.941204	4.25	.058796	52
9	.818247	2.42	.876789	1.85	.941459	4.23	.058541	51
10	.818392	2.40	.876678	1.83	.941713	4.25	.058287	50
11	9.818536	2.42	9.876568	1.85	9.941968	4.25	10.058032	49
12	.818681	2.40	.876457	1.83	.942223	4.25	.057777	48
13	.818825	2.40	.876347	1.85	.942478	4.25	.057522	47
14	.818969	2.40	.876236	1.85	.942733	4.25	.057267	46
15	.819113	2.40	.876125	1.85	.942988	4.25	.057012	45
16	.819257	2.40	.876014	1.83	.943243	4.25	.056757	44
17	.819401	2.40	.875904	1.85	.943498	4.23	.056502	43
18	.819545	2.40	.875793	1.85	.943752	4.25	.056248	42
19	.819689	2.38	.875682	1.85	.944007	4.25	.055993	41
20	.819832	2.40	.875571	1.87	.944262	4.25	.055738	40
21	9.819976	2.40	9.875459	1.85	9.944517	4.23	10.055483	39
22	.820120	2.38	.875348	1.85	.944771	4.25	.055229	38
23	.820263	2.38	.875237	1.85	.945026	4.25	.054974	37
24	.820406	2.40	.875126	1.87	.945281	4.23	.054719	36
25	.820550	2.38	.875014	1.85	.945535	4.25	.054465	35
26	.820693	2.38	.874903	1.87	.945790	4.25	.054210	34
27	.820836	2.38	.874791	1.85	.946045	4.23	.053955	33
28	.820979	2.38	.874680	1.87	.946299	4.25	.053701	32
29	.821122	2.38	.874568	1.87	.946554	4.23	.053446	31
30	.821265	2.37	.874456	1.87	.946808	4.25	.053192	30
31	9.821407	2.38	9.874344	1.87	9.947063	4.25	10.052937	29
32	.821550	2.38	.874232	1.85	.947318	4.23	.052682	28
33	.821693	2.37	.874121	1.87	.947572	4.25	.052428	27
34	.821835	2.37	.874009	1.88	.947827	4.23	.052173	26
35	.821977	2.38	.873896	1.87	.948081	4.23	.051919	25
36	.822120	2.37	.873784	1.87	.948335	4.25	.051665	24
37	.822262	2.37	.873672	1.87	.948590	4.23	.051410	23
38	.822404	2.37	.873560	1.87	.948844	4.25	.051156	22
39	.822546	2.37	.873448	1.88	.949099	4.23	.050901	21
40	.822688	2.37	.873335	1.87	.949353	4.25	.050647	20
41	9.822830	2.37	9.873223	1.88	9.949608	4.23	10.050392	19
42	.822972	2.37	.873110	1.87	.949862	4.23	.050138	18
43	.823114	2.35	.872998	1.88	.950116	4.25	.049884	17
44	.823255	2.37	.872885	1.88	.950371	4.23	.049629	16
45	.823397	2.37	.872772	1.88	.950625	4.23	.049375	15
46	.823539	2.35	.872659	1.87	.950879	4.23	.049121	14
47	.823680	2.35	.872547	1.88	.951133	4.25	.048867	13
48	.823821	2.37	.872434	1.88	.951388	4.23	.048612	12
49	.823963	2.35	.872321	1.88	.951642	4.23	.048358	11
50	.824104	2.35	.872208	1.88	.951896	4.23	.048104	10
51	9.824245	2.35	9.872095	1.90	9.952150	4.25	10.047850	9
52	.824386	2.35	.871981	1.88	.952405	4.23	.047595	8
53	.824527	2.35	.871868	1.88	.952659	4.23	.047341	7
54	.824668	2.33	.871755	1.90	.952913	4.23	.047087	6
55	.824808	2.35	.871641	1.88	.953167	4.23	.046833	5
56	.824949	2.35	.871528	1.90	.953421	4.23	.046579	4
57	.825090	2.33	.871414	1.88	.953675	4.23	.046325	3
58	.825230	2.35	.871301	1.90	.953929	4.23	.046071	2
59	.825371	2.33	.871187	1.90	.954183	4.23	.045817	1
60′	9.825511		9.871073		9.954437		10.045563	0′
131°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	48°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

42°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	137°
0′	9.825511	2.33	9.871073	1.88	9.954437	4.23	10.045563	60′
1	.825651	2.33	.870960	1.90	.954691	4.25	.045309	59
2	.825791	2.33	.870846	1.90	.954946	4.23	.045054	58
3	.825931	2.33	.870732	1.90	.955200	4.23	.044800	57
4	.826071	2.33	.870618	1.90	.955454	4.23	.044546	56
5	.826211	2.33	.870504	1.90	.955708	4.22	.044292	55
6	.826351	2.33	.870390	1.90	.955961	4.23	.044039	54
7	.826491	2.33	.870276	1.92	.956215	4.23	.043785	53
8	.826631	2.32	.870161	1.90	.956469	4.23	.043531	52
9	.826770	2.33	.870047	1.90	.956723	4.23	.043277	51
10	.826910	2.32	.869933	1.92	.956977	4.23	.043023	50
11	9.827049	2.33	9.869818	1.90	9.957231	4.23	10.042769	49
12	.827189	2.32	.869704	1.92	.957485	4.23	.042515	48
13	.827328	2.32	.869589	1.92	.957739	4.23	.042261	47
14	.827467	2.32	.869474	1.90	.957993	4.23	.042007	46
15	.827606	2.32	.869360	1.92	.958247	4.22	.041753	45
16	.827745	2.32	.869245	1.92	.958500	4.23	.041500	44
17	.827884	2.32	.869130	1.92	.958754	4.23	.041246	43
18	.828023	2.32	.869015	1.92	.959008	4.23	.040992	42
19	.828162	2.32	.868900	1.92	.959262	4.23	.040738	41
20	.828301	2.30	.868785	1.92	.959516	4.22	.040484	40
21	9.828439	2.32	9.868670	1.92	9.959769	4.23	10.040231	39
22	.828578	2.30	.868555	1.92	.960023	4.23	.039977	38
23	.828716	2.32	.868440	1.93	.960277	4.22	.039723	37
24	.828855	2.30	.868324	1.92	.960530	4.23	.039470	36
25	.828993	2.30	.868209	1.93	.960784	4.23	.039216	35
26	.829131	2.30	.868093	1.92	.961038	4.23	.038962	34
27	.829269	2.30	.867978	1.93	.961292	4.22	.038708	33
28	.829407	2.30	.867862	1.92	.961545	4.23	.038455	32
29	.829545	2.30	.867747	1.93	.961799	4.22	.038201	31
30	.829683	2.30	.867631	1.93	.962052	4.23	.037948	30
31	9.829821	2.30	9.867515	1.93	9.962306	4.23	10.037694	29
32	.829959	2.30	.867399	1.93	.962560	4.22	.037440	28
33	.830097	2.28	.867283	1.93	.962813	4.23	.037187	27
34	.830234	2.30	.867167	1.93	.963067	4.22	.036933	26
35	.830372	2.28	.867051	1.93	.963320	4.23	.036680	25
36	.830509	2.28	.866935	1.93	.963574	4.23	.036426	24
37	.830646	2.30	.866819	1.93	.963828	4.22	.036172	23
38	.830784	2.28	.866703	1.95	.964081	4.23	.035919	22
39	.830921	2.28	.866586	1.93	.964335	4.22	.035665	21
40	.831058	2.28	.866470	1.95	.964588	4.23	.035412	20
41	9.831195	2.28	9.866353	1.93	9.964842	4.22	10.035158	19
42	.831332	2.28	.866237	1.95	.965095	4.23	.034905	18
43	.831469	2.28	.866120	1.93	.965349	4.23	.034651	17
44	.831606	2.27	.866004	1.95	.965602	4.22	.034398	16
45	.831742	2.28	.865887	1.95	.965855	4.23	.034145	15
46	.831879	2.27	.865770	1.95	.966109	4.22	.033891	14
47	.832015	2.28	.865653	1.95	.966362	4.23	.033638	13
48	.832152	2.27	.865536	1.95	.966616	4.22	.033384	12
49	.832288	2.28	.865419	1.95	.966869	4.23	.033131	11
50	.832425	2.27	.865302	1.95	.967123	4.22	.032877	10
51	9.832561	2.27	9.865185	1.95	9.967376	4.22	10.032624	9
52	.832697	2.27	.865068	1.97	.967629	4.23	.032371	8
53	.832833	2.27	.864950	1.95	.967883	4.22	.032117	7
54	.832969	2.27	.864833	1.95	.968136	4.22	.031864	6
55	.833105	2.27	.864716	1.97	.968389	4.23	.031611	5
56	.833241	2.27	.864598	1.95	.968643	4.22	.031357	4
57	.833377	2.25	.864481	1.97	.968896	4.22	.031104	3
58	.833512	2.27	.864363	1.97	.969149	4.23	.030851	2
59	.833648	2.25	.864245	1.97	.969403	4.23	.030597	1
60′	9.833783		9.864127		9.969656		10.030344	0′
132°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	47°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

43°	Sine.	D. 1″.	Cosine.	D. 1″.	Tang.	D. 1″.	Cotang.	136°
0′	9.833783	2.27	9.864127	1.95	9.969656	4.22	10.030344	60′
1	.833919	2.25	.864010	1.97	.969909	4.22	.030091	59
2	.834054	2.25	.863892	1.97	.970162	4.23	.029838	58
3	.834189	2.27	.863774	1.97	.970416	4.22	.029584	57
4	.834325	2.25	.863656	1.97	.970669	4.22	.029331	56
5	.834460	2.25	.863538	1.98	.970922	4.22	.029078	55
6	.834595	2.25	.863419	1.97	.971175	4.23	.028825	54
7	.834730	2.25	.863301	1.97	.971429	4.22	.028571	53
8	.834865	2.23	.863183	1.98	.971682	4.22	.028318	52
9	.834999	2.25	.863064	1.97	.971935	4.22	.028065	51
10	.835134	2.25	.862946	1.98	.972188	4.22	.027812	50
11	9.835269	2.23	9.862827	1.97	9.972441	4.23	10.027559	49
12	.835403	2.25	.862709	1.98	.972695	4.22	.027305	48
13	.835538	2.23	.862590	1.98	.972948	4.22	.027052	47
14	.835672	2.25	.862471	1.97	.973201	4.22	.026799	46
15	.835807	2.23	.862353	1.98	.973454	4.22	.026546	45
16	.835941	2.23	.862234	1.98	.973707	4.22	.026293	44
17	.836075	2.23	.862115	1.98	.973960	4.22	.026040	43
18	.836209	2.23	.861996	1.98	.974213	4.22	.025787	42
19	.836343	2.23	.861877	1.98	.974466	4.23	.025534	41
20	.836477	2.23	.861758	2.00	.974720	4.22	.025280	40
21	9.836611	2.23	9.861638	1.98	9.974973	4.22	10.025027	39
22	.836745	2.22	.861519	1.98	.975226	4.22	.024774	38
23	.836878	2.23	.861400	2.00	.975479	4.22	.024521	37
24	.837012	2.23	.861280	1.98	.975732	4.22	.024268	36
25	.837146	2.22	.861161	2.00	.975985	4.22	.024015	35
26	.837279	2.22	.861041	1.98	.976238	4.22	.023762	34
27	.837412	2.23	.860922	2.00	.976491	4.22	.023509	33
28	.837546	2.22	.860802	2.00	.976744	4.22	.023256	32
29	.837679	2.22	.860682	2.00	.976997	4.22	.023003	31
30	.837812	2.22	.860562	2.00	.977250	4.22	.022750	30
31	9.837945	2.22	9.860442	2.00	9.977503	4.22	10.022497	29
32	.838078	2.22	.860322	2.00	.977756	4.22	.022244	28
33	.838211	2.22	.860202	2.00	.978009	4.22	.021991	27
34	.838344	2.22	.860082	2.00	.978262	4.22	.021738	26
35	.838477	2.22	.859962	2.00	.978515	4.22	.021485	25
36	.838610	2.20	.859842	2.02	.978768	4.22	.021232	24
37	.838742	2.22	.859721	2.00	.979021	4.22	.020979	23
38	.838875	2.20	.859601	2.02	.979274	4.22	.020726	22
39	.839007	2.22	.859480	2.00	.979527	4.22	.020473	21
40	.839140	2.20	.859360	2.02	.979780	4.22	.020220	20
41	9.839272	2.20	9.859239	2.00	9.980033	4.22	10.019967	19
42	.839404	2.20	.859119	2.02	.980286	4.20	.019714	18
43	.839536	2.20	.858998	2.02	.980538	4.22	.019462	17
44	.839668	2.20	.858877	2.02	.980791	4.22	.019209	16
45	.839800	2.20	.858756	2.02	.981044	4.22	.018956	15
46	.839932	2.20	.858635	2.02	.981297	4.22	.018703	14
47	.840064	2.20	.858514	2.02	.981550	4.22	.018450	13
48	.840196	2.20	.858393	2.02	.981803	4.22	.018197	12
49	.840328	2.18	.858272	2.02	.982056	4.22	.017944	11
50	.840459	2.20	.858151	2.03	.982309	4.22	.017691	10
51	9.840591	2.18	9.858029	2.02	9.982562	4.20	10.017438	9
52	.840722	2.20	.857908	2.03	.982814	4.22	.017186	8
53	.840854	2.18	.857786	2.02	.983067	4.22	.016933	7
54	.840985	2.18	.857665	2.03	.983320	4.23	.016680	6
55	.841116	2.18	.857543	2.02	.983573	4.22	.016427	5
56	.841247	2.18	.857422	2.03	.983826	4.22	.016174	4
57	.841378	2.18	.857300	2.03	.984079	4.22	.015921	3
58	.841509	2.18	.857178	2.03	.984332	4.20	.015668	2
59	.841640	2.18	.857056	2.03	.984584	4.22	.015416	1
60′	9.841771		9.856934		9.984837		10.015163	0′
133°	Cosine.	D. 1″.	Sine.	D. 1″.	Cotang.	D. 1″.	Tang.	46°

TABLE XXIII (*Continued*)

Logarithmic Sines, Cosines, Tangents, and Cotangents

44°	Sine.	D. 1".	Cosine.	D. 1".	Tang.	D. 1".	Cotang.	135°
0'	9.841771	2.18	9.856934	2.03	9.984837	4.22	10.015163	60
1	.841902	2.18	.856812	2.03	.985090	4.22	.014910	59
2	.842033	2.17	.856690	2.03	.985343	4.22	.014657	58
3	.842163	2.18	.856568	2.03	.985596	4.20	.014404	57
4	.842294	2.17	.856446	2.05	.985848	4.22	.014152	56
5	.842424	2.18	.856323	2.03	.986101	4.22	.013899	55
6	.842555	2.17	.856201	2.05	.986354	4.22	.013646	54
7	.842685	2.17	.856078	2.03	.986607	4.22	.013393	53
8	.842815	2.18	.855956	2.05	.986860	4.20	.013140	52
9	.842946	2.17	.855833	2.03	.987112	4.22	.012888	51
10	.843076	2.17	.855711	2.05	.987365	4.22	.012635	50
11	9.843206	2.17	9.855588	2.05	9.987618	4.22	10.012382	49
12	.843336	2.17	.855465	2.05	.987871	4.20	.012129	48
13	.843466	2.15	.855342	2.05	.988123	4.22	.011877	47
14	.843595	2.17	.855219	2.05	.988376	4.22	.011624	46
15	.843725	2.17	.855096	2.05	.988629	4.22	.011371	45
16	.843855	2.15	.854973	2.05	.988882	4.20	.011118	44
17	.843984	2.17	.854850	2.05	.989134	4.22	.010866	43
18	.844114	2.15	.854727	2.07	.989387	4.22	.010613	42
19	.844243	2.15	.854603	2.05	.989640	4.22	.010360	41
20	.844372	2.17	.854480	2.07	.989893	4.20	.010107	40
21	9.844502	2.15	9.854356	2.05	9.990145	4.22	10.009855	39
22	.844631	2.15	.854233	2.07	.990398	4.22	.009602	38
23	.844760	2.15	.854109	2.05	.990651	4.20	.009349	37
24	.844889	2.15	.853986	2.07	.990903	4.22	.009097	36
25	.845018	2.15	.853862	2.07	.991156	4.22	.008844	35
26	.845147	2.15	.853738	2.07	.991409	4.22	.008591	34
27	.845276	2.15	.853614	2.07	.991662	4.20	.008338	33
28	.845405	2.13	.853490	2.07	.991914	4.22	.008086	32
29	.845533	2.15	.853366	2.07	.992167	4.22	.007833	31
30	.845662	2.13	.853242	2.07	.992420	4.20	.007580	30
31	9.845790	2.15	9.853118	2.07	9.992672	4.22	10.007328	29
32	.845919	2.13	.852994	2.08	.992025	4.22	.007075	28
33	.846047	2.13	.852869	2.07	.993178	4.22	.006822	27
34	.846175	2.15	.852745	2.08	.993431	4.20	.006569	26
35	.846304	2.13	.852620	2.07	.993683	4.22	.006317	25
36	.846432	2.13	.852496	2.08	.993936	4.22	.006064	24
37	.846560	2.13	.852371	2.07	.994189	4.20	.005811	23
38	.846688	2.13	.852247	2.08	.994441	4.22	.005559	22
39	.846816	2.13	.852122	2.08	.994694	4.22	.005306	21
40	.846944	2.12	.851997	2.08	.994947	4.20	.005053	20
41	9.847071	2.13	9.851872	2.08	9.995199	4.22	10.004801	19
42	.847199	2.13	.851747	2.08	.995452	4.22	.004548	18
43	.847327	2.12	.851622	2.08	.995705	4.20	.004295	17
44	.847454	2.13	.851497	2.08	.995957	4.22	.004043	16
45	.847582	2.12	.851372	2.10	.996210	4.22	.003790	15
46	.847709	2.12	.851246	2.08	.996463	4.20	.003537	14
47	.847836	2.13	.851121	2.08	.996715	4.22	.003285	13
48	.847964	2.12	.850996	2.10	.996968	4.22	.003032	12
49	.848091	2.12	.850870	2.08	.997221	4.20	.002779	11
50	.848218	2.12	.850745	2.10	.997473	4.22	.002527	10
51	9.848345	2.12	9.850619	2.10	9.997726	4.22	10.002274	9
52	.848472	2.12	.850493	2.08	.997979	4.20	.002021	8
53	.848599	2.12	.850368	2.10	.998231	4.22	.001769	7
54	.848726	2.10	.850242	2.10	.998484	4.22	.001516	6
55	.848852	2.12	.850116	2.10	.998737	4.20	.001263	5
56	.848979	2.12	.849990	2.10	.998989	4.22	.001011	4
57	.849106	2.10	.849864	2.10	.999242	4.22	.000758	3
58	.849232	2.12	.849738	2.12	.999495	4.20	.000505	2
59	.849359	2.10	.849611	2.10	.999747	4.22	.000253	1
60'	9.849485		9 849485		10.000000		10.000000	0'
134°	Cosine.	D. 1".	Sine.	D. 1".	Cotang.	D. 1".	Tang.	45°

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